HOMOLOGICAL ALGEBRA

Contents

Date: ?

0.0.1. *Notation.* Symbol $\square$ means “I said this much and I will say no more”.

5. **Exactness properties and the derived functors**

5.1. **Summary.** For any abelian category $\mathcal{A}$ we can define in the category of complexes $C(\mathcal{A})$ a class of morphisms called *quasi-isomorphisms* (“qis”). Our long term goal is to understand the *derived category* $D(\mathcal{A})$ which is obtained from $C(\mathcal{A})$ by inverting all quasi-isomorphisms.

In particular, when $\mathcal{A}$ and $\mathcal{B}$ are abelian categories we ask which functors $F : \mathcal{A} \to \mathcal{B}$ extend in a useful way to functors between derived categories $D(F) : D(\mathcal{A}) \to D(\mathcal{B})$.

$D(F)$ is called the derived version of F.[(1)]

5.1.1. *The case of exact functors.* Recall that any additive functor $H : \mathcal{U} \to \mathcal{V}$ between additive categories has an obvious extension to complexes, denoted $C(H) : C(\mathcal{U}) \to C(\mathcal{V})$ (we just apply H to all objects and all morphisms in a complex $U \in C(\mathcal{U})$).[(2)] So, for our functor between abelian categories $F : \mathcal{A} \to \mathcal{B}$ we can ask whether the functor $C(F) : C(\mathcal{A}) \to C(\mathcal{B})$ between categories f complexes descends to a functor $D(\mathcal{A}) \to D(\mathcal{B})$ (which we would then call $D(F)$. Clearly, this works iff the functor $C(F)$ preserves quasi-isomorphisms.

A functor $F : \mathcal{A} \to \mathcal{B}$ is worthy of being called an *abelian* functor, i.e., if it is compatible with all structures that appear in the definition of abelian categories (i.e., it is an additive functor and it preserves (co)kernels and (co)images). Equivalently, F should be additive and it should preserve exactness of sequences of maps. So, such functors are called *exact* (rather than “abelian”).

If F is exact then the extension $C(F)$ to complexes will preserve quasi-isomorphisms (since these are defined in terms of cohomology, i.e., of (co)kernels, and (co)images).

So, exact functors F have an obvious extension to derived categories $D(F) : D(A) \to D(\mathcal{B})$ where for $A \in Ob[D(\mathcal{A})] = Ob[C(\mathcal{A})]$, $(DF)A$ is obtained by applying F itself to all ingredients of A. However, exact functors are “simple” and the interesting functors are generally not exact.

5.1.2. *Half exact functors, injectives and projectives.* The key insight is that F may have a useful “extension” to the derived level under some weaker conditions. The basic case is when F sis additive and *left exact* or *right exact* (“half exact”), in the sense that it only preserves exactness of the “left part” or the “right part” of short exact sequences (SES). to a “certain extent”.

[1] We may denote the extension just by F, or more precisely by $D(F)$, or LF or RF, depending on the way we produce the extension.

[2] One often denotes $C(F)$ just by F.

The main ingredient for extending half exact functors to derived categories comes from considering exactness properties of the simplest functors – the ones that come from the category itself. This distinguishes two subclasses of objects of $\mathcal{A}$: the *projective* objects $a \in \mathcal{A}$ are the ones for which the functor $\mathrm{Hom}_{\mathcal{A}}(a,-)$ is exact and *injective* ones for which $\mathrm{Hom}_{\mathcal{A}}(-,a)$ is exact.

5.1.3. *Extension constructions L and R for half-exact functors.* If additive $F:\mathcal{A}\to\mathcal{B}$ is only right exact then one resolves the failure of exactness at the left end by expressing all object in terms of complexes of objects with good exactness properties on the left. This means that we replace any object $a \in \mathcal{A}$ with its left resolutions P by projectives. We denote this construction by

$$(LF)(a) \overset{\text{def}}{=} F(P).$$

So, LF (called the "*left derived functor*" of F) is a construction that starts in $\mathcal{A}$ and its result is in the category $C(\mathcal{B})$ of complexes in $\mathcal{B}$. Notice that LF gives a sequence of constructions $(L^nF)(a) \overset{\text{def}}{=} H^n[(LF)a] \in \mathcal{B}$ for $n \in \mathbb{Z}$.

Remark. We will improve the above LF in stages. Right now it is a construction from $\mathcal{A}$ to $C(\mathcal{B})$ which (!) depends on choices (of projective resolutions). We will make LF into a functor in the section **??** by replacing $C(\mathcal{B})$ by the *homotopy category of complexes* $K(\mathcal{B})$. In ... we will extend this version of LF to a functor between derived categories. $\square$

Dually, for F left exact we use right resolutions $a \to I$ by injectives and this defines $(RF)a \overset{\text{def}}{=} F(I)$ and $(R^nF)a \overset{\text{def}}{=} H^n[(RF)a]$.

5.1.4. *Existence of enough projectives or injectives.* In order to use the extension construction L (resp. R) for a functor F with the source $\mathcal{A}$ one needs that $\mathcal{A}$ has sufficiently many projective (resp. injective) objects, so that any object has a projective (resp. injective) resolution. We will check that for any ring A the category $\mathfrak{m}(A)$ of A-modules has enough projectives and injectives.

5.2. **Exactness properties of functors.**

5.2.1. *Exactness of a sequence of maps.* A sequence of maps in an abelian category $M_a \xrightarrow{\alpha_a} M_{a+1} \to \cdots \xrightarrow{\alpha_{b-1}} M_b$ is said to be *exact at* M_i (for $a < i < b$) if $Im(\alpha_{i-1}) = \mathrm{Ker}(\alpha_i)$. The whole sequence is said to be exact if it is exact at all $M_i,\ a < i < b$. (The sequence may possibly be infinite in one or both directions.)

5.2.2. *Exact functors.* Let $F:\mathcal{A}\to\mathcal{B}$ be an additive functor between abelian categories. (One can think of the case where $\mathcal{A} = \mathfrak{m}(\Bbbk)$ and $\mathcal{B} = \mathfrak{m}(l)$ since the general case works the same.)

We will say that F is exact if it preserves short exact sequences, i.e., for any SES $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ in $\mathcal{A}$, its F-image in $\mathcal{B}$ is exact, i.e., the sequence $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ is a SES in $\mathcal{B}$.

We say, that F is *left exactt* if for any SES its F-image $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'')$ is exact (so, $F(\beta)$ need not be surjective).

F is *right exact* if $F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ is exact ($F(\alpha)$ may fail to be injective).

Lemma. (a) If F is LE it preserves inclusions and kernels. If F is RE it preserves surjections and cokernels. If F is exact then it preserves (co)kernels, (co)images and the cohomology of complexes.

(b) F is LE iff it preserves exactness of sequences of the form

$$0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C''.$$

F is RE iff it preserves exactness of sequences of the form

$$C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0.$$

F is exact iff it preserves exactness of sequences of the form $C' \xrightarrow{\alpha} C \xrightarrow{\beta} C''$.

Proof. Consider the case when F is left exact. Let $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C''$ be an exact sequence (equivalently, $C' \xrightarrow{\alpha} C$ is the kernel of β). This sequence fits into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C' \arrow[r, "\alpha"] & C \arrow[r, "\beta"] & C'' & \\
0 \arrow[r] & C' \arrow[r, "\alpha"] \arrow[u, "="] & C \arrow[r, "\beta'"] \arrow[u, "="] & Im(\beta) \arrow[r] \arrow[u, "i", "\subseteq"'] & 0'
\end{tikzcd}.$$

Since $\mathrm{Ker}(\beta') = \mathrm{Ker}(\beta)$ (here $\beta = i \circ \beta'$ and i is an inclusion!), the second row is also exact.

Now apply F to this diagram:

$$\begin{tikzcd}
0 \arrow[r] & FC' \arrow[r, "F\alpha"] & FC \arrow[r, "F\beta"] & FC'' \\
0 \arrow[r] & FC' \arrow[r, "F\alpha"] \arrow[u, "="] & FC \arrow[r, "F\beta'"] \arrow[u, "="] & F[Im(\beta)]. \arrow[u, "F(i)", "\subseteq"']
\end{tikzcd}$$

Since F is left exact the second row is exact. In particular, $F\alpha$ is again an inclusion.

Because any inclusion $0 \to C' \xrightarrow{\alpha} C$ appears in an exact sequence $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C''$ (for instance we could take $C'' = Im(\alpha)$). this proves that F preserves inclusions.

This property implies that $F(i)$ is an inclusion. Then we see (the same as before), that $\mathrm{Ker}(F\beta) = \mathrm{Ker}(F\beta')$. So, the first row is exact. In particular $F(\mathrm{Ker}(\beta)) = FC' = \mathrm{Ker}(F\beta)$.

Since any map $C \xrightarrow{\beta} C''$ appears in an exact sequence $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C''$ (with $C' = \mathrm{Ker}(\beta)$) this proves that F preserves kernels.

We have proved (a) and (b) for LE functors. The RE case follows by duality. So, if F is exact then we already know that it preserves (co)kernels. Then it preserves (co)images and cohomologies of a complex since these are all defined in terms of (co)kernels, say $H^n(C) = \mathrm{Coker}[C^{n-1} \to \mathrm{Ker}(d^n)]$.

We will check the last claim by thinking of complexes. If F is exact and $C' \xrightarrow{\alpha} C \xrightarrow{\beta} C''$ is exact then $\beta\alpha = 0$ so we can think of it as a complex in degrees $-1, 0, 1$ (put zeros elsewhere). Then for its F-image we have

$$H^0(FC' \xrightarrow{F\alpha} FC \xrightarrow{F\beta} FC'') = F(H^0(C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'']) = F(0) = 0. \quad \square$$

Remark. The properties in the lemma are stronger then the definitions of exactness. However they follow from the definition of exactness properties by a diagram chase which uses the definition of exactness property in two or more places.

5.2.3. *Half exactness of contravariant functors.* This will be needed for the functors $\mathrm{Hom}_{\mathcal{A}}(-, a)$.

We say that a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is *left exact* if it has this property when viewed as an ordinary functor from $\mathcal{A}$ to $\mathcal{B}^o$. So, we ask that if $0 \to a' \xrightarrow{\alpha} a' \xrightarrow{\beta} a''$ is exact in $\mathcal{A}$ then its F-image in $\mathcal{B}$ is exact, i.e., $Fa'' \xrightarrow{F\beta} Fa \xrightarrow{F\alpha} Fa' \to 0$ is exact in $\mathcal{B}$.[3]

The same convention gives the following notion of right exactness of a contravariant F: we ask that for $a' \xrightarrow{\alpha} a \xrightarrow{\beta} a'' \to 0$ in $\mathcal{A}$, its F-image $F(0) \to F(a'') \xrightarrow{F(\beta)} F(a) \xrightarrow{F(\alpha)} F(a')$ is exact.

5.3. **Examples of half exact functors.**

5.3.1. *The basic example of half exact functors.* It involves functors that come from the structure of an abelian category.

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[3] Notice that if we would consider F as a functor $\mathcal{A}^o \to \mathcal{B}$ we would get a different notion of left exactness. The requirement would be that for $0 \to a'' \xrightarrow{\alpha} a \xrightarrow{\beta} a'$ exact in $\mathcal{A}^o$ (i.e., $a' \xrightarrow{\beta} a \xrightarrow{\alpha} a'' \to 0$ exact in $\mathcal{A}$), the F-image $0 \to Fa'' \xrightarrow{F\alpha} Fa \xrightarrow{F\beta} Fa'$ be exact in $\mathcal{B}$.

Lemma. (a) A sequence $b' \xrightarrow{\alpha} b \xrightarrow{\beta} b'' \to 0$ in an abelian category $\mathcal{A}$ is exact iff for any $a \in \mathcal{A}$ the following sequence of Hom' is exact in $\mathcal{A}b$

$$0 \to \mathrm{Hom}(b'', a) \xrightarrow{\beta^*} \mathrm{Hom}(b, a) \xrightarrow{\alpha^*} \mathrm{Hom}(b', a).$$

(b) A sequence $0 \to b' \xrightarrow{\alpha} b \xrightarrow{\beta} b''$ is exact iff for any $a \in \mathcal{A}$ the sequence of Hom' is exact:

$$0 \to \mathrm{Hom}(a, b') \xrightarrow{\beta^*} \mathrm{Hom}(a, b) \xrightarrow{\alpha^*} \mathrm{Hom}(a, b'').$$

□

Proof. By duality, it suffices to prove (a).

$b' \xrightarrow{\alpha} b \xrightarrow{\beta} b'' \to 0$ is exact iff $b \underset{\beta}{\to} b'$ is the cokernel of α (i.e., $b'' = \mathrm{Coker}(\alpha)$). then the lemma implies that the above sequence of Hom's is exact. By definition this means precisely that *any* map $\phi : a \to b$ into b, such that $\alpha^*\phi = \phi\circ\alpha$ is zero, comes from a *unique* map $\psi : a \to b'$ (in the sense that $\phi = \beta^*\psi$, i.e., that $\phi = \psi\circ\beta$. However, this is exactly the exactness of the sequence of Hom's in the lemma for any $a \in \mathcal{A}$! □

Corollary. (a) In an abelian category $\mathcal{A}$ for any $a \in \mathcal{A}$, the functor $\mathrm{Hom}_{\mathcal{A}}(a, -) : \mathcal{A} \to \mathcal{A}b$ is left exact while the functor $\mathrm{Hom}_{\mathcal{A}}(-, a) : \mathcal{A}^o \to \mathcal{A}b$ is right exact!

(b) Left adjoints are right exact!

Proof. The second part of (a) is the claim that for $a \in \mathcal{A}$, applying $\mathrm{Hom}(-, a)$ to any SES $0 \to b' \xrightarrow{\alpha} b \xrightarrow{\beta} b'' \to$ the result $0 \to \mathrm{Hom}(b'', a) \xrightarrow{\beta^*} \mathrm{Hom}(b, a) \xrightarrow{\alpha^*} \mathrm{Hom}(b', a)$ is exact. This is proved in the lemma.[(4)]

(b) Let $F : \mathcal{A} \to \mathcal{B}$ have a right adjoint G. We want F (being a left adjoint of G) to be right exact, i.e., for exact $0 \to a' \xrightarrow{\alpha} a \xrightarrow{\beta} a'' \to 0$ we want $Ga' \xrightarrow{G\alpha} Ga \xrightarrow{G\beta} Ga'' \to 0$ to be exact. However, by the lemma it is equivalent to the claim that for any $b \in \mathcal{B}$

$$0 \to \mathrm{Hom}(Ga'', b) \xrightarrow{\beta^*} \mathrm{Hom}(Ga, b) \xrightarrow{\alpha^*} \mathrm{Hom}(Ga', b)$$

is exact. By adjointness this sequence is the same as □

$$0 \to \mathrm{Hom}(a'', Fb) \xrightarrow{\beta^*} \mathrm{Hom}(a, Fb) \xrightarrow{\alpha^*} \mathrm{Hom}(a', Fb).$$

However, we know that this is exact by the lemma. □

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[4] Here is also a direct proof that $\mathrm{Hom}(a, -)$ is left exact.

(1) α_* is injective. if $a \xrightarrow{\mu} b'$ and $0 = \alpha_*(\mu) \overset{\text{def}}{=} \alpha\circ\mu$, then μ factors through the kernel $\mathrm{Ker}(\alpha)$ (by the definition of the kernel). However, $\mathrm{Ker}(\alpha) = 0$ (by definition of a short exact sequence), hence $\mu = 0$.

(2) $\mathrm{Ker}(\beta^*) = Im(\alpha_*)$. First, $\beta_*\circ\alpha_* = (\beta\circ\alpha)_* = 0_* = 0$, hence $Im(\alpha_*)\subseteq\mathrm{Ker}(\beta^*)$. If $a \xrightarrow{\nu} b$ and $0 = \beta_*(\nu)$, i.e., $0 = \beta\circ\nu$, then ν factors through the kernel $\mathrm{Ker}(\beta)$. But $\mathrm{Ker}(\beta) = a'$ and the factorization now means that ν is in $Im(\alpha_*)$.

Example. A counterexample. Let $\mathcal{A} = \mathcal{A}b$ and apply $\mathrm{Hom}(a, -)$ for $a = \mathbb{Z}/2\mathbb{Z}$ to $0 \to 2\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \to 0$. Then $id_{\mathbb{Z}/2\mathbb{Z}}$ does not lift to a map from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$. So β_* need not be surjective.

Corollary. (a) A sequence $b' \xrightarrow{\alpha} b \xrightarrow{\beta} b \to 0$ in an abelian category $\mathcal{A}$ is exact iff for any $a \in \mathcal{A}$

$$0 \to \mathrm{Hom}(a, b'') \xrightarrow{\beta^*} \mathrm{Hom}(a, b) \xrightarrow{\alpha^*} \mathrm{Hom}(a, b')$$

is exact.

(b) Left adjoints are right exact.

Proof. (a) If $b' \xrightarrow{\alpha} b \xrightarrow{\beta} b \to 0$ is exact then the lemma implies that the above sequence of Hom's is exact.

The claim that the sequence of Hom's is exact for any a is the claim that *any* map $\phi : a \to b$ into b, such that $\alpha^*\phi = \phi\circ\alpha$ is zero, comes from a *unique* map $\psi : a \to b'$ (in the sense that $\phi = \beta^*\psi$, i.e., that $\phi = \psi\circ\beta$. This is precisely the definition of β being a cokernel of α.

5.3.2. *Example. Functoriality of categories of modules over rings.* Recall that for a map of rings $\phi : A \to B$ we have an adjoint triple $(\phi_* = Ind_A^B, \phi^*, \phi_* = Coind_A^B)$ of functors between A and B modules. Therefore, ϕ^* is exact (this is anyway clear from since it does not change the structure of abelian groups and the exactness for modules only involves the level of abelian groups) while $Ind_A^B M = B \otimes_A M$ is right exact and $Coind_A^B M = \mathrm{Hom}_A(B, M$ is left exact. □

5.3.3. *Example: representations of groups.* For a group G let $Rep_{\Bbbk}(G)$ be the category of all representations of G over a commutative ring $\Bbbk$. Here, a representations of G is a pair (V, π) of a $\Bbbk$-modules V and a map of groups $G \xrightarrow{\pi} GL_{\Bbbk}(V)$. We often denote $\pi(g)v$ by gv, and we often omit V or π from the notation.

Consider the functors $I, C : Rep_{\Bbbk}(G) \to \mathfrak{m}(\Bbbk)$ of invariants and coinvariants

$$I(V) \overset{\text{def}}{=} V^G \overset{\text{def}}{=} \{v \in V,\ gv = v,\ g \in G\} \quad \text{and} \quad C(V) \overset{\text{def}}{=} V/[\sum_{g\in G} (g-1)V.$$

is right exact.

Lemma. (a) $Rep_{\Bbbk}(G)$ is an abelian category.

(b) For a subgroup $A \subseteq G$ the forgetful functor $\mathcal{F} : Rep_{\Bbbk}(G) \to Rep_{\Bbbk}(A)$ has adjoints $Ind_A^G, \mathcal{F}, Coind_A^G)$.

(c) Let $\mathfrak{m}(\Bbbk) \xrightarrow{\mathcal{T}} Rep_{\Bbbk}(G)$ be the functor that adds to any $\Bbbk$-module M the trivial action of G, i.e., $g\cdot m = m$ for $m \in M$ and $g \in G$. Then $(C, \mathcal{T}, I)$ is an adjoint triple. The functor I of G-invariants is left exact. The functor C of G-coinvariants is right exact.

Proof. (a) Representations of G are the same as modules for the algebra $\Bbbk[G] \overset{\text{def}}{=} \oplus_{g\in G} \Bbbk g$ (the multiplication is obvious).

(b) Here $\mathcal{F}$ is i^* for for the inclusion of rings $\Bbbk[A]\overset{i}{\subseteq}\Bbbk[G]$. So the left adjoint is $i_*M = \Bbbk[G]\otimes_{\Bbbk[A]}M$ called the inductions and the right adjoint is $i_\star M = \mathrm{Hom}_{\Bbbk[A]}(\Bbbk[G], M)$.

(c) From the categorical point of view these two functors come from the same object of $Rep_\Bbbk(G)$ –the trivial G-modules $\Bbbk$:

$$I(V) \cong \mathrm{Hom}_G(\Bbbk, V) \quad \text{and} \quad C(V) \cong \Bbbk\otimes_{\Bbbk[G]}V.$$

For instance, the canonical isomorphism of functors $I \underset{\cong}{\xrightarrow{\eta}} \mathrm{Hom}_{Rep_\Bbbk(G)}(\Bbbk, -)$ from $Rep_\Bbbk(G)$ to $\mathcal{V}ect_\Bbbk$ consists of maps $V^G \underset{\cong}{\xrightarrow{\eta_V}} \mathrm{Hom}_{Rep_\Bbbk(G)}(\Bbbk, V)$ that sends a G-fixed vector $w \in I(V)$ to a linear map $\eta_V(w) : \Bbbk \to V$, given by multiplying w with scalars: $\Bbbk \ni c \mapsto c{\cdot}v \in V$. One easily checks that η_V is an isomorphism of vector spaces.

Now the adjunction claim is a case of **??** and the exactness claim is a case of the corollary 5.3.1.b. □

Counterexample. It is more interesting to see how exactness fails. For invariants take $\Bbbk = \mathbb{C}$. Then a representation π of $G = \mathbb{Z}$ is the same as a $\mathbb{C}$-vector space V with an invertible linear operator A $(= \pi(1))$. Therefore, $I(V, A)$ is the 1-eigenspace V_1 of A. Short exact sequences in $Rep_\Bbbk(\mathbb{Z})$ are all isomorphic to the ones of the form $0 \to (V', A|V') \to (V, A) \to (V'', \overline{A}) \to 0$, i.e., one has a vector space V with an invertible linear operator A, an A-invariant subspace V' (we restrict A to it), and the quotient space $V'' = V/V'$ (we factor A to it). So the exactness on the right means that any $w \in V/V'$ such that $\overline{A}w = w$ comes from some v in V such that $Av = v$.

If $V = \mathbb{C}e_1\oplus\mathbb{C}e_2$ with $A = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $V' = \mathbb{C}e_1$ then $0 \to I(V', A|V') \to I(V, A) \to I(V'', \overline{A}) \to 0$, is just $0 \to \mathbb{C}e_1 \xrightarrow{id} \mathbb{C}e_1 \xrightarrow{0} V'' \to 0$, and the exactness fails on the right.

5.3.4. *Example: tensoring.*

Lemma. Tensoring is right exact in each argument, i.e., for any left $\Bbbk$-module M the functor $M\underset{\Bbbk}{\otimes}- : \mathfrak{m}^r(\Bbbk) \to \mathcal{A}b$ is right exact, and so is $-\underset{\Bbbk}{\otimes}N : \mathfrak{m}(\Bbbk) \to \mathcal{A}b$ for any right $\Bbbk$-module N.

Proof. Homework. □

5.4. **Projective and injective objects and resolutions.**

5.4.1. *Projectives and injectives.* Let $\mathcal{A}$ be an abelian category. The exactness properties of functors $\mathrm{Hom}_{\mathcal{A}}(a,-)$ and $\mathrm{Hom}_{\mathcal{A}}(-,a)$ lead to the following classes of objects.

We say that $p\in\mathcal{A}$ is a *projective* object if the functor $\mathrm{Hom}_{\mathcal{A}}(p,-):\ \mathcal{A}\to\mathcal{A}b$ is exact.

Dually, we say that $i\in\mathcal{A}$ is an *injective* object if the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-,i):$ $\mathcal{A}\to\mathcal{A}b$ is exact, i.e., the functor $\mathrm{Hom}_{\mathcal{A}}(-,i):\ \mathcal{A}\to\mathcal{A}b^o$ is exact.

Proposition. (a) $p\in\mathcal{A}$ is projective iff for any map $p\xrightarrow{\phi}b''$ from p, and any quotient map $b\xrightarrow{q}b''$ the map β from p to the quotient lifts to a map to b.

(b) $i\in\mathcal{A}$ is injective iff for any map $a'\xrightarrow{\psi}i$ into i, and any inclusion map $a'\hookrightarrow a$, the map α from the subobject to i extends to a map from a.

Proof. (a) Since for any p the functor $\mathrm{Hom}_{\mathcal{A}}(p,-)$ is left exact, what we need is that for any short exact sequence $0\to b'\xrightarrow{\alpha}b\xrightarrow{\beta}b''\to 0$ its image $0\to\mathrm{Hom}_{\mathcal{A}}(p,b')\xrightarrow{\alpha_*}$ $\mathrm{Hom}_{\mathcal{A}}(p,b')\xrightarrow{\beta_*}\mathrm{Hom}_{\mathcal{A}}(p,b'')\to 0$ is exact except possibly at $\mathrm{Hom}_{\mathcal{A}}(p,b'')$. So, exactness is the claim that the map $\mathrm{Hom}(p,b)\xrightarrow{\beta_*}\mathrm{Hom}(p,b'')$ is surjective for any surjective map $b\xrightarrow{q}b''$. This is exactly the above lifting property.

(b) Again, since $\mathrm{Hom}_{\mathcal{A}}(-,i)$ is always a right exact contravariant functor, i.e., , for any short exact sequence $0\to a'\xrightarrow{\alpha}a\xrightarrow{\beta}a''\to 0$, exactness of its $\mathrm{Hom}_{\mathcal{A}}(-,i)$-image $0\to\mathrm{Hom}_{\mathcal{A}}(a'',i)\xrightarrow{\alpha}\mathrm{Hom}_{\mathcal{A}}(a,i)\xrightarrow{\beta}\mathrm{Hom}_{\mathcal{A}}(a',i)\to 0$ may only fail at the right end. So, exactness is equivalent to surjectivity of $\mathrm{Hom}(i,a)\xrightarrow{\alpha^*}\mathrm{Hom}(i,a')$ for $\alpha^*(\phi)=\phi\circ\alpha$. This is exactly the above extension requirement. $\square$

Lemma. (a) $\oplus_{i\in I}\ p_i$ is projective iff all summands p_i are projective.

(b) Product $\prod_{i\in I}\ J_i$ is injective iff all factors J_i are injective.

Proof. (a) The functor $\mathrm{Hom}_{\mathcal{A}}(\oplus_{i\in I}\ p_i,b)$ is isomorphic to $\prod_{i\in I}\ \mathrm{Hom}_{\mathcal{A}}(p_i,b)$. Now the claim follows since in $\mathcal{A}b$ the sequences $0\to B_i\to B_{\to}B_i''\to 0$ are exact for all $i\in I$ iff the product sequence $0\to\prod_{i\in I}B_i\to\prod_{i\in I}B_{\to}\prod_{i\in I}B_i''\to 0$ is exact. $\square$

5.4.2. *Resolutions.* We say that abelian category $\mathcal{A}$ has *enough projectives* if any object is a quotient of a projective object. We say that $\mathcal{A}$ has *enough injectives* if any object is a subobject of an injective object.

A *right resolution* of an object $a\in\mathcal{A}$ is any exact sequence $0\to a\to r^0\to r^1\to\cdots$. Dually, a *left resolution* of an a is any exact sequence $\cdots\to l^{-1}\to l^0\to a\to 0$.

A *projective resolution* of $a\in\mathcal{A}$ is any left resolution $\cdots\to p^{-1}\to p^0\to a\to 0$ where all p^n are projective, A *left resolution* of an a is any exact sequence $0\to a\to i^0\to i^1\to\cdots$ where all i^n are injective.

Lemma. In an abelian category $\mathcal{A}$ the following is equivalent

(a) Any object of $\mathcal{A}$ has a projective resolution iff $\mathcal{A}$ has enough projectives.

(b) Any object of $\mathcal{A}$ has an injective resolution iff $\mathcal{A}$ has enough injectives.

Proof. (b) Let $a \in \mathcal{A}$. If $\mathcal{A}$ has enough injectives a embeds into an injective object $a \xrightarrow{\varepsilon} I^0$. We view the quotient $I^0 \overset{q_0}{\twoheadrightarrow} \mathrm{Coker}(\varepsilon)$ as the error of representing a by the injective object I^0. We choose its embedding $\mathrm{Coker}(\varepsilon) \overset{i_0}{\hookrightarrow} I^1$ into an injective object. This gives exact sequence $0 \to a \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I^1$ for $d^0 \overset{\mathrm{def}}{=} [I^0 \xrightarrow{q_0} \mathrm{Coker}(\varepsilon) \xrightarrow{i_0} I^1]$. Now we embed $\mathrm{Coker}(d^0)$ into an injective I^2 etc.

Conversely, if any $a \in \mathcal{A}$ has an injective resolution I then a embeds into I^0. □

5.4.3. *Adjoints of exact functors.*

Lemma. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Its right adjoint preserves injectives and its left adjoint preserves projectives.

Proof. For the right adjoint G and an injective $b \in \mathcal{B}$, functor $\mathrm{Hom}_{\mathcal{A}}[-, Gb] \cong \mathrm{Hom}_{\mathcal{B}}[F-, b]$ is exact as a composition of two exact functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{\mathrm{Hom}_{\mathcal{B}}(-,b)} \mathcal{A}b$. □

Corollary. For a map of rings $\Bbbk \xrightarrow{\phi} l$ functor $\phi_* : \mathfrak{m}(\Bbbk) \to \mathfrak{m}(l)$, $\phi_*(M) = l \otimes_{\Bbbk} M$ preserves projectivity and $\phi_\star : \mathfrak{m}(\Bbbk) \to \mathfrak{m}(l)$, $\phi_\star(M) = \mathrm{Hom}_{\Bbbk}(l, M)$ preserves injectivity.

Proof. These are the two adjoints of the forgetful functor ϕ^* which is exact. □

5.5. **Projectives and injectives in categories of modules $\mathfrak{m}(A)$.** Let A be a ring. We will show that $\mathfrak{m}(A)$ has enough projectives and enough injectives.

5.5.1. *Projectives.* The following lemma shows that the notion of projectivity is a "categorical extension" of the familiar notion of free modules.

Lemma. (a) An A-module P is projective iff P is a summand of a free module.

(b) Module categories have enough projectives.

Proof. (a') *Summands of free modules are projective.* We first consider the A-module ${}_AA = A$. The functor $\mathfrak{m}(A) \xrightarrow{\mathrm{Hom}_A({}_A,-)} \mathcal{A}b$ is canonically isomorphic to the forgetful functor $\mathfrak{m}(A) \xrightarrow{\mathcal{F}} \mathcal{A}b$ which forgets the action of A. The isomorphism $\mathrm{Hom}_A({}_A, M) \underset{\cong}{\to} M$ is the evaluation at 1_A. $\mathcal{F}$ is clearly exact hence the A-module ${}_AA = A$ is projective.

Now any free A-module is projective by lemma 5.4.1.a and its summands are projective by the same reference.

(a") *Projective modules are summands of free modules.* If P is a projective A-module we choose a free module F and a surjection $F \xrightarrow{q} P$. Since P is projective the map from P given by $P \xrightarrow{1_P} P$ lifts to $\sigma : P \to F$. This splits q and we see that P is a summand of F.

(b) Any module M is a quotient of some free module F (say $F = \oplus_{m\in M}\ Am$). $\square$

5.5.2. *Injectives in $\mathcal{A}b$.* We say an abelian group M is *divisible* if for any $n \in \{1, 2, 3, ...\}$ the multiplication $n : M \to M$ with $n \in \{1, 2, 3, ...\}$ is surjective (i.e., for $a \in M$ there is some $\tilde{a} \in M$ such that $a = n{\cdot}\tilde{a}$).

Lemma. An abelian group I is injective iff it is divisible.

The proof will use the following axiom of set theory[5] which is in some form an essential part of "any" strict definition of set theory:

- $\boxed{\text{"\textit{Zorn lemma}".}}$ Let $(I, \leq)$ be a (non-empty) partially ordered set such that any chain J in I (i.e., any totally ordered subset) is dominated by some element of I (i.e., there is some $i \in I$ such that $i \geq j,\ j \in J$). Then I has a maximal element.

Proof. For any $a \in I$ and $n > 0$ we can consider $\frac{1}{n}\mathbb{Z}\supseteq\mathbb{Z} \xrightarrow{\alpha} I$ with $\alpha(1) = a$. If I is injective then α extends to $\widetilde{\alpha} : \frac{1}{n}\mathbb{Z} \to I$ and then $a = n\widetilde{\alpha}(\frac{1}{n})$.

Conversely, if I is divisible we need to show that it injective, i.e., that if $A\supseteq B$ then any map $B \xrightarrow{\beta} I$ extends to A.

Consider the set $\mathcal{E}$ of all pairs (C, γ) with $B\subseteq C\subseteq A$ and $\gamma : C \to I$ an extension of β. It is partially ordered with $(C, \gamma) \leq (C', \gamma')$ if $C\subseteq C'$ and γ' extends γ. We make the following observations

(1) For any totally ordered subset $\mathcal{E}'\subseteq\mathcal{E}$ there is an element $(C, \gamma) \in \mathcal{E}$ which dominates all elements of $\mathcal{E}'$.
  - This is clear: we take $C = \cup_{(C',\gamma')\in\mathcal{E}'}\ C'$ and then γ is obvious.

(2) If $(C, \gamma) \in \mathcal{E}$ and $C \neq A$ then (C, γ) is not maximal:
  - Choose $a \in A$ which is not in C and let $\widetilde{C} = C + \mathbb{Z}{\cdot}a$. Then $C \cap \mathbb{Z}{\cdot}a$ is of the form $\mathbb{Z}{\cdot}na$ for some $n \geq 0$. If $n = 0$ then $\widetilde{C} = C\oplus\mathbb{Z}{\cdot}a$ and one can extend γ to $\widetilde{C}$ by zero on $\mathbb{Z}{\cdot}a$. If $n > 0$ we remember that I is n-divisible, hence $\gamma(na) \in I$ is of the form nx for some $x \in I$.
    Then one can extend $\gamma : C \to I$ to $\widetilde{\gamma} : \widetilde{C} \to I$ by $\widetilde{\gamma}(a) = x$. We first define a map $C\oplus\mathbb{Z}{\cdot}a \xrightarrow{\widehat{\gamma}} I$ by $\widehat{\gamma}(c\oplus\mathbb{Z}) \overset{\text{def}}{=} \gamma(c) + nx$. Then we descend it to the quotient $\widetilde{C} = (C\oplus\mathbb{Z}a)/\mathbb{Z}{\cdot}(na\oplus - na)$ because $\widehat{\gamma})[na\oplus(-na)] = \gamma(na) - nx = 0$.

The second observation would suffice if A/B would be a finitely generated abelian group (we could extend from B to A in finitely many steps). It is less clear if we want to consider arbitrary large groups. However this set theoretic problem is removed by Zorn lemma.

[5] It is equivalent to the "axiom of choice".

Using the first observation Zorn lemma guarantees that a maximal extension (C, γ) exists. Then (2) implies that $C = A$. □

5.5.3. *Pontryagin duality (representation theory and harmonic analysis) [Can be skipped].* The following is just a motivation for our next tool.

Consider the category $\mathcal{A}b_{lc}$ of locally compact abelian topological groups (with Hausdorff topology), Examples include $\mathbb{R}^n$, discrete abelian groups such as $\mathbb{Z}$ and compact abelian groups such as the circle group $\mathbb{T} \overset{\text{def}}{=} \{z \in \mathbb{C}^*;\ |z| = 1\}$. The duality operation on $\mathcal{A}b_{lc}$ is

$$\widehat{A} \overset{\text{def}}{=} Map_{\mathcal{A}b_{lc}}(A, \mathbb{T}).$$

The elements of $\widehat{M}$ are called *characters* of M.

Theorem. [Pontryagin] (a) For A in $\mathcal{A}b_{lc}$ the dual $\widehat{A}$ is naturally a topological group (for the compact open topology) and it again lies in $\mathcal{A}b_{lc}$.

(b) The biduality map $A \to \hat{\hat{A}}$ is an isomorphism.

(c) There is a canonical Fourier transform isomorphism of Hilbert spaces $\mathcal{F}_A : L^2(A, \mu) \underset{\cong}{\to} L^2(\widehat{A}, \widehat{\mu})$.[6]

(d) A is discrete iff $\widehat{A}$ is compact.

Example. One has $\widehat{R} \cong \mathbb{R}$ and $\widehat{\mathbb{T}} = \mathbb{Z}$. The corresponding Fourier transforms from (c) are the standard Fourier transform for functions on $\mathbb{R}$ and the Fourier series expansion of periodic functions on an interval.

This theorem is a basis of representation theory and harmonic analysis.

5.5.4. *Algebraic duality in abelian groups.* For any abelian group M let us denote

$$\widehat{M} = \operatorname{Hom}(M, \mathbb{Q}/\mathbb{Z}).$$

Remark. Here we have simplified the circle group $\mathbb{T}$ which is a natural object in topological groups to its subgroup $\mu = \{z \in \mathbb{C}^*,\ (\exists n > 0)\ z^n = 1\}$ of roots of unity which is a purely algebraic object. (Notice that $exp(2\pi i -) : \mathbb{Q}/\mathbb{Z} \underset{\cong}{\to} \mu$.)

[6] The measure μ is any *Haar measure* on A, i.e., a measure which is invariant under multiplication functions ("translations") $m_a : A \to A$ by $m_a(b) = ab$. The measure $\widehat{\mu}$ is the Haar measure on $\widehat{A}$determined by the μ.

Lemma. (a) The biduality map $\iota_M : M \to \widehat{\widehat{M}}$ is injective.

(b) Category of abelian groups has enough injectives.

Proof. (a) For $m \in M$, $\chi \in \widehat{M}$, $\iota_M(m)\ (\chi) \stackrel{\text{def}}{=} \chi(m)$. So, $\iota_M(m) = 0$ means that m is killed by each character χ of M.

If $m \neq 0$ then the cyclic group $\mathbb{Z}{\cdot}m$ is $\neq 0$ $\mathbb{Z}{\cdot}m$ is $\neq 0$ and we can easily see that it has a nontrivial character $\chi_0 \neq 0$. (If $|m| = \infty$ then any choice of $\chi_0(m) \in \mathbb{Q}/\mathbb{Z}$ extends uniquely to a character χ_0 of $\mathbb{Z}m$. If $|m| = n < \infty$ then χ_0 can be chosen as an isomorphism with the subgroup $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ of $\mathbb{Q}/\mathbb{Z}$.)

Since $\mathbb{Q}/\mathbb{Z}$ is injective we can extend χ_0 to a character χ of M and $\chi(m) \neq 0$.

(b) To M we associate a huge injective abelian group $I_M = \prod_{x\in\widehat{M}} \mathbb{Q}/\mathbb{Z}{\cdot}x = (\mathbb{Q}/\mathbb{Z})^{\widehat{M}}$, its elements are $\widehat{M}$-families $c = (c_\chi)_{\chi\in\widehat{M}}$ of elements of $\mathbb{Q}/\mathbb{Z}$ (we denote such family also as a (possibly infinite) formal sum $\sum_{\chi\in\widehat{M}} c_\chi{\cdot}\chi$). The canonical map of abelian groups

$$M \xrightarrow{\zeta} I_M, \qquad \zeta(m) \stackrel{\text{def}}{=} (\chi(m))_{\chi\in\widehat{M}} = \sum_{\chi\in\widehat{M}} \chi(m){\cdot}\chi, \quad m \in M;$$

is injective by part (a). □

5.5.5. *Injectives in categories of modules.*

5.5.6. *Theorem.* Module categories $\mathfrak{m}(A)$ have enough injectives.

Proof. The problem will be reduced to the case $A = \mathbb{Z}$ via the canonical map of rings $\mathbb{Z} \xrightarrow{\phi} A$. Any A-module M gives a $\mathbb{Z}$-module ϕ^*M which is M considered as an abelian group. By the lemma 5.5.4.b there is an embedding $M \xrightarrow{\iota} J$ into an injective abelian group.

Moreover, by corollary 5.4.3 $\phi_\star I_M$ is an injective A-module. So it suffices to have an embedding $M \hookrightarrow \phi_\star I_M$.

The adjoint pair $(\phi^*, \phi_\star)$ gives a map of A-modules $M \xrightarrow{\zeta} \phi_\star\phi^*M = \mathrm{Hom}_\mathbb{Z}(A, M)$, by $\zeta(m)\ c = cm$, $m \in M$, $c \in A$. It remains to check that both maps in the composition $M \xrightarrow{\zeta} \phi_\star(M) \xrightarrow{\phi_\star(\iota)} \phi_\star(I_M)$ are injective. For ζ it is obvious since $\zeta(m)\ 1_A = m$. For $\phi_\star(\iota)$ we notice that $\phi_\star = \mathrm{Hom}_\mathbb{Z}(A, -)$ is left exact, so it takes inclusion ι to inclusion $\phi_\star(\iota)$.

5.5.7. *Examples.* (1) An injective resolution of the $\mathbb{Z}$-module $\mathbb{Z}$: $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/Z \to 0$.

(2) Injective resolutions are often big, hence more difficult to use in specific calculations then say, the free resolutions. We will need them mostly for the functor $\Gamma(X, -)$ of global sections of sheaves, and the functors $\mathrm{Hom}_\mathcal{A}(a, -)$.

5.6. **Deriving the half exact functors: the basic construction.** Here we recall (repeat) the constructions of LF and RF from 5.1 but we say things in a different order. We first point out that these constructions make sense for any functor F but then we notice that LF is an extension of F only when F is left exact.

5.6.1. *Left and right derived functors RF, first definition.* Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. If $\mathcal{A}$ has enough injectives we can define the *right derived version RF* of F by replacing each object $a \in \mathcal{A}$ by its chosen injective resolution I. So, RF goes from $\mathcal{A}$ to $C(\mathcal{B})$ by

$$RF(a) \overset{\text{def}}{=} F(I) \quad \text{and} \quad \text{we get } (R^nF)a \overset{\text{def}}{=} H^n[(RF)a] \in \mathcal{B} \ \ (n \in \mathbb{Z}).$$

Dually, if $\mathcal{A}$ has enough projectives we can define the *left derived version LF* of F by replacing $a \in \mathcal{A}$ by its chosen projective resolution P. So,

$$LF(a) \overset{\text{def}}{=} F(P) \quad \text{and} \quad \text{we get } (L^nF)a \overset{\text{def}}{=} H^n[(LF)a] \in \mathcal{B} \ \ (n \in \mathbb{Z}).$$

Notice that $R^nF = 0$ for $n < 0$ and $L^nF = 0$ for $n > 0$.

Remark. The complexes $RF(a)$, $LF(a)$ in general *do* depend on the choice of the resolutions I, P (example: $f = id_{\mathcal{A}}$). It will eventually turn out that their cohomology objects $(R^nF)a$ and $(L^nF)a$ *do not* (section **??**).

Lemma. (a) There is a canonical map $F \to R^0F$.

(b) If the functor $F : \mathcal{A} \to \mathcal{B}$ is left exact, this an isomorphism of functors $F \underset{\cong}{\to} R^oF$.

(c) R^0F is always left exact.

The dual claims hold for L^0F.

Proof. (a) Let $0 \to a \xrightarrow{u} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots$ be an injective resolution of M, so the sequence is exact and all I^p are injective. Applying F gives $0 \to Fa \xrightarrow{F(u)} F(I^0) \xrightarrow{F(d^0)} F(I^1) \cdots$.

On the other hand, $RFa = F(I)$ is the complex $\cdots \to 0 \to F(I^0) \xrightarrow{F(d^0)} F(I^1) \xrightarrow{F(d^1)} F(I^1) \to \cdots$, hence $R^0F(a) = H^0[(RF)a] = \mathrm{Ker}[F(d^0)$. Now, $0 = d^o \circ u$ gives $F(d^o) \circ F(u) = F(d^o \circ u) = F(0) = 0$, hence $F(u) : Fa \to F(I^0)$ factors to a map $Fa \to \mathrm{Ker}[F(d^0)]$.

(b) If F is left exact then $0 \to Fa \xrightarrow{F(u)} F(I^0) \xrightarrow{F(d^0)} F(I^1)$ is exact, hence the map $Fa \to \mathrm{Ker}[F(d^0)]$ is an isomorphism. $\square$

5.6.2. *The case when F is contravariant.* A contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is really a functor $F : \mathcal{A}^0 \to \mathcal{B}$ and as such we know how to derive it.

For, instance if F is left exact then $LF(M) = F(\cdots \to P^{-2} \to P^{-1} \xrightarrow{d^{-1}} P^0 \to 0 \to \cdots)$ equals

$$\cdots \to 0 \to F(P^{-0}) \xrightarrow{d^{-1}} F(P^{-1}) \to F(P^{-2}) \to 0 \to \cdots,$$

and we get $H^0[LF(M)] = \operatorname{Ker}[F(P^{-0}) \xrightarrow{d^{-1}} F(P^{-1})]$. However, applying F to the exact sequence $P^{-1} \xrightarrow{d^{-1}} P^0 \to M \to 0$ gives an exact sequence $0 \to F(M) \to F(P^0) \xrightarrow{F(d^{-1})} F(P^{-1})$. So the canonical map $F(M) \to \operatorname{Ker}[F(P^{-0}) \xrightarrow{d^{-1}} F(P^{-1})]$ is an isomorphism.

5.6.3. *The direct and inverse image functors in algebraic Geometry.* We consider a simple map, the inclusion $0 \overset{i}{\in} \mathbb{A}^n_{\Bbbk}$ of a point in an affine space. It gives a restriction of polynomial functions which is a map of rings $\rho : \mathcal{O}(\mathbb{A}^n) \to \mathcal{O}(0)$, i.e., $\Bbbk[x_1, ..., x_n] \to \Bbbk$ by $x_i \mapsto 0$.

This defines the "forgetful" functor

$$\rho^* : \mathfrak{m}(\Bbbk) = \mathfrak{m}[\mathcal{O}(0)] \to \mathfrak{m}[\mathcal{O}(\mathbb{A}^n)]$$

and its adjoints $(\rho_*, \rho^*, \rho_\star)$.

We will also denote these constructions by $(i^*, i_*, i^!)$.[(7)] Then i_* is called the direct image (*push forward*) functor (along the map i) while i^* is the *inverse image* (*pull back*) functor and $i^!$ is the *extraordinary inverse image* functor.

Here, $i^*M = \rho_*M \overset{\text{def}}{=} \mathcal{O}(0) \otimes_{\mathcal{O}(\mathbb{A}^n)} M = M / \sum_i x_i M$ since $\mathcal{O}(0) = \mathcal{O}(\mathbb{A}^n) / \sum_i x_i \mathcal{O}(\mathbb{A}^n)$. Also,

$$i^!M = \rho_\star M \overset{\text{def}}{=} \operatorname{Hom}_{\mathcal{O}(\mathbb{A}^n)}[\mathcal{O}(0), M] = \{m \in M;\ x_i m = 0\}.$$

The functor i^* is exact so there is no problem driving it. Here, i^* is right exact and $i^!$ is left exact. So, one can consider their derived functors Li^* and $Ri^!$. We will eventually see that both are calculated using the Koszul resolution.

Remark. In general for any embedding of algebraic varieties $i : Y \hookrightarrow X$ one has functors $i_*, i^*, i^!$ between modules for the rings of functions $\mathcal{O}(X)$ and $\mathcal{O}(Y)$. As long as X and Y are smooth the computation of derived inverse images reduces to the Koszul resolution.

[7] The double notation appears because we can think the situation in terms of the map in geometry $i : 0 \in \mathbb{A}^n$ or in terms of the map of rings $\rho : \mathcal{O}(\mathbb{A}^n) \to \mathcal{O}(0)$.