HOMOLOGICAL ALGEBRA

Contents

X. Sheaves

Sheaves are a machinery which addresses an essential problem – the relation between *local and global* information. So they appear throughout mathematics.

0.1. **Presheaves and sheaves.**

0.1.1. *Presheaves.* Consider a topological space (X, τ) where τ is a topology on the set X. A presheaf on X with values in a given category $\mathcal{C}$ consists of

- for each open $U \subseteq X$ an object $\mathcal{S}(U)$ in the category $\mathcal{C}$,
- for each inclusion of open subsets $V \subseteq U \subseteq X$ a morphism $\mathcal{S}(U) \xrightarrow{\rho^U_V} \mathcal{S}(V)$ in $\mathcal{C}$ (we call it the *restriction* map);

such that the trivial restriction ρ^U_U is $1_{\mathcal{S}(U)}$ and for $W \subseteq V \subseteq U$ we have $\rho^U_V \circ \rho^U_V = \rho^U_W$.

Remark. If τ is considered as a poset for $U \leq V$ if $U \supseteq V$ then a presheaf is just a functor $\mathcal{S} : (\tau, \leq) \to \mathcal{C}$.

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Examples. (0) If Y is another topological space we can associate to each open $U\subseteq X$ the set $\mathcal{C}_X(U,Y) \stackrel{\text{def}}{=} Map_{\mathcal{T}op}(U,Y)$ of continuous maps from U to Y. Then $\mathcal{C}_X(-,Y)$ is a presheaf for the operations ρ_V^U given by restrictions of functions.

(1) Let M is any smooth manifold, say $\mathbb{R}^n$. Let V denote $\mathbb{R}$ or $\mathbb{C}$. The notion of V-valued smooth functions on X gives a presheaf $\mathcal{C}_M^\infty(-,V)$ on M with values in the category $Alg_\mathbb{R}$ of $\mathbb{R}$-algebras. To each open $U\subseteq X$ it associates the $\mathbb{R}$-algebra $C^\infty(U,V)$ of smooth functions on U with values in V. The maps ρ_V^U are again the restriction maps.

(gluing) if the functions $f_i \in C^\infty(U_i)$ on open subsets $U_i\subseteq X,\ i\in I$, are compatible in the sense that $f_i = f_j$ on the intersections $U_{ij} \stackrel{\text{def}}{=} U_i\cap U_j$, then they glue into a unique smooth function f on $U = \cup_{i\in I} U_i$.

So, smooth functions can be restricted and glued from compatible pieces.

More generally, if we choose V to be any finite dimensional real vector space V we again get a presheaf $\mathcal{C}_M^\mathcal{Y}(-,V)$ on M but this time with values in the category $\mathcal{V}ec_\mathbb{R}$ of real vector spaces. We can even let V be any smooth manifold then $\mathcal{C}_M^\mathcal{Y}(-,V)$ is a presheaf $\mathcal{C}_M^\mathcal{Y}(-,V)$ withe values in Set.

(3) To a set S one can associated the *constant presheaf* $\underline{S_X}$ on any topological space X — we choose $\underline{S_X}(U)$ to S for any U and all maps ρ_V^U to be identity.

Remark. We will mostly consider categories of *structured sets* , i.e., categories $\mathcal{C}$ whose objects are sets with some extra structure ("of type $\mathcal{C}$"). For instance $\mathcal{C}$ could be one of categories $\mathcal{S}et$, $\mathcal{A}b$, $\mathcal{R}ing$, $\mathfrak{m}(\Bbbk)$ etc.

Then for a $\mathcal{C}$-valued presheaf $\mathcal{S}$ any$\mathcal{S}(U)$ is in particular a set and we can consider its elements. An element $s\in\mathcal{S}(U)$ is called a *section of the presheaf* $\mathcal{S}$ *on* U.

We will say that for $W\subseteq U,V$, two sections $a\in\mathcal{S}(U)$ and $b\in\mathcal{S}(V)$ are *the same on* W if $\rho_W^U a = \rho_W^V b$ in the set $\mathcal{S}(W)$.

0.1.2. *Sheaves on a topological space.* Any presheaf $\mathcal{F}$ is a mechanism that relates some global information $\mathcal{F}(X)$ to local information $\mathcal{F}(U)$ for open U's by restriction maps ρ_V^U. However, we are really interested in a stronger relation.

Let $\mathcal{C}$ be some category of structured sets. A $\mathcal{C}$-valued *sheaf* $\mathcal{F}$ on a topological space $(X,\mathcal{T})$ is a $\mathcal{C}$-valued presheaf that satisfies the following gluing property:

- Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of an open $U\subseteq X$. Let $f_\bullet$ be a family of sections $f_i\in\mathcal{F}(U_i),\ i\in I$, which are compatible in the sense that for any pair $i,j\in I$ the sections f_i and f_j are the same on $U_{ij} \stackrel{\text{def}}{=} U_i\cap U_j$ (i.e., $\rho_{U_{ij}}^{U_i} f_i = \rho_{U_{ij}}^{U_i} f_j$). Then there is a unique $f\in\mathcal{S}(U)$ such that $\rho_{U_i}^U f = f_i$ in $\mathcal{S}(U_i),\ i\in I$.

Examples. (a) *Local property.* The gluing property holds for arbitrary functions: if one knows them on U_i and they agree on intersections then they do define a function of U.

The property remains to hold for classes of functions which are defined by a *local property.* For instance examples $\mathcal{C}_X(-,Y)$ and $\mathcal{C}^\infty(-,V)$ are sheaves because continuity and infinite differentiability of a function can be checked on arbitrary near each point (on an arbitrary small neighborhood). For instance, for a given function f being: (i) a function with values in a given S, (ii) non-vanishing (i.e., invertible), (iii) a solution of a given system $(*)$ of differential equations; these are all local conditions: they can be checked in a neighborhood of each point.

Other classes of functions need not form sheaves. Say, on $\mathbb{R}$ associating to each open U the square integrable functions $L^2(U)$ is a presheaf but not a sheaf since the constant function is locally L^2 but not globally. Another property that is not local is *compact support* (denoted here $C_c(-)$), actually for $U\supseteq V$ one does not even have a restriction map $C_c(U) \to C_c(V)$.[1]

(b) *Sheafification.* The constant presheaf $\underline{S_X}$ is not a sheaf because if $U,V\subseteq X$ are disjoint and $a,b \in S$ are viewed as sections $a \in \underline{S_X}(U)$ and $b \in \underline{S_X}(V)$ then they agree on the intersection $U\cap V = \emptyset$ but can not be glued a section in $\mathcal{S}(U\cup V) = S$. However, it is an obvious improvement $S_X\supseteq\underline{S_X}$ which is a sheaf. The *constant sheaf* S_X on X is defined by

$$S_X(U) \textit{ is the set of } \text{locally } \textit{constant functions from } U \textit{ to } X.$$

[Notice that $\underline{S_X}$ can be viewed as the presheaf of *constant* functions with values in S and "constant" is not a local property.

We will generalize this example by a procedure ("sheafification") which canonically improves any presheaf to a sheaf.

(c) Most any geometric structure can be described as a *ringed space* $(X,\mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X which we call the *structure sheaf* of the geometric object. For instance $\mathcal{O}_X$ can be continuous functions $\mathbb{C}_X(-,\mathbb{R})$ or smooth functions $\mathcal{C}^\infty_X(-,\mathbb{R})$ or holomorphic functions of or "polynomial". In each of these cases the topology on X and the sheaf contain all information on the structure of X.

0.2. **Global sections functor** $\Gamma : \mathcal{S}heaves(X) \to \mathcal{S}et$**.** Elements of $\mathcal{S}(U)$ are called the sections of a sheaf $\mathcal{S}$ on $U\subseteq X$ (this terminology is from classical geometry). By $\Gamma(X,\mathcal{S})$ we denote the set $\mathcal{S}(X)$ of global sections.

The construction $\mathcal{S}\mapsto\Gamma(\mathcal{S})$ means that we are looking at *global objects* in a given class $\mathcal{S}$ of objects. We will see that that the construction Γ comes with a hidden part, the cohomology $\mathcal{S}\mapsto H^\bullet(X,\mathcal{S})$ of sheaves on X.

Γ acquires different meaning when applied to different classes of sheaves. For instance for the constant sheaf pt_X, $\Gamma(X,\mathrm{pt}_X)$ is the set of connected components of X. On any smooth manifold X, $\Gamma(C^\infty) = C^\infty(X)$ is "huge" and there are no higher cohomologies

[1] One does have a map in the wrong direction $C_c(V)\subseteq C_c(U)$, so C_c has a dual property of being a *cosheaf!*.

("nothing hidden"). The holomorphic setting is more subtle in this sense, on a compact connected complex manifold $\Gamma(X, \mathcal{H}_H)$ consists of only the constant functions and a lot of information may be stored in higher cohomology groups.

0.2.1. *Solutions of differential equations.* Solutions of a system $(*)$ of differential equation on X form a sheaf $\mathcal{S}ol_{(*)}$. If X is an interval I in $\mathcal{R}$ and $(*)$ is one equation of the form $y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_0(t) = 0$ with $a_i \in C^\infty(I)$ then any point $c \in I$ gives evaluation isomorphism of vector spaces $\mathcal{S}ol_{(*)}(I) \xrightarrow{E_c} \mathbb{C}^n$ by $E_c(y) = (y(c), ..., y^{(n-1)}(c))$ (solutions correspond to initial conditions!). The sheaf theoretic encoding of this property of the *initial value problem* is :

Lemma. $\mathcal{S}ol_{(*)}$ is a constant sheaf on X.

On the other hand let $(*)$ be the equation $zy' = \lambda y$ considered as equation in holomorphic functions on $X = \mathbb{C}^*$. The solutions are multiples of functions z^λ defined using a branch of logarithm. On any disc $D \subseteq X$, evaluation at a point $c \in D$ still gives $\mathcal{S}ol_{(*)}(D) \xrightarrow{\cong} \mathbb{C}$, so the local behavior of the is simple – it is a locally constant sheaf. However, $\Gamma(X, \mathcal{S}ol_{(*)}) = 0$ if $\lambda \notin \mathbb{Z}$ (any global solution would change by a factor of $e^{2\pi i\lambda}$ as we move once around the origin). So locally there is the expected amount but nothing globally.

0.3. **Projective line $\mathbb{P}^1$ over $\mathbb{C}$.** $\mathbb{P}^1 = \mathbb{C} \cup \infty$ can be covered by $U_1 = U = \mathbb{C}$ and $U_2 = V = \mathbb{P}^1 - \{0\}$. We think of $X = \mathbb{P}^1$ as a complex manifold by identifying U and V with $\mathbb{C}$ using coordinates u, v such that on $U \cap V$ one has $uv = 1$.

Lemma. $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$.

Proof. (1) *Proof using a cover.* A holomorphic function f on X restricts to $f|U = \sum_{n\geq 0} \alpha_n u^n$ and to $f|V = \sum_{n\geq 0} \beta_n v^n$. On $U \cap V = \mathbb{C}^*$, $\sum_{n\geq 0} \alpha_n u^n = \sum_{n\geq 0} \beta_n u^{-n}$, and therefore $\alpha_n = \beta_n = 0$ for $n \neq 0$.

(2) *Proof using maximum modulus principle.* The restriction of a holomorphic function f on X to $U = \mathbb{C}$ is a bounded holomorphic function (since X is compact), hence a constant.

0.4. **Čech cohomology of a sheaf $\mathcal{A}$ with respect to a cover $\mathcal{U}$.** Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X. We will use finite intersections $U_{i_0,\ldots,i_p} \stackrel{\text{def}}{=} U_{i_0} \cap \cdots \cap U_{i_p}$.

0.4.1. *Calculations of global sections using a cover.* Motivated by the calculation of global sections in 0.3, to a sheaf $\mathcal{A}$ on X we associate

- Set $\mathcal{C}^0(\mathcal{U}, \mathcal{A}) \stackrel{\text{def}}{=} \prod_{i\in I} \mathcal{A}(U_i)$ whose elements are systems $f = (f_i)_{i\in I}$ with one section $f_i \in \mathcal{A}(U_i)$ for each open set U_i,
- $\mathcal{C}^1(\mathcal{U}, \mathcal{A}) \stackrel{\text{def}}{=} \prod_{(i,j)\in I^2} \mathcal{A}(U_{ij})$ whose elements are systems $g = (g_{ij})_{I^2}$ of sections g_{ij} on all intersections U_{ij}.

Now, if $\mathcal{A}$ is a sheaf of abelian groups we can reformulate the calculations of global sections of $\mathcal{A}$ in terms of the open cover $\mathcal{U}$

For this we encode the comparison of f_i's on intersections U_{ij} in terms of a map d : $\mathcal{C}^0(\mathcal{U},\mathcal{A}) \xrightarrow{d} \mathcal{C}^1(\mathcal{U},\mathcal{A})$ which sends $f = (f_i)_I \in \mathcal{C}^0$ to $dff \in C^1$ with

$$(dff)_{ij} = \ \rho^{U_j}_{U_{ij}} f_j - \rho^{U_i}_{U_{ij}} f_i.$$

More informally, $(dff)_{ij} = \ f_j|U_{ij} - f_i|U_{ij}$.

Lemma. $\Gamma(\mathcal{A}) \xrightarrow{\cong} \mathrm{Ker}[C^0(\mathcal{U},\mathcal{A}) \xrightarrow{d} C^1(\mathcal{U},\mathcal{A})]$.

0.4.2. *Čech complex* $\mathcal{C}^\bullet(\mathcal{U},\mathcal{A})$. Emboldened, we try more of the same and define the abelian groups

$$\mathcal{C}^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \prod_{(i_0,\dots,i_n)\in I^n} \mathcal{A}(U_{i_0,\dots,i_n})$$

of systems of sections on multiple intersections, and relate them by the maps $\mathcal{C}^n(\mathcal{U},\mathcal{A}) \xrightarrow{d^n} \mathcal{C}^{n+1}(\mathcal{U},\mathcal{A})$ which sends $f = (f_{i_0,\dots,i_n})_{I^n} \in \mathcal{C}^n$ to $d^n \ f \in \mathcal{C}^{n+1}$ with

$$(d^n \ f)_{i_0,\dots,i_{n+1}} = \sum_{s=0}^{n+1} (-1)^s f_{i_0,\dots,i_{s-1},i_{s+1},\dots,i_{n+1}}.$$

Lemma. $(\mathcal{C}^\bullet(\mathcal{U},\mathcal{A}), d^\bullet)$ is a complex, i.e., $d.\circ d^{n-1} = 0$.

0.4.3. *Čech cohomology* $\check{H}^\bullet(X,\mathcal{U};\mathcal{A})$. It is defined as the cohomology of the Čech complex $\mathcal{C}^\bullet(\mathcal{U},\mathcal{A})$. We have already observed that

Lemma. $\check{H}^0(X,\mathcal{U};\mathcal{A}) = \ \Gamma(\mathcal{A})$.

0.4.4. *The "small Čech complex".* If the set I has a complete ordering, we can choose in $\mathcal{C}^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \ \prod_{(i_0,\dots,i_n)\in I^n} \ \mathcal{A}(U_{i_0,\dots,i_n})$ a subgroup $C^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \ \prod_{i_0<\cdots<i_n} \ \mathcal{A}(U_{i_0,\dots,i_n})$. This is what we will usually use in computations since it is smaller but it also computes the Čech cohomology:

Lemma. (a) $C^\bullet(\mathcal{U},\mathcal{A})\subseteq\check{C}^\bullet(\mathcal{U},\mathcal{A})$ is a subcomplex (i.e., it is invariant under the differential).

(b) Map of complexes $C^\bullet(\mathcal{U},\mathcal{A})\subseteq\mathcal{C}^\bullet(\mathcal{U},\mathcal{A})$ is a quasi-isomorphism.

Proof. (a) is clear. (b) is intuitively plausible since the extra data in $\mathcal{C}$ is a duplication of data in $C\bullet$, say $\mathcal{C}^1$ contains $\mathcal{S}(U_{ii}) = \mathcal{S}(U_i)\subseteq C^0$ and for $i < j$ it contains $\mathcal{S}(U_{ij})$ the second time under the name $\mathcal{S}(U_{ji})$.

0.5. **Vector bundles.** We recall the notion of a vector bundle, i.e., a vector space smeared over a topological space. We will be interested in calculating cohomology of sheaves associated to vector bundles.

0.5.1. *Vector bundle over space X.* In general one can extend many notions to the *relative setting* over some *base* X. For instance, a reasonable notion of a "vector space over a set X" is a collection $V = (V_x)_{x\in X}$ of vector spaces, one for each point of X. Then the total space $V = \sqcup_{x\in X} V_x$ maps to X and the fibers are vector spaces. If X is a topological space, we want the family of V_x to be "continuous in x". This leads to the notion of a vector bundle over a topological space.

Le $\Bbbk = \mathbb{R}$ or $\mathbb{C}$. The data for a $\Bbbk$-vector bundle of rank n over a topological space X consists of a map of topological spaces $\pi : V \to X$ and the vector space structures on fibers $V_x = \pi^{-1}x,\ x \in X$. These data should locally be isomorphic to to $X{\times}\Bbbk^n$ in the sense that each point has a neighborhood U such that there exists a homeomorphism $\phi : V|U \overset{\text{def}}{=} \pi^{-1}U \xrightarrow{\cong} U{\times}\Bbbk^n$ such that

$$\begin{array}{ccc} V|U & \xrightarrow{\phi} & U{\times}\Bbbk^n \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle pr_1} \\ X & \xrightarrow{=} & X \end{array}$$

and that the corresponding maps of fibers $V_x \to \Bbbk^n,\ x \in U$, are isomorphisms of vector bundles.

Similarly one defines vector bundle over manifolds or over complex manifolds by requiring that π are local trivialization maps ϕ are smooth or holomorphic.

0.5.2. *Examples.*

(1) The smallest interesting example is the *Moebius strip.* Moebius strip is a line bundle over S^1 (it projects to the central curve S^1 and the fibers are real lines).

(2) *(Co)tangent bundles* On each manifold X there are the tangent and cotangent vector bundles TX, T^*X. In terms of local coordinates x_i at a, the fibers are $T_aX = \oplus\ \mathbb{R}\frac{\partial}{\partial x_i}$ and $T^*_aX = \ \oplus\ \mathbb{R}dx_i$.

(3) Any vector bundle can be obtained by gluing trivial vector bundles $V_i = U_i{\times}\Bbbk^n$ on an open cover $\mathcal{U} = (U_i)_{i\in I}$, The gluing data is given by transition functions

$$\phi_{ij} : U_{ij} \to GL_n(\Bbbk).$$

The corresponding vector bundle is the quotient

$$V \ = \ [\sqcup_{i\in I}\ U_i{\times}\Bbbk^n]/\sim$$

for the equivalence relation given by: $(u_i, z) \in U_i{\times}\Bbbk^n = V_i$ and $(u_j, w) \in U_j{\times}\Bbbk^n = V_j$ are equivalent iff they are related by the corresponding transition function, i.e., $u_i = u_j$ and $z = \phi_{ij}(u_j){\cdot}w$.

0.5.3. *Sheaf $\mathcal{V}$ associated to a vector bundle V.* Let $V \xrightarrow{\pi} M$ be a vector bundle over M. Define the *sections of the vector bundle V* over an open $U \subseteq X$, by

$$\mathcal{V}(U) \stackrel{\text{def}}{=} \{s : U \to V;\ \pi \circ s = id_U\}.$$

More precisely,

If V is obtained by gluing trivial vector bundles $V_i = U_i \times \mathbb{C}^n$ by transition functions ϕ_{ij}, then $\mathcal{V}(U)$ consists of all systems of $f_i \in \mathcal{H}(U_i \cap U, \mathbb{C}^n)$ such that on all intersections $U_{ij} \cap U$ one has $f_i = \phi_{ij} f_j$.

0.6. **Čech cohomology of line bundles on $\mathbb{P}^1$.**

Lemma. $\check{H}^\bullet(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \check{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$.

Proof. Since the cover we use $\mathcal{U} = \{U_1, U_2\}$ has two elements, $C^n = 0$ for $n > 1$. We know $\check{H}^0 = \Gamma(\mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$, so it remains to understand $\check{H}^1 = C^1/dC^0 = \mathcal{O}(U \cap V)/[\mathcal{O}(U) + \mathcal{O}(V)$, i.e., all Laurent series $\phi = \sum_{-\infty}^{+\infty} \gamma_n u^n$ that converge on $\mathbb{C}^*$, modulo the series $\sum_0^{+\infty} \lambda_n u^n$ and $\sum_0^{+\infty} \beta_n u^{-n}$, that converge on $\mathbb{C}$ and on $\mathbb{P}^1 - 0$. However, if a Laurent series $\phi = \sum_{-\infty}^{+\infty} \gamma_n u^n$ converges on $\mathbb{C}^*$,then Laurent series $\phi^+ = \sum_0^{+\infty} \gamma_n u^n$ converges on $\mathbb{C}$, and $\phi^- = \sum_{-\infty}^{-1} \gamma_n u^n$ converge on $\mathbb{C}^* \cup \infty$.

0.6.1. *Line bundles L_n on $\mathbb{P}^1$.* On $\mathbb{P}^1$ let L_n be the vector bundle obtained by gluing trivial vector bundles $U \times \mathbb{C}$, $V \times \mathbb{C}$ over $U \cap V$ by identifying $(u, \xi) \in U \times \mathbb{C}$ and $(v, \zeta) \in V \times \mathbb{C}$ if $uv = 1$ and $\zeta = u^n \cdot \xi$. So for $U_1 = U$ and $U_s = V$ one has $\phi_{12}(u) = u^n,\ U \in U \cap V \subseteq U$. Let $\mathcal{L}_n$ be the sheaf of holomorphic sections of L_n.

Lemma. (a) $\Gamma(\mathbb{P}^1, \mathcal{L}_n) = 0$ for $n < 0$ and for $n \geq 0$ the dimension is $n + 1$ and

$$\Gamma(\mathbb{P}^1, \mathcal{L}_n) \cong \mathbb{C}_{\leq n}[u] \stackrel{\text{def}}{=} \text{ the polynomials in } u \text{ of degree} \leq n$$

$$\cong \mathbb{C}_n[x, y] \stackrel{\text{def}}{=} \text{ homogeneous polynomials in } x, y \text{ of degree } n.$$

(b) $\check{H}^1_{\mathcal{U}}(\mathbb{P}^1; \mathcal{L}_n) = 0$ for $n \geq -1$ and for $d \geq 1$, we have $\dim[\check{H}^1_{\mathcal{U}}(\mathbb{P}^1; \mathcal{L}_{-d})] = d - 1$.

0.6.2. *Sheaves of meromorphic functions associated to divisors.* For distinct points $P_1, ..., P_n$ on $\mathbb{P}^1$, and integers D_i, define the sheaf $\mathcal{L} = \mathcal{O}(\sum D_i P_i)$ by $\mathcal{L}(U) \stackrel{\text{def}}{=}$ "all holomorphic functions f on $U - \{P_1, ..., P_n\}$, such that $\text{ord}_{P_i} f \geq -D_i$. Then

Lemma. $\mathcal{O}(\sum D_i P_i) \cong \mathcal{L}_{\sum D_i}$.

0.7. **Geometric representation theory.** Group $SL_2(\mathbb{C})$ acts on $\mathbb{C}^2$ and therefore on

- (i) polynomial functions $\mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x, y]$,
- (ii) each $\mathbb{C}_n[x, y]$;
- (iii) complex manifold $\mathbb{P}^1$ (the set of all lines in $\mathbb{C}^2$), and less obviously on
- (iv) each $\mathcal{L}_n$, hence also on
- (v) each $H^i(\mathbb{P}^1, \mathcal{L}_n)$.

In fact,

Lemma. $\mathbb{C}_n[x, y] = \Gamma(\mathbb{P}^1, \mathcal{L}_n), n = 0, 1, 2, ..$ is the list of

all irreducible finite dimensional holomorphic representations of SL_2.

By restricting the action to $SU(2)\subseteq SL(2, \mathbb{C})$ we find that this is also the list of all irreducible finite dimensional representations of $SU(2)$ on complex vector spaces.

0.7.1. *Borel-Weil-Bott theorem.* For each semisimple (reductive) complex group G there is a space $\mathcal{B}$ (the flag variety of G) such that all irreducible finite dimensional holomorphic representations of G are obtained as global sections of all line bundles on $\mathcal{B}$.

0.8. **Relation to topology.** Let $\Bbbk$ be any field. The cohomology of the constant sheaf $\Bbbk_X$ on a topological space X coincides with the *cohomology* $H^\bullet(X, \Bbbk)$ of X with coefficients in $\Bbbk$. The cohomology is defined as the dual of homology

$$H^i(X, \Bbbk) \stackrel{\text{def}}{=} H_i(X, \Bbbk)^*.$$

For instance,

Lemma. For $X = S^1$, $\check{H}^*_{\mathcal{U}}(X, \Bbbk_X)$ is dual to $H_*(X, \Bbbk)$.