HOMOLOGICAL ALGEBRA

Contents

0.0.1. *Notation.* Symbol " □" means "I said this much and I will say no more".

4. **Abelian categories**

These are the categories where roughly one can do linear algebra as in categories of modules $\mathfrak{m}(\Bbbk)$ over a ring.

Examples. We will use the following examples: (0) $\mathcal{S}et$, (1) $\mathfrak{m}(\Bbbk)$, (2) its full subcategory $\mathfrak{m}^{free}(\Bbbk)$ of free $\Bbbk$-modules, and (3) $\mathcal{FA}b$ the category of filtered abelian groups.

In case (3) the objects $(A.A_\bullet)$ consist of an abelian group A and a sequence of subgroups $\cdots\subseteq A_{\le -1}\subseteq A_{\le 0}\subseteq A_{\le 1}\subseteq\cdots$. The morphism $\Phi : (A.A_\bullet) \to (B.B_\bullet)$ are given by maps of abelian groups $\phi : A \to B$ such that $\phi(A_{\le n})\subseteq B_{\le n}$ for any n.

4.1. **Additive categories.** Category $\mathcal{A}$ is additive if

- (A0) For any $a, b \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(a, b)$ has a structure of an abelian group such that the compositions are bilinear.
- (A1) $\mathcal{A}$ has a zero object,
- (A2) $\mathcal{A}$ has sums of two objects and $\mathcal{A}$ has products of two objects,

For two additive categories $\mathcal{A}, \mathcal{B}$ a functor $F : \mathcal{A} \to \mathcal{B}$ is said to be *additive* if the maps $\mathrm{Hom}_{\mathcal{A}}(a', a'') \xrightarrow{F_{a',a''}} \mathrm{Hom}_{\mathcal{B}}(Fa', Fa'')$ are always morphisms of abelian groups.

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Lemma. In an additive category $\mathcal{A}$ the sums $a{\sqcup}b$ are canonically isomorphic to products $a{\times}b$, [We usually use the notation $a{\oplus}b$ for either.]

Proof. We always have canonical maps $a_1{\times}a_2 \xrightarrow{p_i} a_i$ and maps $f : x \to a_1{\times}a_2$ correspond to pairs of maps $f_i = p_i{\circ}f :\ x \to a_i$. Therefore, the maps $a_1 \xrightarrow{1} a_1$ and $a_2 \xrightarrow{0} a_1$ give a map $i_1 : a_1 \to a_1{\times}a_2$ (here 0 denotes the zero element in the group $\mathrm{Hom}(a_2, a_1)$). In the same way we get $i_2 : a_2 \to a_1{\times}a_2$.

Dually, we have standard maps $j_i : a_i \to a_1{\sqcup}a_2$ and we construct $q_i : a_1{\sqcup}a_2 \to a_i$.

Now, we get maps $a_1{\sqcup}a_2 \xrightarrow{\phi} a_1{\times}a_2 \xrightarrow{\psi} a_{\sqcup}a_2$. Here, maps $a_1{\sqcup}a_2 \xrightarrow{q_i} a_k \xrightarrow{i_k} a_1{\times}a_2$ give $\phi = i_1{\circ}q_1 + i_2{\circ}q_2$. Dually, we get $\psi = j_1{\circ}p_1 + j_2{\circ}p_2$. Now it remains to check that ϕ and ψ are mutually inverse. □

Example. (0) $\mathcal{S}et$ fails both (A0) and (A1). (1) $\mathfrak{m}(\Bbbk)$ is additive and so is its full subcategory $\mathfrak{m}^{free}(\Bbbk)$. (3) $\mathcal{FA}b$ is again additive, for instance $(A, A_\bullet){\oplus}(B, \bullet)$ is $A{\oplus}B$ with the filtration $(A{\oplus}B)_{\le n} = A_{\le n}{\oplus}B_{\le n}$.

(4) If $\mathcal{A}$ is additive, so is $\mathcal{A}^o$.

(5) If $\mathcal{A}$ is additive then the category $C(\mathcal{A}$ of complexes in $\mathcal{A}$ is defined (definition $d^i{\circ}d^{i-1} = 0$ uses zero maps so it requires (A0). Moreover, $C(\mathcal{A})$ is again additive (with obvious constructions of the zero object and the sum or product).

4.2. **(Co)kernels and (co)images.** In module categories $\mathfrak{m}(\Bbbk)$, a map $\alpha : M \to N$ has kernel $\mathrm{Ker}(\alpha)$, image $Im(\alpha)$ and cokernel $\mathrm{Coker}(\alpha) = B/Im(\alpha)$. To incorporate these notions into our project of defining abelian categories we will find their categorical formulations.

4.2.1. *(Co)kernels.* For this recall that we have noticed that kernels in $\mathfrak{m}(\Bbbk)$ are related to the purely categorical notion of equalizers: for two maps $\alpha, \beta : M \to N$ we have the equalizer $Eq(\alpha, \beta) = \mathrm{Ker}(\alpha - \beta)$. This means that $\mathrm{Ker}(\alpha) = Eq(\alpha, 0)$ for $0 \in \mathrm{Hom}_{\Bbbk}(M, N)$.

So, we define for a map $\alpha : a \to b$ in any additive category $\mathcal{A}$

$$\mathrm{Ker}(\alpha) \overset{\text{def}}{=} Eq(\alpha, 0) \ \text{ and dually, } \ Coker(\alpha) \overset{\text{def}}{=} Coeq(\alpha, 0).$$

This means that a kernel of the map $a \xrightarrow{\alpha} b$ is a *map* $k \xrightarrow{i} a$ which is the final object in the category of all maps $x \xrightarrow{\zeta} a$ such that the two compositions $\alpha{\circ}\zeta$ and $0{\circ}\zeta$ are equal, i.e., such that $\alpha{\circ}ze = 0$.

Remarks. (0) This agrees with our intuition on kernels from categories of type $\mathfrak{m}(\Bbbk)$. Here, For a map of $\Bbbk$-modules $M \xrightarrow{\alpha} N$

- the kernel $\mathrm{Ker}(\alpha)$ is a subobject of M (our analogue is a canonical map $\mathrm{Ker}(\alpha) \xrightarrow{i} a$),
- the restriction of α to the kernel is zero (the analogue is $\alpha\circ i = 0$),
- the kernel is the largest subobject with this property (analogue: the kernel is the final object in the category of all $x \xrightarrow{\zeta} a$ with $\alpha\circ\zeta = 0$).

(2) Recall that being the "final object in the category of all $x \xrightarrow{\zeta} a$ with $\alpha\circ\zeta = 0$" means that any such ζ factors uniquely through $\mathrm{Ker}(\alpha) \xrightarrow{i} a$, i.e., there is a unique $x \xrightarrow{\overline{\zeta}} \mathrm{Ker}(\alpha)$ such that $\zeta = i\circ\overline{\zeta}$. This factorization property describes the functor $\mathrm{Hom}_{\mathcal{A}}(-, \mathrm{Ker}(\alpha))$, so kernel is unique up to a canonical isomorphism.

Another way to say this is that the kernel of $a \xrightarrow{\alpha} b$ is any object that represents the functor

$$\mathcal{A} \ni x \mapsto {}_{\alpha}\mathrm{Hom}_{\mathcal{A}}(x, a) \stackrel{\mathrm{def}}{=} \{\gamma \in \mathrm{Hom}_{\mathcal{A}}(x, a);\ \alpha\circ\gamma = 0\ \}.$$

(3) We denote the kernel by $\mathrm{Ker}(\alpha)$ but as usual one should remember that this is not one specific object ("determined up to a canonical isomorphism") and it is not just a single object of $\mathcal{A}$ – it comes with a map into a.

4.2.2. *Cokernels.* Now it is natural to define cokernels in additive categories as dual to kernels, i.e., we reverse arrows.

The cokernel of $\alpha : a \to b$ is a map $b \xrightarrow{q} c$ which is the initial object of the category of all maps $b \xrightarrow{\xi} y$ such that $\xi\circ\alpha = 0$.

Remarks. (0) Again, this agrees with the definition in $\mathfrak{m}(\Bbbk)$ where the cokernel of $M \xrightarrow{\alpha} N$ is $N/\alpha(M)$. So, $\mathrm{Coker}(\alpha)$ is a quotient of N, the composition with α kills the map $N \to \mathrm{Coker}(\alpha)$, and the cokernel is the largest among such quotients.

(b) In categorical terms for $\alpha : a \to b$ we are interested in the functor

$$\mathcal{A} \to \mathcal{S}et, \quad \text{by} \quad x \mapsto \mathrm{Hom}_{\mathcal{A}}(b, x)_{\alpha} \stackrel{\mathrm{def}}{=} \{\tau \in \mathrm{Hom}_{\mathcal{A}}(b, x);\ \tau\circ\alpha = 0\ \}.$$

Then the cokernel is any object that *corepresents* this functor, i.e.,

$$\mathrm{Hom}_{\mathcal{A}}(b, -)_{\alpha} \cong \mathrm{Hom}_{\mathcal{A}}[\mathrm{Coker}(\alpha), -].$$

4.2.3. *Images and coimages.* Now we can reverse the order of constructions in $\mathfrak{m}(\Bbbk)$ and define image from (co)kernels:

$$Im(a \xrightarrow{\alpha} b) \stackrel{\mathrm{def}}{=} \mathrm{Ker}[b \to \mathrm{Coker}(\alpha)].$$

Here we have followed the the situation in $\mathfrak{m}(\Bbbk)$ where $Im(\xrightarrow{\alpha} N)$ is the kernel of $N \to \mathrm{Coker}(\alpha) \stackrel{\mathrm{def}}{=} N/Im(\alpha)$.

In a general categorical setting it is natural to also define the notion dual to images. The coimage is

$$Coim(a \xrightarrow{\alpha} b) \ \stackrel{\text{def}}{=} \ \text{Coker}[\text{Ker}(\alpha) \to a].$$

Now notice that back in $\mathfrak{m}(\Bbbk)$ the coimage is $M/\text{Ker}(\alpha)$, so there is a canonical isomorphism $Coim(\alpha) \underset{\cong}{\to} Im(\alpha)$. In general categories there is a weaker result:

4.2.4. *Lemma.* If in an additive category $\mathcal{A}$ a map $a \xrightarrow{\alpha} b$ has image and coimage, there is a canonical map $Coim(\alpha) \xrightarrow{\overline{\alpha}} Im(\alpha)$, and it appears in a canonical factorization of α into a composition

$$\begin{array}{ccc} a & \xrightarrow{\alpha} & b \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle i} \\ Coim(\alpha) & \xrightarrow{\overline{\alpha}} & Im(\alpha) \end{array} .$$

Proof. It is easy to construct a canonical map $\overline{\alpha}$. First, since $\alpha \circ (\text{Ker}(\alpha) \to a)$ is zero, we know that α factors uniquely to $\alpha_1 : Coim(\alpha) \to b$. Next $(b \to \text{Coker}(\alpha) \circ \alpha = 0$ implies that $(b \to \text{Coker}(\alpha) \circ \alpha_1 = 0$, and therefore α_1 factors uniquely through q to a map $\alpha_2 : Coim(\alpha) \to \text{Im}(\alpha)$.

This is our $\overline{\alpha}$. However, there is a slight complication since we get another canonical map by factoring α first as $\alpha^1 : a \to Im(\alpha)$ and then α^1 to $\alpha^2 : Coim(\alpha) \to Im(\alpha)$. Now one checks that $\alpha_2 = \alpha^2$ so there is no confusion. □

4.3. **Abelian categories.** Category $\mathcal{A}$ is abelian if it is additive and

- (A3) It has kernels and cokernels (hence in particular it has images and coimages!),
- (A4) The canonical maps $Coim(\phi) \to Im(\phi)$ are isomorphisms.

4.3.1. *Examples.* (1) $\mathfrak{m}(\Bbbk)$ is abelian: the categorical notions of a (co)kernel and image have the usual meaning, and coimages coincide with images.

(2) In $\mathfrak{m}^{free}(\Bbbk)$ kernels and cokernels need not exist (for instance for $\Bbbk = \mathbb{Z}$ the cokernel of $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ is not free).

(3) In $\mathcal{F}\mathcal{A}b$ notice that a filtration $A_\bullet$ on A induces filtrations on each subgroup $A' \subseteq A$ and each quotient A/A' by $A'_{\leq n} \stackrel{\text{def}}{=} A' \cap A_{\leq n}$ and $(A/A)'_{\leq n} \stackrel{\text{def}}{=} Im[A_{\leq n} \to A/A'] \cong A_{\leq n}/A_{\leq n} \cap A' = A_{\leq n}/A'_{\leq n}$.

Now, for $\Phi \in \text{Hom}_{\mathcal{F}\mathcal{A}b}[(A, A_\bullet), (B, B_\bullet)]$ given by $\phi : A \to B$, we have obvious candidates for $\text{Ker}(\Phi)$ and $\text{Coker}(\phi)$ and these are $\text{Ker}(\phi) \subseteq A$ with the filtration induced from A and $\text{Coker}(\phi)$ a quotient of B with the filtration induced from B. These are really the categorical (co)kernels, so (A3) is satisfied.

For (A4) we observe that the canonical map $Coim(\Phi) \to Im(\Phi)$ in $\mathcal{FA}b$ is on the level of abelian groups the map $Coim(\phi) \to Im(\phi)$ in $\mathcal{A}b$ so it is an isomorphism. It remains to compare the filtrations.

For this we will look at the case when $\phi = 1_A$ is a map of filtered abelian groups $\Phi : (A, A'_\bullet) \to (A, A''_\bullet)$ for two filtrations. The condition $\phi(A'_{\leq n}) \subseteq A''_{\leq n})$ means that $A'_{\leq n} \subseteq A''_{\leq n}$.

Since $\mathrm{Ker}(\Phi) = 0$ we have $Coim(\Phi) = A/\mathrm{Ker}(\phi) = A$ with the quotient filtration induced from $(A, A'_\bullet)$, so this just A with filtration $A'_\bullet$. Similarly, since $\mathrm{Coker}(\Phi) = 0$ we have $Im(\Phi) = A$ with the subobject filtration induced from $(A, A''_\bullet)$, so this just A with filtration $A''_\bullet$.

So, in this case the map $Coim \to Im$ is the same as the original map Φ. It is an isomorphism on the level of abelian groups but the filtrations on $Coim$ and Im need not coincide. For instance we could have $A'_n = 0$ for $n \leq 0$ and $A'_n = A$ otherwise, while $A''_n = 0$ for $n < 0$ and $A''_n = A$ otherwise, Then $A'_n \subseteq A''_n$ but this inclusion is not equality.

So, $\mathcal{FA}b$ is not abelian.

(3) If $\mathcal{A}$ is abelian so is $\mathcal{A}^o$.

(4) If $\mathcal{A}$ is abelian so is $\mathcal{C}^\bullet(\mathcal{A})$.

(5) If $\Bbbk$ is noetherian then the full subcategory of finitely generated modules $\mathfrak{m}_{fg}(\Bbbk) \subseteq \mathfrak{m}(\Bbbk)$ is also an abelian category.

4.4. **Exact sequences in abelian categories.** Once we have the notion of kernel and cokernel (hence also of image), we can carry over from module categories $\mathfrak{m}(\Bbbk)$ to general abelian categories our homological train of thought. For instance we say that

- a map $i : a \to b$ makes a into a subobject of b if $\mathrm{Ker}(i) = 0$ (we denote it $a \hookrightarrow b$ or even informally by $a \subseteq b$, one also says that i is a monomorphism or informally that it is an inclusion),
- a map $q : b \to c$ makes c into a quotient of b if $\mathrm{Coker}(q) = 0$ (we denote it $b \twoheadrightarrow c$ and say that q is an epimorphism or informally that q is surjective),
- the quotient of b by a subobject $a \xrightarrow{i} b$ is $b/a \overset{\text{def}}{=} \mathrm{Coker}(i)$,
- a complex in $\mathcal{A}$ is a sequence of maps $\cdots A^n \xrightarrow{d^n} A^{n+1} \to \cdots$ such that $d^{n+1} \circ d^n = 0$, its cocycles, coboundaries and cohomologies are defined by $B^n = Im(d^n)$ is a subobject of $Z^n = \mathrm{Ker}(d^n)$ and $H^n = Z^n/B^n$;
- sequence of maps $a \xrightarrow{\mu} b \xrightarrow{\nu} c$ is exact (at b) if $\nu \circ \mu = 0$ and the canonical map $Im(\mu) \to \mathrm{Ker}(\nu)$ is an isomorphism.

Now with all these definitions we are in a familiar world, i.e., they work as we expect. For instance, sequence $0 \to a' \xrightarrow{\alpha} a \xrightarrow{\beta} a'' \to 0$ is exact iff a' is a subobject of a and a'' is the quotient of a by a', and if this is true then

$$\mathrm{Ker}(\alpha) = 0, \mathrm{Ker}(\beta) = a', \mathrm{Coker}(\alpha) = a'', \mathrm{Coker}(\beta) = 0, \mathrm{Im}(\alpha) = a', \mathrm{Im}(\beta) = a''.$$

4.5. **Abelian categories and abelian groups.** The difference between general abelian categories and module categories is that while in a module category $\mathfrak{m}(\mathbb{k})$ our arguments often use the fact that $\mathbb{k}$-modules are after all abelian groups and sets (so we can think in terms of their elements), the reasoning valid in any abelian category "has to be done" in terms of morphisms in a category since objects need not be sets any more.

Here, the main tool are the factorization properties of (co)kernels and (co)images, i.e., one constructs new maps from known maps by composing and by factoring maps through intermediate objects.

However, it turns out that the set theoretic proofs suffice because

Theorem. [Mitchell] Any abelian category is equivalent to a full subcategory of some category of modules $\mathfrak{m}(\mathbb{k})$.