Homological algebra, Homework 4

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Multilinear algebra

Let $\Bbbk$ be a commutative algebra. Recall that all rings we consider have a unit. By an *algebra* we always mean an associative algebra.

1. Symmetric algebra of a $\Bbbk$-module. *For any $\Bbbk$-module M, the symmetric algebra $S(M) = S_{\Bbbk}(M)$ of M is defined as the $\Bbbk$-algebra generated by M and the commutativity relations $\mathcal{R} = \{x \otimes y - y \otimes x,\ x, y \in M\}$.* Prove that $S(M)$ is a graded algebra, i.e.,

(1) the ideal $I = < \mathcal{R} >$ in $T(M)$ is homogeneous.
(2) $S(M) = T(M)/I$ is isomorphic to $\oplus_{n \geq 0} S^n(M)$ for $S^n(M) \stackrel{\text{def}}{=} T^n(M)/I^n$,
(3) $S(M)$ is a graded algebra.

2. Universal property of the symmetric algebra. Show that

(1) $S(M)$ is commutative.
(2) For any commutative $\Bbbk$-algebra B, there is a canonical isomorphism

$$\mathrm{Hom}_{\Bbbk-algebras}[S(M), B] \cong \mathrm{Hom}_{\Bbbk-modules}(M, B).$$

3. Basic properties of symmetric algebras. Prove that:

(1) $T^0(M) = S^0(M) = \Bbbk$ and $T^1(M) = S^1(M) = M$.
(2) If M is a free $\Bbbk$-module with a basis $e_i,\ i \in I$, then $S^n(M)$ is a free $\Bbbk$-module with a basis $e^J = \prod_{i \in I} e_i^{J_i}$, indexed by all maps $J : I \to \mathbb{N}$ with the integral n.
(3) The algebra of polynomials $\mathbb{C}[x_1, ..., x_n]$ is isomorphic to the symmetric algebra $S(\mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n)$.

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Homological algebra

4. Long exact sequence of cohomology groups. We say that a sequence of maps of complexes of abelian groups $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact if it is exact in each degree, i.e., for each $n \in \mathbb{Z}$ the sequence $0 \to A^n \xrightarrow{f^n} B \xrightarrow{g^n} C^n \to 0$ is exact.

Prove that

(1) Any short exact sequence of complexes gives maps of abelian groups

$$\mathrm{H}^n(C) \xrightarrow{\partial^n} \mathrm{H}^{n+1}(A) \xrightarrow{\mathrm{H}^{n+1}(f)}$$

such that for any $\gamma \in H^n C$ the image $\delta^n \gamma$ is the class α defined by:

 (a) For any $\gamma \in H^n C$ and any cocycle $c \in Z^n(C)$ which represents γ (i.e., $\gamma = [c] \stackrel{\text{def}}{=} c + B^n(C)$), there is some $b \in B^n$ such that $c = g^n b$,

 (b) For such b there exists some $a \in A^{n+1}$ with $db = f^{n+1}a$.

 (c) The class $\alpha = [a] \in \mathrm{H}^{n+1}(A)$ depends only on $[\gamma]$ but not on the choices of c, b and a.

(2) For any short exact sequence of complexes the following long sequence of cohomology groups is exact

$$\cdots \xrightarrow{\partial^{n-1}} \mathrm{H}^n(A) \xrightarrow{\mathrm{H}^n(f)} \mathrm{H}^n(B) \xrightarrow{\mathrm{H}^n(g)} \mathrm{H}^n(C) \xrightarrow{\partial^n} \mathrm{H}^{n+1}(A) \xrightarrow{\mathrm{H}^{n+1}(f)} \mathrm{H}^{n+1}(B) \xrightarrow{\mathrm{H}^{n+1}(f)} \cdots$$

Remark. This observation is *the* basis of calculations in homological algebra.

5. Homotopy. We say that two maps of complexes $A \xrightarrow{\alpha,\beta} B$ are *homotopic* (we denote this $\alpha \mod \beta$), if there is a sequence h of maps $h^n : A^n \to B^{n-1}$, such that

$$\beta - \alpha = dh + hd, \quad i.e., \quad \beta^n - \alpha^n = d_B^{n-1} h^n + h^{n+1} d_A^n.$$

We say that h is a homotopy from α to β.

A map of complexes $A \xrightarrow{\alpha} B$ is said to be a *homotopical equivalence* if there is a map β in the opposite direction such that $\beta \circ \alpha \equiv 1_A$ and $\alpha \circ \beta \equiv 1_B$.

Show that

(1) Homotopic maps are the same on cohomology.
(2) If for a complex C maps $id_C : C \to C$ and $0 : C \to C$ are homotopic then $H^i(C) = 0$ for all i, i.e., C is exact (we also say that C is *acyclic*).
(3) Homotopical equivalences are quasi-isomorphisms.

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Group cohomology

6. G-modules. Let G be a group. A G-module will mean an abelian group M with an action $G\times M \to M$ of the group G by morphisms of abelian groups, i.e., $g(x+y) = gx+gy,\ g\in G,\ x,y\in M$. Let $\mathfrak{m}(G)$ be the category of G-modules (morphisms are maps of abelian groups $A : M \to N$ which preserve G-action: $A(gm) = g(Am)$.)

Any abelian group A can be viewed as a trivial G-module: $ga = a,\ g \in G,\ a \in A$. In particular, we will view $\mathbb{Z}$ as a trivial G-module (unless some other action is specified.)

(1) Let $\mathbb{Z}[G]$ denote the free abelian with a basis G, so the elements are sums $\sum_{g\in G}\ a_g g$ with $a_g \in G$ and only finitely many nonzero coefficients. Show that $\mathbb{Z}[G]$ has a canonical structure of ring. (It is called the *group algebra* of G.)

(2) Prove that G-module is the same as a module for the ring $\mathbb{Z}[G]$. (So, $\mathfrak{m}(G) = \mathfrak{m}(\mathbb{Z}[G])$.)

(3) Show that the following two functors from $\mathfrak{m}(G)$ to abelian groups are isomorphic:
   (a) The functor of G-invariants $I(M) \overset{\text{def}}{=} \{m \in M;\ (\forall g\in G)\ gm = m\}$,
   (b) Functor $M \mapsto \operatorname{Hom}_G(\Bbbk, M)$.

(4) Show that any map of groups $\alpha : G \to H$ makes $\mathbb{Z}[H]$ into a G-module by $g(\sum_{h\in H}\ c_h h) \overset{\text{def}}{=} \sum_{h\in H}\ c_h \alpha(g)h$.

7. The standard resolution of the trivial G-module. For any $n \geq 0$ we consider $\mathbb{Z}[G^n]$ as a G-module via the diagonal homomorphism $G \to G^n$. (For $n = 0$ this means just the usual trivial action of G on $\mathbb{Z}[G^0] = \mathbb{Z}[\{1\}] = \mathbb{Z}$.)

(1) For $n \geq 0$ define $\partial_n : \mathbb{Z}[G^n] \to \mathbb{Z}[G^{n-1}]$ on the $\mathbb{Z}$-basis G^n by

$$\partial_n(g_0, ..., g_{n-1}) \overset{\text{def}}{=} \sum_{0\leq i<n} (-1)^i\ (g_0, ..., \widehat{g_i}, ..., g_{n-1}).$$

Show that ∂_n is a G-map and $\partial_{n-1}\partial_n = 0$ so that

$$\cdots \to \mathbb{Z}[G^n] \to \cdots \to \mathbb{Z}[G^2] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0 \to \cdots$$

is a complex.

(2) Show that the maps $h_n : \mathbb{Z}[G^n] \to \mathbb{Z}[G^{n+1}]$

$$h_n(g_0, ..., g_n) \overset{\text{def}}{=} (1, g_0, ..., g_n)$$

form a homotopy between identity and zero endomorphisms of the complex above.[1]

[1] These are not G-maps!

(3) For $n \geq 0$ let $P^{-n} \stackrel{\text{def}}{=} \mathbb{Z}[G^{n+1}] \in \mathfrak{m}(G)$. Define maps

$$\cdots \to P^{-n} \xrightarrow{d^{-n}} P^{-(n-1)} \to \cdots \to P^0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

in terms of the basis G^{n+1} of P^{-n} by

$$d^{-n}(g_0, ..., g_n) \stackrel{\text{def}}{=} \sum_{i=0}^{n} (-1)^i \ (g_0, ..., \widehat{g_i}, ..., g_n) \quad \text{and} \quad \varepsilon(g_0) \stackrel{\text{def}}{=} 1, \ g_0 \in G.$$

Show that this is a resolution of the trivial G-module $\mathbb{Z}$.

(4) Prove that all P^i are free $\mathbb{Z}[G]$-modules. (So we have a natural resolution by free modules.)