HOMOLOGICAL ALGEBRA

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Date: ?

7. **Bicomplexes and the extension of derived functors to complexes**

For an additive functor $F:\mathcal{A}\to\mathcal{B}$ we have defined its left derived functor $LF:\mathcal{A}\to K^-(\mathcal{B})$ (if $\mathcal{A}$ has enough projectives) by replacing objects with their projective resolutions. However, it is necessary to extend this construction to $LF:\ K^-(\mathcal{A})\to K^-(\mathcal{B})$.[(1)]

This will require the notions of resolutions of complexes. The basic tool will be the construction of injective resolutions of short exact sequences in $\mathcal{A}$ in 7.1.

The natural idea for constructing an injective resolution of a complex A is that one combines injective resolutions of all terms A^n of A. This does indeed provide an injective resolution A of A but this resolution lies in the larger world of *bicomplexes* – the 2-dimensional version of a complexes. Finally, to any bicomplex B one can associate its total complex $Tot(B)$ and $Tot(\mathsf{A})$ will indeed be an injective resolution of the complex A.

In 7.2 we consider bicomplexes and the notions of resolutions of complexes. Here, we also state (along the lines explained above) the three existence theorems for such resolutions. The proofs of these theorems are in **??**.

7.1. **Resolutions of short exact sequences in $\mathcal{A}$.**

Lemma. If abelian category $\mathcal{A}$ has enough injectives any SES in $\mathcal{A}$ can be lifted to a SES of injective resolutions. More precisely,

(a') Any short exact sequence of injective resolutions

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{A} \arrow[r] & \mathsf{B} \arrow[r] & \mathsf{C} \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & a \arrow[r, "\alpha"] \arrow[u, "\iota_a"] & b \arrow[r, "\beta"] \arrow[u, "\iota_b"] & c \arrow[r] \arrow[u, "\iota_c"] & 0 \arrow[u]
\end{tikzcd}$$

necessarily splits degreewise, i.e., $\mathsf{B}^n\cong\mathsf{A}^n\oplus\mathsf{C}^n$.[(2)]

(a") The splittings in (a') form a torsor for the group $\mathrm{Hom}_{\mathcal{A}^\bullet}(\mathsf{A},\mathsf{C})$.

(b) For a short exact sequence $0\to a\to b\to c\to 0$ let A and C be injective resolutions of a and c. Then on the graded object $\mathsf{B}\overset{\mathrm{def}}{=}\mathsf{A}\oplus\mathsf{C}\in\mathcal{A}^\bullet$ there is structure (d_{B},ι_b) of a resolution of b, such that the inclusion and projection $\mathsf{A}\subseteq\mathsf{B}\twoheadrightarrow\mathsf{C}$ form a short exact

[1] For one thing, for calculational reasons we need the property $L(G\circ F)=LG\circ LF$, but this means that we have to apply LG to a complex.

[2] (However, the complex B is not a sum of A and C.)

sequence of complexes:

$$\begin{tikzcd}
\vdots \arrow[r] & \vdots \arrow[r] & \vdots \arrow[r] & \vdots \arrow[r] & \vdots \\
0 \arrow[r] \arrow[u] & \mathsf{A}^2 \arrow[r, "\alpha^2"] \arrow[u, "d^2_{\mathsf{A}}"'] & \mathsf{B}^2 \arrow[r, "\beta^2"] \arrow[u, "d^2_{\mathsf{B}}"'] & \mathsf{C}^2 \arrow[r] \arrow[u, "d^2_{\mathsf{C}}"'] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & \mathsf{A}^1 \arrow[r, "\alpha^1"] \arrow[u, "d^1_{\mathsf{A}}"'] & \mathsf{B}^1 \arrow[r, "\beta^1"] \arrow[u, "d^1_{\mathsf{B}}"'] & \mathsf{C}^1 \arrow[r] \arrow[u, "d^1_{\mathsf{C}}"'] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & \mathsf{A}^0 \arrow[r, "\alpha^0"] \arrow[u, "d^0_{\mathsf{A}}"'] & \mathsf{B}^0 \arrow[r, "\beta^0"] \arrow[u, "d^0_{\mathsf{B}}"'] & \mathsf{C}^0 \arrow[r] \arrow[u, "d^0_{\mathsf{C}}"'] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & a \arrow[r, "\alpha"] \arrow[u, "\iota_A"'] & b \arrow[r, "\beta"] \arrow[u, "\iota_B"'] & c \arrow[r] \arrow[u, "\iota_C"'] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

with α^n the inclusion of the first summand and β^n the projection to the second summand.

Proof. (a) is a degreewise application of the following observation.

> *(a') Any SES* $0 \xrightarrow{\alpha} b \xrightarrow{\beta} c \to 0$ *in* $\mathcal{A}$ *with injective* c *splits.*
> *(a") Its splittings* $\sigma : b \to a$ *form a torsor for* $\mathrm{Hom}_{\mathcal{A}}(a, c)$*.*

First, to check (c') we notice that since a is injective, one can extend the identity map on a to a map $b \xrightarrow{\sigma} a$. This means that $\sigma\circ\alpha = 1_a$, so we get a splitting i.e., a complement $\mathrm{Ker}(\sigma)$ to a in b.

For (c") recall that a *torsor* for a group G is a set S on which G acts simply transitively, i.e., for any $s, s' \in S$ there is a unique $g \in G$ such that $gs = s'$. In our setting $\phi : c \to a$ acts on a splitting $\sigma : b \to a$ by $\sigma \mapsto \sigma + \phi\circ\beta$ (then $(\sigma + \phi\beta)\alpha = \sigma\alpha = 1_a$). Also, for two splittings σ, σ' we have $(\sigma' - \sigma)\alpha = 1_a - 1_a = 0$ hence $\sigma' - \sigma : b \to a$ factors to a map $c \to a$.

(b) We need to define ι_b and d_{B} so that the middle column is a resolution and the diagram commutes. Define $\iota_b : b \longrightarrow \mathsf{B}^0 = \mathsf{A}^0 \oplus \mathsf{C}^0$ by

$$\iota_b(b) \overset{\text{def}}{=} \widetilde{\iota}_a(b) \oplus \iota_c(\beta b)$$

where $\widetilde{\iota}_a : b \longrightarrow \mathsf{A}^0$ is any extension of $\iota_a : a \longrightarrow \mathsf{A}^0$ (it exists since A^0 is injective). This choice ensures that the two squares that contain ι_b commute. Moreover, ι_b is injective since the kernel of the second component $\iota_c\circ\beta$ is $\mathrm{Ker}(\beta) = a$ and the restriction of ι_b to on $a \subseteq b$ is ι_a.

The input of this construction is a pair of injective maps ι_a, ι_b of outer terms of two SES, such that the first component of the second SES is injective. It produced a completion ι_b to a map of SES such that ι_b is injective.

To be able to use it again we will get into this setting again by factoring the differential $d^0_{\mathsf{A}} : \mathsf{A}^0 \to \mathsf{A}^1$ through $\mathrm{Coker}(\iota_a)$ and the same for d^0_{B} and $\mathrm{Coker}(\iota_b)$. According to the following sublemma 7.1.1 this gives gives a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{A}^1 \arrow[r, "\alpha^1"] & \mathsf{A}^1 \oplus \mathsf{C}^1 \arrow[r, "\beta^1"] & \mathsf{C}^1 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & \mathrm{Coker}(\iota_a) \arrow[r, "\alpha^1"] \arrow[u, "\overline{d^0_{\mathsf{A}}}"] & \mathrm{Coker}(\iota_b) \arrow[r, "\beta^1"] & \mathrm{Coker}(\iota_c) \arrow[r] \arrow[u, "\overline{d^0_{\mathsf{C}}}"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & \mathsf{A}^0 \arrow[r, "\alpha^0"] \arrow[u, "q'"] & \mathsf{A}^0 \oplus \mathsf{C}^0 \arrow[r, "\beta^0"] \arrow[u, "q"] & \mathsf{C}^0 \arrow[r] \arrow[u, "q''"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & a \arrow[r, "\alpha"] \arrow[u, "\iota_a"] & b \arrow[r, "\beta"] \arrow[u, "\iota_b"] & c \arrow[r] \arrow[u, "\iota_c"] & 0 \arrow[u]
\end{tikzcd}$$

where $\overline{d^1_{\mathsf{A}}}$ and $\overline{d^1_{\mathsf{C}}}$ are factorizations of d^1_{A} and d^1_{C} through the cokernels.

Now, the same construction gives injective $\overline{d^0_{\mathsf{B}}}$ and then $d_{\mathsf{B}} \overset{\text{def}}{=} \overline{d^0_{\mathsf{B}}} \circ q$. By construction $0 \to b \xrightarrow{\varepsilon_b} \mathsf{B}^0 \xrightarrow{d^0_{\mathsf{B}}} \mathsf{B}^1$ is exact, $\square$

7.1.1. *Sublemma.* (a) If the diagram

$$\begin{tikzcd}
a' \arrow[r] \arrow[d, "\phi'"] & a \arrow[r] \arrow[d, "\phi"] & a'' \arrow[r] & 0 \\
b' \arrow[r] & b \arrow[r] & b'' \arrow[r] & 0
\end{tikzcd}$$

commutes and the rows are exact then there is a unique map $\phi'' : a'' \to b''$ which completes it to a map of short exact sequences.

(b) If the diagram
$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] & & \\ a' \arrow[r] \arrow[d, "\phi'"'] & a \arrow[r] \arrow[d, "\phi"'] & a'' \arrow[r] & 0 \\ b' \arrow[r] \arrow[d, "\psi'"'] & b \arrow[r] \arrow[d, "\psi"'] & b'' \arrow[r] & 0 \\ c' \arrow[r] \arrow[d] & c \arrow[r] \arrow[d] & c'' \arrow[r] & 0 \\ 0 & 0 & & \end{tikzcd}$$
commutes and the rows and columns are exact then the completions from (a) give a SES in the third column.

Proof. Easy. □

Remark. From a category $\mathcal{C}$ we can define a new category $\mathcal{M}or(\mathcal{C})$ whose objects are maps $a \xrightarrow{\alpha} b$ in $\mathcal{C}$ and morphisms $(a' \xrightarrow{\alpha} a) \rightarrow (b' \xrightarrow{\alpha} b)$ are the commutative diagrams
$$\begin{tikzcd} a' \arrow[r, "\alpha"] \arrow[d, "\phi'"'] & a \arrow[d, "\phi"'] \\ b' \arrow[r, "\beta"] & b \end{tikzcd}.$$
Then the part (a) can be restated as "Coker$(-)$ is a functor from $\mathcal{M}or(\mathcal{C})$ to $\mathcal{C}$" and the part (a) as "functor Coker$(-)$ is exact".

7.2. **Bicomplexes and resolutions of complexes.**

7.2.1. *Bicomplexes.* A bicomplex is a bigraded object $B = \oplus_{p.q\in\mathbb{Z}}\ B^{p,q}$ with differentials $B^{p,q} \xrightarrow{d'} B^{p+1,q}$ and $B^{p,q} \xrightarrow{d''} B^{p,q+1}$, such that $d = d' + d''$ is also a differential. This is the same as $d'd'' + d''d' = 0$ since
$$d^2 = (d' + d'')^2 = (d')^2 + d'd'' + d''d' + (d'')^2 = d'd'' + d''d'.$$

7.2.2. *The total complex $Tot(B)$ and cohomology of a bicomplex B.* The total complex of a bicomplex in additive category $\mathcal{A}$ is the complex[(3)]
$$(Tot(B), d) \quad \text{with} \quad Tot(B)^n \stackrel{\text{def}}{=} \oplus_{p+q=n}\ B^{p,q}.$$
The cohomology of a bicomplex B is by definition the cohomology of the total complex $Tot(B)$.

Remark. Occasionally, it is better to use the "completed total complex" $\widehat{Tot}(B)^n = \prod_{p+q=n}\ B^{pq}$.

[3] We need that $\mathcal{A}$ has countable sums.

7.2.3. *Drawing.* We draw a bicomplex as a two dimensional object:

$$\begin{tikzcd}
\cdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \cdots \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & B^{-1,2} \arrow[u, "d''"] \arrow[r, "d'"] & B^{0,2} \arrow[u, "d''"] \arrow[r, "d'"] & B^{1,2} \arrow[u, "d''"] \arrow[r, "d'"] & B^{2,2} \arrow[u, "d''"] \arrow[r, "d'"] & B^{3,2} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & B^{-1,1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{0,1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{1,1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{2,1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{3,1} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & B^{-1,0} \arrow[u, "d''"] \arrow[r, "d'"] & B^{0,0} \arrow[u, "d''"] \arrow[r, "d'"] & B^{1,0} \arrow[u, "d''"] \arrow[r, "d'"] & B^{2,0} \arrow[u, "d''"] \arrow[r, "d'"] & B^{3,0} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & B^{-1,-1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{0,-1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{1,-1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{2,-1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{3,-1} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & \vdots \arrow[u, "d''"] \arrow[r, "d'"] & \vdots \arrow[u, "d''"] \arrow[r, "d'"] & \vdots \arrow[u, "d''"] \arrow[r, "d'"] & \vdots \arrow[u, "d''"] \arrow[r, "d'"] & \vdots \arrow[u, "d''"] \arrow[r, "d'"] & \cdots . \arrow[u, "d''"]
\end{tikzcd}$$

So, B^{pq} has horizontal position p and height q, and d' is a horizontal differential while d'' is a vertical differential.

7.2.4. *The (anti)commutativity of d', d''.* (1) The anti-commutativity relation $d'd''+d''d' = 0$ can be interpreted as commutativity in the correct framework: the super-mathematics.

(2) If d' and d'' commute then we can still get a bicomplex. For that we correct one of these, say we replace d'' by $(\check{d}'')^{p,q} \stackrel{\text{def}}{=} (-1)^p (d'')^{p,q}$. (These anticommute since on the (p,q)-position $d'\check{d}'' + \check{d}''d' = (-1)^p d'd'' + (-1)^{p+1} d''d' = 0$).

7.2.5. *Notions of resolutions of complexes.* A *right resolution* of a complex $A \in C(\mathcal{A})$ is any quasi-isomorphism $A \to I$. An *injective resolution* of a complex $A \in C(\mathcal{A})$ is a right resolution $A \to I$ such that all I^n are injective objects of the abelian category $\mathcal{A}$. The next three theorems will state that injective resolutions of complexes (i) exist (under some conditions), (ii) are unique in a certain sense, and (iii) can be chosen compatibly with short exact sequences of complexes.

A *right bicomplex resolution* of a complex A is a bicomplex $I^{\bullet,\bullet}$ with $I^{pq} = 0$ for $p < 0$, and a map of complexes $\varepsilon : A \to I^{\bullet,0}$ such that in the following diagram

$$\begin{tikzcd}
\cdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \cdots \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & I^{-1,2} \arrow[u, "d''"] \arrow[r, "d'"] & I^{0,2} \arrow[u, "d''"] \arrow[r, "d'"] & I^{1,2} \arrow[u, "d''"] \arrow[r, "d'"] & I^{2,2} \arrow[u, "d''"] \arrow[r, "d'"] & I^{3,2} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & I^{-1,1} \arrow[u, "d''"] \arrow[r, "d'"] & I^{0,1} \arrow[u, "d''"] \arrow[r, "d'"] & I^{1,1} \arrow[u, "d''"] \arrow[r, "d'"] & I^{2,1} \arrow[u, "d''"] \arrow[r, "d'"] & I^{3,1} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & I^{-1,0} \arrow[u, "d''"] \arrow[r, "d'"] & I^{0,0} \arrow[u, "d''"] \arrow[r, "d'"] & I^{1,0} \arrow[u, "d''"] \arrow[r, "d'"] & I^{2,0} \arrow[u, "d''"] \arrow[r, "d'"] & I^{3,0} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & A^{-1} \arrow[u, "d''"] \arrow[r, "d'"] & A^{0} \arrow[u, "d''"] \arrow[r, "d'"] & A^{1} \arrow[u, "d''"] \arrow[r, "d'"] & A^{2} \arrow[u, "d''"] \arrow[r, "d'"] & A^{3} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"]
\end{tikzcd}$$

the columns are resolutions of terms in the complex A. A right bicomplex resolution I is said to be injective if all terms I^{pq} are injective.

Lemma. (a) If A is in $C^-(\mathcal{A})$ then for any projective bicomplex resolution A of A, the complex $Tot(\mathsf{A})$ is a projective resolution of A.

(b) If A is in $C^+(\mathcal{A})$ then for any injective bicomplex resolution A of A, the complex $\widehat{Tot}(\mathsf{A})$ is an injective resolution of A.

Proof. In (a) and (b) respectively, one uses $Tot(\mathsf{A})$ and $\widehat{Tot}(\mathsf{A})$ because projectives are closed under sums and injectives under products. At any rate, considerations of projective or injective resolutions are equivalent by switching from $\mathcal{A}$ to the opposite category $\mathcal{A}^o$ (which takes sums to products).

(b) The map $A \to \widehat{Tot}(I)$ is the composition of maps of complexes $A \xrightarrow{\iota} I^{\bullet,0} \subseteq \widehat{Tot}(I)$.

A direct proof that this map is a quasi-isomorphism is "easy"[(4)] However, we will postpone the proof to the section 8 (see 8.6.1), where it will illustrate the notion of *spectral sequences*, an elegant tool for study of cohomology of bicomplexes (and more). □

7.2.6. *Split injective resolutions.* Let us say that an injective complex K *splits* if the kernels of differentials $Z^n(K)$ are again injective.

[4] It uses the "staircase" pattern typical of bicomplexes.

Lemma. For an injective complex K :

(a) The two short exact sequences

$$0 \to Z^n \to K^n \xrightarrow{\overline{d}_K^n} B^{n+1} \to 0 \quad \text{and} \quad 0 \to B^n \to Z^n \to H^n \to 0$$

both split. So, H^n and B^n are injective.

(b) For any left exact functor $F : \mathcal{A} \to \mathcal{B}$ there is a canonical isomorphism $H^n(FK) \cong F(H^n K)$.

Proof. (a) A SES with the first object injective always splits. So, $0 \to Z^n \to K^n \to B^{n+1} \to 0$ splits and then B^{n+1} is injective (it is a summand of K^n). Now also $0 \to B^n \to Z^n \to H^n \to 0$ has to split etc.

(b) Since F is additive it preserves sums. So, applying F to a split SES we get a split SES. Therefore we get two SES that we call $\mathcal{S}_I$ and $\mathcal{S}_{II}$:

$$0 \to FZ^n \to FK^n \to FB^{n+1} \to 0 \quad \text{and} \quad 0 \to FB^n \to FZ^n \to FH^n \to 0$$

are both (split) SES.

Let us factor $K^n \xrightarrow{d_K^n} K^{n+1}$ as $K^n \overset{\overline{d}_K^n}{\twoheadrightarrow} B^{n+1} \overset{i}{\hookrightarrow} K^{n+1}$. The left exactness of F shows that the map $F(i)$ is injective. This gives the first equality in

$$\operatorname{Ker}[FK^n \xrightarrow{Fd_K^n} FK^{n+1}] \;=\; \operatorname{Ker}[FK^n \xrightarrow{F\overline{d}_K^n} FB^{n+1}] \;=\; FZ^n,$$

while the second comes from the SES $\mathcal{S}_I$.

Similarly, consider the diagram $\begin{array}{ccccc} 0 \to Z^n & \longrightarrow & K^n & \xrightarrow{d_K^n} & K^{n+1} \\ \uparrow{=} & & =\uparrow & & \subseteq\uparrow \\ 0 \to Z^n & \longrightarrow & K^n & \xrightarrow{\widehat{d}_K^n} & B^{n+1} \to 0 \end{array}$, with exact rows and injective columns. Applying F these properties are preserves

$$\begin{array}{ccccc} 0 \to FZ^n & \longrightarrow & FK^n & \xrightarrow{Fd_K^n} & FK^{n+1} \\ =\uparrow & & =\uparrow & & \subseteq\uparrow \\ 0 \to FZ^n & \longrightarrow & FK^n & \xrightarrow{F\widehat{d}_K^n} & FB^{n+1} \to 0; \end{array}$$

because F is left exact and $\mathcal{S}_I$ is exact. Therefore, the image $Im[FK^n \xrightarrow{Fd_K^n} FK^{n+1}]$ is FB^{n+1}.

So, now $H^n(FK) = \operatorname{Ker}(Fd_K^n)/\operatorname{Im}(Fd_K^{n-1}) = F(Z^n)/F(B^n$ and this is $F(H^n)$ according to the SES $\mathcal{S}_{II}$. □

We will says that an injective bicomplex resolution I of A *splits* if there exist injective resolutions $\mathcal{B}^n, \mathcal{H}^n$ of $B^n(A), H^n(A)$ such that as graded objects $I^{n,\bullet}$ are isomorphic to $\mathcal{B}^n \oplus \mathcal{H}^n \oplus \mathcal{B}^{n+1}$. so that the horizontal differential $I^{n,\bullet} \xrightarrow{d'} I^{n+1,\bullet}$ viewed as a map $\mathcal{B}^n \oplus \mathcal{H}^n \oplus \mathcal{B}^{n+1} \to \mathcal{B}^{n+1} \oplus \mathcal{H}^{n+1} \oplus \mathcal{B}^{n+2}$ is the obvious map $b_1 \oplus h \oplus b_2 \mapsto b_2 \oplus 0 \oplus 0$.

7.2.7. In the remainder of this section we list the results on injective resolutions of complexes. (Then the dual statements hold for projective resolution.)

7.2.8. *Existence of injective resolutions of complexes.*

Theorem. If an abelian category $\mathcal{A}$ has enough injectives then any bounded from below complex $A \in C^+(\mathcal{A})$ has an injective resolution. More precisely,

(a) Any $A \in C(\mathcal{A})$ has an injective bicomplex resolution A.

(b) Such bicomplex resolution can be chosen to be *split* in the sense that there exist injective resolutions $\mathsf{B}^n, \mathsf{H}^n$ of $B^n(A), H^n(A)$ such that $I^{n\bullet}$ is isomorphic to $\mathsf{B}^n \oplus \mathsf{H}^n \oplus \mathsf{B}^{n+1}$ in such way that the horizontal component of the differential $d' : I^{n,\bullet} \to I^{n+1,\bullet}$, i.e., $\mathsf{B}^n \oplus \mathsf{H}^n \oplus \mathsf{B}^{n+1} \xrightarrow{d'} \mathsf{B}^{n+1} \oplus \mathsf{H}^{n+1} \oplus \mathsf{B}^{n+2}$, is given by $b' \oplus h \oplus b'' \mapsto b'' \oplus 0 \oplus 0$.

Proof. (a) The first task will be to choose compatible injective resolutions of everything in sight. We start by choosing injective resolutions of coboundaries and cohomologies

$$B^n(A) \xrightarrow{\iota_{B^n(A)}} \mathsf{B}^n = \mathsf{B}^{n,\bullet} \quad \text{and} \quad H^n(A) \xrightarrow{\iota_{H^n(A)}} \mathsf{H}^n = \mathsf{H}^{n,\bullet}.$$

Now, the exact sequences

$$0 \to B^n(A) \to Z^n(A) \to H^n(A) \to 0 \qquad \text{and} \qquad 0 \to Z^n(A) \to A^n \to B^{n+1}(A) \to 0$$

allow us to combine H and B to get injective resolutions of these short exact sequences (lemma 7.1):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{B}^n & \longrightarrow & \mathsf{Z}^n & \longrightarrow & \mathsf{H}^n & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \iota_{B^n(A)}} & & \uparrow{\scriptstyle \iota_{Z^n(A)}} & & \uparrow{\scriptstyle \iota_{H^n(A)}} & & \\ 0 & \longrightarrow & B^n(A) & \longrightarrow & Z^n(A) & \longrightarrow & H^n(A) & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{Z}^n & \longrightarrow & \mathsf{A}^n & \longrightarrow & \mathsf{B}^{n+1} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \iota_{Z^n(A)}} & & \uparrow{\scriptstyle \iota_{A^n}} & & \uparrow{\scriptstyle \iota_{B^{n+1}(A)}} & & \\ 0 & \longrightarrow & Z^n(A) & \longrightarrow & A^n & \longrightarrow & B^{n+1}(A) & \longrightarrow & 0 \end{array}$$

The injective resolutions A^n of A^n will together form a bigraded object A with $\mathsf{A}^{p,q}$ the q^{th} term in the resolution A^p of A^p.

Let us choose the vertical differentials $(d''_{\mathsf{A}})^{p.q} : \mathsf{A}^{p,q} \to \mathsf{A}^{p,q+1}$ as the differentials $d^q_{\mathsf{A}^p}$ in complexes A^p. Since A^n's are injective resolution of A^n's the map $d^n_A : A^n \to A^{n+1}$ lifts to a map of complexes $\widetilde{d^n_A} : \mathsf{A}^n \to \mathsf{A}^{n+1}$. This is a system of maps $\widetilde{d^{n,q}_A} : \mathsf{A}^{n,q} \to \mathsf{A}^{n+1,q}$. If we would choose the horizontal differential $\delta_{\mathsf{A}} : \mathsf{A}^{p,q} \to \mathsf{A}^{p+1,q}$ as $\widetilde{d^{p,q}_A}$ then d', δ would commute. However, as in 7.2.4(2), one can insert signs in the pair d', δ to get an anti-commuting pair d', d''.

Then A is a bicomplex, and its columns are (by definition of A) resolutions of terms of A, i.e., A is a right bicomplex resolution of the complex A. Finally, all terms $\mathcal{A}^{pq}$ are injective objects.

(b) Since B^n is injective the sequence $0 \to \mathsf{B}^n \to \mathsf{Z}^n \to \mathsf{H}^n \to 0$ has a splitting $\sigma_n : \mathsf{H}^n \hookrightarrow \mathsf{Z}^n$. In particular, $\mathsf{Z}^n = \mathsf{B}^n \oplus \sigma(\mathsf{H}^n)$ is injective, Then also $0 \to \mathsf{Z}^n \to \mathsf{A}^n \to \mathsf{B}^{n+1} \to 0$ has a splitting $\tau^n : \mathsf{B}^{n+1}\mathsf{A}^n$, hence $\mathsf{A}^n = \mathsf{Z}^n \oplus \tau(\mathsf{B}^{n+1})$ is injective. So, we get $\mathsf{A}^n = \mathsf{B}^n \oplus \sigma(\mathsf{H}^n) \oplus \tau(\mathsf{B}^{n+1}) \cong \mathsf{B}^n \oplus \mathsf{H}^n \oplus \mathsf{B}^{n+1}$.

Now, a lift $\delta : \mathsf{A}^{n,\bullet} \to \mathsf{A}^{n+1,\bullet}$ of $A^n \xrightarrow{d_A^n} A^{n+1}$, viewed as a map $\mathsf{B}^n \oplus \mathsf{H}^n \oplus \mathsf{B}^{n+1} \xrightarrow{\delta} \mathsf{B}^{n+1} \oplus \mathsf{H}^{n+1} \oplus \mathsf{B}^{n+2}$, can be chosen by $\delta(b' \oplus h \oplus b'') = b'' \oplus 0 \oplus 0$. For this first observe that this δ is a map of complexes as a composition of two maps of complexes $\mathsf{A}^n \to \mathsf{B}^{n+1} \hookrightarrow \mathsf{A}^{n+1}$. Also, δ is compatible with the resolution maps ι_A, i.e., $\delta^{n,0} \circ \iota_{A^n} = \iota_{B^{n+1}(A)} \circ d_A^n$. □

Corollary. (a) If $A \in C^+(\mathcal{A})$ then for any injective bicomplex resolution (I, ι) of A, the canonical map $A \to \widehat{Tot}(I)$ is an injective resolution of A.

(b) If the injective bicomplex resolution I of a complex $A \in C^+(\mathcal{A})$ splits (in the sense of the lemma 7.2.8.b) then

(1) for each q the horizontal line $I^{\bullet q}$ is an injective complexes that splits (in the sense of 7.2.6);
(2) for each p the vertical complexes $('H^p(I^{\bullet\bullet}, d'')$ are injective resolutions of $H^p(A)$.

Proof. (a) follows from the lemma 7.2.8.b. by lemma 7.2.5.

For (b) one recalls from the proof of lemma 7.2.8.b that the horizontal differential in I is (up to signs) of the form $\delta(b' \oplus h \oplus b'') = b'' \oplus 0 \oplus 0$. So, its kernel on height q is $\mathsf{B}^{\bullet q} \oplus \mathsf{H}^{\bullet q}$ which is injective. Also, we see that the p^{th} column of its horizontal cohomology is exactly $\mathsf{H}^{p\bullet}$ which was chosen as an injective resolution of $H^p(A)$. □

7.2.9. *Uniqueness of injective resolutions of complexes.* We extend the uniqueness of injective resolutions from objects of $\mathcal{A}$ to complexes in $C(\mathcal{A})$.

Theorem. (a) If $A \to I$ is an injective resolution and $B \to \mathcal{B}$ is a quasi-isomorphism then then any map $f : B \to A$ in $C(\mathcal{A})$ lifts to a map $F : \mathcal{B} \to I$.

(b) Any two lifts are homotopic.

Corollary. Any two injective resolutions are canonically isomorphic in homotopy category of complexes. So there is a functor of injective resolutions $\mathcal{I} : C^+(\mathcal{A}) \to K(\mathcal{B})$. □

7.2.10. *Injective resolutions of SES of complexes.* Here we extend the uniqueness of injective resolutions of SES of objects in $\mathcal{A}$ to SES of complexes in $C(\mathcal{A})$.

Theorem. For a SES $0 \to A \to B \to C \to 0$ in $C^+(\mathcal{A})$, there is a commutative diagram in $C(\mathcal{A})$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathsf{A} & \longrightarrow & \mathsf{B} & \longrightarrow & \mathsf{C} & \longrightarrow & 0 \end{array}$$

where vertical arrows are injective resolutions and both rows are exact. Moreover, the image of the second row in $K(\mathcal{A})$ is canonical up to a unique isomorphism.

Proof. This is a combination of ideas in proofs of the lemma 7.1 and the theorem 7.2.8.

Corollary. If $\mathcal{A}$ has enough injectives any short exact sequence in $C^+(\mathcal{A})$ defines a distinguished triangle in $K(\mathcal{A})$.

Proof. By the theorem for any SES $0 \to A \to B \to C \to 0$ the SES of injective resolutions $0 \to \mathsf{A} \to \mathsf{B} \to \mathsf{C} \to 0$ is unique up to homotopy. We know that any choice of the SES of injective resolutions defines a distinguished triangle in $K(\mathcal{A})$ (since SES of complexes with injective objects always splits degreewise). Moreover, any homotopy h of two choices of SES of injective resolutions gives a canonical isomorphism of distinguished triangles in $K(\mathcal{A})$. $\square$

8. **Spectral sequences**

In general, spectral sequences[5] are associated to a complex K with a filtration F. The idea is to relate the cohomology of the complex K with the cohomology of a simplified complex $Gr^F(K)$ which may be easier to understand.

8.0.11. *History.* Jean Leray introduced the notion of a sheaf and sheaf cohomology and then he introduced the Leray spectral sequence as a computational technique that gives a relation between cohomology groups of a sheaf and cohomology groups of the pushforward of the sheaf. (All while he was in a prisoner of war camp in Edelbach, Austria from 1940 to 1945.)

This relation involved an infinite process... Soon, spectral sequences were found in diverse situations, and they gave intricate relationships among homology and cohomology groups coming from geometric situations such as fibrations and from algebraic situations involving derived functors. While their theoretical importance has decreased since the introduction of derived categories, they are still the most effective computational tool available. Because of the large amount of information carried in a spectral sequences, they are difficult to grasp and beyond general ideas one has to resort to various tricks.

8.1. **Filtered and graded objects.** A graded object A of $\mathcal{A}$ is a sequence of objects $A^n \in \mathcal{A},\ n \in \mathbb{Z}$. We think of it as a sum $A = \oplus_{\mathbb{Z}} A^n$ (and this is precise if the sum exists in $\mathcal{A}$).

An increasing (resp. decreasing) f *filtration* on an object $a \in \mathcal{A}$ is an increasing (resp. decreasing) sequence of subobjects $a_n \hookrightarrow a,\ n \in \mathbb{Z}$. When we talk of a filtration F we denote a_n by $F_n a$.

A filtration defines the *graded* object $Gr(a)$. If filtration is increasing then $Gr_n(a) \overset{\text{def}}{=} a_n/a_{n-1}$ and if it is decreasing $Gr_n(a) \overset{\text{def}}{=} a_n/a_{n+1}$).

In the opposite direction, a graded object $A = \oplus A^n$ defines a canonical increasing filtration $A_{\leq n} = \oplus_{i \leq n} A^i$ (if the sums exist), and a canonical decreasing filtration $A_{\geq n} = \oplus_{i \geq n} A^i$. Composition *grading* $\mapsto$ *filtration*$\mapsto$*grading* is identity, however, the other composition loses information.

Remark. Increasing filtrations F of a and decreasing filtrations G of a, are in a bijection by $G_i a \overset{\text{def}}{=} F_{-i}a$ and $F_i a \overset{\text{def}}{=} G_{-i}a$. These two operations are mutually inverse since $1-(1-a) = a$. This relation between the corresponding graded objects is

$$Gr^G_i \overset{\text{def}}{=} G_i/G_{i+1} = F_{-i}/F_{-i-1} = Gr^F_{-i}.$$

[5] An approximate quote from Borel:
"*Those were great times when only three of us*[6] *knew spectral sequences. We could prove everything and no one else could prove anything.*"

So, whatever we notice for decreasing filtrations can be translated into a statement for increasing filtrations.

8.1.1. *Filtrations of complexes.* We will be interested in a *decreasing filtrations* F on a complex $(A = \oplus\ A^n, d)$. This is a sequence of subcomplexes F_pA

$$A \supseteq \cdots \supseteq F_pA \supseteq F_{p+1}A \supseteq \cdots.$$

The data are given by terms $(F_pA)^n$ of the subcomplex $F_pA \subseteq A$. These are sub-objects of A^n, so, we also denote then F_pA^n. So, for each n they form a filtration $A^n \supseteq \cdots F_pA^n \supseteq F_{p+1}A^n \supseteq \cdots$ of A^n. Then we ask that each F_pA is a subcomplex of A, i.e., that $d(F_pA^n) \subseteq F_p(A^{n+1})$.

Remark. A filtration F of a complex A gives a filtration F of its cohomology groups. The p^{th} filtered piece of $H^n(A)$ is the image of $H^n(F_pA)$ under the map of cohomologies given by the map of complexes $F_pA \subseteq A$:

$$F_p[\mathrm{H}^n(A)] \overset{\text{def}}{=} \mathrm{Im}[\mathrm{H}^n(F_pA) \to \mathrm{H}^n(A)].$$

Example. Any complex A has the *stupid* decreasing filtration $\sigma_{\leq}$ where the subcomplex $\sigma_{\leq n}A$ is obtained by erasing all terms A^k with $k < n$:

$$\sigma_{\leq n}A \overset{\text{def}}{=} \quad (\cdots \to 0 \to 0 \to A^n \to A^{n+1} \to A^{n+2} \to \cdots).$$

8.1.2. *Decreasing filtrations $'F$ and $''F$ on a bicomplex and on the total complex.* The fact that the complex $Tot(B)$ has come from a bicomplex will be used to produce two decreasing filtrations on the complex $Tot(B)$. Similarly to the stupid filtration on complexes, any bicomplex B has two decreasing filtrations $'F$ and $''F$. The sub-bicomplex $'F_iB$ of a bicomplex B is obtained by erasing the part of B which is on the left from the i^{th} column, and symmetrically, $''F_jB$ is obtained by erasing beneath the j^{th} row. Say, the subbicomplex $'F_iB$ is given by

$$\begin{tikzcd}
\cdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \cdots \\
\cdots \arrow[r, "d'"] \arrow[u, "d''"] & 0 \arrow[r, "d'"] \arrow[u, "d''"] & B^{i,2} \arrow[r, "d'"] \arrow[u, "d''"] & B^{i+1,2} \arrow[r, "d'"] \arrow[u, "d''"] & B^{i+2,2} \arrow[r, "d'"] \arrow[u, "d''"] & B^{i+3,2} \arrow[r, "d'"] \arrow[u, "d''"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[r, "d'"] \arrow[u, "d''"] & 0 \arrow[r, "d'"] \arrow[u, "d''"] & B^{i,1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{i+1,1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{i+2,1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{i+3,1} \arrow[r, "d'"] \arrow[u, "d''"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[r, "d'"] \arrow[u, "d''"] & 0 \arrow[r, "d'"] \arrow[u, "d''"] & B^{i,0} \arrow[r, "d'"] \arrow[u, "d''"] & B^{i+1,0} \arrow[r, "d'"] \arrow[u, "d''"] & B^{i+2,0} \arrow[r, "d'"] \arrow[u, "d''"] & B^{i+3,0} \arrow[r, "d'"] \arrow[u, "d''"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[r, "d'"] \arrow[u, "d''"] & 0 \arrow[r, "d'"] \arrow[u, "d''"] & B^{i,-1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{i+1,-1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{i+2,-1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{i+3,-1} \arrow[r, "d'"] \arrow[u, "d''"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[r, "d'"] \arrow[u, "d''"] & \vdots \arrow[r, "d'"] \arrow[u, "d''"] & \vdots \arrow[r, "d'"] \arrow[u, "d''"] & \vdots \arrow[r, "d'"] \arrow[u, "d''"] & \vdots \arrow[r, "d'"] \arrow[u, "d''"] & \vdots \arrow[r, "d'"] \arrow[u, "d''"] & \cdots . \arrow[u, "d''"]
\end{tikzcd}$$

This then induces filtrations on the total complex, say

$$['F_iTot(B)]^n \overset{\text{def}}{=} Tot('F_iB)^n = \oplus_{p+q=n,\ p\geq i}\ B^{p,q} \subseteq Tot(B)^n \ = \ \oplus_{p+q=n}\ B^{p,q}.$$

Finally, $'F$ and $''F$ induce filtrations on the cohomology

$$'F_p\mathrm{H}^n(Tot\ B) \overset{\text{def}}{=} \mathrm{Im}[\mathrm{H}^n(Tot['F_pB]) \to \mathrm{H}^n(Tot[B])],$$

so the cohomology groups are extensions of pieces

$$'Gr_i[\mathrm{H}^n(Tot\ B)] \overset{\text{def}}{=} \frac{'F_i\mathrm{H}^n(Tot\ B)}{'F_{i+1}\mathrm{H}^n(Tot\ B)}.$$

These pieces will be calculated by the method of spectral sequences in 8.5.

Remark. The only filtrations of complexes that we will seriously use are the above two natural decreasing filtrations on the total complex $Tot(B)$ of a bicomplex B:

8.2. **Spectral sequence of a filtered complex: the idea.** Consider a complex $K \in C(\mathcal{A})$ with a decreasing filtration $F_\bullet$, so $F_pK^n \subseteq K^n$ and the differential $d = d_K$ takes F_pK^n to F_pK^{n+1}. We will associate to the filtration F of K a "spectral sequence" E that converges to (graded) cohomology of K.[(7)]

---

[7] One also says that a spectral sequence "*abuts*" to its limit where the word *abut* means: adjoin, be adjacent to, join, touch, meet, reach, be next to or have a common boundary with. The limit is also called the *abutment* of the sequence. the word means "masonry mass supporting and receiving the thrust of part of an arch". "masonry receiving the arch, beam, truss, etc., at each end of a bridge".

8.2.1. *Sequence E_r of subquotients of K.* For $r \in \mathbb{N}$ define

$$E_r^{p,q} \stackrel{\text{def}}{=} \frac{\{a \in F_pK^{p+q};\ da \in F_{p+r}K^{p+q+1}\} \ + \ (dF_{p+1-r}K^{p+q-1} + F_{p+1}K^{p+q})}{d\ F_{p+1-r}K^{p+q-1} \ + \ F_{p+1}K^{p+q}}.$$

We are really interested in $H^*(K)$, i.e., the classes $[a]$ of cocycles. The filtration can be viewed as specifying a topology on the group K by listing a basis F_pK of neighborhoods of zero. So, "$a \in F_pK$" is interpreted as "a is p-small". Then the elements a of $Z_r^{pq} \stackrel{\text{def}}{=} \{a \in F_pK^{p+q};\ da \in F_{p+r}K^{p+q+1}\}$ can be viewed as r-approximations of cocycles since $da \in F_{p+r}$ is "r-smaller" then $a \in F_p$ (this is our r-approximation of $da = 0$).

8.2.2. *Properties of the sequence E_r.* The following will be proved in the theorem 8.4.2

- (a) $E_0^{pq} \cong Gr_p(K^{p+q})$.
- (b) The differential d on K factors to a differential d_r on E_r of type $(r, 1-r)$. Then
$$H^*(E_r, d_r) \cong E_{r+1}.$$
- (c) Under some conditions on K and F the terms E_r^{pq} stabilize.[8] So, one can define E_∞^{pq} as E_r^{pq} for $r >> 0$. Then
$$E_\infty^{pq} \ \cong \ Gr_p[H^{p+q}(K)].$$

[Compare this with $E_1^{pq} \cong H^{p+q}(Gr_pK)$.]

Remarks. (0) A decreasing filtration F on K allows us to simplify complex K to $E_0 \stackrel{\text{def}}{=} Gr(K)$. We view it as a bigraded object by $E_0^{pq} \stackrel{\text{def}}{=} Gr_p(K^n) = F_pK^n/F_{p+1}K^n$.

It turns out that we can repeat simplifications infinitely many times to get all E_r by observing that d_K induces a differential d_r on E_r and then we take E_{r+1}^{pq} to be $H^*(E_r, d_r)^{pq}$.[9]

(1) While E_0 is of the "same size" as K, each E_{r+1} is a subquotient of E_r so it is potentially smaller.

(2) The sequence E_r stabilizes to $E_\infty^{pq} = Gr_p[H^{p+q}(K)]$. By this method we can not calculate $H^*(K)$ precisely since, to start with, we have simplified K to $E_0 = Gr(K)$. However, we do get the pieces $Gr_p[H^{p+q}(K)]$ whose extension is $H^{p+q}(K)$.

[8] We will see that it suffices that $F_pK = K$ for $p \leq 0$ and $F_pK^n = 0$ for $p > n$.

[9] If one knows E_0 and how to pass from E_r to E_{r+1} one can reinvent the formula for E_r^{pq} that we have started with. One starts by replacing K with its simplification $E_0 = Gr(K)$ which is clearly a bigraded object since two indices appear in $Gr_p(K^n)$. So, all computations will be made in the realm of bigraded objects. Moreover, $E_0^{p,\bullet} = Gr_p(K)$ is a clearly a complex (a quotient of a complex by a subcomplex). This gives E_1 as cohomology of (E_0, d_0).

Now, key observation is that E_1 is again a subquotient of K and the differential on K defines a new differential d_1 on E_1. So, one can define E_2. After repeating this process a few times we get a feeling for what any E_r, d_r will look like (this is the above formula for E_r). This expectation is then verified by straightforward algebra.

(3) The simplest information that a spectral sequence extracts for us are the bounds on "size" of $H^n(K)$ (each E_r^{pq} is an "upper bound" for the piece $Gr_p[H^n(K)]$ which is its subquotient). Of course, we get better information (a finer upper bound) if we can calculate more terms in the spectral sequence.

(4) It is "often" possible to calculate all E_r's (usually in the situation when sequence stabilizes early). A number of deep theorems in mathematics takes form of degeneration of a spectral sequence in E_2 or E_3 term.

(5) This machinery replaces a single calculation of cohomology with respect to a possibly complicated differential d_K, by infinitely many computations of cohomology with respect to simpler differentials d_r.[10]

8.3. **The notion of a spectral sequence.** This notion has several versions useful in different situations. We state here one choice.

A *(cohomological) spectral sequence* in an abelian category $\mathcal{A}$ is a sequence $(E_r, d_r, \iota_r)_{r\in\mathbb{N}}$ such that

(1) E_r is bigraded family of objects of $\mathcal{A}$, $E_r = (E_r^{p,q})_{p,q\in\mathbb{Z}}$
(2) $d_r : E_r \to E_r$ is a differential, i.e., $d_r^2 = 0$, and it has type $(r, 1-r)$, i.e.,

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q+1-r}.$$

(3) ι_r is an isomorphism $E_{r+1} \xrightarrow{\cong} \mathrm{H}(E_r, d_r)$, i.e.,

$$E_{r+1}^{p,q} \xrightarrow{\cong} \mathrm{Ker}(E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q+1-r})/\mathrm{Im}(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q}).$$

Remark. When dealing with spectral sequences one usually draws the $\mathbb{Z}^2$-grid in the plane and then one draws the directions of differentials: d_0 goes vertically by one, d_1 horizontally by one, then d_2 goes two right and one down (the knight move) etc.

8.3.1. *The limit of a spectral sequence.* The limit $E_\infty = \lim E_r$ can be defined for any spectral sequence. We will only consider the most interesting case, i.e., when the spectral sequence stabilizes. We will say that the (p,q)-term *stabilizes* at r if for $s \geq r$ the differentials d_s *from* the (p,q)-term and *into* the (p,q)-term are zero. Then clearly

$$E_r^{p,q} \cong E_{r+1}^{p,q} \cong E_{r+2}^{p,q} \cong \cdots.$$

Then we define $E_\infty^{p,q}$ as $E_r^{p,q}$. We will say that the *spectral sequence stabilizes* (degenerates) in the r^{th} term if all (p,q)-terms do, i.e., if $d_s = 0$ for $s \geq r$. Then E_∞ is just E_r.

8.3.2. *Stabilization criteria.*

[10] Taking cohomology, $K \mapsto H^*(K)$, kills the irrelevant information, and the filtration on K provides a way of doing this in small steps.

Lemma. (a) If $E_r^{p,q}=0$ then $E_{r+1}^{p,q}=0$.

(b) If $E_r^{p-r,q+r-1}=0=E_r^{p+r,q+1-r}$ then $E_{r+1}^{p,q}=E_r^{p,q}$.

(c) If for some r_0 we have that for $r\geq r_0$ and any p,q one of objects $E_r^{p,q}$ or $E_r^{p+s,q+1-s}$ is zero then the spectral sequence degenerates at E_r.

(d) ("First quadrant".) If for some $r_0\geq 0$ there exist $a,b\in\mathbb{Z}$ such that E_{r_0} is supported in the quadrant $\{(p,q)\in\mathbb{Z}^2;\ p\geq a \quad\text{and}\quad q\geq b\}$ then for each pair (p,q) the terms E_r^{pq} stabilize.

(e) ("Parity".) If for some r all $p+q$ with $E_r^{pq}\neq 0$ are of the same parity, then the sequence stabilizes at E_r.

Proof. (a) is obvious since E_{r+1}^{pq} is a subquotient of E_r^{pq}. So are (b-c) since d_s is of type $(s,1-s)$.

(d) For any (p,q) that satisfy the condition $p\geq a,\ q\geq b$, the differentials $d_r: E_r^{pq}\to E_r^{p+r,q+1-r}$ and $d_r: E_r^{p-r,q-1+r}\to E_r^{p,q}$ vanish when the target of the first map or the source of the second do not satisfy these conditions and this happens for $r>>0$.

(e) Since d_s is of type $(s,1-s)$ it changes the total parity $p+q$. So. always either the source or the target of d^s is zero. $\square$

8.4. **The spectral sequence of a filtered complex.** The theorem ... below is the complete version of what has been announced in 8.2. To a complex $K\in C(\mathcal{A})$ with a decreasing filtration F we will associate a spectral sequence E that converges to the graded cohomology of K.

8.4.1. *Filtration on cohomology induced by a filtration of the complex.* The inclusions $F_pK\subseteq K$ are maps of complexes so they gives maps $H^n(F_pK)\to H^n(K)$, and we get the corresponding subobjects of cohomology:

$$F_p\big(H^n(K)\big)\ \stackrel{\text{def}}{=}\ Im[H^n(F_pK)\to H^n(K)]\subseteq H^n(K),\ p\in\mathbb{Z}.$$

We will denote the induced filtrations on subobjects $B^n\subseteq Z^n\subseteq K^n$ by $F_pZ^n\stackrel{\text{def}}{=}\ Z^n\cap F_pK$ and $F_pB^n\stackrel{\text{def}}{=}B^n\cap F_pK$.

Lemma. (a) The filtered pieces are

$$F_p\big(H^n(K)\big)\ =\ \frac{F_pZ^n+B^n}{B^n}\ \cong\ \frac{F_pZ^n}{F_pB^n}.$$

(b) The graded pieces are

$$Gr_p\big(H^n(K)\big)\ \cong\ \frac{F_p(Z^n)}{F_p(B^n)+F_{p+1}(Z^n)}.$$

Proof. (a) The p^{th} filtered piece of $H^n(K)$ is precisely given by classes $a+B^n$ that have a representative $a \in Z^n$ such that a lies in F_pK^n.

(b) $Gr_p\big(H^n(K)\big)$ is

$$\frac{[F_pZ^n+B^n]/B^n}{[F_{p+1}Z^n+B^n]/B^n} = \frac{F_pZ^n+B^n}{F_{p+1}Z^n+B^n} \cong \frac{F_pZ^n}{[F_{p+1}Z^n+B^n]\cap F_pZ^n} \cong \frac{F_pZ^n}{F_{p+1}Z^n+F_pB^n}. \quad \square$$

8.4.2. *The spectral sequence.* For $r \in \mathbb{N}$ denote

$$Z_r^{p,q} \stackrel{\text{def}}{=} \{a \in F_pK^{p+q};\ da \in F_{p+r}K^{p+q+1}\} \quad \text{and} \quad B_r^{pq} \stackrel{\text{def}}{=} dF_{p+1-r}K^{p+q-1}$$

(we have $Z_r^{pq}\supseteq Z_{r+1}^{pq}$ and $B_r^{pq}\subseteq B_{r+1}^{pq}$ but B_r^{pq} need not lie in Z_r^{pq}). Finally, we define[11]

$$E_r^{p,q} \stackrel{\text{def}}{=} \frac{Z_r^{pq}\ +\ (B_r^{pq}\ +\ F_{p+1}K^{p+q})}{B_r^{pq}\ +\ F_{p+1}K^{p+q}} \cong \frac{Z_r^{pq}}{(B_r^{pq}\ +\ F_{p+1}K^{p+q})\cap Z_r^{pq}}$$

Remark. The important part is the coordinate p in E_r^{pq} since this is the choice of the filtered piece, one can ignore the coordinate q (equivalently the coordinate n on K^n) since these are obvious (the differential increases n by 1).

Theorem. A decreasing filtration F on a complex K defines a spectral sequence E with

- (0) $E_0^{p,q} =\ Gr_p(K^{p+q})$
- (1) $E_1^{p,q} =\ H^{p+q}[Gr_p(K)]$
- (2) The differential $d=d_K$ on K induces the a differential d on E_r (which we denote d_r).
- (3) There is a canonical isomorphism $H^*(E_r,d_r)\cong E_{r+1}$.
- (4) Assume that F satisfies: $F_0(K)=K$ and $F_p(K^n)=0$ for $p>n$. Then E is a first quadrant spectral sequence. So each E^{pq} stabilizes.
- (5) The limit is
$$E_\infty^{p,q} =\ Gr_p[\mathrm{H}^{p+q}(K)].$$

8.4.3. *Proof of (0).* Notice that $B_0^{pq}=dF_{p+1}K^{p+q-1}$ lies in $F_{p+1}K^{pq}$. Also, $Z_0^{pq}=\{a \in F_pK^{p+q};\ da \in F_pK^{p+q}\} =\ F_pK^{p+q}$ contains $F_{p+1}K^{pq}$. So, $E_0^{pq}=F_pK^{p+q}/F_{p+1}(K^{p+q}) = Gr_p(K^{p+q}))$.

[11] It is convenient to write a shorthand formula $E_r^{pq}=Z_r^{pq}/[B_r^{pq}\ +\ F_{p+1}K^{p+q}]$ which is not formally correct since the nominator need not contain the denominator. It still indicates that all classes in E_r^{pq} will be represented by elements of Z_r^{pq}.

8.4.4. *Proof of (2).* We need to check that the differential $d = d_K$ takes

- Z_r^{pq} to $Z_r^{p+r,q+1-r}$,
- B_r^{pq} to $B_r^{p+r,q+1-r}$, and
- $F_{p+1}K^{p+q}$ to $F_{p+1}K^{p+q+1}$.

First, $a \in K^{p+q}$ is in Z_r^{pq} if $a \in F_pK^{p+q}$ and $da \in F_{p+r}K^{p+q+1}$, but then certainly $d(da) = 0$ gives $da \in Z_r^{p+r,q+1-r}$.

The second claim is clear since on $B_r^{pq} = dF_{p-r}K^{p+q+1}$ we have $d = 0$, The third claim is just that F_pK is a subcomplex of K.

8.4.5. *Proof of (3).* To check that E_{r+1}^{pq} is the cohomology subquotient of E_r^{pq} we first write E_{r+1}^{pq} as a natural subquotient of E_r^{pq}

$$E_r^{pq} = \frac{Z_r^{pq} + B_r^{pq} + F_{p+1}K^{p+q}}{B_r^{pq} + F_{p+1}K^{p+q}} \supseteq E_{r,r+1}^{pq} \overset{\text{def}}{=} \frac{Z_{r+1}^{pq} + B_r^{pq} + F_{p+1}K^{p+q}}{B_r^{pq} + F_{p+1}K^{p+q}}$$

$$\overset{\beta}{\twoheadrightarrow} \frac{Z_{r+1}^{pq} + B_{r+1}^{pq} + F_{p+1}K^{p+q}}{B_{r+1}^{pq} + F_{p+1}K^{p+q}} = E_{r+1}^{pq}.$$

We first check that

$$\mathrm{Ker}(d_r^{pq}) = E_{r,r+1}^{p,q}.$$

The kernel of $E_r^{p,q} \xrightarrow{d_r^{pq}} E_r^{p+r,q+r-1}$ is given by all classes $[a]_{E_r^{pq}} \in E_r^{pq}$ of elements $a \in Z_r^{pq}$ such that da lies in $B_{p+1-r}^{p+r,q+1-r} + F_{p+1}K^{p+q+1}$, i.e., $da = d\alpha + \beta$ for some $\alpha \in F_{(p+r)+1-r}K^{p+q} = F_{p+1}K^{p+q}$ and $\beta \in F_{(p+r)+1}K^{p+q+1}$. Then $a - \alpha \in F_pK^{p+q}$ is another representative of the class $[a]_{E_r^{pq}}$ such that $d(a-\alpha) = \beta \in F_{p+r+1}K^{p+q+1}$. So, $a-\alpha \in Z_{r+1}^{pq}$ and therefore $[a]_{E_r^{pq}}$ lies in $E_{r,r+1}^{pq}$.

Conversely, any class in $E_{r,r+1}^{pq}$ has a representative $a' \in Z_{r+1}^{pq}$. So, $da' \in F_{p+r+1}K^{p+q}$ and therefore $d_r^{pq}[a'] = [da'] = 0$.

It remains to check that the kernel of the map β in the diagram is the image of the differential

$$d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q}.$$

Notice that $\mathrm{Ker}(\beta)$ consists of all classes $[a]_{E_r^{pq}}$ with $a \in Z_{r+1}^{pq}$ such that a lies in $B_{r+1}^{pq} + F_{p+1}K^{p+q}$. Since one can adjust a representative of a class by anything from $F_{p+1}K^{p+q}$, $\mathrm{Ker}(\beta)$ consists of all classes $[a]_{E_r^{pq}}$ with $a \in Z_{r+1}^{pq} \cap B_{r+1}^{pq}$. The condition is that $a = db$ with $b \in F_{p-r}K^{p+q-1}$ such that $db = a$ lies in F_pK^{p+q}, i.e., $b \in Z_r^{p-r,q+r-1}$. So, These are exactly the classes of the form $[db]_{E_r^{pq}} = d_r[b]_{E_r^{pq}}$ with $[b]_{E_r^{pq}} \in E_r^{pq}$.

8.4.6. *Proof of (1).* Here we deduce the description of E_1^{pq} as $Gr_p[H^{p+q}(K)]$ from (0) and (3).

- (i) In (0) we found a natural identification of the bigraded objects E_0^{pq} and $Gr_pK^{p,q}$.

- (ii) This is really an identification of complexes since the differential d_0 on E_0^{pq} and the differential on $(Gr_pK)^{p,q}$ are both induced from the differential d on K.
- (iii) So, the cohomology of (E_0, d_0) in position (p,q) gets identified with the cohomology $H^{p+q}(Gr_pK)$ of $Gr_p(K)$.

8.4.7. *The convergence (stabilization) conditions on the filtration.* The two conditions $F_0(K) = K$ and together imply that

- (i) $K^n = 0$ for $n < 0$.
- (ii) For $n \geq 0$ the filtration on each K^n lives in the interval $[0,n]$ in the sense that
$$K^n = F_0K^n \supseteq \cdots \supseteq F_nK^n \supseteq F_{n+1}K^n = 0.$$
- (iii) $Gr_p(H^n(K)) = 0$ unless $0 \leq p \leq n$.

In particular the filtration on each K^n is *finite*, *exhaustive*, i.e., $\cup_p\ F_pK^n = K^n$, and *Hausdorff*, i.e., $\cap_p\ F_pK^n = 0$.

Proof. (i) If $n < 0$ then $n+1 \leq 0$ hence $0 = F_{n+1}K^n \subseteq F_0K^n = K^n$ implies that $K^n = 0$. (ii) is obvious and (ii) follows from (ii).

Remark. The convergence conditions above are the *canonical ones* ("canonically bounded filtration"). There are more general (and more complicated) conditions that guarantee convergence to the same E_∞ as above. However, the above case suffices for basic applications and once one understands this particular case, one can try to adjust it according to the needs of a particular application.

8.4.8. *Proof of (4).* E is a first quadrant spectral sequence since $E_0^{pq} = Gr_p(K^{p+q})$ vanishes when either (i) $p < 0$ or (ii) $q < 0$. (This is the the above claim 8.4.7.(iii)).

So, for any (p,q) the sequence E_r^{pq} stabilizes by lemma 8.3.2.4. One can say more precisely when the stabilization has to happen:

$$E_\infty^{pq} \;=\; E_r^{pq} \;\text{ for }\; r = \max(p, q+1) + 1.$$

In other words, if $p, q \geq 0$ and $s > \max(p, q+1)$ then d_s in and out of (p,q)-position is zero because the source and target positions are $(p-s, p+s-1)$ and $(p+s, p+1-s)$ and $p - s < 0 > p+1-s$.

8.4.9. *Proof of (5).* We have $K^n \supseteq Z^n \supseteq B^n$ and we will also denote $Z_\infty^{pq} \overset{\text{def}}{=} F_pZ^{p+q}$. In this notation $\mathcal{E}_\infty^{p,q} \overset{\text{def}}{=} H^{p+q}(Gr_pK)$ is $Z_\infty^{pq}/[B_\infty^{pq} + Z_\infty^{p+1,q-1}]$ (ibid).

($\bullet$) We first write $\mathcal{E}_\infty^{p,q}$ as a natural subquotient of each E_r. We use $Z_r^{pq} \supseteq Z_\infty^{pq}$ and $B_r^{pq} \subseteq B_\infty^{pq}$:

$$E_r^{pq} = \frac{Z_r^{pq} + B_r^{pq} + F_{p+1}K^{p+q}}{B_r^{pq} + F_{p+1}K^{p+q}} \supseteq E_{r,\infty}^{pq} \overset{\text{def}}{=} \frac{F_pZ^{p+q} + \ B_r^{pq} + F_{p+1}K^{p+q}}{B_r^{pq} + F_{p+1}K^{p+q}}$$

$$\twoheadrightarrow \frac{F_pZ^{p+q} + \ B^{p+q} + F_{p+1}K^{p+q}}{B^{p+q} + F_{p+1}K^{p+q}} \ \cong \ \frac{F_pZ^{p+q}}{[B^{p+q} + F_{p+1}K^{p+q}] \cap Z^{p+q} \cap F_pK}.$$

The denominator reduces to

$$[B^{p+q} + F_{p+1}K^{p+q} \cap Z^{p+q}] \cap F_pK^{p+q} \ \cong \ F_pB^{p+q} + F_{p+1}Z^{p+q}.$$

So, we get $Z_\infty^{pq}/[F_pB^{p+q} + F_{p+1}Z^{p+q}]$ which is (by lemma 8.4.1) exactly $H^{p+q}(Gr_pK)$.

($\bullet$) Now for $r >> 0$ notice that $Z_r^{pq} = \{a \in F_pK^{p+q};\ da \in F_{p+r-1}K^{p+q}$ is F_pZ^{p+q} (so that the inclusion is an isomorphism). Also, the kernel of the surjection are the classes in $E_{r,\infty}^{pq}$ of elements $a \in F_pZ^{p+q}$ such that $a \in B^{p+q} + F_{p+1}K^{p+q}$, i.e., for some $b \in K^{p+q-1}$, $a - db$ is in $F_{p+1}K^{p+q}$. ... (so that the surjection is an isomorphism). □

8.4.10. *Example: a case when the spectral sequence calculates H^nK rather then just the graded pieces.* In general $Gr(H^nK) = \ \oplus_{p=0}^n \ E_\infty^{p,n-p}$ (see 8.4.7), so if this sum has at most one nonzero term, then this term is H^nK. For instance,

Lemma. If for some r, E_r is supported in one line $q = q_0$ or a column $p = p = p_0$, then the spectral sequence degenerates (i.e., stabilizes) at E_r. Moreover, then

$$H^nK \ \cong \ E_r^{p_0q_0}$$

for the unique (p_0, q_0) in this line or column, that satisfies $p_0 + q_0 = n$.

8.5. **Spectral sequences of bicomplexes.**

8.5.1. *Partial cohomologies of bicomplexes.* By taking the "horizontal" cohomology we obtain a bigraded object $'\mathrm{H}(B)$ with

$$'\mathrm{H}(B)^{p,q} \ \stackrel{\text{def}}{=} \ \mathrm{H}^p(B^{\bullet,q}) \ = \ \frac{\mathrm{Ker}(B^{p,q} \xrightarrow{d'} B^{p+1,q})}{Im(B^{p-1,q} \xrightarrow{d'} B^{p,q})}.$$

The vertical differential d'' on B factors to a differential on $'\mathrm{H}(B)$ which we denote again by d'':

$$'\mathrm{H}(B)^{p,q} \xrightarrow{d''} \ '\mathrm{H}(B)^{p,q+1}.$$

Next, we take the "vertical" cohomology of $'\mathrm{H}(B)$ (i.e., with respect to the new d''), and get a bigraded object $''\mathrm{H}('\mathrm{H}(B))$ with

$$''('\mathrm{H}(B))^{p,q} \stackrel{\text{def}}{=} \ \mathrm{H}^q('\mathrm{H}(B)^{p,\bullet}) = \ \frac{\mathrm{Ker}['\mathrm{H}(B)^{p,q} \xrightarrow{d''} {'\mathrm{H}(B)^{p,q+1}}]}{Im['\mathrm{H}(B)^{p,q-1} \xrightarrow{d''} {'\mathrm{H}(B)^{p,q}}]}.$$

One defines $''\mathrm{H}(B)$ and $'\mathrm{H}(''\mathrm{H}(B))$ by switching the roles of the first and second coordinates.

8.5.2. *Remark.* Constructions $'\mathrm{H}(B)$ and $''\mathrm{H}(B)$ are upper bounds on the cohomology of the bicomplex, and $'\mathrm{H}(''\mathrm{H}(B))$ and $''\mathrm{H}('\mathrm{H}(B))$ are even better upper bounds. The precise relation is given via the notion of spectral sequences (see 8).

8.5.3. *The two spectral sequences.* Recall that any bicomplex B has two canonical decreasing filtrations $'F$ and $''F$ which induce the corresponding filtrations of the complex $Tot(B)$ (see 8.1.2).

Theorem. For any bicomplex B, filtrations $'F$ and $''F$ of $Tot(B)$ produce two spectral sequences $'E$ and $''E$. The first one satisfies

(1) $'E_0^{p,q} = \ B^{p,q}$
(2) $'E_1^{p,q} = \ ''\mathrm{H}^{p,q}(B)$
(3) $'E_2^{p,q} = \ '\mathrm{H}^p[''\mathrm{H}^{\bullet,q}(B)]$
(4) If B is a first quadrant bicomplex then $'E$ is a first quadrant spectral sequence converging to

$$'E_\infty^{p,q} = \ 'Gr_p[\mathrm{H}^{p+q}(Tot\ B)].$$

The statement for the second filtration is symmetric – one just switches the roles of indices p and q, so in particular $''E_1$ is given by horizontal cohomology $''H$ and $''E_2$ by composition $'H\circ''H$ with the vertical cohomology.

Proof.

□

8.5.4. *Remarks.* (a) The basic consequence is that the piece $'Gr_p[\mathrm{H}^n(Tot\ B)]$ of $\mathrm{H}^n(Tot\ B)$ is a subquotient of $'\mathrm{H}^{n-p}[''\mathrm{H}^{\bullet,q}(B)]$, hence in particular, of $''\mathrm{H}^{p,n-p}(B)$ (since $E_{r+1}^{i,j}$ is always a subquotient of $E_r^{i,j}$). This gives upper bounds on the dimension of $\mathrm{H}^n(Tot\ B)$.

(b) We are fond of bicomplexes such that the first spectral sequence degenerates at $'E_2$, then we recover the constituents of $\mathrm{H}^n(Tot\ B)$ from partial cohomology

$$Gr_\bullet[\mathrm{H}^n(Tot\ B)] \cong \ \oplus_{p+q=n}\ '\mathrm{H}^p[''\mathrm{H}^{\bullet,n-p}(B)].$$

8.6. **Examples of applications.**

8.6.1. *Bicomplex resolutions of complexes give resolutions of complexes.*

Lemma. If a bicomplex B is a right resolution of a complex A and B is a first quadrant bicomplex then the map $A \to Tot(B)$ is a quasi-isomorphism.

Proof. For the first spectral sequence of B we know that $'E_1{}^{pq}$ is the vertical cohomology $''H^{pq}(B)$ of B. Since B is a right resolution of a complex, we see that $'E_1{}^{pq} = 0$ for $q \neq 0$ and $'E_1^{p,0} = A^p$. Moreover, the differential d_1 on $'E_1$ is just the differential of A. Therefore, the second term $'E_2{}^{pq} = \ 'H^p[''H^{\bullet,q}(B)$ is $\delta_{q,0}\ H^p(A)$.

The sequence converges here since the higher differentials d_r for $r \geq 2$ do not preserve the row $q = 0$ row where $\ 'E_2$ is supported.

Now we know that ${}'E_\infty{}^{pq} = {}'Gr_p(H^{p+q}(Tot(B))$ equals ${}'E_2{}^{pq} = \delta_{q,0}\ H^p(A)$. However, by the lemma 8.4.10 we know more. Since E_2 is concentrated in one row we have

$$H^n(Tot(B)) \ = \ {}'E_2^{n,0} = \ H^nA.$$

It remains to check that this isomorphism comes from the above map $A \to Tot(B)$, then we know that this map is a qis. ☐

8.6.2. *Two ways to calculate* Ext *and* Tor. If M, N are modules for a ring A we know that the functor $\mathrm{Hom}_A(-, M) :\ \mathfrak{m}(A)^o \to \mathcal{A}b$ is right exact and the functor $\mathrm{Hom}_A(N, -) :$ $\mathfrak{m}(A) \to \mathcal{A}b$ is left exact. Their derived functors are called Ext-functors :

$$'\mathrm{Ext}^i_A(N, M) \stackrel{\text{def}}{=} L^i\mathrm{Hom}_A(-, M)\ (N) \quad \text{and} \quad ''\mathrm{Ext}^i_A(N, M) \stackrel{\text{def}}{=} R^i\mathrm{Hom}_A(N, -)\ (M).$$

On the other hand if M is a right A-module and N is a left A-module for a ring A, we know that the functors

$$M\otimes_A - :\ \mathfrak{m}^l(A)^o \to \mathcal{A}b \quad \text{and} \quad -\otimes_A N\ :\ \mathfrak{m}^r(A)^o \to \mathcal{A}b$$

are both right exact. Their derived functors are called Tor-functors :

$$'\mathrm{Tor}^A_n(M, N) \ \stackrel{\text{def}}{=}\ R^{-n}(-\otimes_A N)\ (M) \quad \text{and} \quad ''\mathrm{Tor}^A_n(M.N) \ \stackrel{\text{def}}{=}\ R^{-n}(M\otimes_A -)\ (N).$$

Lemma. (a) There is a canonical identification $'\mathrm{Ext} \ \cong\ ''\mathrm{Ext}$ (therefore we will denote either construction just by $\mathrm{Ext}^n_A(M, N)$).

(b) Similarly $'\mathrm{Tor} \stackrel{\text{def}}{=} {}''\mathrm{Tor}$ (so we denote either construction simply by $\mathrm{Tor}^A_n(M, N)$).

Proof. (a) For a projective resolution P of M and an injective resolution I of N one defines a structure of a bicomplex on $\mathrm{Hom}_A(P, Q)$.

Then one uses the two spectral sequences of this bicomplex to construct a canonical isomorphisms $'\mathrm{Ext}^n_A(M, N) \cong'' \mathrm{Ext}^n_A(M, N)$. ☐

8.6.3. F*-acyclic resolutions.* We says that $a \in \mathcal{A}$ is *acyclic* for a left exact functor F : $\mathcal{A} \to \mathcal{B}$ if all higher derived functors of F vanish on a

$$(R^nF)a \ = \ 0\ ,\ n > 0.$$

Example. An injective object $i \in \mathcal{A}$ is acyclic for all left exact functors since it has an injective resolution I of length zero (with $I^n = \delta_{n,0}a$).

Proposition. RF can be calculated using left F-acyclic resolutions.

Proof. Let $0 \to a \to C^0 \to C^1 \cdots$ be a left resolution of a by F-acyclic objects. Let then I be an injective bicomplex resolution of the complex C. Then $a \to C \to Tot(I)$ are quasi-isomorphisms, hence $Tot(I)$ is an injective resolution of both C and a. So, $RF(a)$ is calculated as $F[Tot(I)] = Tot(FI)$.

The vertical cohomology $'E_1{}^{pq} ='' H^{pq}(FI)$ computes the derived functors $(R^qF)C^p$ of terms in the complex C. By assumption, this is zero for $q \neq 0$ and for $q = 0$ we get $R^0F(C^p) = F(C^p)$. So, the first spectral sequence converges and we get that $(R^nF)a = H^n[Tot(FI)]$ is the n^{th} cohomology of the complex $F(C)$. □

Example. For a ring A a right module F is said to be *flat* if the functor $-\otimes_A M : \mathfrak{m}(A) \to \mathcal{A}b$ is exact. The flat A-modules are acyclic for all functors $N\otimes_A-$ of tensoring with right A-modules. Therefore, in order to calculate Tor_A one can use flat resolutions in either variable.

8.6.4. *Grothendieck's spectral sequence.* Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be two left exact functors.

Theorem. (a) $G\circ F$ is a left exact functor.

(b) If $\mathcal{A}, \mathcal{B}$ have enough injectives and F sends injective objects to G-acyclic objects then there is a spectral sequence functorial in $a \in \mathcal{A}$ with

$$E_2^{pq} = R^qG[R^pF(a)] \quad \text{and} \quad E_\infty^{pq} = [R^{p+q}(GF)](a).$$

(c) $R(GF) = RG\circ RF$.

Proof. (a) is obvious. In (c), for any $A \in C^+(\mathcal{A})$ we can choose an injective resolution $A \to I$. Then $R(GF)\ A = (GF)(I) = G(FI)$. According to the lemma 8.6.1 this is the same as $RG(FI)$ since FI is a complex whose terms are G-acyclic (so it is its own G-acyclic resolution). So, $R(GF)\ A \cong (RG)[F(I)] = (RG)[(RF)A]$.

(b) As above, for $A \in C^+(\mathcal{A})$ we compute $RF(A)$ as $F(I)$ for come injective resolution I of A. We now consider an injective bicomplex resolution J of FI. It gives a bicomplex GJ in $\mathcal{C}$. We will now get the desired information by considering *both* spectral sequences of this bicomplex.

(I) The total cohomology of $G(J)$ calculates $R^\bullet(GF)$:

$$H^n[Tot(GJ)] = H^n[G\ Tot(J)] \overset{(*)}{\cong} H^n[G\ (FI)] = H^n[(GF)I] = [R^n(GF)]A$$

where the isomorphism $(*)$ comes from the lemma 8.6.1 which says that one can compute $RG\ (FI)$ either from an injective resolution $Tot(J)$ of FI or from a G-cyclic resolution such as FI itself.

Notice that we have really used the first spectral sequence of GJ for this calculation of $H^n[Tot(GJ)]$ since the proof of the lemma 8.6.1 comes from the first spectral sequence.

(II) The spectral sequence E in the theorem will be the second spectral sequence $''E$ of the bicomplex GJ.

First, $''E_1{}^{pq}$ is the horizontal cohomology $'H^{pq}(GJ) = H^q[GJ^{p,\bullet}]$ of GJ. We can choose the bicomplex resolution J of the complex FI to be *split* (theorem 7.2.8.b), and then its horizontal lines $J^{p,\bullet}$ are split injective complexes (corollary 7.2.8.b). Then the lemma 7.2.6 shows that $'\mathrm{H}^{pq}(GJ) = H^p[GJ^{\bullet,q}]$ is canonically identified with $G[H^p(J^{\bullet,q})]$.

Moreover, corollary 7.2.8.b guarantees that (since J is a split bicomplex resolution of FI), for any p the vertical line $'\mathrm{H}^p(J^{\bullet,\bullet})]$ is an injective resolution of $H^q(FI^p) = (R^qF)a$. Now,

$$''E_2{}^{pq} \;=\; ''\mathrm{H}^p[{}'\mathrm{H}^{\bullet,q}(GJ)] \;=\; ''\mathrm{H}^q[G[{}'\mathrm{H}^p(J^{\bullet,\bullet})]] \;=\; (R^qG)[(R^pF)a].$$

. $\square$