Homological algebra, Homework 1

Multilinear Algebra I

This is mostly an exposition material S.Lang's *Algebra* written as a sequence of problems. A more advanced version is in N.Bourbaki's *Algebra* etc.

A. Tensor product of modules over a ring

Let A be a ring with a unit 1.

1. Tensor product of A-modules. *For a left A-module U and a right A-module V, we define a free abelian group $F = F_{U,V}$, with a basis $U\times V$: $F = \oplus_{u\in U,\ v\in V}\ \mathbb{Z}\cdot(u,v)$. The tensor product of U and V is the abelian group $U\underset{A}{\otimes}V$ defined as a quotient $U\underset{A}{\otimes}V\overset{def}{=}F/R$ of F by the subgroup R generated by the elements of one of the following forms (here $u, u_i \in U,\ v, v_i \in V,\ a \in A$):*

(1) $(u_1+u_2, v)-(u_1, v)-(u_2, v)$, (2) $(u, v_1+v_2)-(u, v_1)-(u, v_2)$, (3) $(u{\cdot}a, v)-(u, a{\cdot}v)$, .

The image of $(u, v) \in F$ in $U\underset{A}{\otimes}V$ is denoted $u\otimes v$. Let $\pi : U\times V\to U\underset{A}{\otimes}V$ be the composition of maps $U\times V\hookrightarrow F\to U\underset{A}{\otimes}V$, so that $\pi(u, v) = u\otimes v$.

(a) Show that

(a_1) $(u_1 + u_2)\otimes v = u_1\otimes v + u_2\otimes v$,

(a_2) $u\otimes(v_1 + v_2) = u\otimes v_1 + u\otimes v_2$,

(a_3) $(u{\cdot}a)\otimes v = u\otimes(a{\cdot}v)$.

(b) Show that each element of $U\underset{A}{\otimes}V$ is a finite sum of the form $\sum_{i=1}^{n}\ u_i\otimes v_i$, for some $u_i \in U,\ v_i \in V$.

2. The universal property of the tensor product. *For U, V as above we say that a map $\phi : U\times V\to H$ with values in an abelian group H, is A-*balanced *if it satisfies the conditions: $\phi(u_1 + u_2, v) = \phi(u_1, v) + \phi(u_2, v)$, $\phi(u, v_1 + v_2) = \phi(u, v_1) + \phi(u, v_2)$, $\phi(u{\cdot}a, v) = \phi(u, a{\cdot}v)$, for $u, u_i \in U,\ v, v_i \in V,\ a \in A$.*

Show that the balanced maps $\phi : U\times V\to H$ are in a one-to-one correspondence with the morphisms of abelian groups $\psi : U\underset{A}{\otimes}V\to H$, by $\psi\mapsto\phi\overset{\text{def}}{=}\psi\circ\pi$.

[*The above notion of "balanced maps" is a version of the notion of bilinear maps which makes sense even for non-commutative rings A. One direction of the one-to-one correspondence above says that the tensor product reformulates balanced maps in terms of linear maps. In the opposite direction, one constructs maps from a tensor product $U\underset{A}{\otimes}V$ by constructing balanced maps from the product $U\times V$.*]

3. Functoriality. Let $U_0 \xrightarrow{\alpha} U_1$ be a map of right A-modules and $V_0 \xrightarrow{\beta} V_1$ be a map of left A-modules.

(a) A map of abelian groups $U_0 \otimes_A V_0 \xrightarrow{\gamma} U_1 \otimes_A V_1$ is well defined by $\gamma(u \otimes v) = \alpha(u) \otimes \beta(v),\ u \in U_0,\ v \in V_0$. [This map is usually denoted by $\alpha \otimes \beta$ though one need not think of it as an element of some tensor product.]

(b) $1_U \otimes 1_V = 1_{U \otimes V}$.

(c) If one also has maps $U_1 \xrightarrow{\alpha'} U_2$ of right modules and $V_1 \xrightarrow{\beta'} V_2$ of left A-modules, then

$$(\alpha' \otimes \beta') \circ (\alpha \otimes \beta) = (\alpha' \circ \alpha) \otimes (\beta' \circ \beta).$$

4. Additivity. Let $V = \oplus_{i \in I} V_i$ be a direct sum of left A-modules, then $U \underset{A}{\otimes} V = \oplus_{i \in I} U \underset{A}{\otimes} V_i$.

5. Free Modules. If V is a free left A-module with a basis $v_i,\ i \in I$, then $U \underset{A}{\otimes} V \cong U^I$, i.e., it is a sum of I copies of U.

6. Cancellation. Show that the map $U \underset{A}{\otimes} A \xrightarrow{\alpha} U,\ \alpha(\sum\ u_i \otimes a_i) = \sum u_i \cdot a_i$, is (i) well defined, (ii) an isomorphism of abelian groups.

7. Quotient by relations interpreted as tensoring. For each left ideal I in A, map $U \underset{A}{\otimes} A/I \xrightarrow{\alpha} U/U \cdot I,\ \alpha[\sum\ u_i \otimes (a_i + I)] = (\sum u_i \cdot a_i) + U \cdot I$; is (i) well defined, (ii) an isomorphism of abelian groups.

8. Tensoring of bimodules. If U is a bimodule for a pair of rings (R, A) and V is a bimodule for a pair of rings (A, S), show that $U \underset{A}{\otimes} V$ is a bimodule for (R, S).

9. Right exactness of tensor products. (a) Let A be a ring, L a right A-module and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of left A-modules. Show that there is a short exact sequence of abelian groups $L \underset{A}{\otimes} M' \to L \underset{A}{\otimes} M \to L \underset{A}{\otimes} M'' \to 0$. Find an example when the sequence $0 \to L \underset{A}{\otimes} M' \to L \underset{A}{\otimes} M \to L \underset{A}{\otimes} M'' \to 0$ is not exact, i.e., $L \underset{A}{\otimes} M' \not\subseteq L \underset{A}{\otimes} M$.

(b) Let A be a ring, L a right A-module and M a left A-module. Show that any algebra morphism $\phi : B \to A$, gives a surjective map $L \underset{B}{\otimes} M \to L \underset{A}{\otimes} M$, with the kernel generated by elements of the form $x \cdot a \otimes y - x \otimes a \cdot y,\ x \in L,\ y \in M,\ a \in A$.