HOMOLOGICAL ALGEBRA

Contents

9. **Derived categories of abelian categories**

9.1. **Summary.** The course covers three main constructions

(1) The cohomology functors $C(\mathcal{A}) \xrightarrow{H^n} \mathcal{A}$ for an abelian category $\mathcal{A}$. One tool for computing cohomology are spectral sequences.
(2) The homotopy category of complexes $K(\mathcal{A})$ for an additive category $\mathcal{A}$. These are used to make injective/projective resolutions canonical and therefore to make sense of the derived functors. If $\mathcal{A}$ is abelian with enough injectives then to an additive functor $F : A \to \mathcal{B}$, we associate its right derived functor $K^+(\mathcal{A}) \xrightarrow{RF} K^+(\mathcal{B})$ by $RF(A) \overset{\text{def}}{=} F(I)$ for any injective resolution I of A. One can say that this construction makes manifest some information hidden in the functor F itself.
(3) The derived category of complexes $D(\mathcal{A})$ for an abelian category $\mathcal{A}$. It works much the same as one produces derived functors $D^+(\mathcal{A}) \xrightarrow{RF} D^+(\mathcal{B})$ by the same formulas $RF(A) \overset{\text{def}}{=} F(I)$ for any injective resolution I of A. However, the same idea of de4rived functors works technically better in this setting.

Date: ?

The treatment of Homological Algebra in this course is mostly "classical" in the sense that we concentrate on defining functors $R^nF : \mathcal{A} \to \mathcal{A}$ and we use spectral sequences to calculate these. In the modern approach the main heroes are the derived categories and the derived functors. We will see a sketch of how this is used in ...

9.1.1. *History.* The notion of triangulated categories has been discovered independently in

- In algebra it appeared as the (temporarily) ultimate setting for *Homological Algebra*, i.e., for calculus of complexes in abelian categories. The introduction by Verdier and Grothendieck has been motivated largely by the needs of algebraic geometry. The first complete treatment of a mathematical field by derived categories may have been in topology under a name of *Verdier duality*.
- In topology derived categories appeared independently in *stable homotopy theory*. This is the category $\mathcal{H}$ whose objects are topological spaces but the choice of morphisms is more subtle. The first ingredient is use of homotopy and the "stable" refers to a a process of "linearization" of spaces (an abstract version of the procedure of taking tangent spaces or deformations to normal cones).

9.1.2. *Stable homotopy theory.* Surprisingly, the structure of the stable category $\mathcal{H}$ of topological spaces is again that of a triangulated category, the same as for a derived category $D(\mathcal{A})$. In $\mathcal{H}$ the shift is giving by the *suspension* operation while the distinguished triangles come from the topological construction of the mapping cone C_f of a map $f : X \to Y$ of topological spaces.[(1)]

Today, the homotopy theory is viewed as the *derived set theory.* The *parallel*:

- One only considers topological spaces representable by CW complexes and these can be described in terms of *simplicial sets.* This simplicial sets are related to complexes of abelian groups.
- In the setting of simplicial sets the correct version of constructions is the *derived one.* This means that one uses the analogues of injective/projective resolutions which are here called the *(co)fibrant replacements.*

9.1.3. *Derived categories and the derived algebraic geometry.* A recent step by Lurie and others was a development of the *derived version of category theory.* It is usually called the "$(\infty, 1)$-*categories*", or less precisely the "*infinity categories*".

Subsequently, in hist second book Lurie developed the algebra on the level of derived categories.

[1] Topologists did not notice the Octahedron axiom. Also, there may be no published/readable presentation of the construction of the triangulated category of topological spaces.

On the other hand it is still not known whether the Octahedron axiom is a consequence of other axioms. It requires existence of an exact triangle such that five squares commute. Existence of the triangle and commutativity of 4 squares is a consequence of the remaining axioms.

In the third book (under construction) he constructs the derived version of algebraic geometry.

Remark. Here we will only be concerned with the standard appearance of derived categories in algebra.

9.2. **Localization of categories.** For a class $\mathcal{S}$ of morphisms in a category $\mathcal{C}$, the *localization* $\mathcal{C}[\mathcal{S}^{-1}]$ is the category obtained by inverting all morphisms in $\mathcal{S}$.

More precisely, consider all functors $\mathcal{C} \xrightarrow{\lambda} \mathcal{D}$ such that for each $s \in \mathcal{S}$ the morphism $\lambda(s)$ in $\mathcal{D}$ is invertible. Such pairs $(\mathcal{D}, \lambda)$ form a category and the localization is defined as the *initial* object $\mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$ in this category.

We will assume that $\mathcal{S}$ contains all units $\mathcal{S}$ is closed under composition.[2]

9.2.1. *Existence and uniqueness of localization.*

Lemma. $\mathcal{C}[\mathcal{S}^{-1}]$ exists. It can be constructed in the following way. The objects are the same as for $\mathcal{C}$ and for $a, b \in \mathcal{C}$ the morphisms $\mathrm{Hom}_{\mathcal{C}[\mathcal{S}^{-1}]}(a,b)$ are *equivalence classes* of diagrams in $\mathcal{C}$ with $s_i \in \mathcal{S}$

$$a \xrightarrow{\phi_1} a_1 \xleftarrow{s_1} b_1 \xrightarrow{\phi_2} a_2 \xleftarrow{s_2} \cdots \xrightarrow{\phi_{n-1}} a_{n-1} \xleftarrow{s_{n-1}} b_{n-1} \xrightarrow{\phi_n} a_n \xleftarrow{s_n} b_n = b.$$

Remarks. (0) The composition of such diagrams is given by concatenation.

(1) The functor $\iota : \mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$ takes a map $a \xrightarrow{\phi} b$ in $\mathcal{C}$ to the diagram $\iota(\phi) \overset{\text{def}}{=} (a \xrightarrow{\phi} b \xleftarrow{1_b} b)$.

(2) I do not describe the equivalence relation completely. However, one does impose that for any $s : a \to b$ in $\mathcal{S}$ the diagram $a \xrightarrow{s} b \xleftarrow{s} a$ is equivalent to $a \xrightarrow{1_a} a \xleftarrow{1_a} a$. This has the effect of inverting $\iota(s)$, the inverse is $b \xrightarrow{1_b} b \xleftarrow{s} a$.

Therefore, a long diagram as above will represent the morphism $\iota(s_n)^{-1}\, \iota(\phi_n) \cdots \iota(s_1)^{-1}\, \iota(\phi_1)$ in $\mathcal{C}[\mathcal{S}^{-1}]$ in $\mathcal{C}[\mathcal{S}^{-1}]$.

9.2.2. *Calculation of localization under favorable circumstances.* As we see the general construction of localization in 9.2.1 is "easy" but it is difficult to control what are the morphisms in $\mathcal{S}[\mathcal{S}^{-1}]$. Here is a case where the construction is much simpler.

[2] This can be achieved by enlarging $\mathcal{S}$ without changing the problem , i.e., the desired object $\mathcal{C}[\mathcal{S}^{-1}]$.

Lemma. Suppose that $\mathcal{S}$ also satisfies the condition

- $(\star)$ A diagram $a \xrightarrow{\phi} x \xleftarrow{s} b$ with $s \in \mathcal{S}$ can be canonically completed to a commutative diagram $\begin{array}{ccc} y & \xrightarrow{\psi} & b \\ \downarrow{\scriptstyle u} & & {\scriptstyle s}\downarrow \\ a & \xrightarrow{\phi} & x \end{array}$ with $u \in \mathcal{S}$.

  Also, one can also canonically complete u, ψ with $u \in \mathcal{S}$ to the commutative diagram above with $s \in \mathcal{S}$.

Then $\mathrm{Hom}_{\mathcal{C}[\mathcal{S}^{-1}]}(a, b)$ can be described as equivalence classes of diagrams in $\mathcal{C}$ of the form

$$a \xrightarrow{\phi} x \xleftarrow{s} b, \qquad \text{with } s \in \mathcal{S};$$

where, two diagrams $a \xrightarrow{\phi_i} x_i \xleftarrow{s_i} b$, $i = 1, 2$; are equivalent iff they are "$\mathcal{S}$-quasi-isomorphic" to a third diagram, in the sense that there are maps $x_i \xrightarrow{u_i} x$, $i = 1, 2$ in $\mathcal{S}$ such that

$$\begin{array}{rcl} u_1 \circ \phi_1 = & u_2 \circ \phi_2 \\ u_1 \circ s_1 = & u_2 \circ s_2 \end{array}, \qquad \text{i.e.,} \qquad \begin{array}{ccccc} a & \xrightarrow{\phi_1} & x_1 & \xleftarrow{s_1} & b \\ {\scriptstyle =}\downarrow & & {\scriptstyle u_1}\downarrow & & {\scriptstyle =}\downarrow \\ a & \xrightarrow{\phi} & x & \xleftarrow{s} & b \\ {\scriptstyle =}\uparrow & & {\scriptstyle u_2}\uparrow & & {\scriptstyle =}\uparrow \\ a & \xrightarrow{\phi_2} & x_2 & \xleftarrow{s_2} & b. \end{array}$$

Remark. The composition of (equivalence classes of) diagrams is based on the property $(\star)$. We want to compose the morphisms $a \xrightarrow{\alpha_1} b \xrightarrow{\alpha_2} c$ represented by the diagrams $a \xrightarrow{\phi_1} x_1 \xleftarrow{s_1} b$ and $b \xrightarrow{\phi_2} x_2 \xleftarrow{s_2} c$ with $s_i \in \mathcal{S}$.

First, one puts the diagrams together into a longer diagram $a \xrightarrow{\phi_1} x_1 \xleftarrow{s_1} b \xrightarrow{\phi_2} x_2 \xleftarrow{s_2} c$. Then the property $(\star)$ gives a canonical replacement for the "inner" part $x_1 \xleftarrow{s_1} b \xrightarrow{\phi_2} x_2$, by a diagram of the form $x_1 \xrightarrow{\phi} x \xleftarrow{s} x_2$ with $s \in \mathcal{S}$. This gives a diagram $a \xrightarrow{\phi_1} x_1 \xrightarrow{\phi} x \xleftarrow{s} x_2 \xleftarrow{s_2} c$. Then the composition $\alpha_2 \circ \alpha_1$ is represented by

$$a \xrightarrow{\phi \circ \phi_1} x \xleftarrow{s \circ s_2} c.$$

9.3. **The derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$.** We define $D(\mathcal{A})$ as the localization of $K(\mathcal{A})$ by quasi-isomorphisms

$$D(\mathcal{A}) \stackrel{\text{def}}{=} K(\mathcal{A})[qis^{-1}].$$

Lemma. $K(\mathcal{A})$ satisfies the property $(\star)$.

9.3.1. *Remarks.* So, objects in $D(\mathcal{A})$ are (again) the complexes in $\mathcal{A}$, while morphisms $\alpha \in \mathrm{Hom}_{D(\mathcal{A})}(A, B)$ are as in lemma 9.2.2, i.e., the equivalence classes of diagrams in $K(\mathcal{A})$ of the form $A \xrightarrow{\phi} X \xleftarrow{s} B$ with s a quasi-isomorphism.

9.3.2. *$D(\mathcal{A})$ as a localization of $C(\mathcal{A})$.* The following is just a philosophical remark.

Lemma. (a) The localization of $C(\mathcal{A})$ by the class of morphisms that are homotopy equivalences is exactly $K(\mathcal{A})$.

(b) The localization of $K(\mathcal{A})$ by quasi-isomorphisms is exactly $D(\mathcal{A})$.

Proof. (a) We have defined $K(\mathcal{A})$ from $C(\mathcal{A})$ by identifying homotopic morphisms. We have noticed that all homotopy equivalences are inverted in $K(\mathcal{A})$.

One can also see that the effect of the process of inverting homotopy equivalences is exactly that one identifies homotopic morphisms.

(b) follows from (a). $K(\mathcal{A})$ is obtained from $C(\mathcal{A})$ by inverting some quasi-isomorphisms (the homotopy equivalences) and then $D(\mathcal{A})$ is obtained from $K(\mathcal{A})$ by inverting the remaining quasi-isomorphisms. So. all together $D(\mathcal{A})$ comes from $C(\mathcal{A})$ by inverting all quasi-isomorphisms. □

Remark. A reason to pass from $C(\mathcal{A})$ to $D(\mathcal{A})$ in stages (via $(\mathcal{A})$) is that the direct construction $D(\mathcal{A}) = C(\mathcal{A})[qis^{-1}]$ is difficult to understand because the property $(\star)$ does not hold in $C(\mathcal{A})$.

9.3.3. *The triangulated category structure on $D(\mathcal{A})$.*

Theorem. $D(\mathcal{A})$ acquires from $K(\mathcal{A})$ a triangulated category structure. (So, it is an additive category while shifts and the distinguished triangles are defined as images of such objects in $K(\mathcal{A})$.)

Remark. The origin of exact triangles in $D(\mathcal{A})$ is as in $K(\mathcal{A})$. We know the following constructions of the exact (=distinguished) triangles in $D(\mathcal{A})$:

1. from maps of complexes,
2. from short exact sequences of complexes that splits on each level,
3. from SES of complexes in $C^+(\mathcal{A})$, if $\mathcal{A}$ has enough injectives,
4. from SES of complexes in $C^-(\mathcal{A})$, if $\mathcal{A}$ has enough projectives.

9.4. **Truncations.**

9.4.1. *Cohomology functors.*

Lemma. The cohomology functors descend from $K(A)$ to $H^n: \ D(\mathcal{A}) \to \mathcal{A}$.

Proof. $H^n(A \xrightarrow{\phi} X \xleftarrow{s} B) \overset{\text{def}}{=} \ H^n(s)^{-1} \circ H^n(\phi)$. $\square$

9.4.2. *The triangulated subcategories $D^?(\mathcal{A}) \subseteq D(\mathcal{A})$ for $? \in \{+, b, -\}$.* These are defined as full subcategories of all complexes A such that the cohomology satisfies the property $/$, i.e., that $H^\bullet(A)$ lies in $C^?(\mathcal{A})$ (with zero differential). Also, for $\mathcal{Z} \subseteq \mathcal{Z}$ one defines a full subcategory $D^{\mathcal{Z}}(\mathcal{A})$ by asking that $H^\bullet(A) \in \ C^{\mathcal{Z}}(\mathcal{A})$.

9.4.3. *Truncations.* These are the functors

$$D^{\leq n}(\mathcal{A}) \xleftarrow{\tau_{\leq n}} D(\mathcal{A}) \xrightarrow{\tau_{\leq n}} D^{\geq n},$$

that come with canonical maps $\tau_{\leq n} A \to A \to \tau_{\geq n} A$. They are defined on objects by

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \longrightarrow & Z^n(A) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots & \tau_{\leq n}(A) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & A^{n+2} & \longrightarrow \cdots & A \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A^n/B^n(A) & \longrightarrow & A^{n+1} & \longrightarrow & A^{n+2} & \longrightarrow \cdots & \tau_{\geq n}(A)
\end{array}$$

Lemma. (a) $\tau_{\leq n}$ is the left adjoint to the inclusion $D^{\leq n}(\mathcal{A}) \subseteq D(\mathcal{A})$, and $\tau_{\geq n}$ is the right adjoint to the inclusion $D^{\geq n}(\mathcal{A}) \subseteq D(\mathcal{A})$.

(b) $\mathrm{H}^i(\tau_{\leq n} A) = \left\{ \begin{smallmatrix} \mathrm{H}^i(A) & i \leq n \\ 0 & i > n \end{smallmatrix} \right\}$, and $\mathrm{H}^i(\tau_{\geq n} A) = \left\{ \begin{smallmatrix} \mathrm{H}^i(A) & i \geq n \\ 0 & i < n \end{smallmatrix} \right\}$.

9.4.4. *Inclusion $\mathcal{A} \hookrightarrow D(\mathcal{A})$.* We can consider Each $A \in \mathcal{A}$ defines a complex concentrated in degree zero, usually again denote A. This gives functors $\mathcal{A} \xrightarrow{\iota} C(\mathcal{A}) \to K(\mathcal{A}) \to D(\mathcal{A})$.

The functor $\mathcal{A} \to D(\mathcal{A})$ factors through the full subcategory $D^0(\mathcal{A}) \subseteq D(\mathcal{A})$ consisting of all complexes A with $H^i(A) = 0$ for $i \neq 0$.

Lemma. (a) By interpreting each $A \in \mathcal{A}$ as a complex concentrated in degree zero, one gets an isomorphism of categories from $\mathcal{A}$ to a full subcategory of $C(\mathcal{A})$, $K(\mathcal{A})$ or $D(\mathcal{A})$.

(b) The functor $\mathcal{A} \to D^0(\mathcal{A})$ is an equivalence of categories.

Proof. In (a) we need to see that for A, B in $\mathcal{A}$ and $\mathcal{X} \in \{C, K, D\}$ the canonical map $\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{X}(\mathcal{A})}(A, B$ is an isomorphism.

When $\mathcal{X} = K$ this is true since any homotopy between two complexes concentrated in degree zero is clearly 0 (recall that $h^n : A^N \to B^{n-1}$),

When $\mathcal{X} = D$ then one shows that for $A, B \in \mathcal{A}$

(1) a diagram in $K(\mathcal{A})$ of the form $A \xrightarrow{\phi} X \xleftarrow{s} B$ with s a quasi-isomorphism, gives (apply H^0) a diagram in $\mathcal{A}$ $A \xrightarrow{\mathrm{H}^0(\phi)} \mathrm{H}^0(X) \xleftarrow{\mathrm{H}^0(s)} B$. This gives a map $A \xrightarrow{\mathrm{H}^0(s)^{-1}\mathrm{H}^0(\phi)} B$ in $\mathcal{A}$,

(2) Two diagrams of the form $A \xrightarrow{\alpha_i} B \xleftarrow{1_B} B$, are equivalent iff $\alpha_1 = \alpha_2$. □

Remarks. (0) Here we have here two ways of including $\mathcal{A}$ into $D(\mathcal{A})$ as a full subcategory $\mathcal{A}\subseteq D(\mathcal{A})$ or as a subcategory $D^0(\mathcal{A})$ which is only equivalent to $\mathcal{A}$. However, the second approach is more natural from the point of view of $D(\mathcal{A})$ since $D^0(\mathcal{A})$ is closed under isomorphisms inside $D(\mathcal{A})$ while $\mathcal{A}$ is not.

(1) Also, $D^-(\mathcal{A})$ is a "description of $\mathcal{A}$ inside $D(\mathcal{A})$" (up to equivalence), which only uses the functors H^i on $D(\mathcal{A})$ (rather than the construction of $D(\mathcal{A})$ from $\mathcal{A}$).

9.4.5. Ext *functors via derived categories.*

Lemma. For $a, b \in \mathcal{A}$ and $n > 0$,

$$\mathrm{Ext}^n_{\mathcal{A}}(a,b) \;\cong\; \mathrm{Hom}_{D(\mathcal{A})}(a, b[n]).$$

This is also the set of isomorphism class of diagrams of the form ...

9.5. **Homotopy category descriptions of derived categories.** The following provides a "down to earth" description of the derived category and gives us a way to calculate in the derived category.

Theorem. Let $\mathcal{I}_{\mathcal{A}}$ be the full subcategory of $\mathcal{A}$ consisting of all injective objects.

(a) $\mathcal{I}_{\mathcal{A}}$ is an additive subcategory.

(b) If $\mathcal{A}$ has enough injectives the canonical functors

$$K^+(\mathcal{I}_{\mathcal{A}}) \xrightarrow{\sigma} D^+(\mathcal{I}_{\mathcal{A}}) \xrightarrow{\tau} D^+(\mathcal{A})$$

are equivalences of categories.

Proof. (a) is clear. We consider (b) in steps.

(1) $K^+(\mathcal{I}) \to D^+(\mathcal{I})$ is an equivalence.

This based on the observation that

- (i) $D^+(\mathcal{I})$ is obtained from $K^+(\mathcal{I})$ by inverting quasi-isomorphisms.
- (ii) Any quasi-isomorphism between complexes in $C^+(\mathcal{I})$ is actually a homotopy equivalence.

Here, (ii) implies that that quasi-isomorphisms in $K^+(\mathcal{I})$ are already invertible in $K^+(\mathcal{I})$, so the passage to $D^+(\mathcal{I})$ is trivial.

(2) $D^+(\mathcal{I}) \xrightarrow{\tau} D^+(\mathcal{A})$ is essentially surjective.

We know that any complex A in $C^+(\mathcal{A})$ is quasi-isomorphic to its injective resolution $I \in D^+(\mathcal{I})$. Also any map $A' \to A''$ is quasi-isomorphic to a map of their injective resolutions $I' \to I''$. This is the surjectivity of τ on objects and morphisms.

(3) $D^+(\mathcal{I}) \xrightarrow{\tau} D^+(\mathcal{A})$ is injective on morphisms. This is the remaining observation, that For any $I, J \in D^+(\mathcal{I})$ the map

$$\mathrm{Hom}_{D(\mathcal{I})}(I, J) \to \mathrm{Hom}_{D(\mathcal{A})}(I, J)$$

is injective. □

9.6. **Derived functors** $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$**.** Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.

9.6.1. *Summary of known constructions based on* $K(\mathcal{A})$. We know that if $\mathcal{A}$ has enough injectives we can define the right derived functors $RF : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ by $(RF)A \stackrel{\text{def}}{=} F(I)$ for any injective resolution I of A and then also $R^nF : K^+(\mathcal{A}) \to \mathcal{B}$ by $(R^nF)(A) \stackrel{\text{def}}{=} H^n[(RF)A]$. We know that if F is left exact then $R^0F \cong F$.

Lemma. Any short exact sequence $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ in $\mathcal{A}$ defines a long exact sequence of derived functors

$$0 \to R^0F(A') \xrightarrow{R^0F(\alpha)} R^0F(A) \xrightarrow{RF^n(\beta)} R^0F(A'') \xrightarrow{\partial^0} \cdots \xrightarrow{\partial^{n-1}} R^nF(A') \xrightarrow{R^nF(\alpha)} R^nF(A) \xrightarrow{RF^n(\beta)} R^nF(A'')$$

Proof. We know that the short exact sequence $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ has an injective resolution

$$0 \to I' \xrightarrow{\alpha} I \xrightarrow{\beta} I'' \to 0$$

which is a SES of complexes. Since the terms in I' are injective this SES splits degreewise. Therefore, its F-image

$$0 \to F(I') \xrightarrow{\alpha} F(I) \xrightarrow{\beta} F(I'') \to 0$$

is again exact. additive. The long exact sequence of cohomologies of this SES of complexes is the desired sequence of derived functors R^nF. □

Remark. Even if F is not left exact the above construction produces a left exact functor R^0F.

9.6.2. *Derived functors* $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$. We will define the right derived functor RF as the universal one among all extensions of $\mathcal{A} \xrightarrow{F} \mathcal{B}$ to $D^+(\mathcal{A}) \to D^+(\mathcal{B})$. Then we see that the "replacement by injective resolution" construction satisfies this universality property. This gives both existence of RF (when there are enough injectives) and its concrete description.

(1) A functor $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ between two triangulated categories is said to be a *morphism of triangulated categories* (a *triangulated functor* or a *∂-functor*), if it preserves all relevant structure. In other words Φ it is additive, it intertwines shifts, i.e., $F(A)[n] \cong F(A[n])$ and it preserves exact triangles.

(2) A *(right) extension* of of an additive functor $F : \mathcal{A} \to \mathcal{B}$ to triangulated categories is a pair $(\mathcal{F}, \xi)$ of a triangulated functor $\mathcal{F} : D^+(\mathcal{A}) \to D^+(\mathcal{B})$, and a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ i_{\mathcal{A}}\big\downarrow & & i_{\mathcal{B}}\big\downarrow \\ D^+(\mathcal{A}) & \xrightarrow{\mathcal{F}} & D^+(\mathcal{B}) \end{array} .$$

Here, a commutativity is not a property of the diagram, rather it is the datum of a morphism of functors

$$i_{\mathcal{B}} \circ F \xrightarrow{\xi} RF \circ i_{\mathcal{A}}.$$

(We call ξ a "commutativity constraint" for the diagram.[3] Notice that such extensions form a category.

(3) Finally, the right derived functor of F is the initial object (RF, ζ) in the category of all extensions. This means that

for any right extension $(\mathcal{F}, \xi)$, the morphism ξ factors uniquely through ζ, i.e., there is a unique morphism of functors $RF \xrightarrow{\mu} \mathcal{F}$ such that $\xi = \mu \circ \zeta$, i.e.,

- $i_{\mathcal{B}} \circ F \xrightarrow{\xi} \mathcal{F} \circ i_{\mathcal{A}}$ *factors as* $i_{\mathcal{B}} \circ F \xrightarrow{\xi} RF \circ i_{\mathcal{A}} \xrightarrow{\mu \circ 1_{\mathcal{A}}} \mathcal{F} \circ i_{\mathcal{A}}$.

Remarks. Though the definition does not require ζ to be an isomorphism, in practice it will be an isomorphism).

9.6.3. *Construction of the derived functor.* The simplest case is when F is exact, then then one can define RF simply as F acting on complexes. However, in general, the most useful criterion for existence of RF is

Theorem. (a) Suppose that $\mathcal{A}$ has enough injectives. Then for any additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ its right derived extension $RF : D^+(\mathcal{A}) \to D(\mathcal{B})$ exists.

(b) Moreover, for any complex $A \in D^+(\mathcal{A})$, there is a canonical isomorphism $(RF)A \cong F(I)$ for any injective resolution I of A.

[3] For $A \in \mathcal{A}$, $i_{\mathcal{A}}(A)$ is A viewed as a complex. The most obvious meaning of "$\mathcal{F}$ extends F" is that the objects $RF(i_{\mathcal{A}}A)$ and $i_{\mathcal{B}}(FA)$ of $D^+(\mathcal{B})$ should be equal. However, this never happens in practice. We have to weaken this requirement which is too strict to be useful. Even if we replace it with "there is a canonical isomorphism $\xi_A : RF(i_{\mathcal{A}}A) \xrightarrow{\cong} i_{\mathcal{B}}(FA)$" this is still too strict.

Remark. Part (b) says that the new notion of the derived functor RF on $D^+(\mathcal{A})$ is compatible with the old notion of RF on $K^+(\mathcal{A})$ (since they are given by the same formula). So, one can calculate derived functors either in homotopy categories of complexes or in the derived categories of complexes.

Proof. Recall from 9.5 that one has an equivalence $K^+(\mathcal{I}_\mathcal{A}) \xrightarrow{\tau\sigma} D^+(\mathcal{A})$, and that an additive functor $F_\mathcal{I} \stackrel{\text{def}}{=} (\mathcal{I}_\mathcal{A}\subseteq\mathcal{A} \xrightarrow{F} \mathcal{B})$ has a canonical extension $K(F_\mathcal{I}) : K(\mathcal{I}_\mathcal{A}) \to K(\mathcal{B})$. So we can define RF as the composition

$$RF \stackrel{\text{def}}{=} \ [D^+(\mathcal{A}) \xrightarrow{(\tau\circ\sigma)^{-1}} K^+(\mathcal{I}_\mathcal{A}) \xrightarrow{K(F_\mathcal{I})} K^+(\mathcal{B}) \to D^+(\mathcal{B})].$$

This construction also satisfies the formula for RF in (b). For this we notice that an injective resolution $A \xrightarrow{i} I$ in $C^+(\mathcal{A})$ gives a canonical isomorphism $(\tau\sigma)^{-1}A\xrightarrow{\cong}I$. Therefore, $RF(A) \stackrel{\text{def}}{=} [K(F_\mathcal{I})\circ(\tau\circ\sigma)^{-1}]\ A \ \cong\ K(F_\mathcal{I})\ I \ =\ F(I)$.

So,it remains to check that the RF that we have now defined is the universal extension.

RF preserves shifts by its definition. Any exact triangle in $D^(\mathcal{A})$ is isomorphic to a triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ that comes from an exact sequence $0 \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\gamma} C \to 0$ in $C(\mathcal{A})$ that splits on each level. Moreover, we can replace the exact sequence $0 \xrightarrow{\alpha} A' \xrightarrow{\beta} A \xrightarrow{\gamma} A'' \to 0$ with an isomorphic (in $D(\mathcal{A})$) short exact sequence of injective resolutions $0 \to I' \xrightarrow{a} I \xrightarrow{b} I'' \to 0$. Since it also splits on each level its F-image $0 \to F(I') \xrightarrow{F(a)} F(I) \xrightarrow{F(b)} F(I'') \to 0$ is an exact sequence in $C(\mathcal{B})$. Therefore it defines an exact triangle $F(I') \xrightarrow{F(a)} F(I) \xrightarrow{F(b)} F(I'') \xrightarrow{\widetilde{\gamma}} F(I')[1]$ in $D(\mathcal{A})$. To see that this is the triangle we observe that by definition $F(I') = RF(A')$ $F(I) = RF(A)$ $F(I'') = RF(A'')$, and also $F(a) = RF(\alpha)$ and $F(b) = RF(\beta)$. It remains to see that $\widetilde{\gamma} = RF(\gamma)$. For this recall that γ and $\widetilde{\gamma}$ are defined using splittings of the first and second row of

$$\begin{array}{ccccccccc}
0 & \xrightarrow{\alpha} & A' & \xrightarrow{\beta} & A & \xrightarrow{\gamma} & A'' & \longrightarrow & 0 \\
 & & {\scriptstyle\iota'}\downarrow & & {\scriptstyle\iota}\downarrow & & {\scriptstyle\iota''}\downarrow & & \\
0 & \longrightarrow & I' & \xrightarrow{a} & I & \xrightarrow{b} & I'' & \longrightarrow & 0.
\end{array}$$

So, we need to be able to choose the splitting of the second row compatible with the one in the first row. However, this is clearly possible by the construction of the second row.

9.7. **Usefulness of the derived category.** We list a few more reasons to rejoice in derived categories.

9.7.1. *Do not take cohomology too early.* When we pass from a complex A to its cohomology sequence $H^n(A)$ we are distilling the most obvious information contained in the complex A. We also loose some information – how the cohomologies are glued together into object A.

This becomes a problem only if the complex A is not the end object but the first step in some consideration. Then the next step may not be doable at all because of the lost information. For this reason it is better not to discard the complex A but to keep all relevant information it contains.

The computation with complexes is optimally performed in the setting of derived categories.

Example. The derived functor RF has values in complexes so it contains more information than its cohomologies R^nF. An example where the loss of information from taking cohomologies is felt is the relation between derived functors of F, G and the composition $G{\circ}F$. While one has $R(G{\circ}F) \cong\ RG{\circ}RF$ under natural conditions, if we instead work with R^iF, R^jG and $R^n(G{\circ}F)$ the above formula degenerates to a relation which is weak (some information gets lost) and complicated (uses language of spectral sequences). So, a seemingly more complicated construction RF is more natural and has better properties then a bunch of functors R^iF.

Example. If one is interested in a composition of two functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ where F is left exact while G is right exact, we have a functor $D^b(\mathcal{A}) \xrightarrow{LG\circ RF} D^b(\mathcal{C})$ which is often very useful. However, it has no obvious analogue as a family of functors from $\mathcal{A}$ to $\mathcal{C}$ since the composition $G{\circ}F$ need not be neither left nor right exact.[(4)]

9.7.2. *Essential objects and tools that exist only in derived categories.* Some examples are the *dualizing sheaves* in topology and algebraic geometry, the *perverse sheaves* in topology (more recently also in algebraic geometry).

9.7.3. *Hidden relations between abelian categories.* It turns out that many deep relations between abelian categories $\mathcal{A}$ and $\mathcal{B}$ become understandable only on the level of derived categories. For instance $D(\mathcal{A})$ and $D(\mathcal{B})$ are sometimes equivalent though the abelian categories $\mathcal{A}$ and $\mathcal{B}$ are very different. This observation has revolutionized several areas of mathematics and physics.

9.8. **Four functors formalism in geometry.** Consider a category $\mathcal{G}$ of geometric nature, i.e., a category of spaces in some class and the corresponding maps of spaces. So, $\mathcal{G}$ could be topological spaces, smooth manifolds, complex manifolds, stratified spaces, algebraic varieties, schemes, semialgebraic varieties over $\mathbb{R}$ etc.

We will consider one standard method of studying a geometry like $\mathcal{G}$. We can call it *linearization* since it encodes the geometric information into linear algebra information. Traditionally (say in harmonic analysis), this was done by considering the space $\mathcal{O}(S)$ of

[4] One of the famous EGA books (Elements de Geometrie algebrique, the foundations of the contemporary language of algebraic geometry) deals just with the construction of the derived functor of the bifunctor $\Gamma(X, \mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{B})$ which (there were no derived categories at the time), they had to do in a very explicit and involved way since Γ is left exact and tensoring is right.

functions on an interesting space S. In the contemporary version one often works locally, so the vector space of functions $\mathcal{O}(S)$ is replaced by the sheaf of functions $\mathcal{O}_S$. We will see how linearization leads to homological algebra.

9.8.1. *Structure sheaves, abelian and triangulated categories.* Each space $S \in \mathcal{G}$ comes with the *structure sheaf* $\mathcal{O}_S$ given by functions on S that are appropriate for the setting $\mathcal{G}$. For topological spaces these would be continuous real valued functions $\mathcal{O}_S = \mathcal{C}_S$ or locally constant functions $\mathbb{R}_S$, for smooth manifolds we have the smooth (i.e., infinitely differentiable) functions etc.

Since $\mathcal{O}_S$ is a sheaf of algebras one can next associate to S the abelian category $\mathcal{S}h_S$ of $\mathcal{O}_S$-modules and then the triangulated category $D_S \overset{\text{def}}{=} D(\mathcal{S}h_S)$ of complexes of $\mathcal{O}_S$-modules.

9.8.2. *The system of triangulated categories $D_S,\ S \in \mathcal{G}$.* I will now explain geometric features of this system while neglecting the technical details that may depend on the choice of $\mathcal{G}$. For instance, instead of all $\mathcal{O}_S$-modules one can consider a particular class of interest or instead of the derived category $D(\mathcal{S}h_S)$ one may sometimes use its bounded versions $D^?(\mathcal{S}h_S)$.

When these are taken care of, to each map map of spaces $f : X \to Y$ in $\mathcal{G}$ one can attach four functors relating categories D_X and D_Y the two direct image (pushforward) functors two and two inverse image (pullback) functors

$$D_X \xrightarrow{f_!, f^*} D_Y \xrightarrow{f^!, f^*} D_X.$$

Remark. There is a little bit more. The *inner Hom* functor $\mathcal{H}\text{om} :\ D_X^o \times D_X \to\ D_X$ and the *duality* functor $\mathbb{D}_X = \mathcal{H}\text{om}(-, \omega_X) :\ D_X \to D_X^o$. The last one uses a special object $\omega_X \in\ D_X$ called the *dualizing sheaf.*

Sometimes there is also the so called *nearby cycles* functor ψ and the *vanishing cycles* functor ψ, which are analogues of the limit and the derivative in calculus.

9.8.3. *Toy example: $\mathcal{G}$ is the category of finite sets.* For a finite set S the algebra $\Bbbk[S]$ of functions on S with values in a commutative ring $\Bbbk$ has a basis by δ-functions $\delta_s,\ s \in S$, where $\delta_s(x) = \delta_{s,x}$ $(x \in S)$. Since S is discrete the sheaf $\mathcal{O}_S$ of $\Bbbk$-valued functions on S is the same as the single algebra $\Bbbk[S]$. Therefore, $\mathcal{S}h_S = \mathfrak{m}(\Bbbk[S])$ consists of $\Bbbk$-modules M with an S-decomposition $M = \oplus_{s \in S}\ M_s$ where M_s is the part supported at $s \in S$.

On the level of functions a map $f : X \to Y$ gives two $\Bbbk$-linear maps

$$\Bbbk[X] \xrightarrow{f_!} \Bbbk[Y] \xrightarrow{f^*} \Bbbk[X] \ \ where\ \ (f_!\alpha)(y) = \sum_{x \mapsto y} \alpha(x) \quad \text{and} \quad (f^*\beta)(x) = \beta(f(x)).$$

The second map is a map of algebras.

This system of linear maps between spaces of functions is the linearization of combinatorics (i.e., of the world of finite sets), to the level of vector spaces if $\Bbbk$ is a field.

The linearization to the level of abelian categories is given by functors that we are familiar with

$$\mathfrak{m}(\Bbbk[X]) \xrightarrow{f_!, f^*} \mathfrak{m}(\Bbbk[Y]) \xrightarrow{f^!, f^*} \mathfrak{m}(\Bbbk[X]).$$

Here, for N a $\Bbbk[Y]$-module, $f^*N = \Bbbk[X] \otimes_{\Bbbk[Y]} N$ while for M a $\Bbbk[X]$-module, $f_! M$ is M considered as a $\Bbbk[Y]$-module via the map of algebras $\Bbbk[Y] \xrightarrow{f^*} \Bbbk[X]$.

Remarks. (0) Roughly, for any geometry $\mathcal{G}$ the properties of functors $f_!, f^*$ will be "essentially" the same as the properties that are true in this combinatorial example. In other words, "$f_!, f^*$ will behave in a set-theoretic way".

(1) One way this combinatorial example is actually much simpler than the general case. When $\mathcal{G}$ is the category of finite sets the derived categories do not offer anything new. In the non discrete case the derived categories are essential. For instance only, only three of the four functors are seen on the level of abelian categories. Say, in topology the functors $f_!, f_*, f^*$ on derived categories are the derived functors of the corresponding functors defined between abelian categories but $f^!$ is not.

(2) Another way the combinatorial case is simpler is that here one has $f_! = f_*$ and $f^! = f^*$.

In general there is only a map of functors $f_! \to f_*$ which is an isomorphism when the map f is proper (meaning roughly that the fibers are compact), while $f^!$ and f^* are essentially the same when the map f is smooth (meaning roughly that the fibers are smooth). So, the subtleties appear in the continuous setting, in particular at singularities of spaces.

(2)

9.8.4. *Properties of functors.* Philosophically, these functors provide a unified point on view on studies of various classes of geometries $\mathcal{G}$. They encode the traditional issues (we will see the examples of sheaf cohomology and algebraic topology in 9.9) and they provide the most flexible calculational tool. Here are some of the basic rules for calculation:

- The duality is involutive: $\mathbb{D}_X^2 \cong id$;
- The "extraordinary" operations $f_!, f^!$ are obtained from the 'ordinary" operations f_*, f^* by conjugating with duality, say

$$^{\mathbb{D}}(f_*) \stackrel{\text{def}}{=} \mathbb{D}_Y \circ f_* \circ \mathbb{D}_X \ \cong \ f_!;$$

- Transitivity $(g \circ f)_? = g_? f_?$ and $f \circ f)^? = f^? g^?$ for $? = \ *, !$. (Also, $(id_X)_? = id_{D_X} = ((id_X)^?$.)

- [Base Change] For a Cartesian square $\begin{tikzcd} X & \Sigma \arrow[l, "p"'] \arrow[d, "q"] \\ Z \arrow[from=u, "f"'] & Y \arrow[l, "g"] \end{tikzcd}$, i.e., when Σ is the fibered product $X \times_Z Y$, one has
$$g^* f_! \;\cong\; q_! p^*.$$
- [Gluing] If $Y \overset{i}{\subseteq} X$ is closed and $U = X - Y \overset{j}{\subseteq} X$ is open in X then for any $\mathcal{F} \in D_X$ one has the canonical exact triangles[(5)]
$$i_! i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F} \quad \text{and} \quad j_! j^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}.$$
- Projection formula and
- more.

9.9. **The topological context.** We will start with the category $\mathcal{G} = \mathcal{T}op$ of topological spaces and then specialize to spaces of particular interest (manifolds and stratified spaces).

For any space topological $S \in \mathcal{T}op$ any abelian group A defines the *constant sheaf* A_S over S (the sheaf of locally constant functions into $\mathbb{Z}$). We will choose the structure sheaf $\mathcal{O}_S$ as the constant sheaf $\mathbb{Z}_S$ (or $\Bbbk_S$ for some commutative ring $\Bbbk$). The category $\mathfrak{m}(Z_S)$ of all $\mathbb{Z}_S$-modules is the category $\mathcal{A}b_X$ of all sheaves of abelian groups over S.

9.9.1. *Cohomology of sheaves.* For any map $f : X \to Y$ we have the operation of direct image of sheaves, we denote it here by $f_*^{\mathcal{A}b} : \mathcal{A}b_X \to f_* Y$. For instance, when we choose the map as $a : X \to \mathrm{pt}$ the direct image $a_*^{\mathcal{A}b} :\ \mathcal{A}b_X \to \mathcal{A}b_{\mathrm{pt}} = \mathcal{A}b$ is exactly the operation $\Gamma(X, -)$ of taking the global sections of a sheaf.

The direct image functor has a right derived functor
$$f_* \ \overset{\text{def}}{=}\ R(f_*^{\mathcal{A}b}) :\ D^+(\mathcal{A}b_X) \to D^+(\mathcal{A}b_Y).$$
For the map $a : X \to \mathrm{pt}$ the derived direct image is called the *sheaf cohomology.* As a derived functor it is of the form
$$a_* :\ D^+(\mathcal{A}b_X) \to D^+(\mathcal{A}b_{\mathrm{pt}}) = D(\mathcal{A}b)$$
and for a sheaf $\mathcal{F}$ by taking the cohomology groups of $a_*\mathcal{F}$ we get the usual sheaf cohomology groups
$$H^n(X, \mathcal{F}) \;=\; H^n(a_* \mathcal{F}).$$
So, $H^n(X, \mathcal{F})$ is just the n^{th} derived functor $(R^n a_* \mathcal{A}b)\mathcal{F}$ for the ordinary direct image of sheaves $a_*^{\mathcal{A}b}$.

[5] So, $\mathcal{F}$ is glued from its restrictions to Y and U. This provides another way to break complicated tasks (like computing cohomology) into simpler ones.

9.9.2. *Cohomology of spaces.* A special case of this is when $\mathcal{F}$ is a constant sheaf A_X. Then we recover the algebraic topology since the sheaf cohomology groups $H^n(X, A_X)$ of the constant sheaf A_X are just the the cohomology groups $H^n(X, A)$ of the space X with the coefficients in A, the ones that we learn about in algebraic topology. So,

$$H^n(X,A) \ = \ H^n(a_*A_X) \ = \ (R^n a_*^{\mathcal{A}b})A_X.$$

Now one can calculate the cohomology $H^*(X,A)$ of the space X in stages, using the flexibility of the sheaf theory. Choose a convenient space Y to which X maps, so we have $X \xrightarrow{\pi} Y \xrightarrow{b} \mathrm{pt}$ and

$$a_*A_X = \ (b\pi)_*A_X = \ b_*(\pi_*A_X).$$

So, if maps b, π are simpler then the map a we can hope to calculate first π_*A_X and then its b_*-image. In this way we get a_*A_X which contains all cohomology groups $H^n(X,A)$.

9.9.3. *Stratified spaces.* A stratification $\mathcal{S}$ of a space X is a family of subspaces $S\subseteq X$ such that $X = \ \bigsqcup_{S\in\mathcal{S}} S$ and the closure of each stratum $S \in \mathcal{S}$ is a union of some of the strata.

To fix terminology we will now specialize to the category $\mathcal{G}$ of complex algebraic varieties (one could use real semialgebraic varieties or smooth manifolds etc). For $X \in \mathcal{S}$ there exists a stratification $\mathcal{S}$ such that all strata $S \in \mathcal{S}$ are smooth subvarieties.

Now one defines the *constructible* part D^c_X of the derived category $D(\mathcal{A}b_X)$ of sheaves of abelian groups on X, to consist of all complexes $\mathcal{F} \in D^b(\mathcal{A}b_X)$ such that there exists a stratification $\mathcal{S}$ by smooth subvarieties such that for each stratum $S\overset{i}{\subseteq}X$, the complexes $i^*\mathcal{F}$ and $i^!\mathcal{F}$ in $D(\mathcal{A}b_S)$ are *smooth* in the sense that all cohomology sheaves $H^n(i^*\mathcal{F}) \in \mathcal{A}b_X$ and $H^n(i^!\mathcal{F}) \in \mathcal{A}b_X$ are *locally constant* sheaves ("*local systems*").

In this context – of triangulated categories D^c_X of constructible sheaves – one has the complete formalism of functors $f_!, f_*, f^!, f^*, \mathbb{D}_X, \mathcal{H}\mathrm{om}, \phi, \psi$.

9.9.4. *The 4 standard classes of (co)homology theories of spaces.* We note that the formalism of four functors organizes together the standard constructions. First, for $X \xrightarrow{a} \mathrm{pt}$ the constant sheaf is $\mathbb{Z}_X = a^*A$ while the dualizing sheaf ω_X is $a^!\mathbb{Z}$. Then the cohomology and *compactly supported cohomology* are given by

$$H^*(X,\mathbb{Z}) \ \stackrel{\bullet}{=} \ a_*\mathbb{Z}_X = \ a_*a^*\mathbb{Z}, \quad H^*_c(X,\mathbb{Z}) \ \stackrel{\bullet}{=} \ a_!\mathbb{Z}_X = \ a_*a^*\mathbb{Z},$$

and similarly for homology and *Borel-Moore homology*

$$H_*(X,\mathbb{Z}) \ \stackrel{\bullet}{=} \ a_!\omega_X = \ a_!a^!\mathbb{Z}, \quad \mathcal{H}_*(X,\mathbb{Z}) \ \stackrel{\bullet}{=} \ a_*\omega_X = \ a_*a^!\mathbb{Z}.$$

Actually, these are not quite isomorphisms since the objects on the right hand side are smarter (they are complexes while there are no differential on the LHS). The symbol $\stackrel{\bullet}{=}$ here means that we get the LHS by taking all cohomology groups of complexes on the RHS.