HOMOLOGICAL ALGEBRA

Contents

Date: ?

0.0.1. *Notation.* Symbol □ means "I said this much and I will say no more".

6. **Abelian category of sheaves of abelian groups**

For a topological space X we will denote by $\mathcal{S}h(X) = \mathcal{S}heaves(X, \mathcal{A}b)$ the category of sheaves of abelian groups on X. Since a sheaf of abelian groups is something like an abelian group smeared over X we hope to $\mathcal{S}h(X)$ is again an abelian category. When attempting to construct cokernels, the first idea does not quite work – it produces something like a sheaf but without the gluing property. This forces us to

- (i) generalize the notion of sheaves to a weaker notion of a presheaf,
- (ii) find a canonical procedure that improves a presheaf to a sheaf.

(We will also see that a another example that requires the same strategy is the pull-back operation on sheaves.)

Now it is easy to check that we indeed have an abelian category. What allows us to compute in this abelian category is the lucky break that one can understand kernels, cokernels, images and exact sequences just by looking at the stalks of sheaves.

6.1. **Categories of sheaves.**

A presheaf of sets $\mathcal{S}$ on a topological space $(X, \mathcal{T})$ consists of the following data:

- for each open $U \subseteq X$ a set $\mathcal{S}(U)$,
- for each inclusion of open subsets $V \subseteq U \subseteq X$ a map $\mathcal{S}(U) \xrightarrow{\rho_V^U} \mathcal{S}(V)$ (called the restriction map);

and these data are required to satisfy

- (Sh0)(Transitivity of restriction) $\rho_V^U \circ \rho_V^U = \rho_W^U$ for $W \subseteq V \subseteq U$

A sheaf of sets on a topological space $(X, \mathcal{T})$ is a presheaf $\mathcal{S}$ which also satisfies

- (Sh1) (Gluing) Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of an open $U \subseteq X$ (We denote $U_{ij} = U_i \cap U_j$ etc.). We ask that any family of compatible sections $f_i \in \mathcal{S}(U_i),\ i \in I$, glues uniquely. This means that if sections f_i agree on intersections in the sense that $\rho_{U_{ij}}^{U_i} f_i = \rho_{U_{ij}}^{U_i} f_j$ in $\mathcal{S}(U_{ij})$ for any $i, j \in I$; then there is a unique $f \in \mathcal{S}(U)$ such that $\rho_{U_i}^{U} f = f_i$ in $\mathcal{S}(U_i),\ i \in I$.
- $\mathcal{S}(\emptyset)$ is a *point*.

6.1.1. *Remarks.* (1) Presheaves of sets on X form a category $pre\mathcal{S}heaves(X, \mathcal{S}et)$ when $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ consists of all systems $\phi = (\phi_U)_{U \subseteq X \ open}$ of maps $\phi_U : \mathcal{A}(U) \to \mathcal{B}(U)$ which are compatible with restrictions, i.e., for $V \subseteq U$

$$\begin{array}{ccc} \mathcal{A}(U) & \xrightarrow{\phi_U} & \mathcal{B}(U) \\ \rho^U_V \big\downarrow & & \rho^U_V \big\downarrow \\ \mathcal{A}(V) & \xrightarrow{\phi_V} & \mathcal{B}(V) \end{array} .$$

(One reads the diagram above as : "the diagram ... commutes".) The sheaves form a full subcategory $pre\mathcal{S}heaves(X, \mathcal{S}et)$ of $\mathcal{S}heaves(X, \mathcal{S}et)$.

(2) We can equally define categories of sheaves of abelian groups, rings, modules, etc. For a sheaf of abelian groups we ask that all $\mathcal{A}(U)$ are abelian groups, all restriction morphisms are maps of abelian groups, and we modify the least interesting requirement (Sh2): $\mathcal{S}(\phi)$ is the trivial group $\{0\}$. In general, for a category $\mathcal{A}$ one can define categories $pre\mathcal{S}heaves(X, \mathcal{A})$ and $\mathcal{S}heaves(X, \mathcal{A})$ similarly (the value on $\emptyset$ should be the final object of $\mathcal{A}$).

6.2. **Sheafification of presheaves.** We will use the wish to pull-back sheaves as a motivation for a procedure that improves presheaves to sheaves.

6.2.1. *Functoriality of sheaves.* Recall that for any map of topological spaces $X \xrightarrow{\pi} Y$ one wants a pull-back functor $\mathcal{S}heaves(Y) \xrightarrow{\pi^{-1}} \mathcal{S}heaves(X)$. As we have seen in the definition of a stalk of a sheaf (pull ,back to a point), the natural formula is

$$\underline{\pi^{-1}}(\mathcal{N})\ (U) \stackrel{\text{def}}{=} \lim_{\substack{\rightarrow \\ V \supseteq \pi(U)}} \mathcal{N}(V),$$

where limit is over open $V \subseteq Y$ that contain $\pi(U)$, and we say that $V' \leq V''$ if V'' better approximates $\pi(U)$, i.e., if $V'' \subseteq V'$.

6.2.2. *Lemma.* This gives a functor of presheaves $pre\mathcal{S}heaves(X) \xrightarrow{\underline{\pi^{-1}}} pre\mathcal{S}heaves(Y)$.

Proof. For $U' \subseteq U$ open, $\underline{\pi^{-1}}\mathcal{N}(U') = \lim_{\rightarrow\ V \supseteq \pi(U')} \mathcal{N}(V)$ and $\underline{\pi^{-1}}\mathcal{N}(U) = \lim_{\rightarrow\ V \supseteq \pi(U)} \mathcal{N}(V)$ are limits of inductive systems of $\mathcal{N}(V)$'s, and the second system is a subsystem of the first one, this gives a canonical map $\underline{\pi^{-1}}\mathcal{N}(U) \rightarrow \underline{\pi^{-1}}\mathcal{N}(U')$.

6.2.3. *Remark.* Even if $\mathcal{N}$ is a sheaf, $\underline{\pi^{-1}}(\mathcal{N})$ need not be sheaf.

For that let $Y = pt$ and let $\mathcal{N} = S_Y$ be the constant sheaf of sets on Y given by a set S. So, $S_Y(\emptyset) = \emptyset$ and $S_Y(Y) = S$. Then $\underline{\pi^{-1}}(S_Y)\ (U) = \begin{cases} \emptyset & if\ U = \emptyset, \\ S & U \neq \emptyset \end{cases}$. We can say: $\underline{\pi^{-1}}(S_Y)\ (U) =$ constant functions from U to S. However, we have noticed that constant functions do not give a sheaf, so we need to correct the procedure $\underline{\pi^{-1}}$ to get sheaves from sheaves. For that remember that for the presheaf of constant functions there is a related sheaf S_X of *locally constant* functions.

Our problem is that the presheaf of constant functions is defined by a global condition (constancy) and we need to change it to a local condition (local constancy) to make it into a sheaf. So we need the procedure of

6.2.4. *Sheafification.* This is a way to improve any presheaf of sets $\mathcal{S}$ into a sheaf of sets $\widetilde{\mathcal{S}}$. We will imitate the way we passed from constant functions to locally constant functions. More precisely, we will obtained the sections of the sheaf $\tilde{\mathcal{S}}$ associated to the presheaf $\mathcal{S}$ in two steps:

(1) we glue systems of local sections s_i which are compatible in the weak sense that they are <u>*locally*</u> the same, and
(2) we identify two results of such gluing if the local sections in the two families are <u>*locally*</u> the same.

Formally these two steps are performed by replacing $\mathcal{S}(U)$ with the set $\widetilde{\mathcal{S}}(U)$, defined as the set of all equivalence classes of systems $(U_i, s_i)_{i\in I}$ where

(1) Let $\widehat{\mathcal{S}}(U)$ be the class of all systems $(U_i, s_i)_{i\in I}$ such that
    - $(U_i)_{i\in I}$ is an open cover of U and s_i is a section of $\mathcal{S}$ on U_i,
    - sections s_i are *weakly compatible* in the sense that they are locally the same, i.e., for any $i', i'' \in I$ sections $s_{i'}$ and $s_{i''}$ are the same near any point $x \in U_{i'i''}$. (Precisely, this means that there is neighborhood W such that $s_{i'}|W = s_{i''}|W$.)
(2) We say that two systems $(U_i, s_i)_{i\in I}$ and $(V_j, t_j)_{j\in J}$ are $\equiv$, iff for any $i \in I,\ j \in J$ sections s_i and t_j are weakly equivalent (i.e., for each $x \in U_i \cap V_j$, there is an open set W with $x \in W \subseteq U_i \cap V_j$ such that "$s_i = t_j$ on W" in the sense of restrictions being the same).

6.2.5. *Remark.* The relation $\equiv$ on $\widehat{\mathcal{S}}(U)$ really says that $(U_i, s_i)_{i\in I} \equiv (V_j, t_j)_{j\in J}$ iff the disjoint union $(U_i, s_i)_{i\in I} \sqcup (V_j, t_j)_{j\in J}$ is again in $\widehat{\mathcal{S}}(U)$.

6.2.6. *Lemma.* (a) $\equiv$ is an equivalence relation.

(b) $\widetilde{\mathcal{S}}(U)$ is a presheaf and there is a canonical map of presheaves $\mathcal{S} \xrightarrow{q} \widetilde{\mathcal{S}}$.

(c) $\widetilde{\mathcal{S}}$ is a sheaf.

Proof. (a) is obvious.

(b) The restriction of a system $(U_i, s_i)_{i\in I}$ to $V \subseteq U$ is the system $(U_i \cap V, s_i|U_i \cap V)_{i\in I}$. The weak compatibility of restrictions $s_i|U \cap V$ follows from the weak compatibility of sections s_i. Finally, restriction is compatible with $\equiv$, i.e., if $(U'_i, s'_i)_{i\in I}$ and $(U''_j, s''_j)_{j\in J}$ are $\equiv$, then so are $(U'_i \cap V, s'_i|U'_i \cap V)_{i\in I}$ and $(U''_j \cap V, s''_j|U''_j \cap V)_{j\in J}$.

The map $\mathcal{S}(U) \to \widetilde{\mathcal{S}}(U)$ is given by interpreting a section $s \in \mathcal{S}(U)$ as a (small) system: open cover of $(U_i)_{i\in\{0\}}$ is given by $U_0 = U$ and $s_0 = s$.

(c') Compatible systems of sections of the presheaf $\widetilde{\mathcal{S}}$ glue. Let $V^j,\ j \in J$, be an open cover of an open $V \subseteq X$, and for each $j \in J$ let $\sigma^j = [(U^j_i, s^j_i)_{i\in\ I_j}]$ be a section of $\widetilde{\mathcal{S}}$ on V_j. So, σ^j is an equivalence class of the system $(U^j_i, s^j_i)_{i\in\ I_j}$ consisting of an open cover $U^j_i,\ i \in\ I_j$, of V_j and weakly compatible sections $s^i_j \in \mathcal{S}(U^i_j)$.

Now, if for any $j,k \in J$ sections $\sigma^j = [(U^j_p, s^j_p)_{p\in\ I_j}]$ and $\sigma^k = [(U^k_q, s^k_q)_{q\in\ I_k}]$ of $\widetilde{\mathcal{S}}$ on V^j and V^k, agree on the intersection V^{jk}. This means that for any j,k $\sigma^j|V^{jk} = \sigma^k|V^{jk}$, i.e.,

$$(U^j_p \cap V^{jk}, s^j_p|U^j_p \cap V^{jk})_{p\in\ I_j} \quad\equiv\quad (U^k_q \cap V^{jk}, s^k_q|U^k_q \cap V^{jk})_{q\in\ I_k}.$$

This in turn means that for $j,k \in J$ and any $p \in I_j,\ q \in I_k$, sections s^j_p and s^k_q are weakly compatible. Since all sections $s^j_p,\ j \in J,\ p \in I_j$ are weakly compatible, the disjoint union of all systems $(U^j_i, s^j_i)_{i\in\ I_j},\ j \in J$ is a system in $\widehat{\mathcal{S}}(V)$. Its equivalence class σ is a section of $\widetilde{\mathcal{S}}$ on V, and clearly $\sigma|V^j = \sigma^j$.

(c") Compatible systems of sections of the presheaf $\widetilde{\mathcal{S}}$ glue uniquely. If $\tau \in \widetilde{\mathcal{S}}(V)$ is the class of a system $(U_i, s^i)_{i\in I}$ and $\tau|V^j = \sigma^j$ then σ's are compatible with all s^j_p's, hence $(U_i, s^i)_{i\in I} \equiv \sqcup_{j\in J}(U^j_i, s^j_i)_{i\in\ I_j}$, hence $\tau = \sigma$.

6.2.7. *Sheafification as a left adjoint of the forgetful functor.* As usual, we have not invented something new: it was already there, hidden in the more obvious forgetful functor

6.2.8. *Lemma.* Sheafification functor $pre\mathcal{S}heaves \ni \mathcal{S} \mapsto \widetilde{\mathcal{S}} \in \mathcal{S}heaves$, is the left adjoint of the inclusion $\mathcal{S}heaves \subseteq pre\mathcal{S}heaves$, i.e, for any presheaf $\mathcal{S}$ and any sheaf $\mathcal{F}$ there is a natural identification

$$\mathrm{Hom}_{\mathcal{S}heaves}(\widetilde{\mathcal{S}}, \mathcal{F}) \xrightarrow[\cong]{} \mathrm{Hom}_{pre\mathcal{S}heaves}(\mathcal{S}, \mathcal{F}).$$

Explicitly, the bijection is given by $(\iota_{\mathcal{S}})_* \alpha = \alpha \circ \iota_{\mathcal{S}}$, i.e., $(\widetilde{\mathcal{S}} \xrightarrow{\alpha} \mathcal{F}) \mapsto (\mathcal{S} \xrightarrow{\iota_{\mathcal{S}}} \widetilde{\mathcal{S}} \xrightarrow{\alpha} \mathcal{F})$.

6.3. **Inverse and direct images of sheaves.**

6.3.1. *Pull back of sheaves (finally!)* Now we can define for any map of topological spaces $X \xrightarrow{\pi} Y$ a pull-back functor

$$\mathcal{S}heaves(Y) \xrightarrow{\pi^{-1}} \mathcal{S}heaves(X), \qquad \pi^{-1}\mathcal{N} \stackrel{\text{def}}{=} \widetilde{\underline{\pi^{-1}}\mathcal{N}}.$$

6.3.2. *Examples.* (a) A point $a \in X$ can be viewed as a map $\{a\} \xrightarrow{\rho} X$. Then $\rho^{-1}\mathcal{S}$ is the stalk $\mathcal{S}_a$.

(b) Let $a : X \to \mathrm{pt}$, for any set S one has $S_X = a^{-1}S$.

6.3.3. *Direct image of sheaves.* Besides the pull-back of sheaves which we defined in 6.3.1, there is also a much simpler procedure of the push-forward of sheaves:

6.3.4. *Lemma.* (Direct image of sheaves.) Let $X \xrightarrow{\pi} Y$ be a map of topological spaces. For a sheaf $\mathcal{M}$ on X, formula

$$\pi_*(\mathcal{M})\ (V) \stackrel{\text{def}}{=} \mathcal{M}(\pi^{-1}V),$$

defines a sheaf $\pi_*\mathcal{M}$ on Y, and this gives a functor $\mathcal{S}heaves(X) \xrightarrow{\pi_*} \mathcal{S}heaves(Y)$.

6.3.5. *Adjunction between the direct and inverse image operations.* The two basic operations on sheaves are related by adjunction:

Lemma. For sheaves $\mathcal{A}$ on X and $\mathcal{B}$ on Y one has a natural identification

$$\text{Hom}(\pi^{-1}\mathcal{B}, \mathcal{A}) \cong \text{Hom}(\mathcal{B}, \pi_*\mathcal{A}).$$

Proof. We want to compare $\beta \in \text{Hom}(\mathcal{B}, \pi_*\mathcal{A})$ with α in

$$\text{Hom}_{\mathcal{S}h(X)}(\pi^{-1}\mathcal{B}, \mathcal{A}) = \text{Hom}_{\mathcal{S}h(X)}(\widetilde{\underline{\pi^{-1}}\mathcal{B}}, \mathcal{A}) \cong \text{Hom}_{pre\mathcal{S}h(X)}(\underline{\pi^{-1}}\mathcal{B}, \mathcal{A}).$$

α is a system of maps

$$\lim_{\rightarrow\ V\supseteq\pi(U)} \mathcal{B}(V) = \underline{\pi^{-1}}\mathcal{B}(U) \xrightarrow{\alpha_U} \mathcal{A}(U), \quad \text{for } U \text{ open in } X,$$

and β is a system of maps

$$\mathcal{B}(V) \xrightarrow{\beta_V} \mathcal{A}(\pi^{-1}V), \quad \text{for } V \text{ open in } Y.$$

Clearly, any β gives some α since

$$\lim_{\rightarrow\ V\supseteq\pi(U)} \mathcal{B}(V) \xrightarrow{\underset{\rightarrow}{\lim}\ \beta_V} \lim_{\rightarrow\ V\supseteq\pi(U)} \mathcal{A}(\pi^{-1}V) \rightarrow \mathcal{A}(U),$$

the second map comes from the restrictions $\mathcal{A}(\pi^{-1}V) \rightarrow \mathcal{A}(U)$ defined since $V\supseteq\pi(U)$ implies $\pi^{-1}V\supseteq U$.

For the opposite direction, any α gives for each V open in Y, a map $\lim_{\rightarrow\ W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W) = \underline{\pi^{-1}}\mathcal{B}(\pi^{-1}V) \xrightarrow{\alpha_{\pi^{-1}V}} \mathcal{A}(\pi^{-1}V)$. Since $\mathcal{B}(V)$ is one of the terms in the inductive system we have a canonical map $\mathcal{B}(V) \rightarrow \lim_{\rightarrow\ W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W)$, and the composition with the first map $\mathcal{B}(V) \rightarrow \lim_{\rightarrow\ W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W) \xrightarrow{\alpha_{\pi^{-1}V}} \mathcal{A}(\pi^{-1}V)$, is the wanted map β_V.

6.3.6. *Lemma.* (a) If $X \xrightarrow{\pi} Y \xrightarrow{\tau} Z$ then

$$\tau_*(\pi_*\mathcal{A}) \cong (\tau\circ\pi)_*\mathcal{A} \quad \text{and} \quad \tau_*(\pi_*\mathcal{A}) \cong (\tau\circ\pi)_*\mathcal{A}.$$

(b) $(1_X)_*\mathcal{A} \cong \mathcal{A} \cong (1_X)^{-1}\mathcal{A}$.

Proof. The statements involving direct image are very simple and the claims for inverse image follow by adjunction.

6.3.7. *Corollary.* (Pull-back preserves the stalks) For $a \in X$ one has $(\pi^{-1}\mathcal{N})_a \cong \mathcal{N}_{\pi(a)}$.

This shows that the pull-back operation which was difficult to define is actually very simple in its effect on sheaves.

6.4. **Stalks.** Part (a) of the following lemma is the recollection of the description of inductive limits of abelian groups from the remark **??**.

6.4.1. *Lemma.* (Inductive limits of abelian groups.) (a) For an inductive system of abelian groups (or sets) A_i over $(I, \leq)$, inductive limit $\varinjlim A_i$ can be described by

- for $i \in I$ any $a \in A_i$ defines an element $\overline{a}$ of $\varinjlim A_i$,
- all elements of $\varinjlim A_i$ arise in this way, and
- for $a \in A_i$ and $b \in A_j$ one has $\overline{a} = \overline{b}$ iff for some $k \in I$ with $i \leq k \geq j$ one has $a = b$ in A_k.

(b) For a subset $K\subseteq I$ one has a canonical map $\varinjlim_{i\in K} A_i \longrightarrow \varinjlim_{i\in I} A_i$.

Proof. In general (b) is clear from the definition of $\varinjlim$, and for abelian groups also from (a).

6.4.2. *Stalks of (pre)sheaves.* Remember that the stalk of a presheaf $\mathcal{A}$b at a point x is $\mathcal{A}_x \stackrel{\text{def}}{=} \varinjlim \mathcal{A}(U)$, the limit over (diminishing) neighborhoods u of x. This means that

- any $s \in \mathcal{A}(U)$ with $U \ni x$ defines an element s_x of the stalk,
- all elements of $\mathcal{A}_x$ arise in this way, and
- For $s' \in \mathcal{A}(U')$ and $s'' \in \mathcal{A}(U'')$ one has $s'_x = s''_x$ iff for some neighborhood W of x in $U' \cap U''$ one has $s' = s''$ on W.

6.4.3. *Lemma.* For a presheaf $\mathcal{S}$, the canonical map $\mathcal{S} \to \widetilde{\mathcal{S}}$ is an isomorphism on stalks.

Proof. We consider a point $a \in X$ as a map $\text{pt} = \{a\} \xrightarrow{i} X$, so that $\mathcal{A}_x = i^{-1}\mathcal{A}$. For a sheaf B on the point

$$\text{Hom}_{\mathcal{S}h(pt)}(i^{-1}\widetilde{\mathcal{S}}, \mathcal{B}) \cong \text{Hom}_{\mathcal{S}h(X)}(\widetilde{\mathcal{S}}, i_*\mathcal{B}) \cong \text{Hom}_{pre\mathcal{S}h(X)}(\mathcal{S}, i_*\mathcal{B})$$
$$\cong \text{Hom}_{pre\mathcal{S}h(pt)}(i^{-1}\mathcal{S}, \mathcal{B}) = \text{Hom}_{\mathcal{S}h(pt)}(i^{-1}\mathcal{S}, \mathcal{B}).$$

6.4.4. *Germs of sections and stalks of maps.* For any neighborhood U of a point x we have a canonical map $\mathcal{S}(U) \to \lim_{\to\, V\ni x} \mathcal{S}(V) \stackrel{\text{def}}{=} \mathcal{S}_x$ (see lemma 6.4.1), and we denote the image of a section $s\in\Gamma(U,\mathcal{S})$ in the stalk $\mathcal{S}_x$ by s_x, and we call it the germ of the section at x. The germs of two sections are the same at x if the sections are the same on some (possibly very small) neighborhood of x (this is again by the lemma 6.4.1).

A map of sheaves $\phi:\mathcal{A}\to\mathcal{B}$ defines for each $x\in M$ a map of stalks $\mathcal{A}_x\to\mathcal{B}_x$ which we call ϕ_x. It comes from a map of inductive systems given by ϕ, i.e., from the system of maps $\phi_U:\mathcal{A}(U)\to\mathcal{B}(U),\ U\ni x$ (see **??**) ; and on germs it is given by $\phi_x(a_x) = [\phi_U(a)]_x,\ a\in\mathcal{A}(U)$.

For instance, let $\mathcal{A}=\mathcal{H}_{\mathbb{C}}$ be the sheaf of holomorphic functions on $\mathbb{C}$. Remember that the stalk at $a\in\mathbb{C}$ can be identified with all convergent power series in $z-a$. Then the germ of a holomorphic function $f\in\mathcal{H}_{\mathbb{C}}(U)$ at a can be thought of as the power series expansion of f at a. An example of a map of sheaves $\mathcal{H}_{\mathcal{C}}\xrightarrow{\Phi}\mathcal{H}_{\mathbb{C}}$ is the multiplication by an entire function $\phi\in\mathcal{H}_{\mathbb{C}}(\mathbb{C})$, its stalk at a is the multiplication of the the power series at a by the power series expansion of ϕ at a.

6.4.5. The following lemma from homework shows how much the study of sheaves reduces to the study of their stalks.

Lemma. (a) Maps of sheaves $\phi,\psi:\mathcal{A}\to\mathcal{B}$ are the same iff the maps on stalks are the same, i.e., $\phi_x=\psi_x$ for each $x\in M$.

(b) Map of sheaves $\phi:\mathcal{A}\to\mathcal{B}$ is an isomorphism iff ϕ_x is an isomorphism for each $x\in M$.

6.4.6. *Sheafifications via the etale space of a presheaf.* We will construct the sheafification of a presheaf $\mathcal{S}$ (once again) in an "elegant" way, using the etale space $\dot{\mathcal{S}}$ of the presheaf. It is based on the following example of sheaves

Example. Let $Y\xrightarrow{p}X$ be a continuous map. For any open $U\subseteq X$ the elements of

$$\Sigma(U) \stackrel{\text{def}}{=} \{s:U\to Y,\ s \text{ is continuous and } p\circ s = 1_u\}$$

are called the (continuous) sections of p over U. Σ is a sheaf of sets.

To apply this construction we need a space $\dot{\mathcal{S}}$ that maps to X:

- Let $\dot{\mathcal{S}}$ be the union of all stalks $\mathcal{S}_m,\ m\in X$.
- Let $p:\dot{\mathcal{S}}\to X$ be the map such that the fiber at m is the stalk at m.
- For any pair (U,s) with U open in X and $s\in\mathcal{S}(U)$, define a section $\tilde{s}$ of p over U by

$$\tilde{s}(x) \stackrel{\text{def}}{=} s_x\in\mathcal{S}_x\subset\dot{\mathcal{S}},\qquad x\in U.$$

Lemma. (a) If for two sections $s_i \in \mathcal{S}(U_i),\ i=1,2;$ of $\mathcal{S}$, the corresponding sections $\tilde{s}_1$ and $\tilde{s}_2$ of p agree at a point then they agree on some neighborhood of of this point (i.e., if $\tilde{s}_1(x)=\tilde{s}_2(x)$ for some $x\in U_{12}\overset{\text{def}}{=}U_1\cap U_2$, then there is a neighborhood W of x such that $\tilde{s}_1=\tilde{s}_2$ on W).

(b) All the sets $\tilde{s}(U)$ (for $U\subseteq X$ open and $s\in\mathcal{S}(U)$), form a basis of a topology on $\overset{\bullet}{\mathcal{S}}$. Map $p:\overset{\bullet}{\mathcal{S}}\to M$ is continuous.

(c) Let $\tilde{\mathcal{S}}(U)$ denote the set of continuous sections of p over U. Then $\tilde{\mathcal{S}}$ is a sheaf and there is a canonical map of presheaves $\iota:\mathcal{S}\to\tilde{\mathcal{S}}$.

Remark. Moreover, p is "etale" meaning "locally an isomorphism", i.e., for each point $\sigma\in\overset{\bullet}{\mathcal{S}}$ there are neighborhoods $\sigma\in W\subseteq\overset{\bullet}{\mathcal{S}}$ and $p(\sigma)\subseteq U\subseteq X$ such that $p|W$ is a homeomorphism $W\underset{\cong}{\to}U$.

Lemma. The new $\widetilde{\mathcal{S}}$ and the old $\widetilde{\mathcal{S}}$ (from 6.2.4) are the same sheaves (and the same holds for the canonical maps $\iota:\mathcal{S}\to\widetilde{\mathcal{S}}$).

Proof. Sections of p over $U\subseteq X$ are the same as the equivalence classes of systems $\widehat{\mathcal{S}}/\equiv$ defined in 6.2.4.

6.5. **Abelian category structure.** Let us fix a map of sheaves $\mathcal{A}\xrightarrow{\alpha}\mathcal{B}$ since the non-trivial part is the construction of (co)kernels. Consider the example where the space is the circle $X=\{z\in\mathbb{C},\ |z|=1\}$ and $\mathcal{A}=\mathcal{B}$ is the sheaf $\mathcal{C}_X^\infty$ of smooth functions on X, and the map α is the differentiation $\partial=\frac{\partial}{\partial\theta}$ with respect to the angle θ. For $U\subseteq X$ open, $\mathrm{Ker}(\partial_U):\ \mathcal{C}_X^\infty(U)\to\mathcal{C}_X^\infty(U)$ consists of locally constant functions and the cokernel $\mathcal{C}_X^\infty(U)/\partial_U\mathcal{C}_X^\infty(U)$ is

- zero if $U\neq X$ (then any smooth function on U is the derivative of its indefinite integral defined by using the exponential chart $z=e^{i\theta}$ which identifies U with an open subset of $\mathbb{R}$),
- one dimensional if $U=X$ – for $g\in C^\infty(X)$ one has $\int_X\ \partial g=0$ so say constant functions on X are not derivatives (and for functions with integral zero the first argument applies).

So by taking kernels at each level we got a sheaf but by taking cokernels we got a presheaf which is not a sheaf (local sections are zero but there are global non-zero sections, so the object is not controlled by its local properties).

6.5.1. *Subsheaves.* For (pre)sheaves $\mathcal{S}$ and $\mathcal{S}'$ we say that $\mathcal{S}'$ is a sub(pre)sheaf of $\mathcal{S}$ if $\mathcal{S}'(U)\subseteq\mathcal{S}(U)$ and the restriction maps for $\mathcal{S}'$, $\mathcal{S}'(U)\xrightarrow{\rho'}\mathcal{S}'(V)$ are restrictions of the restriction maps for $\mathcal{S}$, $\mathcal{S}(U)\xrightarrow{\rho}\mathcal{S}(V)$.

6.5.2. *Lemma.* (Kernels.) Any map $\mathcal{A} \stackrel{\alpha}{\to} \mathcal{B}$ has a kernel and $\mathrm{Ker}(\alpha)(U) = \mathrm{Ker}[\mathcal{A}(U) \xrightarrow{\phi(U)} \mathcal{B}(U)]$ is a subsheaf of $\mathcal{A}$.

Proof. First, $\mathcal{K}(U) \stackrel{\text{def}}{=} \mathrm{Ker}[\mathcal{A}(U) \xrightarrow{\phi(U)} \mathcal{B}(U)]$ is a sheaf, and then a map $\mathcal{C} \stackrel{\mu}{\to} \mathcal{A}$ is killed by α iff it factors through the subsheaf $\mathcal{K}$ of $\mathcal{A}$.

Lemma. (Cokernels.) Any map $\mathcal{A} \stackrel{\alpha}{\to} \mathcal{B}$ defines a presheaf $C(U) \stackrel{\text{def}}{=} \mathcal{B}(U)/\alpha_U(\mathcal{A}(U))$, the associated sheaf $\mathcal{C}$ is the cokernel of α.

Proof. For a sheaf $\mathcal{S}$ one has

$$\mathrm{Hom}_{\mathcal{S}heaves}(\mathcal{B}, \mathcal{S})_\alpha \cong \mathrm{Hom}_{pre\mathcal{S}heaves}(C, \mathcal{S}) \cong \mathrm{Hom}_{\mathcal{S}heaves}(\mathcal{C}, \mathcal{S}).$$

The second identification is the adjunction. For the first one, a map $\mathcal{B} \stackrel{\phi}{\to} \mathcal{S}$ is killed by α, i.e., $0 = \phi \circ \alpha$, if for each U one has $0 = (\phi \circ \alpha)_U \mathcal{A}(U) = \phi_U(\alpha_U \mathcal{A}(U))$; but then it gives a map $C \stackrel{\overline{\phi}}{\to} \mathcal{S}$, with $\overline{\phi}_U : C(U) = \mathcal{B}(U)/\alpha_U \mathcal{A}(U) \to \mathcal{S}(U)$ the factorization of ϕ_U. The opposite direction is really obvious, any $\psi : C \to \mathcal{S}$ can be composed with the canonical map $\mathcal{B} \to C$ (i.e., $\mathcal{B}(U) \to \mathcal{B}(U)/\alpha_U \mathcal{A}(U)$) to give map $\mathcal{B} \to \mathcal{S}$ which is clearly killed by α.

6.5.3. *Lemma.* (Images.) Consider a map $\mathcal{A} \stackrel{\alpha}{\to} \mathcal{B}$.

(a) It defines a presheaf $I(U) \stackrel{\text{def}}{=} \alpha_U(\mathcal{A}(U)) \subseteq \mathcal{B}(U)$ which is a subpresheaf of $\mathcal{B}$. The associated sheaf $\mathcal{I}$ is the image of α.

(b) It defines a presheaf $c(U) \stackrel{\text{def}}{=} \mathcal{A}(U)/\mathrm{Ker}(\alpha_U)$, the associated sheaf is the coimage of α.

(c) The canonical map $Coim(\alpha) \to Im(\alpha)$ is isomorphism.

Proof. (a) $Im(\alpha) \stackrel{\text{def}}{=} \mathrm{Ker}[\mathcal{B} \to \mathrm{Coker}(\alpha)]$ is a subsheaf of $\mathcal{B}$ and $b \in \mathcal{B}(U)$ is a section of $Im(\alpha)$ iff it becomes zero in $\mathrm{Coker}(\alpha)$. But a section $b + \alpha_U \mathcal{A}(U)$ of C on U is zero in $\mathcal{B}$ iff it is locally zero in C, i.e., there is a cover U_i of U such that $b|U_i \in \alpha_{U_i} \mathcal{A}(U_i)$. But this is the same as saying that b is locally in the subpresheaf I of $\mathcal{B}$, i.e., the same as asking that b is in the corresponding presheaf $\mathcal{I}$ of $\mathcal{B}$.

(b) The coimage of α is by definition $Coim(\alpha) \stackrel{\text{def}}{=} \mathrm{Coker}[\mathrm{Ker}(\alpha) \to \mathcal{A}]$, i.e., the sheaf associated to the presheaf $U \mapsto \mathcal{A}(U)/\mathrm{Ker}(\alpha)(U) = c(U)$.

(c) The map of sheaves $Coim(\alpha) \to Im(\alpha)$ is associated to the canonical map of presheaves $c \to I$, however already the map of presheaves is an isomorphism: $c(U) = \mathcal{A}(U)/\mathrm{Ker}(\alpha)(U) \cong \alpha_U \stackrel{\text{def}}{=} \mathcal{A}(U) = I(U)$.

6.5.4. *Stalks of kernels, cokernels and images; exact sequences of sheaves.*

6.5.5. *Lemma.* For a map of sheaves $\mathcal{A}\xrightarrow{\alpha}\mathcal{B}$ and $x\in X$

- (a) $\mathrm{Ker}(\mathcal{A}\xrightarrow{\alpha}\mathcal{B})_x = \mathrm{Ker}(\alpha_x:\ \mathcal{A}_x \to \mathcal{B}_x)$,
- (b) $\mathrm{Coker}(\mathcal{A}\xrightarrow{\alpha}\mathcal{B})_x = \mathrm{Coker}(\alpha_x:\ \mathcal{A}_x \to \mathcal{B}_x)$,
- (c) $Im(\mathcal{A}\xrightarrow{\alpha}\mathcal{B})_x = Im(\alpha_x:\ \mathcal{A}_x \to \mathcal{B}_x)$.

Proof. (a) Let $x\in U$ and $a\in\mathcal{A}(U)$. The germ a_x is killed by α_x if $0=\alpha_x(a_x)\stackrel{\text{def}}{=}(\alpha_U(a))_x$, i.e., iff $\alpha_U(a)=0$ on some neighborhood U' of x in U. But this is the same as saying that $0=\alpha_U(a)|U' = \alpha_{U'}(a|U')$, i.e., asking that some restriction of a to a smaller neighborhood of x is a section of the subsheaf $\mathrm{Ker}(\alpha)$. And this in turn, is the same as saying that the germ a_x lies in the stalk of $\mathrm{Ker}(\alpha)$.

(b) Map $\mathcal{B}\xrightarrow{q}\mathrm{Coker}(\alpha)$ is killed by composing with α, so the map of stalks $\mathcal{B}_x\xrightarrow{q_x}\mathrm{Coker}(\alpha)_x$ is killed by composing with α_x.

To see that q_x is surjective consider some element of the stalk $\mathrm{Coker}(\alpha)_x$. It comes from a section of a presheaf $U\mapsto\mathcal{B}(U)/\alpha_U\mathcal{A}(U)$, so it is of the form $[b+\alpha_U(\mathcal{A}(U))]_x$ for some section $b\in\mathcal{B}(U)$ on some neighborhood U of x. Therefore it is the image $\alpha_x(b_x)$ of an element b_x of $\mathcal{B}_x$.

To see that q_x is injective, observe that a stalk $b_x\in\mathcal{B}_x$ (of some section $b\mathcal{B}(U)$), is killed by q_x iff its image $\alpha_x(b_x) = [b+\alpha_U(\mathcal{A}(U))]_x$ is zero in $\mathrm{Coker}(\alpha)$, i.e., iff there is a smaller neighborhood $U'\subseteq U$ such that the restriction $[b+\alpha_U(\mathcal{A}(U))]|U' = b|U'+\alpha_{U'}(\mathcal{A}(U'))$ is zero, i.e., $b|U'$ is in $\alpha_{U'}\mathcal{A}(U')$. But the existence of such U' is the same as saying that b_x is in the image of α_x.

(c) follows from (a) and (b) by following how images are defined in terms of kernels and cokernels.

6.5.6. *Corollary.* A sequence of sheaves is exact iff at each point the corresponding sequence of stalks of sheaves is exact.