HOMOLOGICAL ALGEBRA

Contents

6. **Homotopy category of complexes** $K(\mathcal{A})$

6.0. **Summary.** On the way to inverting quasi-isomorphisms, in the first step we will invert a special kind of quasi-isomorphisms – the homotopy equivalences. This is achieved by passing from the category of complexes $C(\mathcal{A})$ to the so called "homotopy category of complexes" $K(\mathcal{A})$.

6.0.1. *The triangulated structure of* $K(\mathcal{A})$. Here, $K(\mathcal{A})$ is defined for any additive category $\mathcal{A}$ and it has extra properties if $\mathcal{A}$ is abelian. The new category $K(\mathcal{A})$ is not an abelian category (even if $\mathcal{A}$ is abelian!). However, it always has a similar if less familiar structure of a *triangulated category.*

The similarity that I am talking about is that in an abelian category $\mathcal{C}$ one has the notion of short exact sequences.[1] The analogue of short exact sequences which works in $K(\mathcal{A})$ (for additive $\mathcal{A}$) is the notion of *distinguished triangles*, also called "*exact triangles*".

We will formalize the properties of exact triangles in $K(\mathcal{A})$ into the concept of a *triangulated category* which turns out to be the standard framework for homological algebra.

Date: ?

[1] This is essential since a short exact sequence $0 \to a \to b \to c \to 0$ can be viewed as describing b as a combination of two simpler objects a and c.

6.0.2. *$K(\mathcal{A})$ and derived functors.* The first use of the homotopy category of complexes is that:

Any two projective resolutions of an object $a \in \mathcal{A}$ are canonically isomorphic in $K(\mathcal{A})$.

So, in the setting of homotopy categories of complexes the derived functor constructions LF, RF that we have introduced earlier will be actual functors (because $K(\mathcal{A})$ will remove the dependence on the choice of a projective resolution).

6.0.3. *Distinguished triangles and triangulated categories.* We start by listing in 6.1 the special structures that the categories of complexes $C(\mathcal{A})$ have. The most substantial ones are the Hom-complex and the mapping cone ("distinguished triangles"). For us the Hom-complex will motivate the introduction of homotopy category. The properties of the distinguished triangles in $C(\mathcal{A})$ will improve when we pass from to $K(\mathcal{A})$. The result will be axiomitized into the notion of a *triangulated category.* So $K(\mathcal{A})$ will be our first example of this notion.

6.1. **Structures carried by categories $C(\mathcal{A})$ of complexes.** For any additive category $\mathcal{A}$ we will observe here certain structures and properties of the category $C(\mathcal{A})$.

6.1.1. *Structures on the category $C(\mathcal{A})$.* Here we list the relevant structures and then we will explain them at length.

(A) Structures for additive category $\mathcal{A}$.

(1) *Shift functors.* For any integer n there is a functor $[n]: C(\mathcal{A}) \to C(\mathcal{A})$ which shifts complexes n places to the left.
(2) *The* Hom-*complex.* For any $a, b \in \mathcal{A}$, the abelian group $\mathrm{Hom}_{\mathcal{A}}(a,b)$ is naturally upgraded to a complex $\mathrm{Hom}^{\bullet}_{\mathcal{A}}(a,b)$ in $\mathcal{A}b$.
(3) *Mapping cones.* Any map $\alpha \in \mathrm{Hom}_{C(\mathcal{A})}(A,B)$ defines a complex $C_\alpha \in C(\mathcal{A})$ called the *mapping cone* of α.
(4) *The (distinguished) triangles.* Triangles in $C(\mathcal{A})$ are the diagrams of the form $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$. We will see that any map $\alpha \in \mathrm{Hom}_{\mathcal{A}}(a,b)$ defines a triangle $a \xrightarrow{\alpha} b \xrightarrow{\dot{\alpha}} C_\alpha \xrightarrow{\ddot{\alpha}} a[1]$ involving the mapping cone. The triangles of this form are called the *distinguished triangles.*
(5) *Subcategories $C^b(\mathcal{A})$ etc.* If ? is one of the symbols $b, +, -$ we define a full subcategory $C^?(\mathcal{A})$ of $C(\mathcal{A})$, consisting respectively of *bounded* complexes: $A^n = 0$ for $|n| >> 0$; complexes *bounded from below*: $A^n = 0$ for $n << 0$ (hence allowed to stretch in the $+$ direction), complexes bounded from above (so they may stretch in the $-$ direction).

(B) The additional structures when $\mathcal{A}$ is abelian

(1) The *cohomology* functors $\mathrm{H}^i : C(\mathcal{A}) \to \mathcal{A},\ i \in \mathbb{Z}$.

(2) The *quasi-isomorphisms* are a special class of morphisms (related to cohomology functors).
(3) *Truncation functors* τ. For any integer n a complex $A \in \mathbb{C}(\mathcal{A})$ has two truncation's: $\tau^{\leq n}A$ that lives in degrees $\leq n$ and $\tau^{\geq n}A$ which lives in degrees $\geq n$.[(2)]

6.1.2. *Remark on signs.* In homological algebra formulas often incorporate some convenient choices of *signs* $(-1)^?$. We remark here that these can all be explained (or reinvented) by systematical use of the *super-commutativity rule.* It says that when x of y are objects of degrees a and b then the natural notion of commutativity in such *graded* setting is that

$$yx \textit{ should equal to } (-1)^{ab}xy.$$

In other words, "when y jumps over x this introduces the sign $(-1)^{ab}$ ".

6.1.3. *The shift functors* $[n]: \ C(\mathcal{A}) \to C(\mathcal{A})$ *for* $n \in \mathbb{Z}$. They act on graded objects by the rule that the p^{th} term in $A[n]$ is $(A[n])^p \stackrel{\text{def}}{=} A^{n+p}$, while on $A[n]$ the differential $(A[n])^p \xrightarrow{d_{A[n]}} A[n])^{p+1}$ is given as $A^{p+n} \xrightarrow{(-1)^n d_A^{p+n}} A^{p+1+n}$.

6.1.4. *The category* $\mathcal{A}^\bullet$ *of* graded $\mathcal{A}$-objects. The objects are the sequences $A = (A^n)_{n\in\mathbb{Z}}$ of objects A^n in $\mathcal{A}$.[(3)] The morphisms $f \in \text{Hom}_{\mathcal{A}^\bullet}(A,B)$ are families $f = (f^n)_{n\in\mathbb{Z}}$ with $f^n \in \text{Hom}_{\mathcal{A}}(A^n,B^n)$. So, $\text{Hom}_{\mathcal{A}^\bullet}(A,B)$ is itself a sequence of abelian groups $\text{Hom}_{\mathcal{A}}(A^n,B^n)$, i.e., $\text{Hom}_{\mathcal{A}^\bullet}(A,B)$ lies in $\mathcal{A}b^\bullet$.

6.1.5. *The* Hom-*complex* $\text{Hom}^\bullet(a,b)$. For two complexes $A,B \in C(\mathcal{A})$, their Hom-complex has terms $\text{Hom}^n(A,B) \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{A}^\bullet}(A,B[n]) \in \mathcal{A}b$ and the differential $\text{Hom}^n(A,B) \xrightarrow{d^n_{\text{Hom}}} \text{Hom}^{n+1}(A,B)$ is[(4)]

$$\text{Hom}_{\mathcal{A}}(A,B[n]) \xrightarrow{d^n_{\text{Hom}}} \text{Hom}_{\mathcal{A}}(A,B[n+1]), \quad d^n_{\text{Hom}}(f) \stackrel{\text{def}}{=} d_B \circ f + (-1)^{n+1} f \circ d_A.$$

So, $\text{Hom}^0(A,B)$ is just $\text{Hom}_{\mathcal{A}^\bullet}(A,B)$, i.e., all systems $f = (f^n)_{n\in\mathbb{Z}}$ of $f^n: A^n \to B^n$. We call elements $h = (h^n)_{n\in\mathbb{Z}}$ of $\text{Hom}^{-1}_{\mathcal{A}}(A,B) = \text{Hom}_{\mathcal{A}^\bullet}(A,B[-1])$ the *homotopies from* A *to* B.

We say that two maps of complexes $\alpha_1,\alpha_2 : A \to B$ are *homotopic* if $\alpha_2 - \alpha_1$ is of the form $d_{\text{Hom}}(h)$ for some homotopy $h \in \text{Hom}_{\mathcal{A}^\bullet}(A,B[1])$. This means that $\alpha_2 - \alpha_1 = d_B \circ h + h \circ d_A$. We denote this relation by $\alpha_2 \equiv \alpha_2$.

---

[2] These functors will be introduced later.
[3] If $\mathcal{A}$ has countable sums we can think of A as a single object $\prod_{n\in\mathbb{Z}} A^n$ in $\mathcal{A}$).
[4] The sign comes from the super-commutativity rule!

Lemma. (a) d_{Hom} is a differential.
(b) $Z^0[\mathrm{Hom}^\bullet_{\mathcal{A}}(A,B)] \;=\; \mathrm{Hom}_{C(\mathcal{A})}(A,B)$.
(c) $H^0[\mathrm{Hom}^\bullet_{\mathcal{A}}(A,B)] \;=\; \mathrm{Hom}_{C(\mathcal{A})}(A,B)/\equiv$.

Proof. (a) We check that

$$\begin{aligned}(d_{\mathrm{Hom}})^2(f) = \; & d^{n+1}_{\mathrm{Hom}}(d^n_{\mathrm{Hom}}\,(f)) = \; d[df+(-1)^{n+1}fd] + (-1)^{n+2}[df+(-1)^{n+1}fd]d \\ = \; & d_B^2\circ f + \;(-1)^{n+1}dfd + (-1)^{n+2}dfd + \;(-1)^{2n+3}f\cdot d^2 = 0.\end{aligned}$$

(b) The LHS consists of all $f\in\mathrm{Hom}^0(A,B)=\mathrm{Hom}_{\mathcal{A}^\bullet}(A,B)$ such that $d_{\mathrm{Hom}}(f)=d_Bf+(-1)^1fd_A$ is zero, i.e., such that f is a morphism of complexes. (c) is now clear. $\square$

Remark. $\mathrm{Hom}^\bullet_{\mathcal{A}}$ gives another category structure on complexes since one has composition $\mathrm{Hom}^m_{\mathcal{A}}(B,C)\times\mathrm{Hom}^n_{\mathcal{A}}(A,B)\to\mathrm{Hom}^{m+n}_{\mathcal{A}}(A,C)$ by $(g,f)\mapsto\big(A\xrightarrow{f}B[n]\xrightarrow{g[n]}C[n+m]\big)$.

6.1.6. *The mapping cone C_α.* The idea is that the cone of a map of complexes $A\xrightarrow{\alpha}B$ is another complex $\boldsymbol{C}_\alpha$ which measures how far α is from being an isomorphism.

For us the mapping cones will eventually become an expression of the idea of a short exact sequences of complexes (SES), which is meaningful even when such SES are not defined, i.e., when $\mathcal{A}$ is additive but maybe not abelian.

Any map $A\xrightarrow{\alpha}B$ of complexes defines a complex C_α called the *cone* of α with the terms $C^n_\alpha = B^n\oplus A^{n+1}$ and the differential $d^n_C:\; B^n\oplus A^{n+1}\to B^{n+1}\oplus A^{n+2}$ that combines the differentials in A and B and the map α by:

$$d^n_{C_\alpha}(b^n\oplus a^{n+1}) \;\stackrel{\mathrm{def}}{=}\; (d^n_Bb^n+\alpha^{n+1}a^{n+1})\oplus\; -d^{n+1}_Aa^{n+1},\quad\text{i.e.,}\quad d_C = \;(d_B+\alpha)\oplus(-d_A).$$

The fact that this is a differential is a part of the following lemma (here $\mathsf{A}=A[1]$ hence $d_{\mathsf{A}}=-d_A$).

Lemma. Let A,B be complexes in an additive category $\mathcal{A}$ and let C be the graded $\mathcal{A}$-object $B\oplus\mathsf{A}$. There is a canonical bijection between

- (i) Differentials d_C on C such that the inclusion and projection $B\overset{\phi}{\hookrightarrow}C\overset{\psi}{\twoheadrightarrow}\mathsf{A}$ are both maps of complexes.
- (ii) Maps of complexes $\alpha\in\mathrm{Hom}_{C(\mathcal{A})}(\mathsf{A}[1],B)$.

Here,

- α gives $d_C=(d_B+\alpha)\oplus d_{\mathsf{A}}$ [5]

[5] $C^n\xrightarrow{d^n_C}C^{n+1}$ is the map $B^n\oplus\mathsf{A}^n\xrightarrow{d^n_C}B^{n+1}\oplus\mathsf{A}^{n+1}$ given at $b^n\in B^n$ and $a^n\in\mathsf{A}^n$ by

$$d(b^n\oplus a^n)\;\stackrel{\mathrm{def}}{=}\;(d^n_Bb^n+\alpha^na^n)\oplus\,d^n_{\mathsf{A}}a^n.$$

- d_C recovers α by composing with the other pair of inclusions and projections: $\alpha = [A \overset{\sigma[-1]}{\hookrightarrow} C[-1] \xrightarrow{d_C} C \overset{\tau}{\twoheadrightarrow} B]$.

Proof. We have denoted the component maps of the splitting $C^n = B^n \oplus \mathsf{A}^n$ by ϕ, ψ, σ, τ so that $\phi\tau + \sigma\psi = 1_C$. The elements of C^n are of the form $b \oplus a = \phi b + \sigma a$ with $b \in B^n$ and $a \in \mathsf{A}^n$.

Now, we will get a bijection between maps of graded objects $d_C : C \to C[1]$ such that $d_C\phi = \phi d_B$ and $d_\mathsf{A}\psi = \psi d_C$ and maps of graded objects $\mathsf{A}[1] \xrightarrow{\alpha} B$. Starting with such d_C we get $\alpha \overset{\text{def}}{=} [\mathsf{A}[1] \xrightarrow{\sigma[1]} C[1] \xrightarrow{d_C} C \xrightarrow{\tau} B]$ and one recovers d_C from this α by (since $\psi\sigma = 1_{A[1]}$):

$$d_C(\phi b + \sigma a) = \ d_C\phi\ b + 1_C d_C \sigma\ a = \ \phi d_B\ b + \ (\phi\tau + \sigma\psi) d_C \sigma\ a$$

$$= \ \phi d_B\ b + \ \phi\tau d_C\sigma\ a + (\sigma d_\mathsf{A})(\psi\sigma)\ a = \ \phi(d_B\ b + \ \alpha\ a) + \sigma d_\mathsf{A}\ a = \ (d_B\ b + \ \alpha\ a) \oplus\ d_\mathsf{A}\ a.$$

Finally, d_C is a differential iff $\alpha : \mathsf{A}[1] \to B$ is a map of complexes:

$$d_C^2 \ = \ d_C \circ [(d_B + \alpha) \oplus\ d_\mathsf{A}] = \ [d_B^2 + d_B \circ \alpha + \alpha \circ (-d_{\mathsf{A}[1]})] \oplus\ (d_\mathsf{A})^2 = 0. \quad \square$$

6.1.7. *Terminology.* If $\mathcal{A}$ is abelian then the data in (ii) are precisely the data of a SES of complexes $0 \to B \to C \to \mathsf{A} \to 0$ with a degreewise splitting, i.e., in each degree n the exact sequence $0 \to B^n \to C^n \to \mathsf{A}^n \to 0$ in $\mathcal{A}$ is given a splitting $C^n \cong B^n \oplus \mathsf{A}^n$.

We now stretch the terminology a bit. If $\mathcal{A}$ is only additive (so the exact sequences are not defined) we can define *split SES sequences* in $\mathcal{A}$ as sequences $0 \to a \xrightarrow{\alpha} c \xrightarrow{\beta} b \to 0$ of the form $0 \to a \hookrightarrow a \oplus b \twoheadrightarrow b \to 0$, i.e., wit a chosen isomorphism $c \cong a \oplus b$ such that α is inclusion and β is projection. Then one defines *degreewise split SES* of complexes as sequences of complexes the form $0 \to B \xrightarrow{\phi} C \xrightarrow{\psi} \mathsf{A} \to 0$ where in each degree $0 \to B^n \xrightarrow{\phi^n} C^n \xrightarrow{\psi^n} \mathsf{A}^n \to 0$ is a split SES sequence in $\mathcal{A}$. In other words as a graded object C is identified with $B \oplus \mathsf{A}$ in such way that ϕ and ψ are the inclusion and the projection.

Now we can restate the lemma 6.1.6 as the following being equivalent: (i) a degreewise split SES in $C(\mathcal{A})$, (ii) a morphisms of complexes in $C(\mathcal{A})$ and (iii) a cone triangle in $C(\mathcal{A})$.

6.1.8. *The distinguished triangles.* For a map of complexes $A \xrightarrow{\alpha} B$ its *cone triangle* is

$$A \xrightarrow{\alpha} B \xrightarrow{\dot{\alpha}} C_\alpha \xrightarrow{\ddot{\alpha}} A[1], \qquad \text{with} \quad \dot{\alpha}^n(b^n) = \ b^n \oplus 0 \quad \text{and} \quad \ddot{\alpha}^n(b^n \oplus a^{n+1}) \overset{\text{def}}{=} \ a^{n+1}.$$

In other words, $\dot{\alpha}$ and $\ddot{\alpha}$ are the inclusion and projection for the two summands of the graded object $C_\alpha = B \oplus A[1]$. (Then $\dot{\alpha}$ and $\ddot{\alpha}$ are maps of complexes by the lemma 6.1.6.) The distinguished triangles are defined as the ones isomorphic to cone triangles. Now we restate the lemma 6.1.6 using new terminology:

Lemma. The distinguished triangles are equivalent to degreewise split short exact sequences of complexes. The two inverse constructions are given by

- Any cone triangle $A \xrightarrow{\alpha} B \xrightarrow{\dot\alpha} C_\alpha \xrightarrow{\ddot\alpha} A[1]$ contains a degreewise split SES of complexes
$$0 \to B \xrightarrow{\dot\alpha} C_\alpha \xrightarrow{\ddot\alpha} A[1] \to 0.$$
- To any degreewise split SES of complexes
$$0 \to B \xrightarrow{\phi} C \xrightarrow{\psi} \mathsf{A} \to 0 \quad \text{with} \quad C^n \cong\ B^n \oplus \mathsf{A}^n,$$
one associates the cone triangle of the map of complexes $A \xrightarrow{\alpha} B$ where $A = \mathsf{A}[-1]$ and for any $a \in A$ element $\alpha(a) \in B$ is the B component of the element $d_C(0_B \oplus a) \in C$.

Example. For a degreewise split SES $0 \to B \xrightarrow{\phi} C \xrightarrow{\psi} \mathsf{A} \to 0$ the triangle
$$\mathsf{A}[-1] \xrightarrow{\alpha} B \xrightarrow{\phi} C \xrightarrow{\psi} \mathsf{A}$$
is canonically isomorphic in $C(\mathcal{A})$ to the cone triangle $\mathsf{A}[-1] \xrightarrow{\alpha} B \xrightarrow{\dot\alpha} C_\alpha \xrightarrow{\ddot\alpha} \mathsf{A}$ of the map α.

Proof. This is just the lemma 6.1.6 with the notation $A = \mathsf{A}[-1]$. $\square$

6.1.9. *Subcategories* $C^?(\mathcal{A}) \subseteq C(\mathcal{A})$. For $\mathcal{Z} \subseteq \mathbb{Z}$ we can denote by $C^{\mathcal{Z}}(\mathcal{A})$ the full subcategory consisting of complexes A such that $A^n = 0$ when $n \notin \mathcal{Z}$. For instance one has $C^{\leq n} \stackrel{\text{def}}{=}\ C^{(-\infty,n]}$ and $C^{\geq n} \stackrel{\text{def}}{=}\ C^{([n,\infty)}$, as well as $C^{\{0\}}(\mathcal{A})$ which is equivalent to $\mathcal{A}$.

6.2. **Properties of cohomology functors.** Here $\mathcal{A}$ is necessarily abelian.

Lemma. (a) $C(\mathcal{A})$ is again an abelian category and a sequence of complexes is exact iff it is exact on each level!

Proof. For a map of complexes $A \xrightarrow{\alpha} B$ we can define $K^n = \mathrm{Ker}(A^n \xrightarrow{\alpha^n} B^n)$ and $C^n = A^n/\alpha^n(B^n)$. This gives complexes since d_A induces a differential d_K on K and d_B a differential d_C on C. Moreover, it is easy to check that in category $C(\mathcal{A})$ one has $K = \mathrm{Ker}(\alpha)$ and $C = \mathrm{Coker}(\alpha)$. Now one finds that $Im(\alpha)^n =\ Im(\alpha^n) =\ \alpha^n(A^n)$ and $Coim(\alpha)^n =\ Coim(\alpha^n) =\ A^n/\mathrm{Ker}(\alpha^n)$, so the canonical map $Coim \to Im$ is given by isomorphisms $A^n/\mathrm{Ker}(\alpha^n) \xrightarrow[\cong]{} \alpha^n(A^n)$. Now the exactness claim is clear. $\square$

6.2.1. *Lemma.* A short exact sequence of complexes $0 \to A \to B \to C \to 0$ gives a long exact sequence of cohomologies.

$$\cdots \xrightarrow{\partial^{n-1}} \mathrm{H}^n(A) \xrightarrow{\mathrm{H}^n(\alpha)} \mathrm{H}^n(B) \xrightarrow{\mathrm{H}^n(\beta)} \mathrm{H}^n(C) \xrightarrow{\partial^n} \mathrm{H}^{n+1}(A) \xrightarrow{\mathrm{H}^{n+1}(\alpha)} \mathrm{H}^{n+1}(B) \xrightarrow{\mathrm{H}^{n+1}(\beta)} \cdots$$

Proof. We need to construct for a class $\gamma \in \mathrm{H}^n(C)$ a class $\partial\gamma \in \mathrm{H}^{n+1}$. So if $\gamma = [c]$ is the class of a cocycle c, we need

(1) From a cocycle $c \in Z^n(C)$ a cocycle $a \in Z^{n+1}$.
(2) Independence of $[a]$ on the choice of c or any other auxiliary choices.
(3) The sequence of cohomology groups is exact.

Recall that a sequence of complexes. $0 \to A \to B \to C \to 0$ is a short exact sequence if for each integer n the sequence $0 \to A^n \to B^n \to C^n \to 0$ is exact.

The following calculation is in the set-theoretic language appropriate for module categories but can be rephrased in the language of abelian categories (and also the result for module categories implies the result for abelian categories since any abelian category is equivalent to a full subcategory of a module category).

(1) Since β^n is surjective, $c = \beta^n b$ for $b \in B^n$, Now $db = \alpha^{n+1}a$ for some $a \in A^{n+1}$ since $\beta^{n+1}(db) = d\beta^{n+1}b = dc = 0$. Moreover, a is a cocycle since $\alpha^{n+2}(d^{n+1}a) = d^{n+1}(\alpha^{n+1}a) = d^{n+1}(d^n b) = 0$.

(2) So we want to associate to $\gamma = [c]$ the class $\alpha = [a] \in \mathrm{H}^{n+1}(A)$. For that $[a]$ should be independent of the choices of c, B and a. So let $[c] = [c']$ and $c' = \beta^n b'$ with $b' \in B^n$, and $db' = \alpha^{n+1}a'$ for some $a' \in A^{n+1}$.

Since $[c] = [c']$ one has $c' = c+dz$ for some $z \in C^{n-1}$. Choose $y \in B^{n-1}$ so that $z = \beta^{n-1}y$, then

$$\beta^n b' = c' = c + dz = \beta^n b + d(\beta^{n-1}y) = \beta^n b + \beta^n dy = \beta^n(b + dy).$$

The exactness at B now shows that $b' = b + dy + \alpha^n x$ for some $x \in A^n$. So,

$$\alpha^{n+1}a' = db' = db + d\alpha^n x = \alpha^{n+1}a + \alpha^n(dx) = \alpha^{n+1}(a + dx).$$

Exactness at A implies that actually $a' = a + dx$.

(3) I omit the easier part: the compositions of any two maps are zero.

<u>*Exactness at $H^n(B)$.*</u> Let $b \in Z^n(B)$, then $\mathrm{H}^n(\beta)[b] = [\beta^n b]$ is zero iff $\beta^n b = dz$ for some $z \in C^{n-1}$. Let us lift this z to some $y \in B^{n-1}$, i.e., $z = \beta^{n-1}y$. Then $\beta^n(b-dy) = dz-dz = 0$, hence $b - dy = \alpha^n a$ for some $a \in A^n$. Now a is a cocycle since $\alpha^n(da) = d(b - dy) = 0$, and $[b] = [b - dy] = \mathrm{H}^n(\alpha)[a]$.

<u>*Exactness at $H^n(A)$.*</u> Let $a \in Z^n(A)$ be such that $\mathrm{H}^n(\alpha)[a] = [\alpha^n a]$ is zero, i.e., $\alpha^n a = db$ for some $b \in B^{n-1}$. Then $c = \beta b$ is a cocycle since $dc = \beta^n(db) = \beta^n\alpha^n a = 0$; and by the definition of the connecting morphisms (in (1)), $[a] = \delta^{n-1}[c]$.

Exactness at $H^n(C)$. Let $c \in Z^n(C)$ be such that $\partial^n[c] = 0$. Remember that this means that $c = \beta b$ and $db = \alpha a$ with $[a] = 0$, i.e., $a = dx$ with $x \in A^{n-1}$. But then $db = \alpha(dx) = d(\alpha x)$, so $b - \alpha x$ is a cocycle, and then $c = \beta(b) = \beta(b - \alpha(x))$ implies that $[c] = \mathrm{H}^n(\beta)[b - \alpha(x)]$.

6.3. **The homotopy category $K(\mathcal{A})$ of complexes in $\mathcal{A}$.** The following definitions are just repeated from 6.1.5.

6.3.1. *The homotopy relation.* We say that two maps of complexes $A \xrightarrow{\alpha,\beta} B$ are *homotopic* (we denote this $\alpha \equiv \beta$), if there is a sequence h of maps $h^n:\ A^n \to B^{n-1}$, such that

$$\beta - \alpha = dh + hd, \quad \textit{i.e.,} \quad \beta^n - \alpha^n = d_B^{n-1}h^n + h^{n+1}d_A^n.$$

Then we say that h is a *homotopy* from α to β.

A map of complexes $A \xrightarrow{\alpha} B$ is said to be a *homotopical equivalence* if there is a map β in the opposite direction such that $\beta\circ\alpha \equiv 1_A$ and $\alpha\circ\beta \equiv 1_B$. Then we write $A \equiv B$ and we say that β is a *homotopy inverse* of α.

Lemma. (a) Homotopic maps are the same on cohomology.

(b) Homotopical equivalences are quasi-isomorphisms.

(c) A complex A is homotopy equivalent to the zero object iff $1_A = hd + dh$. Then the complex A is acyclic, i.e., $\mathrm{H}^*(A) = 0$.

(d) $\alpha \equiv \beta$ implies $\mu\circ\alpha \equiv \mu\circ\beta$ and $\alpha\circ\nu \equiv \beta\circ\nu$.

Proof. (a) Denote for $a \in Z^n(A)$ by $[a]$ the corresponding cohomology class in $\mathrm{H}^n(A)$ and by $\mathrm{H}^*(\alpha) : \mathrm{H}^*(A) \to \mathrm{H}^*(B)$ the action of α on cohomology classes. Then $\mathrm{H}^*(\beta)[a] - \mathrm{H}^*(\alpha)[a] = [(\beta - \alpha)a] = [d_B^{n-1}h^n(a) + h^{n+1}d_A^n(a)] = [d_B^{n-1}(h^n a)] = 0$.

(b) If $\beta\circ\alpha \equiv 1_A$ and $\alpha\circ\beta \equiv 1_B$ then $\mathrm{H}^*(\beta)\circ\mathrm{H}^*(\alpha) = \mathrm{H}^*(\beta\circ\alpha) = \mathrm{H}^*(1_A) = 1_{\mathrm{H}^*(A)}$ etc.

(c) A map of complexes $0 \xrightarrow{\alpha} A$ is necessarily $\alpha = 0$. It is a homotopy equivalence if there is a map $A \xrightarrow{\beta} 0$ (then necessarily $\beta = 0$) such that $\beta\circ\alpha \equiv 1_0$ and $\alpha\circ\beta \equiv 1_A$. The first equation is $0 = 0$ and in the second, the LHS is always zero, the condition is that on A we have $1_A \equiv 0$.

(d) If $\beta - \alpha = d_B\circ h + h\circ d_A$ then for $X \xrightarrow{\nu} A \xrightarrow{\alpha} B \xrightarrow{\mu} Y$ one has $\mu\circ\beta - \mu\circ\alpha = d_C\circ(\mu\circ h) + (\mu\circ h)\circ d_A$ etc.

6.3.2. *Homotopy category $K(\mathcal{A})$.* The objects are again just the complexes but the maps are the homotopy classes $[\phi]$ of maps of complexes ϕ (for the second equality see 6.1.5):

$$\mathrm{Hom}_{K(\mathcal{A})}(A,B) \overset{\text{def}}{=} \mathrm{Hom}_{C(\mathcal{A})}(A,B)/\equiv \quad = \quad H^0[\mathrm{Hom}^\bullet_{\mathcal{A}}(A,B)].$$

Now, the identity morphisms for A in $K(\mathcal{A})$ is the class $[1_A]$ and the composition is defined by $[\beta]\circ[\alpha] \overset{\text{def}}{=} [\beta\circ\alpha]$ (this makes sense by the part (d) of the lemma 6.3!).

6.3.3. *Remarks.* (1) Observe that for a homotopy equivalence $\alpha : A \rightarrow B$ the corresponding map in $K(\mathcal{A})$, $[\alpha] : A \rightarrow B$ is an isomorphism. So we have accomplished a part of our long term goal – we have inverted some quasi-isomorphisms: the homotopy equivalences!

(2) More precisely, we know what are isomorphisms in $K(\mathcal{A})$. The homotopy class $[\alpha]$ of a map of complexes α, is an isomorphism in $K(\mathcal{A})$ iff α is a homotopy equivalence!

Lemma. (a) $K(\mathcal{A})$ is an additive category.

(b) The shift functors $[n]$ on $C(\mathcal{A})$ descend to functors on $K(\mathcal{A})$.

Proof. (a) By its definition $\mathrm{Hom}_{K(\mathcal{A})}(A, B)$ is an abelian group. $K(\mathcal{A})$ gets zero object and finite sums(=products) from $C(\mathcal{A})$. Claim (b) follows from the action of shifts on homotopies. $\square$

6.3.4. *Homotopy in topology and algebra.* Historically, the homotopy for complexes has been introduced based on the notion of homotopy for maps of topological spaces. The relation is that a (geometric) homotopy H between two maps $F_0, F_1 : Y \to X$ of topological spaces gives an (algebraic) homotopy h between the corresponding morphisms $C_*(\alpha_i)$ of complexes of singular chains.[6]

6.4. **Triangulated category structure on $K(\mathcal{A})$.**

6.4.1. *Shifts, triangles, rotations.* We will say that a *shift* functor on a category $\mathcal{T}$ is an action of $\mathbb{Z}$ on $\mathcal{T}$, i.e., a collection of functors $[n] : \mathcal{T} \to \mathcal{T}$ for $n \in \mathbb{Z}$, such that $[m]\circ[n] = [m+n]$ and $[0] = id_{\mathcal{T}}$. Then the functor $[1]$ is itself called the *shift.*

A *triangle* in a category $\mathcal{T}$ with a shift is a diagram on $\mathcal{T}$ of the form $a \to b \to c \to a[1]$. If $\mathcal{T}$ is additive we define the *rotation* operation on triangles that takes $a \xrightarrow{\alpha} b \xrightarrow{\beta} c \xrightarrow{\gamma} a[1]$ to

$$b \xrightarrow{\beta} c \xrightarrow{\gamma} a[1] \xrightarrow{-\alpha[1]} b[1].$$

[6] Such H is a continuous family of continuous maps $H : [0,1] \to Map(Y, X)$ such that $H(i) = F_i$ for $i = 0, 1$, i.e., , H is a single continuous map $H : [0,1]\times Y \to X$ such that $H(i, x) = F_i(x)$ for $i = 0, 1$ and $x \in X$. Such H indeed gives an (algebraic) homotopy $h_n : C_n(Y) \rightarrow\!> C_{n+1}(X)[1]$ between the two morphisms of complexes of singular chains $C_*(Y) \xrightarrow[C_*(F_i)]{}\!> C_*(X)$ given by the maps F_i.

6.4.2. *Triangulated categories.* These are additive categories with a shift functor and a class $\mathcal{D}$ of triangles (called *distinguished* or *exact* triangles); that satisfy the following conditions

- (T0) The class $\mathcal{D}$ is closed under isomorphisms.
- (T1) Any map α inn $\mathcal{T}$ appears as the first map in some distinguished triangle.
- (T2) For any object $A \in \mathcal{T}$, the triangle $A \xrightarrow{1_A} A \xrightarrow{0} 0 \to A[1]$ is distinguished.
- (T3) (Rotation) A triangle is in $\mathcal{D}$ iff its rotation is in $\mathcal{D}$.
- (T4) Any diagram with distinguished rows

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\mu"'] & B \arrow[r, "\beta"] \arrow[d, "\nu"'] & C \arrow[r, "\gamma"] & A[1] \\ A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r, "\gamma'"] & A'[1] \end{tikzcd}$$

can be completed to a morphism of triangles

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\mu"'] & B \arrow[r, "\beta"] \arrow[d, "\nu"'] & C \arrow[r, "\gamma"] \arrow[d, "\eta"'] & A[1] \arrow[d, "\mu[1]"'] \\ A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r, "\gamma'"] & A'[1]. \end{tikzcd}$$

- (T5) (Octahedral axiom) If maps $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ and the composition $A \xrightarrow{\gamma=\beta\circ\alpha} C$, appear in distinguished triangles
  (1) $A \xrightarrow{\alpha} B \xrightarrow{\alpha'} C_1 \xrightarrow{\alpha''} A[1]$,
  (2) $B \xrightarrow{\beta} C \xrightarrow{\beta'} A_1 \xrightarrow{\beta''} B[1]$,
  (3) $A \xrightarrow{\gamma} C \xrightarrow{\gamma'} B_1 \xrightarrow{\gamma''} C[1]$; then there is a distinguished triangle

$$C_1 \xrightarrow{\phi} B_1 \xrightarrow{\psi} A_1 \xrightarrow{\chi} C_1[1]$$

that fits into the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "="'] & B \arrow[r, "\alpha'"] \arrow[d, "\beta"'] & C_1 \arrow[r, "\alpha''"] \arrow[d, "\phi"'] & A[1] \arrow[d, "="'] \\ A \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & C \arrow[r, "\gamma'"] \arrow[d, "="'] & B_1 \arrow[r, "\gamma''"] \arrow[d, "\psi"'] & A[1] \arrow[d, "\beta[1]"'] \\ B \arrow[r, "\beta"] & C \arrow[r, "\beta'"] & C_1 \arrow[r, "\beta''"] \arrow[d, "\chi"'] & B[1] \arrow[d, "="'] \\ & & C_1[1] & B_1[1] \arrow[l, "\alpha'[1]"'] \end{tikzcd}$$

6.4.3. *Remarks.* (0) The notion of triangulated category appears in the thesis of Verdier. His advisor was Grothendieck.

(1) Axiom (T5) is most complicated and the least used. It is called "octahedral" because it can naturally be drawn on an octahedron. It asserts that the distinguished triangles for $\alpha, \beta, \beta\circ\alpha$ are related by three maps ϕ, ψ, χ that form another distinguished triangle and satisfy 5 commutativity conditions given by 5 squares.[7]

It is not known whether (T5) follows from other axioms.

To see the intuitive meaning of (T5) one may consider what it says when $\mathcal{T} = K(\mathcal{A})$ for an abelian $\mathcal{A}$. One can choose complexes A, B, C to live in degree zero, i.e., so that these are objects of $\mathcal{A}$ and the maps α, β are inclusions $A \hookrightarrow B \hookrightarrow C$ in $\mathcal{A}$.

(2) In (T4), the map γ is not unique nor is there a canonical choice. This is a source of some difficulties in using triangulated categories.[8] This is resolved by upgrading triangulated categories to the level of *differential graded categories*. The starting point here is the above construction of the Hom-complex.

6.4.4. *The triangulated structure on* $K(\mathcal{A})$. Though $K(\mathcal{A})$ is not an abelian category it has a structure that allow us to make similar computations. First, recall that the shift functor [1] on $C(\mathcal{A})$ factors to a shift functor $[1] : K(\mathcal{A}) \to K(\mathcal{A})$. Now we define the class $\mathcal{D}$ of distinguished triangles in $K(\mathcal{A})$ as all triangles in $K(\mathcal{A})$ isomorphic to the image $A \xrightarrow{[\alpha]} B \xrightarrow{[\overset{\bullet}{\alpha}]} C_\alpha \xrightarrow{[\overset{\bullet\bullet}{\alpha}]} A[1]$ in $K(\mathcal{A})$ of some cone triangle in $C(\mathcal{A})$ (equivalently, to the image of some distinguished triangle in $C(\mathcal{A})$).

Theorem. For any additive category $\mathcal{A}$, $K(\mathcal{A})$ is a triangulated category (for the standard notion of shifts and distinguished triangles).

Proof. Property (T0) comes from the definition of $\mathcal{D}$.

(T1) Any map $\phi \in \mathrm{Hom}_{K(\mathcal{A})}(A, B)$ in $K(\mathcal{A})$ is a homotopy class $\phi = [\alpha]$ of some map of complexes $\alpha \in \mathrm{Hom}_{C(\mathcal{A})}(A, B)$. Then we have in $C(\mathcal{A})$ the cone triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ and its image in $K(\mathcal{A})$ is a distinguished triangle $A \xrightarrow{[\alpha]} B \xrightarrow{[\beta]} C \xrightarrow{[\gamma]} A[1]$ in $K(\mathcal{A})$ which starts with $[\alpha]$.

(T2) means that the cone $C_{1_A} = C$ of the identity map on A is isomorphic in $K(\mathcal{A})$ to the zero complex, i.e., that the cone C_{1_A} is homotopically equivalent to the zero complex. For

[7] Actually, maps ϕ, ψ such that the upper 4 squares (out of 5) commute exist by (T4). However, one needs to be able to choose such ϕ, ψ so that they can be completed to a distinguished triangle ϕ, ψ, χ for some χ such that the lowest square commutes.

[8] One problem is that in the present formalism there are *no cones for maps of functors*. This claim concerns the situation where $F \xrightarrow{\eta} G$ is a morphisms between two functors $F, G : \mathcal{C} \to \mathcal{T}$ and $\mathcal{T}$ is triangulated. Then for each $c \in \mathcal{C}$ there exists a distinguished triangle $F(c) \xrightarrow{\eta_c} G(c) \xrightarrow{\mu_c} C(c) \xrightarrow{\nu_c} F(c)[1]$, however there is in general no way to choose $C : Ob(\mathcal{C}) \to Ob(\mathcal{T})$ to be a functor (and we would also want μ, η to be morphisms of functors).

this we need a homotopy between maps $1_C, 0$ in $\mathrm{End}_{C(\mathcal{A})}(A)$. We choose $h^n : C^n_{1_\mathcal{A}} \to C^{n-1}_{1_\mathcal{A}}$ to be identity on the common summand and zero on its complement, i.e.,

$$A^n \oplus A^{n+1} \xrightarrow{h^n} A^{n-1} \oplus A^n, \qquad h^n(a^n \oplus a^{n+1}) \stackrel{\text{def}}{=} 0 \oplus a^n.$$

Then indeed

$$(d_C^{n-1} h^n + h^{n+1} d_C^n)(a^n \oplus a^{n+1}) = d_C^{n-1}(0 \oplus a^n) + h^{n+1}[(d_A a^n + 1_A a^{n+1}) \oplus - d_A^{n+1} a^{n+1})]$$
$$= (1_A a^n \oplus - d_A a^n) + 0 \oplus (d_A a^n + 1_A a^{n+1}) = a^n \oplus a^{n+1} = (1_C - 0)(a^n \oplus a^{n+1}).$$

Requirement (T3) says that if one applies rotation to any cone triangle in $C(\mathcal{A})$, $A \xrightarrow{\alpha} B \xrightarrow{\beta = \dot{\alpha}} C_\alpha \xrightarrow{\ddot{\alpha}} A[1]$, then for $\beta \stackrel{\text{def}}{=} \dot{\alpha}$, the rotated triangle

$$B \xrightarrow{\beta} C_\alpha \xrightarrow{\ddot{\alpha}} A[1] \xrightarrow{-\alpha[1]} B[1]$$

is isomorphic in $K(\mathcal{A})$ to the cone triangle

$$B \xrightarrow{\beta} C_\alpha \xrightarrow{\dot{\beta}} C_\beta \xrightarrow{\ddot{\beta}} B[1].$$

This requires a map of complexes $A \xrightarrow{\zeta} C_\beta$ such that the following diagram commutes <u>in $K(\mathcal{A})$</u>:

$$\begin{array}{ccccccc}
B & \xrightarrow{\beta} & C_\alpha & \xrightarrow{\gamma} & A[1] & \xrightarrow{-\alpha[1]} & B[1] \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle \zeta}\downarrow & & {\scriptstyle =}\downarrow \\
B & \xrightarrow{\beta} & C_\alpha & \xrightarrow{\dot{\beta}} & C_\beta & \xrightarrow{\ddot{\alpha}} & B[1]
\end{array}$$

and that ζ has a homotopy inverse ξ. Notice that $C^n_\alpha = B^n \oplus A^{n+1}$ and $C^n_\beta = C^n_\alpha \oplus B^{n+1} = (B^n \oplus A^{n+1}) \oplus B^{n+1}$ We define the maps $A^{n+1} \xrightarrow{\zeta^n} C^n_\beta \xrightarrow{\xi^n} A^{n+1}$ by

$$A^{n+1} \xrightarrow{\zeta^n} B^n \oplus A^{n+1} \oplus B^{n+1} \xrightarrow{\zeta^n} A^{n+1}, \ \zeta(a^{n+1}) \stackrel{\text{def}}{=} 0 \oplus a^{n+1} \oplus - \alpha a^{n+1}, \ \xi(b^n \oplus a^{n+1} \oplus b^{n+1}) \stackrel{\text{def}}{=} a^{n+1}.$$

It suffices to check that

(1) ζ and ξ are maps of complexes,
(2) ζ and ξ are inverse homotopy equivalences, precisely

$$\xi \circ \zeta = 1_{A[1]} \quad \text{and} \quad \zeta \circ \xi \equiv 1_{C_\beta}.$$

(3) The diagram commutes, i.e., $\zeta \circ \gamma = \phi$ and $\mu \circ \zeta = -\alpha[1]$.

All conditions except one are straightforward from formulas. However, the homotopy relation $\zeta \circ \xi \equiv 1_{C_\beta}$ in $\mathrm{End}_{\mathcal{A}^\bullet}(C_\beta)$ requires a choice of homotopy $h^n : C^n_\beta \to C^{n-1}_\beta$ such that

$$1_{C_\beta} - \zeta \circ \xi = d_{C_\beta} h + h d_{C_\beta}.$$

This works for the maps

$$h^n : B^n \oplus A^{n+1} \oplus B^{n+1} \to B^{n-1} \oplus A^n \oplus B^n, \quad h^n(b^n \oplus a^{n+1} \oplus b^{n+1}) = 0 \oplus 0 \oplus b^n.$$

(T4) One can certainly replace the rows with isomorphic ones which are images in $K(\mathcal{A})$ of cone triangles for two maps of complexes α_0, α_0' in $C(\mathcal{A})$. Then the diagram is the image in $K(\mathcal{A})$ of a diagram in $C(\mathcal{A})$

$$\begin{tikzcd} A \arrow[r, "\alpha_0"] \arrow[d, "\mu_0"'] & B \arrow[r, "\overset{\bullet}{\alpha}_0"] \arrow[d, "\nu_0"'] & C_\alpha \arrow[r, "\overset{\bullet\bullet}{\alpha}_0"] & A[1] \\ A' \arrow[r, "\alpha_0'"] & B' \arrow[r, "\overset{\bullet}{\alpha}{}_0'"] & C_{\alpha'} \arrow[r, "\overset{\bullet\bullet}{\alpha}{}_0'"] & A'[1] \end{tikzcd}$$

for any representatives μ_0, ν_0 of homotopy classes μ, ν.

However, the square in this diagram need not commute in $\mathbb{C}(\mathcal{A})$.(9) So, we only have the homotopical commutativity $[\nu]\circ[\alpha] = [\alpha']\circ[\mu]$ which means that there exist homotopy maps $h^n : A^n \to (B')^{n-1}$ such that

$$\nu_0\circ\alpha_0 - \alpha_0'\circ\mu_0 = d_{B'}h + hd_A.$$

Now, it turns out to be possible to construct a map of complexes $C_\alpha \xrightarrow{\eta} C_{\alpha'}$ such that the diagram

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[d, "\nu_0"'] & C_\alpha \arrow[r, "\gamma"] \arrow[d, "\eta_0"'] & A[1] \arrow[d, "\mu_0[1]"'] \\ B' \arrow[r, "\beta'"] & C_{\alpha'} \arrow[r, "\gamma'"] & A'[1] \end{tikzcd}$$

commutes in $C(\mathcal{A})$. This will incorporate the above homotopy correction to commutativity:

$$\eta^n : B^n \oplus A^{n+1} \to (B')^n \oplus (A')^{n+1}, \qquad \eta(b^n \oplus a^{n+1}) \overset{\text{def}}{=} (\nu_0 b^n + h^{n+1}a^{n+1}) \oplus \mu_0 a^{n+1}.$$

One just has to check that η is a map of complexes and that the two squares commute.

(T5) We omit verification of this property which is a description of a certain “complicated” relation between compositions of maps and cones of maps. □

Lemma. The construction of the homotopy category works the same for any of the subcategories $C^?(\mathcal{A}) \subseteq C(\mathcal{A})$ where ? is one of the symbols $b, +, -$. We get full subcategories $K^?(\mathcal{A})$ of $K(\mathcal{A})$.

6.5. Distinguished triangles in $K(\mathcal{A})$, SES in $C(\mathcal{A})$ and LES of cohomologies.

---

[9] If we could choose representatives μ_0, ν_0 so that the diagram in $C(\mathcal{A})$ still commutes, then a representative η_0 of η would simply come from the functoriality (“naturality”) of the cone construction.

6.5.1. *A short passage from degreewise split SES in $C(\mathcal{A})$ to distinguished triangles in $K(\mathcal{A})$.* We know that in $C(\mathcal{A})$ there is an equivalence between distinguished triangles $B \to C \to \mathsf{A}$ and degreewise split SES $0 \to A \to B \to C \to 0$. This simple relation uses a "rotation" since $\mathsf{A} = A[1]$. However, when triangles are considered in $K(\mathcal{A})$ one can state the correspondence in a simpler way.

Lemma. Let $\mathcal{A}$ be an additive category.

(a) Any degreewise split SES in $C(\mathcal{A})$

$$0 \to P \to Q \to R \to 0$$

gives a canonical distinguished triangle in $K(\mathcal{A})$ of the form

$$P \xrightarrow{\phi} Q \xrightarrow{\psi} R \xrightarrow{-\alpha} P[1].$$

Here, the map $R \xrightarrow{\alpha} P$ is the component of the differential d_Q given by the splittings $Q^n \cong P^n \oplus R^n$, i.e., $\alpha^n \stackrel{\text{def}}{=} [R^n \hookrightarrow Q^n \xrightarrow{d_Q^n} Q^{n+1} \twoheadrightarrow P^{n+1}]$.

(b) Any distinguished triangle in $K(\mathcal{A})$ is isomorphic to one that comes from a degreewise split SES by the construction in (a).

Proof. (b) (a) We know by lemma 6.1.8 that for SES as above, with degreewise splittings $Q^n \cong P^n \oplus R^n$, the triangle

$$R[-1] \xrightarrow{\alpha} P \xrightarrow{\phi} Q \xrightarrow{\psi} (R[-1])[1]$$

(with α defined as stated in the present lemma), is isomorphic in $C(\mathcal{A})$ to the cone triangle for α, so it is distinguished.

Since we are in $K(\mathcal{A})$ we can rotate this triangle using the property (T3) to get an exact triangle $P \xrightarrow{\phi} R \xrightarrow{\psi} Q \xrightarrow{\alpha[1]} P[1]$.

(b) follows because the relation between degreewise split SES and cone triangles in $C(\mathcal{A})$ is an equivalence[(10)] and so is the rotation operation in $\mathcal{D}$. $\square$

Remark. This simpler correspondence is however not an equivalence as SES are considered in $C(\mathcal{A})$ and distinguished triangles in $K(\mathcal{A})$.

6.5.2. *Cohomologies.* Here $\mathcal{A}$ must be abelian so that cohomology is defined. Notice that the cohomology functors $\mathrm{H}^i : C(\mathcal{A}) \to \mathcal{A}$ factor to $\mathrm{H}^i : K(\mathcal{A}) \to \mathcal{A}$. (This is the claim that for a map of complexes $\alpha : A \to B$ the map $\mathrm{H}^i(\alpha) : \mathrm{H}^i(A) \to \mathrm{H}^i(B)$ depends on on the class $[\alpha]$. This is the lemma 6.3.1.a.)

[10] Here word "equivalence" can be interpreted as "equivalent data" or as "bijection of isomorphism classes" or as "equivalence of categories" if one defines the natural categories for these two kinds of objects.

Corollary. If $\mathcal{A}$ is abelian, any distinguished triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$ in $K(\mathcal{A})$ gives a *long exact sequence* (LES) of cohomologies

$$\cdots \to \mathrm{H}^i(X) \xrightarrow{\mathrm{H}^i(\alpha)} \mathrm{H}^i(Y) \xrightarrow{\mathrm{H}^i(\beta)} \mathrm{H}^i(Z) \xrightarrow{\mathrm{H}^i(\gamma)} \mathrm{H}^{i+1}(X) \to \cdots.$$

Proof. Notice that the maps in our sequence of cohomologies are well defined since $\mathrm{H}^{(\frac{1}{2})}$: $\mathrm{H}^i(Z) \to \mathrm{H}^i(X[1]) = \mathrm{H}^{i+1}X$. For exactness recall that in $K(\mathcal{A})$, our distinguished triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$ is isomorphic to a triangle of the form $P \xrightarrow{[\phi]} Q \xrightarrow{\psi]} R \xrightarrow{[-\alpha]} P[1]$, associated to some degreewise split SES $0 \to P \xrightarrow{\phi} Q \xrightarrow{\psi} R \to 0$ in $C(\mathcal{A})$. We know that a SES of complexes does indeed provide a long exact sequence of cohomologies.

It remain to check that the cohomology objects in these two long sequences are the same and that the maps between cohomologies in two sequences are the same. □

6.5.3. *Acyclic complexes.* We say that a complex A is acyclic if all cohomologies vanish.

Corollary. For an abelian category $\mathcal{A}$ A map $\alpha : A \to B$ in $C(\mathcal{A})$ is a qis iff the complex C_α is acyclic.

Proof. This is clear from the long exact sequence of cohomologies for the cone triangle. □

6.5.4. *Extension of additive functors to homotopy categories.* Here we consider the trivial extensions $C(F)$ and $K(F)$ of an additive functor $F : \mathcal{A} \to \mathcal{B}$ between two additive categories. We will use this to construct the interesting extensions LF, RF in 6.6.3.

Lemma. (a) There is a canonical functor $C(\mathcal{A}) \to K(\mathcal{A})$ which sends each complex A to itself and each map of complexes ϕ to its homotopy class $[\phi]$.

(b) Any additive functor between additive categories $\mathcal{A} \xrightarrow{F} \mathcal{B}$ has canonical extensions to (homotopic) categories of complexes such that

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\subseteq"'] \arrow[d, "F"] & C(\mathcal{A}) \arrow[r, "q_{\mathcal{A}}"] \arrow[d, "C(F)"'] & K(\mathcal{A}) \arrow[d, "K(F)"'] \\
\mathcal{B} \arrow[r, "\subseteq"'] & C(\mathcal{B}) \arrow[r, "q_{\mathcal{A}}"] & K(\mathcal{B})
\end{tikzcd}$$

commutes.[11]

(c) F extends to a functor $\mathcal{A}^\bullet \xrightarrow{F^\bullet} \mathcal{B}^\bullet$ and for any $A, B \in C(\mathcal{A})$ this gives a map between Hom-complexes $\mathrm{Hom}^\bullet_{\mathcal{A}}(A,B) \to \mathrm{Hom}^\bullet_{\mathcal{B}}(FA, FB)$.

Proof. (a) is clear.

[11] $K(F)$ is uniquelly determined by the commutativity of the second square.

In (c) functor $F^\bullet$ is just F in each degree, i.e., $[F^\bullet A]^n = F(A^n)$. We get maps $F^n : \mathrm{Hom}^n_{\mathcal{A}}(A,B) \to \mathrm{Hom}^n_{\mathcal{B}}(FA,FB)$ as $\mathrm{Hom}_{\mathcal{A}^\bullet}(A,B[n]) \xrightarrow{F^\bullet} \mathrm{Hom}_{\mathcal{B}^\bullet}(FA,FB[n])$. These maps clearly intertwine the differentials on Hom-complexes.

In (b), for any $A \in C(\mathcal{A})$ we define the complex $C(F)A$ as the graded object $F^\bullet A$ above with the differentials $F(d^n_A)$. For $C(F)$ to factor to $K(F)$ we need that $C(F)$ is compatible with homotopies, this follows from (c) since a map of complexes $F^\bullet : \mathrm{Hom}^\bullet_{\mathcal{A}}(A,B) \to \mathrm{Hom}^\bullet_{\mathcal{B}}(FA,FB)$ gives a map of 0^{th} cohomologies $H^0(F^\bullet) : \mathrm{Hom}_{K(\mathcal{A})}(A,B) \to \mathrm{Hom}_{K(\mathcal{B})}(FA,FB)$. $\square$

6.6. **Derived functors and functorial projective resolutions.**

6.6.1. *Lifts of maps to projective resolutions.*

Lemma. Consider two complexes

$$\cdots \to P^{-n} \to \cdots \xrightarrow{d_P^{-1}} P^0 \xrightarrow{\varepsilon_a} a \to 0 \to \cdots$$

$$\cdots \to B^{-n} \to \cdots \xrightarrow{d_B^{-1}} B^0 \xrightarrow{\varepsilon_B} b \to 0 \to \cdots$$

such that all P^k are projective and the second complex is exact. Then any map $\alpha : a \to b$ in $\mathcal{A}$ lifts to a map complexes $P \xrightarrow{\phi} B$ which makes the following diagram commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-1} \arrow[r, "d_P^{-1}"] \arrow[d, "\phi^{-1}"'] & P^0 \arrow[r, "\varepsilon_a"] \arrow[d, "\phi^0"'] & a \arrow[r] \arrow[d, "\alpha"'] & 0 \arrow[r] \arrow[d, "="'] & \cdots \\
\cdots \arrow[r] & B^{-1} \arrow[r, "d_B^{-1}"] & B^0 \arrow[r, "\varepsilon_b"] & b \arrow[r] & 0 \arrow[r] & \cdots.
\end{tikzcd}$$

(b) Any two such lifts are canonically homotopic.

Proof. (a) The map $P^0 \xrightarrow{\alpha\varepsilon_a} b$ factors through ε_b since P^0 is projective and $\varepsilon_B : B^0 \to B$ is surjective. This gives a map $P^0 \xrightarrow{\phi^0} B^0$ such that $\varepsilon_b \circ \phi^0 = \alpha \circ \varepsilon_a$.

Notice that the composition $P^{-1} \xrightarrow{d_P^{-1}} P^0 \xrightarrow{\phi^0} B^0$ goes to $\mathrm{Ker}(\varepsilon_b) \subseteq B^0$ since

$$\varepsilon_B(\phi^0 d_P^{-1}) = (\alpha\varepsilon_a) d_P^{-1} = \alpha \circ 0 = 0$$

because the first row is a complex.

Now, exactness of the second row shows that $d_B^{-1} : B^{-1} \to B^0$ factors through the surjection $d_B^{-1} : P^{-1} \twoheadrightarrow \mathrm{Ker}(\varepsilon_b)$. So, since P^{-1} is projective the map $\phi^0 \circ d_P^{-1} : P^{-1} \to \mathrm{Ker}(\varepsilon_b)$ factors through $d_B^{-1} : B^{-1} \to \mathrm{Ker}(\varepsilon_b)$, giving a map $\phi^{-1} : P^{-1} \to B^{-1}$, such that $\phi^0 \circ d_P^{-1} = d_B^{-1} \circ \phi^{-1}$.

In this way we construct all ϕ^n inductively.

(b) If we have lifts $\phi_i : P \to B$ of maps $\alpha_i : a \to b$ then $\phi_2 - \phi_1$ lifts $\alpha_2 - \alpha_1$. So, two lifts ϕ_i of α give a lift $\phi = \phi_2 - \phi_1$ of $0 : a \to b$. So, we need to see that any a lift $\phi : P \to B$ of $0 : a \to b$ is of the form $dh + hd$ for $h^n : P^n \to B^{n-1}$.

The proof is the same as in (a). First, $\phi^0 : P^0 \to B^0$ goes to $\mathrm{Ker}(\varepsilon_B)$ since the lifting relation is $\varepsilon_B \phi^0 = 0 \circ \varepsilon_A = 0$. Since P^0 is projective and $d_B^{-1} : B^0 \to \mathrm{Ker}(\varepsilon_b)$ is surjective, the map $\phi^0 : P^0 \to \mathrm{Ker}(\varepsilon_B)$ lifts to a map $h^0 : P^0 \to B^{-1}$. This h^0 satisfies

$$\phi^0 = d_B^{-1} \circ h^0$$

(and this is $d_B^{-1} \circ h^0 + h^1 \circ d_B^0$ since the two factors in the last term are both zero).

Now one continuous similarly

$$d_B^{-1} \circ \phi^{-1} = \phi^0 \circ d_P^{-1} = d_B^{-1} \circ h^0 \circ d_P^{-1},$$

hence the map $\phi^{-1} - d_B^{-1} \circ h^0 : P^{-1} \to B^{-1}$ goes to $\mathrm{Ker}(d_B^{-1})$. Since $d_B^{-2} : B^{-2} \mathrm{Ker}(d_B^{-1})$ is surjective, map $\phi^{-1} - d_B^{-1} \circ h^0$ lifts to a map $h^{-1} : P^{-1} \to B^{-2}$. This means that

$$\phi^{-1} - d_B^{-1} \circ h^0 = h^{-1} \circ d_P^{-2}.$$

Etc. □

Corollary. (a) If P and Q are projective resolutions of objects a and b in $\mathcal{A}$, then any map $a \to b$ lifts uniquely to a map $P \to Q$.

(b) Any two projective resolutions of the same object of $\mathcal{A}$ are canonically isomorphic in $K(\mathcal{A})$.

6.6.2. *Projective resolution functor.* The last corollary can be restated (in shorthand) as

Corollary. If abelian $\mathcal{A}$ has enough projectives then then there is a canonical projective resolution functor $\mathcal{P} : \mathcal{A} \to K^-(\mathcal{A})$.

Proof. The precise meaning is the following. The first claim is that there is a functor $\mathcal{P} : \mathcal{A} \to K^-(\mathcal{A})$ such that for each $a \in \mathcal{A}$, $\mathcal{P}_a$ is a projective resolution of a. In order to construct it we need to choose for each $a \in \mathcal{A}$ a projective resolution $\mathcal{P}_a$ of a. Then by the lemma any map $\alpha : a \to b$ in $\mathcal{A}$ has a unique lift $\mathcal{P}(\alpha) : \mathcal{P}_a \to \mathcal{P}_b$ (by a lift we mean a map of complexes such that the diagram as in lemma commutes). Uniqueness now implies that $\mathcal{P}$ is a functor – for instance for $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ we have $\mathcal{P}(\beta) \circ \mathcal{P}(\alpha) = \mathcal{P}(\beta\alpha)$ since both sides are lifts of $\beta\alpha$.

The second information is that though $\mathcal{P}$ is not literally unique (it depends on our choices) any two versions $\mathcal{P}', \mathcal{P}''$ are *canonically isomorphic*. The reason is that for any $a \in \mathcal{A}$ lemma tells us that the map 1_a has unique lifts $\zeta_a : \mathcal{P}'_a \to \mathcal{P}''_a$ and $\xi_a : \mathcal{P}''_a \to \mathcal{P}'_a$. Then $\xi_a \circ \zeta_a = id_{\mathcal{P}'_a}$ since again both sides lift 1_a. □

6.6.3. *Derived functors $LF : \mathcal{A} \to K^-(\mathcal{B})$ and $RG : \mathcal{A} \to K^+(\mathcal{B})$.*

Lemma. Let $\mathcal{A}$ be abelian and $\mathcal{B}$ additive. If $\mathcal{A}$ has enough projectives then for any additive functor $F : \mathcal{A} \to \mathcal{B}$ its left derived functor $LF : \mathcal{A} \to K^{\leq 0}(\mathcal{B})$ is well defined by replacing objects with their projective resolutions

$$LF(a) \stackrel{\text{def}}{=} C(F)(P)$$

where P is any projective resolution of a.

Proof. We choose a projective resolution functor $\mathcal{P} : \mathcal{A} \to K^-(\mathcal{A})$ and define LF as a composition of functors $\mathcal{A} \xrightarrow{\mathcal{P}} K(\mathcal{A}) \xrightarrow{K(F)} K(\mathcal{B})$. Then for any projective resolution P of a we have canonical isomorphism $P \underset{\cong}{\to} \mathcal{P}_a$ hence $K(F)(P) \underset{\cong}{\to} (LF)a$. $\square$

Remark. Remember that $\mathrm{H}^0[(LF)(A) \cong F(A)$.

6.7. **Appendix.** Here are some useful claims which we do not cover.

Lemma. A. In a triangulated category $(\mathcal{T}, [1], \mathcal{D})$ the composition of any two maps in a distinguished triangle is zero.

Lemma. B. A homotopy h between two maps $\alpha_1, \alpha_2 \in \mathrm{Hom}_{C(\mathcal{A})}(A, B)$ gives an isomorphism of the corresponding mapping cones $C_{\alpha_1} \underset{\cong}{\to} C_{\alpha_2}$ in $C(\mathcal{A})$.

Lemma. C. [dsSES and exact triangles in $K(\mathcal{A})$.][12] Consider a sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ in $C(\mathcal{A})$.

(a) If $\beta\alpha = 0$ then β canonically factors through the cone of α, i.e., there is a canonical μ that makes the following commutative (actually, $\mu(b^n \oplus a^{n+1}) = \beta(b)$):

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "="'] & B \arrow[r, "\dot{\alpha}"] \arrow[d, "="'] & C_\alpha \arrow[d, "\mu"'] \\
A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C.
\end{tikzcd}$$

(b) If the sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is *exact* then μ is a qis.

(c) If the sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is *exact* and *degreewise split* then μ is a homotopy equivalence.

Remark. Eventually, any SES in $C(\mathcal{A})$ will give a distinguished triangle in $D(\mathcal{A})$, at if outer terms have injective or projective resolution.

[12] This is a version of our lemmas on the same subject.