Homological algebra, Homework 6

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Koszul complex

1. Koszul complex. For a vector space V over a field $\Bbbk$ consider the sequence of maps

$$\cdots \xrightarrow{d^{-k-1}} \overset{k}{\wedge} V \otimes S(V) \xrightarrow{d^{-k}} \cdots \xrightarrow{d^{-3}} \overset{2}{\wedge} V \otimes S(V) \xrightarrow{d^{-2}} \overset{1}{\wedge} V \otimes S(V) \xrightarrow{d^{-1}} \overset{0}{\wedge} V \otimes S(V) \to 0 \to \cdots$$

where

$$d^{-k}(v_1 \wedge \cdots \wedge v_k \otimes f) = \sum_1^k (-1)^{k-i} v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k \otimes v_i f.$$

Here, $v_i \in V$, $f \in S(V)$ and $\widehat{\ }$ means that we are omitting this term.

(a) Show that the maps d^{-k} above are well defined.

(b) Show that this is a complex.

(c) Show that if $\dim(V) = n < \infty$ the complex is finite and of the form

$$\cdots \to 0 \to \overset{n}{\wedge} V \otimes S(V) \xrightarrow{d^{-n}} \cdots \xrightarrow{d^{-3}} \overset{-2}{\wedge} V \otimes S(V) \xrightarrow{d^{-2}} V \otimes S(V) \xrightarrow{d^{-1}} V \otimes S(V) \to 0 \to \cdots.$$

(d) Consider the Koszul complex for $V = \mathbb{C}^2$. Where have you used it before?

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Sheafification

If, when playing with sheaves, we get lost and find ourselves in a larger world of *presheaves* (and these are less interesting objects), we need to find our way home. This is the main technical step[(1)] in making sheaves useful.

By a presheaf we mean the same structure as a sheaf, except that we do not require the *gluing* property. For instance while locally constant functions are a sheaf, constant functions are just a presheaf. Presheaves are by themselves not so interesting because lack of gluing means that they do not relate local and global information well. However, presheaves are not avoidable, for instance we will see that applying some basic constructions to sheaves results in presheaves.

Sheafification is a way to improve any presheaf of sets $\mathcal{S}$ into a sheaf of sets $\widetilde{\mathcal{S}}$. We will obtain the sections of the sheaf $\tilde{\mathcal{S}}$ associated to a presheaf $\mathcal{S}$ in two steps:[2]

(1) *add more sections* by gluing systems of local sections s_i which are compatible in the sense that they are locally the same, and
(2) *cut down on sections* by identifying two results of such gluing procedures when the local sections in the two families are locally the same.

[1]Also, the most painful.
[2]You can think that we are imitating the passage from constant functions to locally constant functions.

In the first step for each open $U \subseteq X$ we replace $\mathcal{S}(U)$ by a larger set $\hat{\mathcal{S}}(U)$, and in the second step we decrease this to $\widetilde{\mathcal{S}}(U)$, a quotient of $\widehat{\mathcal{S}}(U)$ by an equivalence relation $\equiv$. Here are the definitions:

(1) For any presheaf $\mathcal{A}$ we say that sections $a \in \mathcal{A}(U),\ b \in \mathcal{A}(V)$ are locally the same if for any point $x \in U \cap V$ they are the same near x, i.e.,

There is neighborhood W of x in $U \cap V$ such that $a|W = b|W$.

(Here $a|W$ denotes the restriction $\rho^U_W a$.) We say that a family of sections $a_i \in \mathcal{A}(U_i),\ i \in I$, is locally the same if any two sections in the family are locally the same.

(2) $\widehat{\mathcal{S}}(U)$ consists of all families $(U_i, s_i)_{i \in I}$ such that
 (a) $(U_i)_{i \in I}$ is an open cover of U and $s_i \in \mathcal{S}(U_i)$ is a section of $\mathcal{S}$ on U_i,
 (b) sections $s_i, i \in I$, are locally the same.

(3) We say that two systems $(U_i, s_i)_{i \in I}$ and $(V_j, t_j)_{j \in J}$ are $\equiv$ if for any $i \in I,\ j \in J$, the sections s_i and t_j are locally the same.

2. Prove that

- (a) The relation $\equiv$ on $\widehat{\mathcal{S}}(U)$ really says that $(U_i, s_i)_{i \in I} \equiv (V_j, t_j)_{j \in J}$ iff

  the disjoint union $(U_i, s_i)_{i \in I} \sqcup (V_j, t_j)_{j \in J}$ is again in $\widehat{\mathcal{S}}(U)$.
- (b) $\equiv$ is an equivalence relation on $\widehat{\mathcal{S}}(U)$.
- (c) $\widetilde{\mathcal{S}}(U)$ is a presheaf.

3. Prove that $\widetilde{\mathcal{S}}$ is a sheaf.

0.0.1. *Remark.* The same construction has a categorical formulation in the sense that it works for presheaves $\mathcal{S}$ on X with values in in any category $\mathcal{C}$ that has all necessary limits.[3]

• This homework is finally "real" stuff. The problems 2 and 3 together are an elementary but thoughtful exercise in topology which is absolutely essential for dealing with sheaves. The other problems are more routine (since a lot of hints are supplied). • Before you start working on a problem read any comments placed after the problem.

[3] Here is a sketch that omits one essential element ("locally the same"), You may want to correct it at some point:

For each open $U \subseteq X$ we have the indexing category $\mathcal{I} = \mathcal{OC}_U$ of open covers of U. Here morphisms are refinements, i.e., a map of covers $(U_i)_{i \in I} \xrightarrow{\iota} (U_j)_{j \in J}$ is a map $\iota : J \to I$ such that for any $j \in J$ $V_j \subseteq U_{\iota(j)}$. If $\mathcal{C}$ has products then $\mathcal{S}$ defines a functor $\mathcal{S}_0(U) : \mathcal{OC}_U \to \mathcal{C}$ by $\mathcal{S}_0(U_i,\ i \in I) \stackrel{\text{def}}{=} \prod_{i \in I} \mathcal{S}(U_i)$. If $\mathcal{C}$ has inductive limits then we have $\widetilde{\mathcal{S}}(U) \stackrel{\text{def}}{=} \lim_{\to\ \mathcal{OC}_U} \widetilde{\mathcal{S}}(U_i,\ i \in I)$. Then $\widetilde{\mathcal{S}}$ is a sheaf.

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Group cohomology II

Recall that $P^{-n} \stackrel{\text{def}}{=} \mathbb{Z}[G^{n+1}]$ form a free resolution $P^\bullet$ of the trivial G-module $\mathbb{Z}$

$$\cdots \to P^{-n} \xrightarrow{d^{-n}} P^{-(n-1)} \to \cdots \to P^0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \quad \text{with}$$

$$d^{-n}(g_0, ..., g_n) \stackrel{\text{def}}{=} \sum_{i=0}^{n} (-1)^i \ (g_0, ..., \widehat{g_i}, ..., g_n) \quad \text{and} \quad \varepsilon(g_0) \stackrel{\text{def}}{=} 1, \ g_0 \in G.$$

4. The standard complex of a G-module. We will write it in three equivalent forms. Show that for any G-module M :

(1) The structure of a complex of G-modules on $P^\bullet$ gives a canonical structure of a complex of abelian groups on $\mathfrak{C}^\bullet(G, M) \stackrel{\text{def}}{=} \text{Hom}_G(P^\bullet, M)$ with

$$\cdots \to 0 \to \mathfrak{C}^0(G, M) \xrightarrow{\partial^0_{\mathfrak{C}}} \mathfrak{C}^1(G, M) \to \cdots \to \mathfrak{C}^n(G, M) \xrightarrow{\partial^n_{\mathfrak{C}}} \mathfrak{C}^{n+1}(G, M) \to \cdots.$$

Here, $\partial^n_{\mathfrak{C}}$ comes from $P^{-(n+1)} \xrightarrow{d^{-(n+1)}} P^{-n}$ as the adjoint $\text{Hom}_G(P^{-n}, M) \xrightarrow{(d^{-(n+1)})^*} \text{Hom}_G(P^{-(n+1)}, M)$. This amounts to, for $\phi \in \text{Hom}_G(P^{-n}, M)$),

$$(\partial^n_{\mathfrak{C}} \phi)(g_0, ..., g_{n+1}) = \sum_{0}^{n+1} (-1)^i \cdot \phi(g_0, ...\widehat{g_i}, ..., g_{n+1}).$$

(2) (**Homogeneous complex.**) There is a natural complex $\mathcal{C}^\bullet(G, M)$ with

$$\mathcal{C}^n(G, M) \stackrel{\text{def}}{=} \ Map_G(G^{n+1}, M) \stackrel{\text{def}}{=} \{\phi : G^{n+1} \to M; \ \phi(gg_0, ..., gg_n) = g \ \phi(g_0, ..., g_n) \ .\}$$

$$\text{and} \quad (\partial^n_{\mathcal{C}} \phi)(g_0, ..., g_{n+1}) = \sum_{0}^{n+1} (-1)^i \cdot \phi(g_0, ..., \widehat{g_i}, ..., g_{n+1}).$$

(3) (**Inhomogeneous complex.**) There is a natural complex $C^\bullet(G, M)$ with $C^n(G, M) \stackrel{\text{def}}{=} \ Map(G^n, M)$ and the differential

$$(\partial^n \phi)(g_1, ..., g_{n+1}) = \ g_1\big(\phi(g_2, ..., g_{n+1})\big) + \sum_{1}^{n} (-1)^i \cdot \phi(g_1, ..., g_i g_{i+1}, ..., g_{n+1})$$

$$+ \ (-1)^{n+1} \cdot \phi(g_1, ..., g_n).$$

(4) Prove that the three complexes above are naturally isomorphic.

0.0.2. *Remarks.* (1) The cohomology $H^\bullet(G,M)$ of any of the three (isomorphic) complexes above is called the group cohomology of G with coefficients in M.

(2) In problem 4, the idea is not at all to construct separately three complexes and then relate them. Rather, $\mathfrak{C}$ is the easiest to construct and it is natural from abstract point of view, while C is the most complicated to construct but the most useful for calculations in low degrees. So, one constructs the easy complex and then finds different points of view on it:

- (i) One obtains $\mathfrak{C}^\bullet$ from $P^\bullet$, (need to check that the formula for $\partial_{\mathfrak{C}}$ is really the adjoint of the d in $P^\bullet$ (then $d \circ d = 0$ will imply $\partial_{\mathfrak{C}} \circ \partial_{\mathfrak{C}} = 0$);
- (ii) One notices that $G^{n+1} \subseteq \mathbb{Z}[G^{n+1}]$ gives a restriction ρ^n of G-module maps from $P^{-n} = \mathbb{Z}[G^{n+1}]$ to M, to G-maps from G^{n+1} to M, which is an isomorphism $\rho^n : \mathfrak{C}^n \xrightarrow{\cong} C^n$ (then one also needs to check that this isomorphism identifies the maps $\partial_{\mathfrak{C}}$ and $\partial_{\mathcal{C}}$, this will show that $\mathcal{C}$ is a complex);
- One checks that the homogenization map $C^\bullet \ni f \mapsto f_H \in \mathcal{C}^\bullet$,
$$f_H(g_0,...,g_n) = \ g_0 f({g_0}^{-1}g_1, {g_1}^{-1}g_2, ..., {g_{n-1}}^{-1}g_n)$$
is an isomorphism of $\mathcal{C}^n$ and C^n and it identifies $\partial_{\mathcal{C}}$ with ∂ (so $C^\bullet$ is also a complex).

5. Interpretations in low degrees. Prove that

(1) $H^0(G,M) = Z^0(G,M) = M^G$ (the G-]invariants in M).

(2) $Z^1(G,M)$ consists of all $f : G \to M$ with
$$f(ab) = \ f(a) + \ {}^a\big(f(b)\big).$$
$B^1(G,M)$ consists of all $\partial m,\ m \in M = C^0(G,M)$, where
$$(\partial m)(a) = \ {}^a m - m.$$
If the action of G on M is trivial, $H^1(G,M) = \mathrm{Hom}(G,M)$.

(3) $Z^2(G,M)$ consists of all $f : G^2 \to M$ with
$$f(a,bc) + \ {}^b\big(f(a,c)\big) \ = \ f(ab,c) + f(a,c),$$
and for $\phi : G \to M$ in $C^1(G,M)$,
$$\partial\phi(a,b) \ = \ {}^a\phi(b) - \phi(ab) + \phi(b).$$

0.0.3. *Remark.* Here I denote the action gm of $g \in G$ on $m \in M$ by ${}^g m$.