Homological algebra, Homework 2

Multilinear Algebra II

1. Tensoring over a field. If A is a field then A-modules are vector spaces over A. Let $u_i, i \in I$, $v_j, j \in J$, be bases of U and V, show that $u_i \otimes v_j,\ i \in I, j \in J$; is a basis of $U \underset{A}{\otimes} V$ and $\dim(U \underset{A}{\otimes} V) = \dim(U) \cdot \dim(V)$.

2. Tensoring of finite abelian groups over $\mathbb{Z}$. Show that $\mathbb{Z}_n \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_m \cong \mathbb{Z}_k$ for some k and calculate k.

3. Tensoring of algebras. Let A be a commutative ring and let B and C be A-algebras. Then $B \otimes_A C$ has a canonical structure of an algebra such that

$$(b_1 \otimes c_1) \cdot (b_2 \otimes c_)) = \ b_1 b_2 \otimes c_1 c_2.$$

Multiple tensor products,

When one has a sequence of bimodules $M_1, ..., M_n$ where each M_i is an (A_{i-1}, A_i)-bimodule $(1 \le i \le n)$ one defines the multiple tensor product $M_1 \otimes_{A_1} M_2 \otimes_{A_2} \cdots \otimes_{A_{n-1}} M_n$ as a quotient F/R of a free abelian group F with a basis $M_1 \times \cdots \times M_n$, by the subgroup $\mathcal{A}$ generated by the elements of the following forms (here $1 \le i \le n$ and $m_i \in M_i,\ a_i \in A_i$):

(1) $(m_1, ..., m_i' + m_i'', ..., m_n) - (m_1, ..., m_i', ..., m_n) - (m_1, ..., m_i'', ..., m_n)$,
(2) $(m_1, ..., m_{i-1} \cdot a_i, ..., m_n) - (m_1, ..., a_i \cdot m_i, ..., m_n)$

The image of $(m_1, ..., m_n) \in F$ in $M_1 \otimes_{A_1} M_2 \otimes_{A_2} \cdots \otimes_{A_n} M_n$ is denoted $m_1 \otimes \cdots \otimes m_n$. Denote by $\pi : M_1 \times \cdots \times M_n \to M_1 \otimes_{A_1} M_2 \otimes_{A_2} \cdots \otimes_{A_{n-1}} M_n,\ \pi(m_1, ..., m_n) = m_1 \otimes \cdots \otimes m_n$ the composition $M_1 \times \cdots \times M_n \hookrightarrow F \twoheadrightarrow M_1 \otimes_{A_1} M_2 \otimes_{A_2} \cdots \otimes_{A_{n-1}} M_n$.

4. Multiple tensor products. (a) Show that in $M_1 \otimes_{A_1} M_2 \otimes_{A_2} \cdots \otimes_{A_{n-1}} M_n$

(1) $m_1 \otimes \cdots \otimes m_i' + m_i'' \otimes \cdots \otimes m_n = m_1 \otimes \cdots \otimes m_i' \otimes \cdots \otimes m_n + m_1 \otimes \cdots \otimes m_i'' \otimes \cdots \otimes m_n$, ,
(2) $m_1 \otimes \cdots \otimes m_{i-1} \cdot a_i \otimes m_i \otimes \cdots \otimes m_n = m_1 \otimes \cdots \otimes m_{i-1} \otimes a_i \cdot m_i \otimes \cdots \otimes m_n$.

(b) Each element of $M_1 \otimes_{A_1} M_2 \otimes_{A_2} \cdots \otimes_{A_{n-1}} M_n$ is a finite sum of the form $\sum_{k=1}^{p}\ m_{k,1} \otimes \cdots \otimes m_{k,n}$ with $m_{k,i} \in M_i$.

5. Bimodule structure. Show that $M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$ is an (A_0, A_n)-bimodule.

6. Universal property. Let M_i be an (A_{i-1}, A_i)-bimodule ($1 \le i \le n$), and H an abelian group. We say that a map $\phi : M_1 \times \cdots \times M_n \to H$ is balanced if it is additive in each factor and balanced in rings $A_1, ..., A_{n-1}$, i.e.,

(1) $\phi(m_1, ..., m_i' + m_i'', ..., m_n) = \phi(m_1, ..., m_i' + m_i'', ..., m_n) + \phi(m_1, ..., m_i' + m_i'', ..., m_n)$, and
(2) $\phi(m_1, ..., m_{i-1}, a_i m_i, ..., m_n) = \phi(m_1, ..., m_{i-1} a_i, m_i, ..., m_n)$.

Show that (a) Map π is balanced. and (b) Map π gives a bijection between all balanced maps $\phi : M_1 \times \cdots \times M_n \to H$ and all maps of abelian groups $\psi : M_1 \otimes_{A_1} \cdots \otimes_{A_{n-1}} M_n \to H$, by $\psi \mapsto \psi \circ \pi$. [1]

7. Associativity. This definition is associative in the sense that there are canonical isomorphisms

$$(M_1 \underset{A_1}{\otimes} \cdots \underset{A_{p_1-1}}{\otimes} M_{p_1}) \underset{A_{p_1}}{\otimes} (M_{p_1+1} \underset{A_{p_1+1}}{\otimes} \cdots \underset{A_{p_1+p_2-1}}{\otimes} M_{p_1+p_2}) \underset{A_{p_1+p_2}}{\otimes} \cdots \otimes (M_{p_1+\cdots+p_{k-1}+1} \otimes \cdots \otimes M_{p_1+\cdots+p_k})$$
$$\cong M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{p_1+\cdots+p_k-1}}{\otimes} M_{p_1+\cdots+p_k}.$$

8. Two factors. For $n = 2$ this notion of a tensor product agrees with the one introduced previously.

9. Commutative rings. If the rings A_i are all commutative explain how the above construction defines a multiple tensoring operation $M_1 \otimes_{A_1} M_2 \otimes_{A_2} \cdots \otimes_{A_{n-1}} M_n$ for when each M_i is a left module for A_{i-1} and A_i, and these two actions commute.

Duality of modules over a ring

D1. Dualizing maps. Let $\Bbbk$ be a ring. For a left $\Bbbk$-module $M = \Bbbk \in \mathfrak{m}^l(\Bbbk)$ show that

(a) The map that assigns to $a \in \Bbbk$ the operator of right multiplication $R_a x \stackrel{\text{def}}{=} x \cdot a$, gives an isomorphism of right $\Bbbk$-modules $\Bbbk \xrightarrow{R} \Bbbk^*$. [One could denote it more precisely by $\Bbbk_R \xrightarrow{R} ({}_{\Bbbk}\Bbbk)^*$.

(b) $\iota_{\Bbbk}$ is an isomorphism.

D2. The dual of the structure sheaf of a point in a plane. For the point Y in a plane $\mathbb{A}^2$ show that

$$L\mathbb{D}[\mathcal{O}(Y)] \cong \mathcal{O}(Y)[-2]$$

(i.e., calculate the cohomology of $\mathbb{D}P^\bullet$ for the resolution $P^\bullet$ from **??**).

[1] We say that π is universal among all balanced maps from $M_1 \times \cdots \times M_n$ to an abelian group, because every such map ϕ factors through π.