Homological algebra, Homework 10

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Ext and Tor

Problem 1. If M, N are modules for a ring A we know that the functor $\mathrm{Hom}_A(-, M) : \mathfrak{m}(A)^o \to \mathcal{A}b$ is right exact and the functor $\mathrm{Hom}_A(N, -) : \mathfrak{m}(A) \to \mathcal{A}b$ is left exact. Their derived functors are called Ext-functors :

$$'\mathrm{Ext}^i_A(N, M) \overset{\mathrm{def}}{=} L^i\mathrm{Hom}_A(-, M)\ (N) \quad \text{and} \quad ''\mathrm{Ext}^i_A(N, M) \overset{\mathrm{def}}{=} R^i\mathrm{Hom}_A(N, -)\ (M).$$

Prove that $'\mathrm{Ext} =''\mathrm{Ext}$ (then we will denote either construction just by $\mathrm{Ext}^n_A(M, N)$):

(1) For a projective resolution P of M and an injective resolution I of N define a structure of a bicomplex on $\mathrm{Hom}_A(P, Q)$.
(2) Use the two spectral sequences of this bicomplex to construct canonical isomorphisms $'\mathrm{Ext}^n_A(M, N) \cong'' \mathrm{Ext}^n_A(M, N)$.

Problem 2. Let M be a right A-module and N a left A-module for a ring A. We know that the functors

$$M\otimes_A - :\ \mathfrak{m}^l(A)^o \to \mathcal{A}b \quad \text{and} \quad - \otimes_A N\ :\ \mathfrak{m}^r(A)^o \to \mathcal{A}b$$

are both right exact. Their derived functors are called Tor-functors :

$$'\mathrm{Tor}^A_n(M, N) \ \overset{\mathrm{def}}{=}\ R^{-n}(-\otimes_A N)\ (M) \quad \text{and} \quad ''\mathrm{Tor}^A_n(M.N) \ \overset{\mathrm{def}}{=}\ R^{-n}(M\otimes_A -)\ (N).$$

Prove that $'\mathrm{Tor} ='' \mathrm{Tor}$ using spectral sequences. (Then we will denote either construction simply by $\mathrm{Tor}^A_n(M, N)$.)

Group cohomology IV

Recall how we have introduced the group cohomology $H^*(G, M)$ for a group G and a representation $M \in Rep_{\mathcal{A}b}(G)$ of G in the category of abelian groups. First we have though of representations of G as modules for the group algebra $\mathbb{Z}[G]$. Then we chose a specific resolution P of the the trivial G-module $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules. This gives a complex $\mathrm{Hom}_G(P, M)$ in abelian groups and $H^*(G, M)$ was defined as the cohomology of the complex $\mathrm{Hom}_G(P, M)$.

Problem 3. Prove that

(1) In terms of the above *Ext*-terminology we can now write $H^n(G, M)$ as $'\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$.
(2) $''\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ are the derived functors $(R^n I)(M)$ of the left exact functor of invariants

$$I : Rep_{\mathcal{A}b}(G) \to \mathcal{A}b, \quad I(M) \stackrel{\text{def}}{=} \{m \in M;\ gm = m \ for\ all\ g \in G\}.$$

(3) The group cohomology functors $H^n(G, -)$ are the derived functors $R^n I$ of the functor I of G-invariants.

Now we define the *group homology* $H_n(G, M)$ as the left derived functors $(L^{-n}C)M$ of the functor of *coinvariants*

$$C : Rep_{\mathcal{A}b}(G) \to \mathcal{A}b, \quad C(M) \stackrel{\text{def}}{=} \frac{M}{\sum_{g \in G} (g-1)M}.$$

Problem 4. Prove that

(1) The functor C of G-coinvariants is isomorphic to the functor $\mathbb{Z} \otimes_{\mathbb{Z}(G)} -$.
(2) C is right exact.
(3) The group homology functors $H_n(G, M)$ defined above can be computed as

$$H^{-n}(P \otimes_{\mathbb{Z}[G]} M)$$

for any free resolution P of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$.

Problem 5. Prove that the first group homology functor $H_1(G, -)$ applied to the trivial G-module $\mathbb{Z}$ gives the maximal abelian quotient G^{ab} of the group G.