Homological algebra, Homework 3

Multilinear algebra III

Tensor algebras $T_A(M)$ of modules over commutative rings

Let A be a commutative ring with a unit. For any A-module M we will denote the by

$$T_A^n(M) \overset{\text{def}}{=} \overset{n}{\otimes}_A M \overset{\text{def}}{=} M^{\otimes n},\ n\geq 0.$$

the n-tuple tensor product $M\underset{A}{\otimes}\cdots\underset{A}{\otimes}M$. For $n=0$ this is – by definition – A itself, so it does not depend on M. For $n=1$ this is the module M. For $n>1$ we use the above construction of multiple tensor products.

1. Tensor algebra $T_A(M)$ of the A-module M. Show that the sum $T(M)\overset{\text{def}}{=}\sum_{n\geq 0} T^n(M)$ has a unique structure of an associative A-algebra, such that for all $p,q\geq 0$ and $m_i,n_j\in M$, the mutiplication in $T_A(M)$ is the cocatenation[1]

$$(m_1\otimes\cdots\otimes m_p)\cdot(n_1\otimes\cdots\otimes n_q) = m_1\otimes\cdots\otimes m_p\otimes n_1\otimes\cdots\otimes n_q.$$

Remark. For this algebra structure structure pure tensors, i.e., those of the form $m_1\otimes\cdots\otimes m_p$, are simply product $m_1\cdots m_p$ of $m_i\in M=M^{\otimes 1}\subseteq TM$.

2. Universal property of tensor algebras. For each A-algebra B restriction

$$\text{Hom}_{assoc.\ A-alg.\ with\ 1}(TM,B)\ni\phi\mapsto\phi|M\in\text{Hom}_{A-modules}(M,B),$$

is a bijection.

Remark. We say that $T(M)$ is the *A-algebra defined (generated) by the A-module M*, or that $T(M)$ is *universal* among all A-algebras B endowed with a map of A-modules $M\to B$.

Let $\Bbbk$ be a commutative algebra. Recall that all rings we consider are associative and have a unit. By an *algebra* we will always mean an associative algebra.

3. Algebras generated by generators and relations. To a $\Bbbk$-module M and a set $\mathcal{R}$ of algebraic relations between some elements of M, we will associate the *universal* algebra $\Bbbk(M;\mathcal{R})$ in which these relations are satisfied.

What we mean by an algebraic relations between elements $m_{i,j}$ of M are *intuitively* the conditions of type

$$(*)\quad \sum_{i=1}^{k} a_i m_{i,1}\cdots m_{i,n_i} = 0$$

[1]The meaning of this formula when $p=0$ is the action of A on its module $M^{\otimes q}$ and the same when $q=0$.

for some $a_i \in \Bbbk$. The precise meaning of such relations is that the expression on the left hand side defines an element $r = \sum_{i=1}^k \ a_i{\cdot}m_{i,1}{\otimes}\cdots{\otimes}m_{i,n_i}$ of the tensor algebra $T(M)$. So any set of such relations defines

- (i) a subset $\mathcal{R}{\subseteq}T(M)$,
- (ii) a $\Bbbk$-algebra $\Bbbk(M;\mathcal{R}) \stackrel{\text{def}}{=} T(M)/ <\mathcal{R}>$ where $<\mathcal{R}>$ denotes the 2sided ideal in $T(M)$ generated by $\mathcal{R}$,
- (iii) a canonical map of $\Bbbk$-modules $\iota \stackrel{\text{def}}{=} [M{\subseteq}T(M){\twoheadrightarrow}\Bbbk(M;\mathcal{R})]$.

(a) Show that for each $\Bbbk$-algebra B, $\text{Hom}_{\Bbbk-algebras}[\Bbbk(M;\mathcal{R}),B]$ is naturally identified with the set of all $\beta \in \text{Hom}_{\Bbbk-modules}(M,B)$, such that for all $r = \sum_{i=1}^k \ a_i{\cdot}m_{i,1}{\otimes}\cdots{\otimes}m_{i,n_i}$ in $\mathcal{R}$, the following relation between (images of) elements of M holds in B: $\sum_{i=1}^k \ a_i{\cdot}\beta(m_{i,1}){\cdots}\beta(m_{i,n_i}) = 0$.

Remark. Let us summarize. An algebraic relation of type $(*)$ between elements of M acquires meaning in any $\Bbbk$-algebra B supplied with a map of $\Bbbk$-modules $M{\to}B$. Algebra $\Bbbk(M;\mathcal{R})$ is universal among all such $\Bbbk$-algebras B that satisfy relations from $\mathcal{R}$.

Exterior algebras of modules over commutative rings

4. Exterior algebra of a $\Bbbk$-module. *For any $\Bbbk$-module M, the exterior algebra $\overset{\bullet}{\wedge}M = \overset{\bullet}{\underset{\Bbbk}{\wedge}}M$ is defined as the $\Bbbk$-algebra generated by M and by anti-commutativity relations $\mathcal{R} = \{x{\otimes}y + y{\otimes}x,\ x,y \in M\}$. The multiplication operation in $\overset{\bullet}{\wedge}M$ is denoted $\wedge$, so that the image of $m_1{\otimes}\cdots{\otimes}m_n \in T(M)$ in $\overset{\bullet}{\wedge}M$ is denoted $m_1{\wedge}\cdots{\wedge}m_n \in \overset{n}{\wedge}M$.*

(a) $\overset{\bullet}{\wedge}M$ is a graded algebra. Show that

(1) the ideal $I = <\mathcal{R}>$ in $T(M)$ is homogeneous, i.e., $I = \oplus_{n\geq 0}\ I^n$ for $I^n \stackrel{\text{def}}{=}\ I{\cap}T^nM$.
(2) $\overset{\bullet}{\wedge}M = T(M)/I$ is isomorphic to $\oplus_{n\geq 0}\ \overset{n}{\wedge}M$ for $\overset{n}{\wedge}M \stackrel{\text{def}}{=} T^n(M)/I^n$.
(3) $\overset{\bullet}{\wedge}M$ is a graded algebra, i.e., $\overset{n}{\wedge}M\ {\cdot}\overset{m}{\wedge}M{\subseteq}\overset{n+m}{\wedge}M,\ n,m \geq 0$.

5. Universal property of the exterior algebra. Show that for each $\Bbbk$-algebra B, $\text{Hom}_{\Bbbk-algebras}[\overset{\bullet}{\wedge}M,B]$ can be identified with the set of all $\phi : \text{Hom}_{\Bbbk-modules}(M,B)$, such that the ϕ-images of elements of M *anti-commute* in B, i.e.,

$$\phi(y)\phi(x) = -\phi(x)\phi(y),\ x,y \in M.$$

6. Basic properties of exterior algebras.

(1) **Low degrees.** $\overset{0}{\wedge}M = T^0(M) = \Bbbk$ and $\overset{1}{\wedge}M = T^1(M) = M$.

(2) **Bilinear forms extend to exterior algebras.** For any $\Bbbk$-modules L and M, and any $\Bbbk$-bilinear map $<\ ,\ >: L\times M\to\Bbbk$ (i.e., linear in each variable); there is a unique $\Bbbk$-bilinear map $<\ ,\ >: \overset{n}{\wedge}L\times\overset{n}{\wedge}M\to\Bbbk$, such that

$$< l_1\wedge\cdots l_n, m_1\wedge\cdots\wedge m_n > = \det(< l_i, m_j >), \quad l_i \in L,\ m_j \in M.$$

(3) **Free modules.** If M is a free $\Bbbk$-module with a basis $e_1, ..., e_d$, then $\overset{k}{\wedge}M$ is a free $\Bbbk$-module with a basis $e^J = e_{j_1}\wedge\cdots\wedge e_{j_k}$, indexed by all subsets $J = \{j_1 < \cdots < j_n\}\subseteq I$ with k elements.

(4) **Dimension.** $\dim(\overset{\bullet}{\wedge}\ \mathbb{C}^n) = 2^n$.

Categories

C1. Yoneda lemma. Let $\mathcal{C}$ be a category.

(1) Any object $a \in \mathcal{C}$ defines a functor $F_a : \mathcal{C}^o \to \mathcal{S}et$ by
$$F_a(c) \overset{\text{def}}{=} \text{Hom}_{\mathcal{C}}(c, a).$$
(2) Explain what does a "morphism of functors $\Phi : F_a \to F_b$" consists of (which data and which properties)? What is a composition $\Psi \circ \Phi$ of morphisms of functors $\Phi : F_a \to F_b$ and $\Psi : F_b \to F_c$? What is id_{F_a} ?
(3) Any morphism of objects $\phi : a \overset{\cong}{\to} b$ in $\mathcal{C}$ defines a morphism of functors $\widetilde{\phi} : F_a \to F_b$.
(4) $\widetilde{\psi \circ \phi} = \widetilde{\psi} \circ \widetilde{\phi}$ and $\widetilde{id_a} = id_{F_a}$. If ϕ is isomorphism in $\mathcal{C}$ then $\widetilde{\phi}$ is an isomorphism of functors.
(5) Any morphism of functors $\Phi : F_a \to F_b$. defines a morphism of objects $\overline{\Phi} \in \text{Hom}_{\mathcal{C}}(a, b)$.
(6) $\overline{\Psi \circ \Phi} = \overline{\Psi} \circ \overline{\Phi}$ and $\overline{id_{F_a}} = id_a$. If Φ is isomorphism of functors then $\widetilde{\phi}$ is an isomorphism in $\mathcal{C}$.
(7) The two procedures above give inverse bijections between $\text{Hom}_{\mathcal{C}}(a, b)$ and the set of morphisms of functors $F_a \to F_b$.
(8) Functor $\text{Hom}_{\mathcal{C}}(-, a)$ determines a up to a unique isomorphism.

Sheaves

S1. Let $X = \mathbb{P}^1{}_{\mathbb{C}}$.

(1) Show that the tangent sheaf $\mathcal{T}_X$ is isomorphic to $\mathcal{O}(2)$.
(2) Show that the cotangent sheaf $\mathcal{T}_X^*$ is isomorphic to $\mathcal{O}(-2)$.
(3) Find the dimensions of the Čech cohomology groups of $\mathcal{T}_X$ and $\mathcal{T}_X^*$. (For the standard open cover of X.)