HOMOLOGICAL ALGEBRA

Date: ?

Contents

0.0.1. *Notation.* Symbol $\square$ meas "I said this much and I will say no more".

3. **Categories**

We will use the language of categories seriously on several levels. Some examples:

- *Abelian categories.* This is a basic setting for homological algebra. It explains why we can calculate with sheaves "the same" as with abelian groups.
- *Triangulated categories.* This is the "optimal" setting for homological algebra needed for more subtle calculations and constructions.
- *Study of sheaves.* Categories appear from the beginning since we are interested in sheaves with values in a certain category. The notion of a stalk of a sheaf, i.e., a restriction of a sheaf is an instance of a notion of a limit in a category.

3.0.2. *Why categories?* Set theory studies groups objects. category theory studies groups of *related* objects. The interpretation of "related" in category theory is that it makes sense to go from one such object to another via something (a "morphism"). Since this is indeed what we usually do, the *language* of categories is convenient.

3.0.3. *From sets to categories.* For a set A the basic question is "what are its elements?". For a category $\mathcal{C}$ the basic question is "how are its elements related?".

So, the change of point of view from set theory to categories is that "what is this object?" is replaced by "how does this object relate or interact with others?".[(1)]

[1] The latter point of view is standard in physics where one does not hope to know what elementary particles are but all experiments are done to see how particles interact.

Remark. Historically, language of sets was used in mathematics to describe any mathematical object precisely – as a system of sets. This was very useful but it also sometimes made mathematics cumbersome as objects of natural interest may have lengthy descriptions.[2]

3.0.4. *Upgrading concepts from set theory to category theory.* As the categorical way of thinking is more subtle a single notion in sets may have several upgrades. Some examples:

- empty set $\mapsto$ initial object;
- union of sets $\mapsto$ sum of objects, more generally a direct (inductive) limit of objects;
- product of sets, $\mapsto$ product of objects or more generally "projective limit of objects";
- abelian groups $\mapsto$ additive category, abelian category, triangulated categories;
- maps between sets $\mapsto$ functors between categories;
- subsets $\mapsto$ subcategories, full subcategories;
- bijections $\mapsto$ equivalences of categories.

The process of constructing a (useful) categorical upgrade is called **categorification** (this is now by itself an exciting branch of mathematics).

Remark. This enriched language of categories has been recognized as fundamental for describing various complicated phenomena in mathematics. In particular, the study of *special classes of categories* mushroomed somewhat similarly as the study of special classes of functions in analysis (continuous, smooth, analytic, 3.17 times differentiable, p-integrable etc.).

3.1. **Categories.** To define a category one use the notion of a *class*. There is a number of versions of foundational aspects of mathematics and the precise meaning of "class" depends on the choice you make. We will use a flexible and imprecise idea that a *class* is any "collection of mathematical objects that is well defined by certain property.[3]

A *category* $\mathcal{C}$ consists of

(1) a class $Ob(\mathcal{C})$, its elements are called *objects* of $\mathcal{C}$,
(2) for any $a, b \in Ob(\mathcal{C})$ a class $\mathrm{Hom}_{\mathcal{C}}(a, b)$ whose elements are called *morphisms* ("maps") from a to b in $\mathcal{C}$,
(3) for any $a, b, c \in Ob(\mathcal{C})$ a function $\mathrm{Hom}_{\mathcal{C}}(b, c) \times \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}}(a, c)$, called *composition*,
(4) for any $a \in Ob(\mathcal{C})$ an element $1_a \in \mathrm{Hom}_{\mathcal{C}}(a, a)$,

such that

[2] Physicists excel at finding colorful replacements for "cumbersome" in this sentence.

[3] One role for foundations of mathematics is to pinpoint a class of constructions which is large enough for your purposes and small enough that it does not allow contradictions that come from lumping together objects of all sizes (say, "set of all sets"). However, such concerns rarely come up in practice.

- the composition is associative and
- 1_a is a neutral element for composition.

Instead of $a \in \mathrm{Ob}(\mathcal{C})$ we will usually just say that $a \in \mathcal{C}$.

Remarks. A category is said to be *small* if the class $Ob(\mathcal{C})$ is a set. One often considers categories such that all classes $\mathrm{Hom}_{\mathcal{C}}(a, b)$ are sets.(4)

3.1.1. *Examples.*

(1) Any type of structure S on sets defines a category $\mathcal{S}$: its objects are sets that carry such structure and morphisms are maps of sets that are compatible with this structure. This is how category $\mathcal{S}et$ of sets leads to the categories $\mathcal{A}b$ of abelian groups, $\mathfrak{m}(\Bbbk)$ of modules for a ring $\Bbbk$ (denoted also $\mathcal{V}ec(\Bbbk)$ if $\Bbbk$ is a field), $\mathcal{G}roup$ of groups, $\mathcal{R}ing$ of rings, $\mathcal{T}op$ of topological spaces etc.

(2) There are also purely categorical constructions of new categories from old ones. For instance, to a category $\mathcal{C}$ one attaches the opposite category $\mathcal{C}^o$ so that objects are the same but the "direction of arrows reverses":
$$\mathrm{Hom}_{\mathcal{C}^o}(a, b) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(b, a).$$

(3) Some familiar structures in a set A make A into a category. For instance a partial order $\leq$ on a set I defines a category with $Ob = I$ while $\mathrm{Hom}(a, b)$ is a point if $a \leq b$ (call this point (a, b)), and $\emptyset$ otherwise. Similarly for an equivalence relation on a set I.

(4) If $\mathcal{C}$ is a category, any topological space X defines the category $Sh(X, \mathcal{C})$ of sheaves on X with values in the category $\mathcal{C}$.

(5) A *subcategory* of a category $\mathcal{C}$ is a category $\mathcal{C}'$ such that $Ob(\mathcal{C}')\subseteq\mathrm{Ob}(\mathcal{C})$, for any $a, b \in \mathbb{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(a, b)$ is a subset of $\mathrm{Hom}_{\mathcal{S}}(a, b)$ and the composition and units are as in $\mathcal{C}$.

3.1.2. *Notation.* One denotes $f \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ graphically by $f : a \to b$ or $a \xrightarrow{f} b$. One calls f a morphism or just a *map* in $\mathcal{C}$. We write $f : a \underset{\cong}{\to} b$ if f has an inverse. One sometimes denotes $\mathrm{Hom}_{\mathcal{C}}(a, b)$ by $\mathcal{C}(a, b)$

3.2. **Functors.** A functor F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of

(1) for each object $a \in \mathcal{A}$ an object $F(a) \in \mathcal{B}$;
(2) for each morphism $\alpha \in \mathrm{Hom}_{\mathcal{A}}(a', a'')$ in $\mathcal{A}$ a morphism $F(\alpha) \in \mathrm{Hom}_{\mathcal{B}}(Fa', Fa'')$ in $\mathcal{B}$;

such that

- (i) F preserves compositions and units, i.e., $F(\beta\circ\alpha) = F(\beta)\circ F(\alpha)$ and

4 This version does not suffice for the lemma 3.6.2 below.

- (ii) $F(1_a) = 1_{Fa}$.

Remark. A *functor* means a *natural construction*, i.e., a way of constructing from each object $A \in \mathcal{A}$ some object $F(a) \in \mathcal{B}$, which is sufficiently natural so that it extends to "relations between objects", i.e., to morphisms in $\mathcal{A}$.

Examples. (0) For any category $\mathcal{A}$ there is the identity functor $1_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. Two functors $\mathcal{A} \xrightarrow{F} \mathcal{B}$ and $\mathcal{B} \xrightarrow{G} \mathcal{C}$ can be composed to a functor $\mathcal{A} \xrightarrow{G \circ F} \mathcal{C}$.

(1) An object $a \in \mathcal{A}$ defines two functors,

$$\mathrm{Hom}_{\mathcal{A}}(a, -) : \ \mathcal{A} \to \mathcal{S}et \quad \text{and} \quad \mathrm{Hom}_{\mathcal{A}}(-, a) : \ \mathcal{A}^o \to \mathcal{S}et.$$

Moreover, $\mathrm{Hom}_{\mathcal{A}}(-,-)$ is a functor from $\mathcal{A}^o \times \mathcal{A}$ to sets![(5)]

(2) For a ring $\Bbbk$, tensoring is a functor $- \otimes_{\Bbbk} - : \ \mathfrak{m}^r(\Bbbk) \times \mathfrak{m}^l(\Bbbk) \to \mathcal{A}b$.

(3) As we see in these examples, when the extension of the functor from objects to morphisms is obvious one often omits the details. (However, this extension is sometimes highly nontrivial.)

3.2.1. *The direct and inverse image of modules.* Any map of rings $\Bbbk \xrightarrow{\phi} l$ gives two functors between their categories of modules

- the *pull-back* (or *inverse image*) functor $\phi^* : \mathfrak{m}(l) \to \mathfrak{m}(\Bbbk)$, where $\phi^* N$ is the same as N as a set or an abelian group, but now it is considered as module for $\Bbbk$ via ϕ.
- the *push-forward* (or *direct image*) functor $\phi_* : \ \mathfrak{m}(\Bbbk) \to \mathfrak{m}(l)$ where $\phi_* M \overset{\text{def}}{=} l \otimes_{\Bbbk} M$. (This is also called *change of coefficients* from $\Bbbk$ to l).

Proof. To see that these are functors, we need to define them also on maps. So, a map $\beta : N' \to N''$ in $\mathfrak{m}(l)$ gives a map $\phi^*(\beta) : \phi^*(N') \to \phi^*(N'')$ in $\mathfrak{m}(\Bbbk)$ which as a function between sets is really just $\beta : N' \to N''$. On the other hand, $\alpha : M' \to M''$ in $\mathfrak{m}(\Bbbk)$ gives $\phi_*(\alpha) : \phi_*(M') \to \phi_*(M'')$ in $\mathfrak{m}(l)$, this is just the map $1_l \otimes \alpha : \ l \otimes_{\Bbbk} M' \to l \otimes_{\Bbbk} M''$, $c \otimes x \mapsto c \otimes \alpha(x)$. We will skip checking the property required for a functor. $\square$

Remark. Here we see a general feature of the subject that

functors often come in pairs ("adjoint pairs of functors"*, see 3.7) and often one of them is "stupid" and the other one an "interesting" construction.*

[5] Exercise. Define the product $\mathcal{A} \times \mathcal{B}$ of categories $\mathcal{A}$ and $\mathcal{B}$.

3.2.2. *Contravariant functors.* This is just a terminology meaning "going the wrong way". We say that a *contravariant functor* F from $\mathcal{A}$ to $\mathcal{B}$ is given by assigning to any $a \in \mathcal{A}$ some $F(a) \in \mathcal{B}$, and for each map $\alpha \in \mathrm{Hom}_{\mathcal{A}}(a', a'')$ in $\mathcal{A}$ a map $F(\alpha) \in \mathrm{Hom}_{\mathcal{B}}(Fa'', Fa')$, such that $F(\beta \circ \alpha) = F(\alpha) \circ F(\beta)$ and $F(1_a) = 1_{Fa}$.

This is not really a new notion since a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is the same as a functor F from $\mathcal{A}$ to $\mathcal{B}^o$ (or a functor F from $\mathcal{A}^o$ to $\mathcal{B}$).

3.3. **Constructions of objects in a category: limits.** We start with specific constructions such as initial object in a category or product of objects in a category. This is then generalized to the notion of limits in categories which is the general framework for combining objects into a new object. We find the formula for limits in category of sets and its generalization to arbitrary categories.

3.3.1. *Some special objects and maps.* We say that $i \in \mathcal{C}$ is an *initial* object if for any $a \in \mathcal{C}$ set $\mathrm{Hom}_{\mathcal{C}}(i, a)$ has precisely one element "it is a point"). Also, $t \in \mathcal{C}$ is a *terminal* or *final* object if for any $a \in \mathcal{C}$ set $\mathrm{Hom}_{\mathcal{C}}(a, t)$ is a point. We say that $z \in \mathcal{C}$ is a *zero* object if it is both initial and terminal.

A map $\phi \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ is said to be an *isomorphism* if it is invertible, i.e., if there is a $\psi \in \mathrm{Hom}_{\mathcal{C}}(b, a)$ such that ...

Examples. (i) In $\mathcal{S}et$ empty set is the only initial object while terminal objects are precisely the one-point sets, and so there are no zero objects.

(ii) In $\mathcal{A}b$ initial, terminal and zero objects coincide – these are the zero groups 0 (groups with one element).

(iii) In $\mathcal{R}ing$, $\mathbb{Z}$ is the initial object and the terminal object is the one element ring $0 = \{0\}$. If the subcategory $\mathcal{R}ing'$ where we require that $1_R \neq 0_R$ there is no terminal object.

Lemma. Initial, final or zero object in $\mathcal{C}$ (if it exists) is unique up to a canonical isomorphism. This means that for any two initial objects i, j in $\mathcal{C}$ we have a canonical isomorphism $\alpha : i \underset{\cong}{\rightarrow} j$. (The same for terminal and zero objects.)

Proof. For two final objects f_1, f_2 in $\mathcal{C}$ set $\mathrm{Hom}_{\mathcal{C}}(f_i, f_j)$ has one elementα_{ji}. This uniqueness implies that $\alpha_{ii} = 1_{f_i}$ and then that $\alpha_{ij} \circ \alpha_{ji}$ is 1_{f_i}. So, $\alpha_{ji} : i \to j$ is an isomorphism and it is canonical (no choices). $\square$

3.3.2. *Which notion of "equality" is useful in categories?* This (trivial) "philosophical" remark explains much about how one navigates in categories.

Two elements a, b of a set A are either equal or not. If a, b are two objects in a category $\mathcal{C}$ the situation is richer, they can be :

- (i) the same $a = b$,
- (ii) isomorphic $a \cong b$ (meaning that there exists and isomorphism from a to b),
- (iii) isomorphic by a canonical (given) isomorphism.

It turns out that (i) is too restrictive, (ii) is too lax and (iii) is the most useful – the correct analogue of equality of elements of a set. In practice this means that we will often be imprecise and use a <u>shorthand</u> "$a = b$" while we really mean that "*I have in mind a specific isomorphism* $\phi : a \underset{\cong}{\longrightarrow} b$".

Example. With (i) the problem is that in practice it "never happens", for instance in the category $\mathcal{V}ec_{\Bbbk}^{fd}$ of finite dimensional vector spaces over a field $\Bbbk$ the double dual V^{**} is not literally the same vector space as V but it is natural to identify them. With (ii) the problem is that if a and b are isomorphic but I have not made a choice of an isomorphism $\alpha : a \underset{\cong}{\longrightarrow} b$ then I can not replace a with b in computations where it would be convenient. The convention (iii) provides (according to the lemma) a shorthand "terminal object in a category $\mathcal{C}$ is unique" for the precise statement that "any two terminal objects in $\mathcal{C}$ are canonically isomorphic".[(6)]

3.3.3. *Products of objects.* A *product* of objects a and b in $\mathcal{C}$ is a triple (Π, p, q) where $\Pi \in \mathcal{C}$ is an object while $p \in \mathrm{Hom}_{\mathcal{C}}(\Pi, a)$, $q \in \mathrm{Hom}_{\mathcal{C}}(\Pi, b)$ are maps such that for any $x \in \mathcal{C}$ the function

$$\mathrm{Hom}_{\mathcal{C}}(x, \Pi) \ni \phi \mapsto (p \circ \phi, q \circ \phi) \in \mathrm{Hom}_{\mathcal{C}}(x, a) \times \mathrm{Hom}_{\mathcal{C}}(x, b)$$

is a bijection. In shorthand, a map into Π is "the same" as a pair of maps into a and into b (i.e., there is a canonical bijection between these two kinds of data).

Remarks. (0) Clearly, the above categorical notion of a product is just the abstract formulation of properties of the product of sets.

(1) From our experience we expect that a product of a and b should be a specific object built from a and b. However, this is not what the categorical definition above says. For given a, b there may be many triples (Π, p, q) satisfying the product property. However it is easy to see that any two such (Π_i, p_i, q_i), $i = 1, 2$; are related by a *canonical isomorphism* $\phi : \Pi_1 \underset{\cong}{\longrightarrow} \Phi_2$ provided by the the defining property of the product. This is another example of 3.3.2.

(2) In some categories the categorical notion of the product has a canonical realization. For instance in $\mathcal{S}et$ we have the *usual* notion of the product $A_1 \times A_2$ of two sets (the set of pairs (a, b) with $a_i \in A_i$), is a realization of the *categorical* notion of the product.[(7)]

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[6] Another poetic wording is that the terminal object is "a well defined object with possibly many realizations".

[7] Set $A_1 \times A_2$ comes with two projections $pr_i : A_1 \times A_2 \to A_i$ and $(A_1 \times A_2, pr_1, pr_2)$ is a categorical product since we know a map into $A_1 \times A_2$ is the same as a pair of maps into A_1 and A_2 !

(3) The advantage of the categorical notion of the product is that it is *more flexible* – it works uniformly in many settings where the set theoretic construction as a set of pairs does not make sense.

Example. A product of two objects a and b in a given category $\mathcal{C}$ need not exist! (For an example show that in a poset the product means supremum!)

3.3.4. *Sums.* A *sum* of objects a and b in $\mathcal{C}$ is a triple (Σ, i, j) where $\Sigma \in \mathcal{C}$ is an object while $i \in \mathrm{Hom}_{\mathcal{C}}(a, \Sigma),\ j \in \mathrm{Hom}_{\mathcal{C}}(b, \Sigma)$ are maps such that for any $x \in \mathcal{C}$ the function

$$\mathrm{Hom}_{\mathcal{C}}(\Sigma, x) \ni \phi \mapsto (\phi \circ i, \phi \circ j) \in \mathrm{Hom}_{\mathcal{C}}(a, x) \times \mathrm{Hom}_{\mathcal{C}}(b, x)$$

is a bijection.

Example. In $\mathcal{S}et$ the sums exist and the sum of a and b is the disjoint union $a \sqcup b$.

3.3.5. *Sums and products of families of objects.* This is the same as for two objects. A *product in* $\mathcal{C}$ of a family of objects $a_i \in \mathcal{C},\ i \in I$, is a pair $(P, (p_i)_{i \in I}))$ where $P \in \mathcal{C}$ and $p_i : P \to a_i$ are such that the map

$$\mathrm{Hom}_{\mathcal{C}}(x, P) \ni \phi \mapsto (p_i \circ \phi)_{i \in I} \in \Pi_{i \in I} \mathrm{Hom}_{\mathcal{C}}(x, a_i)$$

is a bijection.

A sum of $a_i \in \mathcal{C},\ i \in I$ is a pair $(S, (j_i)_{i \in I}))$ where $j_i : a_i \to S$ gives a bijection

$$\mathrm{Hom}_{\mathcal{C}}(S, x) \ni \phi \mapsto (\phi \circ j_i)_{i \in I} \in \Pi_{i \in I} \mathrm{Hom}_{\mathcal{C}}(a_i, x).$$

Remark. The notation for products is $\prod_{i \in I} a_i$. and for sums $\sqcup_{i \in I} a_i$ or $\oplus_{i \in I} a_i$. Sums are are also called *coproducts* (here "co" indicates that this is the dual notion to products, i.e., a coproduct in $\mathcal{C}$ is a product in $\mathcal{C}^o$).

Example. In the category $\mathcal{S}et$ the categorical products are just the usual products, i.e., $\prod_{i \in I} M_i$ consists of all families $m = (m_i)_{i \in I}$ with $m_i \in M_i,\ i \in I$. The sums are the disjoint unions.

Lemma. For a ring $\Bbbk$ the category $\mathfrak{m}(\Bbbk)$ has sums and products.

(1) As a set, the product $\prod_{i \in I} M_i$ of modules is the same as the product in sets.[8]
(2) A sum of modules $\oplus_{i \in I} M_i$ is the submodule of $\prod_{i \in I} M_i$ consisting of all "finite" families $m = (m_i)_{i \in I}$ (i.e., $m_i = 0$ for all but finitely many $i \in I$).[9]

Remarks. This is how we get familiar with categorical constructions: by checking what they mean in familiar categories.

[8] However, now a family $m = (m_i)_{i \in I} \in \prod_{i \in I} M_i$ is often written as a (possibly infinite) sum $\sum_{i \in I} m_i \stackrel{\text{def}}{=} (m_i)_{i \in I}$).

[9] So, sums in modules and in sets are not compatible!

3.4. **Limits in categories.** Categorical thinking allows us to extend the idea of *limits* from analysis to many other settings.[10]

Example. In some instances it is clear what one should mean by a limit of a family of objects. Consider a sequence of *increasing subsets* $A_0 \subseteq A_1 \subseteq \cdots$ of a set A, we will say that its limit $\lim\limits_{\rightarrow} A_i$ is the subset $\cup_{i\geq 0} A_i$ of A. Similarly, the limit of a *decreasing sequence* of subsets $B_0 \supseteq B_1 \subseteq \cdots$ of A, will be the subset $\lim\limits_{\leftarrow} B_i \stackrel{\text{def}}{=} \cap_{i\geq 0} B_i$ of A.

In general there will be two kinds of limits in categories, extending these two examples.

3.4.1. *Inductive limits.* An *inductive system* of objects of $\mathcal{C}$ over a poset[11] $(I, \leq)$, consists of

(1) a family of objects $a_i \in \mathcal{C},\ i \in I$; and a
(2) system of maps $\phi_{ji} : a_i \to a_j$ for all $i \leq j$ in I;

such that

- $\phi_{ii} = 1_{a_i},\ i \in I$ and
- $\phi_{kj} \circ \phi_{ji} = \phi_{ki}$ when $i \leq j \leq k$.

A short way to say this is that an inductive system a (consisting of a_i's and α_{ji}'s) is a functor $a : (I, \leq) \to \mathcal{C}$ defined on a poset.

Remark. A "candidate" for the limit of a_i as i becomes large in $(I, \leq)$ should be be an object $L \in \mathcal{C}$ that should lie "beyond all a_i's" and be "compatible with the transition maps $\alpha_{ji} : a_{\rightarrow} a_j$ in the system". The meaning of the first idea is that L should be related to a_i's by some maps $\lambda_i : a_i \to L$ and of the second requirement that for $i \leq j$ we have $\lambda_j \circ \alpha_{ji} = \lambda_i$.

Let us extend the poset I to a poset $I \sqcup \infty$ where $i \leq \infty$ for $i \in I$. Then a "candidate" is exactly the data for extending the functor $a : (I, \leq) \to \mathcal{C}$ to a functor $\widehat{a} : (I \sqcup \infty, \leq) \to \mathcal{C}$ (the extension is by $\widehat{a}(\infty) = L$. □Now we can state

An *limit* of an inductive system a is an extension (L, λ) of a to $I \sqcup \infty$ (i.e., $L \in \mathcal{C}$ and $\lambda_i : a_i \to L,\ i \in I$, with $\lambda_j \circ \alpha_{ji} = \lambda_i$ for $i \leq j$), such that (L, λ) is the *universal* extension in the sense that

for any extension (B, β) (here $\beta_i : a_i \to B$), there is a unique map $f : L \to B$ such that $\beta_i = f \circ \lambda_i,\ i \in I$.

[10] This is one of the two most useful ideas in category theory. The other one is the notion of *adjoint functors*, see 3.7.

The derivatives of functions extend to a notion of derivatives of functors but we will not cover that.

[11] Meaning a *partially ordered set.*

Informally, i.e., in shorthand, we write $\lim_{\to\, I,\leq} a_i = L$ (while we remember maps λ_i).

Remark. Notice that all extensions (L, λ) of a from I to $I \sqcup \infty$ naturally form a category $\mathcal{E}$ where morphisms $(L', \lambda') \to (L'', \lambda'')$ in $\mathcal{E}$ are simply morphisms $L' \xrightarrow{f} L''$ in $\mathcal{C}$ which are compatible with structure maps λ' and λ'', i.e.,

$$\begin{array}{ccc} a'_p & \xrightarrow{\mu_p} & a''_{\iota(p)} \\ {\scriptstyle \alpha'_{qp}}\big\downarrow & & {\scriptstyle \alpha''_{qp}}\big\downarrow \\ a'_q & \xrightarrow{\mu_q} & a''_{\iota(q)} \end{array} .$$

So, any inductive system a defines a category $\mathcal{E}$ and a limit of a is the same as an initial object in $\mathcal{E}$.

Corollary. The limit object $\lim_{\to\, I,\leq} a_i$ in $\mathcal{C}$ is well defined up to a canonical isomorphism.

Proof. First, any two limits $(L^k, \lambda^k), k = 1, 2$, are initial objects in $\mathcal{E}$ so there is a canonical isomorphism $\zeta : (L^1, \lambda^1) \underset{\cong}{\to} (L^2, \lambda^2)$ in $\mathcal{E}$. However, a morphism $\phi : (L^1, \lambda^1) \to (L^2, \lambda^2)$ in $\mathcal{E}$ is the same as a morphisms $\phi : L^1 \to L^2$ in $\mathcal{C}$ which intertwines λ^2 and λ^2. So, ζ is in particular a (canonical) isomorphism $\zeta : L^1 \underset{\cong}{\to} L^2$ in $\mathcal{C}$. $\square$

3.4.2. *Projective limits. Projective systems* in $\mathcal{C}$ can be defined as inductive systems in $\mathcal{C}^o$. So, a *projective system* of objects of $\mathcal{C}$, over a poset $(I, \leq)$, consists of

(1) a family of objects $a_i \in \mathcal{C},\ i \in I$; and
(2) for all $i \leq j$ in I a map $\alpha_{ji} : a_j \to a_i$

such that $\alpha_{ii} = 1_{a_i}$ and $\alpha_{ji} \circ \alpha_{kj} = \alpha_{ki}$ when $i \leq j \leq k$.

Its limit is a pair $(L, \lambda$ of $L \in \mathcal{C}$ and maps $\lambda_i : L \to a_i$ such that

- $\alpha_{ji} \circ \lambda_j = \lambda_i$ for $i \leq j$, is universal in the sense that for any (C, γ) of the same form there is a unique map $f : C \to L$ such that $\beta_i = \lambda_i \circ f,\ i \in I$.

This means that the limit of a projective system is the final object of the category of extensions of $a : (I, \leq) \to \mathcal{C}$ to $I \sqcup \infty$. Again, informally, we write $\lim_{\leftarrow\, I,\leq} a_i = L$.

3.4.3. *Limits are functorial.* Let $\mathcal{IS}(\mathcal{C})$ be the category of inductive systems in $\mathcal{C}$. Objects are pairs (I, a) consisting of a poset $(I, \leq)$ and an inductive system $a : (I. \leq) \to \mathcal{C}$ given by a system of a_i's and α_{ji}'s.

A morphism $(I', a') \to (I'', a'')$ consists of a map of posets $\iota : I' \to I''$ and a system of maps $\mu_k : a'_k \to a''_{\iota(k)},\ i \in I'$, compatible with the structure maps of the two inductive systems i.e., for $p \leq q$ in I' the following diagram commutes:

$$\begin{array}{ccc} a'_p & \xrightarrow{\mu_p} & a''_{\iota(p)} \\ {\scriptstyle \alpha'_{qp}}\downarrow & & {\scriptstyle \alpha''_{qp}}\downarrow \\ a'_q & \xrightarrow{\mu_q} & a''_{\iota(q)} \end{array}.$$

Lemma. If limits of both systems exist then a map (ι, μ) of systems defines a map

$$\underset{\rightarrow\ k\in I'}{\lim}\ a'_k \xrightarrow{\underset{\rightarrow}{\lim}\ \mu_i} \underset{\rightarrow\ l\in I''}{\lim}\ a''_l.$$

Proof. By definition of $\underset{\rightarrow}{\lim}\ a'_k$ we know how to construct a map from it to $\underset{\rightarrow}{\lim}\ a''_l$. □

Corollary. If $J\subseteq I$ is a *final subset*, i.e., for any $i \in I$ there is some $j \in J$ with $i \leq j$ then there is a canonical isomorphism $\underset{\rightarrow\ j\in J}{\lim}\ a_j \underset{\cong}{\rightarrow} \underset{\rightarrow\ i\in I}{\lim}\ a_i$. In particular, if $j \in I$ is the largest element of I then $a_j \underset{\cong}{\rightarrow} \underset{\rightarrow\ i\in I}{\lim}\ a_i$. □

3.5. **Limits II: formulas.** We say that a category $\mathcal{C}$ *has inductive limits* if any inductive system in $\mathcal{C}$ has a limit. Having *countable* or *finite* inductive limits means the same statement where the posets I are only allowed to be countable or finite.

3.5.1. *Limits in sets.* Next we will see that in the category $\mathcal{S}et$ one has inductive and projective limits (i.e., each inductive or projective system has a limit):

Lemma. $\mathcal{S}et$ has inductive and projective limits and they are given by

(a) For any inductive system of sets $\underset{\rightarrow}{\lim}\ A_i$ is the quotient $[\sqcup_{i\in I}\ A_i]/\sim$ of the disjoint union by the equivalence relation generated by $a \sim \alpha_{ji}a$ for any $a \in A_i$ and $j \geq i$.[(12)]

(b) For any projective system of sets $\underset{\leftarrow}{\lim}\ A_i$ is the subset $\Sigma\subseteq \prod_{i\in I}\ A_i$ consisting of all families $a = (a_i)_{i\in I}$ in the product, such that $\alpha_{ji}a_j = a_i$ for $i \leq j$.

Proof. (a) For any set B, a map $[\sqcup_{i\in I}\ A_i]/\sim \xrightarrow{f} B$ is the same as a map $\sqcup_{i\in I}\ A_i \xrightarrow{F} B$ such that $a \sim b$ implies $F(a) = F(b)$. Notice that this condition is equivalent to $i \leq j$ implies that for $a \in A_i$ one has $F(a) = F(\alpha_{ji}a)$.

However, a map $\sqcup_{i\in I}\ A_i \xrightarrow{F} B$ is the same as a family of maps $A_i \xrightarrow{\beta_i} B$ for $i \in I$ where β_i is the restriction of F to A_i. In terms of the maps β_i the condition on F is that for $i \leq j$ and any $a \in A_i$ one has $\beta_i(a) = \beta_j(\alpha_{ji}a)$, i.e., that $\beta_j\circ\alpha_{ji} = \beta_i$.

[12] The relation $\sim$ is described by $a \in A_I$ is equivalent to $b \in A_j$ iff there is $k \geq i, j$ such that $\alpha_{ki}a = \alpha_{kj}b$.

So, we see that the maps from $[\sqcup_{i\in I}\ A_i]/\sim$ to a set B are the same as an extension structure β on B. This is precisely the condition for $[\sqcup_{i\in I}\ A_i]/\sim$ to be the limit of the inductive system.

(b) For any set B, a map $B\to\Sigma$ is the same as a map $B\xrightarrow{F}\prod_{i\in I}\ A_i$ such that for $i\le j$ $pr_i\circ F=\alpha_{ji}\circ pr_jF$. Again, a map $B\xrightarrow{F}\prod_{i\in I}\ A_i$ is the same as a family of maps $B\xrightarrow{\beta_i}A_i$ for $i\in I$, where $\beta_i=pr_i\circ F$. Then, in terms of the maps β_i the condition on F is that for $i\le j$ and any $a\in A_i$ one has $\beta_i=\alpha_{ji}\circ\beta_j$. □

Remark. $\lim\limits_{\rightarrow} A_i$ can be described in English:

- for $i\in I$, any $a\in A_i$ defines an element $\overline{a}$ of $\lim\limits_{\rightarrow} A_i$,
- all elements of $\lim\limits_{\rightarrow} A_i$ arise in this way, and
- for $a\in A_i$ and $b\in A_j$ one has $\overline{a}=\overline{b}$ iff for some $k\in I$ with $i\le k\ge j$ one has "$a=b$ in A_k".(13)

3.5.2. *Basic examples.*

Corollary. (a) If poset $(I,\le)$ is discrete (i.e., $i\le j\Leftrightarrow i=j$ then $\lim\limits_{\rightarrow} A_i=\ \sqcup_{i\in I}\ A_i$ and $\lim\limits_{\leftarrow} A_i=\ \prod_{i\in I}\ A_i$.

Proof. For discrete I an I-inductive system or and I-projective system are both just a family $A_i,\ i\in I$. Then our formulas are a case of the lemma 3.5.1. □

Remarks. (0) Actually, the same is true in any category: the discrete projective limits are the same as products. Proof is obvious.

(1) In general, limits of system need not exist in $\mathcal{C}$. For instance, we have already noticed that the special cases of sums and products need not exist in a category given by a poset (in this setting a sum=infimum and product=supremum).(14) □

Now we consider a simple non-discrete example. Let I be given by $i,j\le k$. Then a projective I-system in $\mathcal{C}$ is given by a diagram $a\xrightarrow{p}c\xleftarrow{q}b$. Its projective limit is called the fibered product of a and b above c and denoted $a\times_c b$.

Corollary. (b) For sets $A\xrightarrow{p}C\xleftarrow{q}B$. the fibered product $A\times_C B$ is the set of all pairs $(a,b)\in A\times B$ such that $p(a)=q(b)$. □

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13 Here $\overline{a}$ denotes the image of $a\in A_i$ in the quotient $[\sqcup_{i\in I}\ A_i]/\sim$ and "$a=b$ in A_k" means$\alpha_{ki}a=\alpha_{kj}b$.

14 However an inductive system in a subcategory $\mathcal{C}'\subseteq\mathcal{C}$ which does not have a limit in $\mathcal{C}'$ still may have a limit in a larger category $\mathcal{C}$.

Example. If $A \overset{\subseteq}{\to} C \overset{\supseteq}{\leftarrow} B$ are inclusions of subsets then the fibered product $A\underset{C}{\times}B$ is just the intersection $A \cap B$.

3.5.3. *Exercises.*

3.5.4. *Stalks of a presheaf.* We want to restrict a sheaf $\mathcal{F}$ on a topological space X to a point $a \in X$. The restriction $\mathcal{F}|a$ is a sheaf on a point, so it just one set $\mathcal{F}_a \overset{\text{def}}{=} (\mathcal{F}|a)(\{a\})$ called the stalk of $\mathcal{F}$ at a. It will give us one of fundamental intuitions about sheaves.

What should $\mathcal{F}_a$ be? It has to be related to all $\mathcal{F}(U)$ where $U\subseteq X$ is is open and contains a, and $\mathcal{F}(U)$ should be closer to $\mathcal{F}_a$ when U is a smaller neighborhood. A formal way to say this is that

- (i) the set $\mathcal{N}_a$ of neighborhoods of a in X is partially ordered by $U \leq V$ if $V\subseteq U$,
- (ii) the values of $\mathcal{F}$ on neighborhoods $(\mathcal{F}(U))_{U\in\mathcal{N}_a}$ form an inductive system,
- (iii) we define the stalk by $\mathcal{F}_a \overset{\text{def}}{=} \underset{\substack{\to \\ U\in\mathcal{N}_a}}{\lim} \ \mathcal{F}(U)$.

The basic examples are given by

Lemma. (a) The stalk of a constant sheaf of sets $S_{\mathbb{R}^n}$ at any point is canonically identified with the set S.

(b) The stalk of a the sheaf $\mathcal{H}_{\mathbb{C}}$ of holomorphic functions at the origin is canonically identified with the ring of convergent power series. ("Convergent" means that the series converges on *some* disc around the origin.)

3.5.5. *Limits in related categories.* Here is how functors act on inductive systems. For an inductive system $a : (I, \leq) \to \mathcal{A}$ in $\mathcal{A}$, any functor $F : \mathcal{A} \to \mathcal{B}$ moves a (which is a system of a_i's and α_{ji}'s), to an inductive system $F(a)$ in $\mathcal{B}$ which is the system of $F(a_i)$'s and $F(\alpha_{ji})$'s.

Lemma. If $\underset{\to}{\lim}^{\mathcal{A}} a_i$ exists in $\mathcal{A}$ and If $\underset{\to}{\lim}^{\mathcal{B}} F(a_i)$ exists in $\mathcal{B}$ then there is a canonical ("comparison") map in $\mathcal{B}$

$$\underset{\to}{\lim}^{\mathcal{B}} \ F(a_i) \ \xrightarrow{\zeta} \ F(\underset{\to}{\lim}^{\mathcal{A}} \ a_i) \quad \square.$$

If this is an isomorphism we say that F is compatible with the limit.

Remark. This need not happen. We have already noticed that the forgetful functor $\mathcal{F} : \mathfrak{m}(\Bbbk) \to \mathcal{S}et$ does not preserve sums. $\square$

Lemma. (a) The category $\mathcal{T}op$ has projective and inductive limits and these are compatible with the ones in sets.

(b) The projective limits in $\mathfrak{m}(\Bbbk)$ are compatible with the ones in sets. $\square$

Proof. (a) We have the forgetful functor $\mathcal{F} : \mathcal{T}op \to \mathcal{S}et$. The claim is that say, for a projective system of topological spaces (X_i, τ_i) (where τ_i is the topology on X_i), The projective limit $\underleftarrow{\lim}^{\mathcal{T}op}\ (X_i, \tau_i)$ is as a set the projective limit in sets $X = \underleftarrow{\lim}^{\mathcal{S}et}\ X_i$ with some topology τ.

From the formula for X as a subset of $\prod_{i\in I}\ X_i$ one can make the guess that the topology τ is the restriction of the product topology on $\prod_{i\in I}\ X_i$ to the subset X.

So, now it remains to check that for this choice of τ, (X, τ) really is the projective limit of (X_i, τ_i) in $\mathcal{T}op$. $\square$

Remark. This is an indication of how one constructs limits in complicated categories by using limits in simpler settings.

3.5.6. *Limits in arbitrary categories: Existence and Construction.* Now we extend the formulas for limits from lemma 3.5.1 to arbitrary categories. We know that products and disjoint unions in sets have analogues in arbitrary category: products and coproducts. However, in set theoretic formulas 3.5.1 we also had to impose some equalities and this has been done in two ways – by taking a subset or a quotient set. The categorical version of these operations are *equalizers* and *coequalizers*.

Once we introduce these, the construction and proof proof in a general category will be "the same" as in sets.

- The *equalizer* $Eq(\alpha, \beta)$ of a pair of maps $a \xrightarrow{\alpha,\beta} b$ from a to b is the final object among all maps $e \xrightarrow{i} a$ such that $\alpha\circ i = \beta\circ i$.

- The *coequalizer* $Coeq(\alpha, \beta)$ is the initial object among all maps $b \xrightarrow{q} c$ such that $q\circ\alpha = q\circ\beta$.

Example. (a) In $\mathcal{S}et$ the equalizer of maps $\alpha, \beta : A \to B$ is the subset $\{a \in A;\ \alpha(a) = \beta(a)\}$ of A. The coequalizer is the quotient $B/\sim$ by the equivalence relation on B generated by $\alpha(a) \sim \beta(a)$ for all $a \in A$.

(b) In $\mathfrak{m}(\Bbbk)$ for $\alpha, \beta : M \to N$ we have $Eq(\alpha, \beta) = \mathrm{Ker}(\beta - \alpha)$ and $Coeq(\alpha, \beta) = \mathrm{Coker}(\beta - \alpha) \overset{\text{def}}{=} N/Im(\beta - \alpha)$. $\square$

3.5.7. *Lemma.* (a) If a category $\mathcal{C}$ has products (of families of objects) and *equalizers* then $\mathcal{C}$ has projective limits, and these can be described in terms of products and equalizers.

(b) Dually, if a category $\mathcal{C}$ has sums and coequalizers then $\mathcal{C}$ has inductive limits, and these can be described in terms of sums and coequalizers.

Corollary. For instance $\mathcal{A}b$ and $\mathcal{T}op$ have both kinds of limits.

3.5.8. *More general kind of limits.* For a category $\mathcal{I}$ an *inductive*$\mathcal{I}$-system in a category $\mathcal{C}$ means a functor $a:\mathcal{I}\to\mathcal{C}$. Now one can again formulate the notion of a limit. This time we use the extensions of $\mathcal{I}$ to a larger category $\mathcal{I}_+=\mathcal{I}\sqcup\infty$ which is defined by asking that ∞ is the final object in $\mathcal{I}_+$.

Example. The equalizer of maps $\alpha,\beta:a\to b$ is a projective limit over the category $\mathcal{I}$ given by objects $1,2$ and two arrows from 1 to 2. (This is not a poset!)

3.5.9. *Exercises.* (**1**) Let $(I,\le)$ be $\{1,2,3,...\}$ with the order $i\le j$ if $i|j$, i.e., i divides j. In $\mathcal{A}b$ let $A_i=\mathbb{Q}/\mathbb{Z}$ for all $i\in I$, and let ϕ_{ji} be the multiplication by j/i when i divides j. This is an inductive system and $\lim\limits_{\to}\ A_i=?.$[15]

(**2**) Let $(I,\le)$ be $\mathbb{N}=\{0,1,...\}$ with the standard order. In $\mathcal{R}ing$ let $\Bbbk_n=\mathbb{C}[x]/x^{n+1}$ and for $i\le j$ let ϕ_{ij} be the obvious quotient map. This is a projective system and $\lim\limits_{\leftarrow}\ \Bbbk_n=?.$

(**3**) (a) For $\mathbb{N}=\{0,1,...\}$ with the standard order, for a positive integer p consider the projective system of rings $A_n=\mathbb{Z}/p^n\mathbb{Z}$ (under the natural quotient maps). Its limit is denoted $\mathbb{Z}_p$ and called the ring of p-padic integers. Show that $\mathbb{Z}_p$ can be identified as a set with the set of all symbols of the form $\sum_{i=0}^\infty\ a_ip^i$ with $a_i\in\{0,..,p-1\}$. What is the ring structure on these symbols?

(b) For $I=\{1,2,3,...\}$ with the order $i|j$ the rings $A_n=\mathbb{Z}/n\mathbb{Z}$ again form a projective system under the natural quotient maps. Show that the ring $\widehat{\mathbb{Z}}\stackrel{\text{def}}{=}\ \lim\limits_{\leftarrow}\ \mathbb{Z}_{/}n\mathbb{Z}$ is isomorphic to the product over the set $\mathcal{P}$ of primes $\prod_{p\in\mathcal{P}}\ \mathbb{Z}_p$.

3.6. **Functors II :.** We start with the standard categorical versions of basic set theoretic notions (3.6.1.

3.6.1. *A categorification of notions of set theory.* We consider categorical versions of being a subset, injection, surjection and bijection. In the more subtle world of categories, some notions have multiple generalizations.

(**1**) [Subsets $A\subseteq B$.] A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ is given by a subclass $Ob(\mathcal{C}')\subseteq Ob(\mathcal{C})$ and for any $a,b\in Ob(\mathcal{C}')$ a subclass $\mathrm{Hom}_{\mathcal{C}'}(a,b)\subseteq\mathrm{Hom}_{\mathcal{C}}(a,b)$ such that $\mathrm{Hom}_{\mathcal{C}'}(a,a)\ni 1_a,\ a\in\mathcal{C}'$, and the sets $\mathrm{Hom}_{\mathcal{C}'}(a,b),\ a,b\in\mathcal{C}'$ are closed under the composition in $\mathcal{C}$.

A *full subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ is a subcategory $\mathcal{C}'$ such that for any $a,b\in\mathcal{C}'$ one has $\mathrm{Hom}_{\mathcal{C}'}(a,b)=\ \mathrm{Hom}_{\mathcal{C}}(a,b)$. Notice that choosing a full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is the same as choosing a subclass $Ob(\mathcal{C}')\subseteq Ob(\mathcal{C})$.

[15] rem In our basic example of an inductive system of increasing sets $A_2\subseteq A)2\subseteq\cdots$ the limit is the really the union. However, the present example shows that $\lim\limits_{\to}$ does not always mean growth.

Examples. (1) The free modules $\mathcal{F}ree(\Bbbk)$ naturally form a full subcategory of the category $\mathfrak{m}(\Bbbk)$ of all modules. (2) Category $\mathcal{C}$ defines subcategory $\mathcal{C}^*$ where objects are the same and morphisms are the isomorphisms from $\mathcal{C}$ (not full!).

(**1'**) [Injections $A \overset{f}{\hookrightarrow} B$.] A functor $F : \mathcal{A} \hookrightarrow \mathcal{B}$ is

- *faithful* (also called an *embedding of categories*), if all maps $\mathrm{Hom}_{\mathcal{A}}(a', a'') \to \mathrm{Hom}_{\mathcal{B}}(Fa', Fa''),\ a', a'' \in \mathcal{A}$ are injective.
- *fully faithful* (or a *full embedding*) if all $\mathrm{Hom}_{\mathcal{A}}(a', a'') \to \mathrm{Hom}_{\mathcal{B}}(Fa', Fa'')$ are bijections.

(**2**) [Surjections $A \overset{f}{\twoheadrightarrow} B$.] Functor $F : \mathcal{A} \to \mathcal{B}$ is said to be *essentially surjective*, if it is surjective on isomorphism classes of objects, i.e., any $b \in \mathcal{B}$ is isomorphic to Fa for some $a \in \mathcal{A}$.

(**3**) [Bijections $A \underset{\cong}{\overset{f}{\to}} B$.] Functor $F : \mathcal{A} \to \mathcal{B}$ is said to be an *equivalence of categories* if it is essentially surjective and fully faithful.

Remark. It turns out that injectivity has been transplanted on morphisms and surjectivity on objects, and we have used "bijective=injective+surjective.

Examples. (3) The forgetful functor $\mathcal{T}op \xrightarrow{\mathcal{F}} \mathcal{S}et$ is faithful and essentially surjective.

(4) For a map of rings $\phi : A \to B$ the functor $\phi^* : \mathfrak{m}(B) \to \mathfrak{m}(A)$ is always faithful[16] but it need not be fully faithful (for $\mathbb{R} \subseteq \mathbb{R}[x]$) nor essentially surjective (for $\mathbb{R} \subseteq \mathbb{C}$).

(5) [Equivalent approaches to linear algebra.] Let $\Bbbk$ be a field and $\mathcal{V}_{\Bbbk}$ the category such that $Ob(\mathcal{V}_k) = \mathbb{N}$ and $\mathrm{Hom}(n, m) = M_{mn}$, the matrices with m rows and n columns (the composition is matrix multiplication). Let $\mathcal{V}ec_{\Bbbk}^{fd}$ be the category of finite dimensional vector spaces. Consider the functor $\mathcal{V}_{\Bbbk} \xrightarrow{\iota} \mathcal{V}ec_{\Bbbk}^{fd}$ given by $\iota(n) = \Bbbk^n$ while for a matrix $\alpha \in M_{mn}$, $\iota_\alpha :\ \Bbbk^m \to \Bbbk^n$ is the operator of multiplication by α.

This is an equivalence of categories. Notice that the categories $\mathcal{V}_{\Bbbk}$ and $\mathcal{V}ec_{\Bbbk}^{fd}$ have very different objects (only the first one is small)., However, their content is the same – linear algebra. One of these categories is more convenient for computation and the other for thinking. Historically, equivalence ι is roughly the observation that one can do linear algebra without always choosing coordinates (i.e., a basis of a vector space).

3.6.2. *Natural transformations of functors ("morphisms of functors").* A way to relate two functors $F, G : \mathcal{A} \to \mathcal{B}$ is to relate their values. A *natural transformation* from F to G consists of "comparison" morphisms in $\mathcal{B}$ $\eta_a \in \mathrm{Hom}_{\mathcal{B}}(Fa, Ga)$ of values on arbitrary

[16] So, "faithful" is a useful but not very strong property.

$a \in \mathcal{A}$, which is compatible with maps in the sense that for any map $\alpha : a' \to a''$ in $\mathcal{A}$ the following diagram commutes

$$\begin{array}{ccc} F(a') & \xrightarrow{F(\alpha)} & F(a'') \\ {\scriptstyle\eta_{a'}}\downarrow & & {\scriptstyle\eta_{a''}}\downarrow \\ G(a') & \xrightarrow{G(\alpha)} & G(a'') \end{array}, \quad i.e., \quad \eta_{a''}\circ F(\alpha) = G(\alpha)\circ\eta_{a'}.$$

So, η relates values of functors on objects in a way compatible with the values of functors on maps.

Remark. The compatibility property is often called "naturality". In practice, any "natural" choice of maps η_a will have automatically have the compatibility property.

3.6.3. *Example.* For the functors $\phi_* M = l \otimes_{\Bbbk} M$ and $\phi^* N = N$ from 3.2.1(1), there are canonical morphisms of functors

$$\alpha : \ \phi_*\circ\phi^* \to 1_{\mathfrak{m}(l)}, \qquad \phi_*\circ\phi^*(N) = l\otimes_{\Bbbk} N \xrightarrow{\alpha_N} N = 1_{\mathfrak{m}(l)}(N)$$

is the action of l on N and

$$\beta : \ 1_{\mathfrak{m}(\Bbbk)} \to \phi^*\circ\phi_*, \qquad \phi^*\circ\phi_*(M) = l\otimes_{\Bbbk} M \xleftarrow{\beta_M} M = 1_{\mathfrak{m}(\mathbb{M})}(M)$$

is the map $m \mapsto 1_l \otimes m$.

3.6.4. *Lemma.* For two categories $\mathcal{A}, \mathcal{B}$, the functors from $\mathcal{A}$ to $\mathcal{B}$ form a category $Fun(\mathcal{A}, \mathcal{B})$.

Proof. For $F, G : \mathcal{A} \to \mathcal{B}$ one defines $\mathrm{Hom}_{Fun(\mathcal{A},\mathcal{B})}(F, G)$ as the set of natural transforms from F to G. The remaining e structure is routine: any functor $F : \mathcal{A} \to \mathcal{B}$ has $1_F : F \to F$ (with $(1_F)_a = 1_{Fa} : Fa \to Fa$) and for three functors F, G, H from $\mathcal{A}$ to $\mathcal{B}$ one can compose morphisms $\mu : F \to G$ and $\nu : G \to H$ to $\nu\circ\mu : F \to H$. $\square$

Remarks. (0) This is an improvement over functions from a set A to a set B – functors can be compared (related). (1) The lemma indicates that the reasonable relation between two functors is not equality but a canonical isomorphism of functors!

3.7. **Adjoint functors.** This is often the most useful categorical idea. A pair of functions in opposite directions $A \xrightarrow{f} B \xrightarrow{g} A$ may have a property that $g\circ f = id$ or $f\circ g = id$ or both. The relation between a pair of functors is much richer, in particular it involves an extra structure (not just a property).

3.7.1. *Adjoint pairs.* An adjointness structure on pair of functors (F,G) where $\mathcal{A}\xrightarrow{F}\mathcal{B}\xrightarrow{G}\mathcal{A}$ is an isomorphism of two functors from $\mathcal{A}^o\times\mathcal{B}$ to $\mathcal{S}et$

$$\zeta:\ \mathrm{Hom}_{\mathcal{B}}(F-,-)\underset{\cong}{\rightarrow}\mathrm{Hom}_{\mathcal{A}}(-,G-).$$

This means that for any $a\in\mathcal{A},\ b\in\mathcal{B}$ we have a "natural identification"

$$\zeta_{a,b}:\ \mathrm{Hom}_{\mathcal{B}}(Fa,b)\underset{\cong}{\rightarrow}\mathrm{Hom}_{\mathcal{A}}(a,Gb),.$$

Here, "naturality" means compatibility with morphisms $(a,b)\xrightarrow{(\alpha,\beta)}(a',b')$ in $\mathcal{A}^o\times\mathcal{B}$, i.e., compatibility with morphisms $a'\xrightarrow{\alpha}a$ in $\mathcal{A}$ and $b\xrightarrow{\beta}b'$ in $\mathcal{B}$.[17]

We say that F is the *left adjoint* of G and that that G is *right adjoint* of F (in the above Hom's F appears on the left side and G on the right).

Lemma. An adjointness structure ζ for (F,G) gives canonical maps of functors

$$u:\ id_{\mathcal{A}}\rightarrow GF\quad\text{and}\quad c:\ FG\rightarrow id_{\mathcal{B}},$$

called the *unit* and the *counit* of adjunction.

Proof. The special case when $b=Fa$ gives $\zeta_{a,Fa}:\ \mathrm{Hom}_{\mathcal{B}}(Fa,Fa)\underset{\cong}{\rightarrow}\mathrm{Hom}_{\mathcal{A}}(a,GFA)$, hence

$$u_a\overset{\text{def}}{=}\zeta_{a,Fa}(1_{Fa}):\ a\rightarrow GFa.$$

Similarly, $a=Gb$ gives $\zeta_{Gb,b}:\ \mathrm{Hom}_{\mathcal{B}}(FGb,b)\underset{\cong}{\rightarrow}\mathrm{Hom}_{\mathcal{A}}(Gb,Gb)$, hence

$$c_b\overset{\text{def}}{=}\zeta_{Gb,b}{}^{-1}(1_{Gb}):\ FGb\rightarrow b.$$

□

Remark. One can restate the notion of adjoints completely in terms of units and counits.

3.7.2. *Example: functoriality of modules for rings.* For a map of rings $\phi: A\rightarrow B$ there is a canonical adjoint triple $(\phi_*,\phi^*,\phi_\star)$ of functors between A and B modules where for $M\in\mathfrak{m}(A)$ and $N\in\mathfrak{m}(B)$, the pull-back ϕ^*N is N viewed as an A-module via ϕ, while

$$\phi_*M\overset{\text{def}}{=}B\otimes_A M\quad\text{and}\quad\phi_\star M\overset{\text{def}}{=}\mathrm{Hom}_A({}_AB,M).$$

Here, $b\in B$ takes $\phi\in\mathrm{Hom}_A({}_AB,M)$ to $b\phi$ such that on $y\in B$ one has $(b\phi)(y)\overset{\text{def}}{=}\phi(yb)$ (see also 3.2.1). The adjoint structure on $(\phi_*,\phi^*,\phi_\star)$ First, the adjoint structure for (ϕ_*,ϕ^*) is the canonical identification

$$\mathrm{Hom}_{\mathfrak{m}(B)}(B\otimes_A M,N)\xrightarrow{\eta_{M,N}}\mathrm{Hom}_{\mathfrak{m}(A)}(M,N)$$

such that for two maps $B\otimes_A M\xrightarrow{\sigma}N$ and $M\xrightarrow{\tau}N$ in the LHS and RHS sets one has

$$[\eta(\sigma)](m)\ \overset{\text{def}}{=}\ \sigma(1\otimes m)\quad\text{and}\quad\eta^{-1}(\tau)(b\otimes m)=b\tau(m),\quad m\in M,\ b\in B.$$

[17] Here, α gives maps between Hom-sets $F(\alpha)^*$ and α^* and β gives maps β_* and $G(\beta)_*$.

Then, the adjointness for $(\phi^*, \phi_\star)$ is the identification

$$\mathrm{Hom}_{\mathfrak{m}(A)}(N, M) \xrightarrow{\zeta_{N,M}} \mathrm{Hom}_{\mathfrak{m}(B)}[N, \mathrm{Hom}_A(B, M)]$$

is given on maps $N \xrightarrow{\tau} M$ and $\mathrm{Hom}_A(B, M) \xrightarrow{\rho} N$ in the LHS and RHS sets by

$$\big[[\zeta(\tau)](n)\big](b) \stackrel{\text{def}}{=} \tau(bn) \quad \text{and} \quad [\zeta^{-1}(\rho)](n) \stackrel{\text{def}}{=} [\rho(n)](1_B), \quad n \in N,\ b \in B.$$

Remark. As in this example, often an adjoint pair appears in the following way: there is an obvious functor F (so obvious that we usually do not pay it any attention), but it has an adjoint G which is an interesting construction. The fact that this "interesting construction" G is intimately tied to the original "stupid" construction F allows one to deduce properties of G from the properties of the simpler construction F.

3.8. **Description of objects as representable functors.** The Yoneda lemma below says that passing from an object $a \in \mathcal{A}$ to the corresponding functor $\mathrm{Hom}_{\mathcal{A}}(-, a)$ does not loose any information – a can be recovered from the functor $\mathrm{Hom}_{\mathcal{A}}(-, a)$. This has the following applications:

(1) One can describe an object a by describing the corresponding functor $\mathrm{Hom}_{\mathcal{A}}(-, a)$. This turns out to be **the** *most natural* description of a.
(2) One can start with a functor $F : \mathcal{A}^o \to \mathcal{S}et$ and ask whether it comes from some objects of a. (Then we say that a represents F and that F is representable).
(3) Functors $F : \mathcal{A}^o \to \mathcal{S}et$ behave somewhat alike the objects of $\mathcal{A}$, and we can think of their totality as a natural enlargement of $\mathcal{A}$ (as one completes $\mathbb{Q}$ to $\mathbb{R}$).

3.8.1. *Category* $\widehat{\mathcal{A}}$. To a category $\mathcal{A}$ one can associate a category

$$\widehat{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{F}un(\mathcal{A}^o, \mathcal{S}et)$$

of contravariant functors from $\mathcal{A}$ to sets.

Theorem. (Yoneda lemma)
(a) Construction ι is a functor $\iota : \mathcal{A} \to \widehat{\mathcal{A}}$.

(b) For any functor $F \in \widehat{\mathcal{A}} = \mathcal{F}un(\mathcal{A}^o, \mathcal{S}et)$ and any $a \in \mathcal{A}$ there is a canonical identification

$$\mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, F) \cong F(a).$$

Proof. (b) A map of functors $\eta : \iota_a \to F$ gives $\overline{\eta} \in F(a)$ simply by evaluating at a and then at 1_a. First, the evaluation of η at a gives $\eta_a : \iota_a(a) \to F(a)$. Now, since $\iota_a(a) = \mathrm{Hom}_{\mathcal{A}}(a, a)$ contains 1_a, we get an element $\overline{\eta} \stackrel{\text{def}}{=} \eta_a(1_a)$ of $F(a)$.

In the opposite direction, a choice of $f \in F(a)$, gives for any $x \in \mathcal{A}$ the composition of functions where the last step is the evaluation on f

$$\widetilde{f}_x \stackrel{\text{def}}{=} [\iota_a(x) = \mathrm{Hom}_{\mathcal{A}}(x, a) = \mathrm{Hom}_{\mathcal{A}^o}(a, x) \xrightarrow{F} \mathrm{Hom}_{\mathcal{S}et}[F(a), F(x)] \xrightarrow{ev_f} F(x)].$$

Now one checks that

- (i) $\widetilde{f}$ is a map of functors $\iota_a \to F$, and
- (ii) procedures $\eta \mapsto \overline{\eta}$ and $f \mapsto \widetilde{f}$ are inverse functions between $\mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, F)$ and $F(a)$.

□

Corollary. (a) Yoneda functor $\iota : \mathcal{A} \to \widehat{\mathcal{A}}$ is a full embedding of categories, i.e., for any $a, b \in \mathcal{A}$ the map

$$\iota : \mathrm{Hom}_{\mathcal{A}}(a, b) \ \to \ \mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, \iota_b),$$

given by the functoriality of ι, is an isomorphism.

(b) Functor $\mathrm{Hom}_{\mathcal{A}}(-, a) = \iota_a$ determines a up to a unique isomorphism, i.e., an isomorphism of functors $\phi : \iota_a \underset{\cong}{\to} \iota_b$ in $\widehat{\mathcal{A}}$ gives an isomorphism $a \underset{\cong}{\to} b$ in $\mathcal{A}$.

Proof. (a) follows the part (b) of the Yoneda lemma (take $F = \iota_b$).

Then (b) follows from (a) since $\phi : \iota_a \underset{\cong}{\to} \iota_b$ is of the form $\iota_{\overline{\phi}} = (\overline{\phi})_*$ for a unique $\overline{\phi} : a \to b$. (To see that $\overline{\phi}$ is an isomorphism we construct $\overline{\phi^{-1}}$ in the same way and show that it is the inverse of $\overline{\phi}$,) □

3.8.2. *Examples of "Categorical Thinking" ("Interaction Principle").* Let us describe *Categorical Thinking* as the approach where objects are described by how they interact with others.

This is different from the set theory which describes objects by <u>*what they are*</u> (i.e., certain systems of sets). It works in settings where we do not know what is the exact nature of objects but one can measure their interactions.

This includes the physics of elementary particles where one measures the *scattering amplitudes* of collisions of particles – the probability that the collision will will have certain outcome.

A classical mathematical example are the distributions. Among functions one can not find beauties like the very useful δ-functions δ_a ($a \in \mathbb{R}$), however one knows how δ_a interacts with functions – the interaction with $f \in C_c^\infty(\mathbb{R})$ is $\langle \delta_a, f \rangle = \int_{\mathbb{R}} f\delta_a \overset{\text{def}}{=} f(a)$. So, one extends the notion of of functions by adding distributions as (continuous) linear functionals on the vector space of of (nice) functions: $C_c^\infty(\mathbb{R})$.

The categorical example is Yoneda's lemma. In a category $\mathcal{A}$ the interactions of $a \in \mathcal{A}$ with all objects are encoded in the functor $\mathrm{Hom}_{\mathcal{A}}(-, a)$. So, Yoneda says that if you know the interactions of a you know a.

3.8.3. *Representable functors.* A functor $F \in \widehat{\mathcal{A}}$, i.e., $F : \mathcal{A}^o \to \mathcal{S}et$, is *representable* if there is some $a \in \mathcal{A}$ and an isomorphism of functors

$$\eta : \ \mathrm{Hom}_{\mathcal{A}}(-,a) \xrightarrow[\cong]{} F.$$

Then we say that a represents F.

Example. In the category of schemes, the double point scheme $\mathrm{Spec}(\mathbb{Z}[x]/x^2)$ is represented by the functor on the category $\mathcal{CR}ing$

$$A \mapsto \{a \in A;\ a^2 = 0\}.$$

3.8.4. *Non-representable functors.* What is new in $\widehat{\mathcal{A}}$ that may not be in $\mathcal{A}$? One can describe the completion of $\mathcal{A}$ to $\widehat{\mathcal{A}}$ as *adding to* $\mathcal{A}$ *all limits of* inductive systems *in* $\mathcal{A}$. (Just as one constructs $\mathbb{R}$ from $\mathbb{Q}$ by adding limits of sequences.)

Lemma. Any inductive system in $\mathcal{A}$ has a limit in the larger category $\widehat{\mathcal{A}}$

Proof. Any inductive system $\boldsymbol{a} = (a_i)_I$ in $\mathcal{A}$ *always* defines a functor $\widetilde{\boldsymbol{a}}$ in $\widehat{\mathcal{A}}$, by

$$\widetilde{\boldsymbol{a}}(c) \overset{\text{def}}{=} \lim_{\to\ I} \ \iota_{a_i}(c) = \lim_{\to\ I} \ \mathrm{Hom}_{\mathcal{A}}(c, a_i) \in \ \mathcal{S}et.$$

The reason why this works is that in the category $\mathcal{S}et$ all inductive limits exist!

Now it is easy to check that the functor $\widetilde{\boldsymbol{a}}$ is indeed the limit of the inductive system $\iota_{\boldsymbol{a}} = (\iota_{a_i})_I$ in the category $\widehat{\mathcal{A}}$. □

Remark. Inductive systems in $\mathcal{A}$ are called *ind-objects of* $\mathcal{A}$. A precise meaning of that is that an inductive system $\boldsymbol{a}$ in $\mathcal{A}$ gives an object $\widetilde{\boldsymbol{a}}$ in a larger category $\widehat{\mathcal{A}}$. (18)

3.8.5. *Uniqueness and existence of adjoints.* As an application of Yoneda lemma we find that adjoints are unique and we examine when they exist.

Lemma. (a) If functor F has a right adjoint then for each $b \in \mathcal{B}$ the functor

$$\mathrm{Hom}_{\mathcal{B}}(F-, b) : \mathcal{A}^o \to \mathcal{S}et, \ a \mapsto \mathrm{Hom}_{\mathcal{B}}(Fa, b)$$

is representable.

(b) Suppose that for each $b \in \mathcal{B}$ the functor $\mathrm{Hom}_{\mathcal{B}}(F-, b) : \mathcal{A} \to \mathcal{S}et$ is representable. For each $b \in B$ choose a representing object $Gb \in \mathcal{A}$, then G has a canonical extension to a functor from $\mathcal{B}$ to $\mathcal{A}$ and this is the right adjoint of F.

(c) The right adjoint of F, if it exists, is unique up to a canonical isomorphism.

Proof. (a) If G is a right adjoint of F then the functor $\mathcal{A}^o \ni a \mapsto \mathrm{Hom}_{\mathcal{B}}(Fa, b) \cong \mathrm{Hom}_{\mathcal{A}}(a, Gb)$ is represented by Gb.

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18 Similarly one calls projective systems *pro-objects* of $\mathcal{A}$.

(b) A map $\beta : b' \to b''$ in $\mathcal{B}$ gives a map of functors $\mathrm{Hom}_{\mathcal{B}}(Fa, b') \xrightarrow{\beta_*} \mathrm{Hom}_{\mathcal{B}}(Fa, b'')$, hence a map of isomorphic functors $\mathrm{Hom}_{\mathcal{A}}(-, Gb)' \xrightarrow{\beta_*} \mathrm{Hom}_{\mathcal{A}}(-, Gb'')$. By Yoneda lemma this comes from a unique map $Gb' \to Gb''$ in $\mathcal{A}$.

(c) Since Gb represents the functor $\mathcal{A}^o \ni a \mapsto \mathrm{Hom}_{\mathcal{B}}(Fa, b)$, it is unique up to a canonical isomorphism. □

Of course the symmetric claims hold for left adjoints.

3.8.6. *Example: Left adjoints of some forgetful functors.* Now that we know that the adjoint is unique we consider more examples of how interesting functors are found as adjoints of trivial functors.

We say that a functor $\mathcal{F}$ is *forgetful* if it amounts to dropping a part of the structure. Bellow we will denote its left adjoint by $\mathcal{G}$. Standard construction (that add to the structure of an object), are often adjoints of forgetful functors

(1) If $\mathcal{F} : \mathfrak{m}(\Bbbk) \to \mathcal{S}et$ then $\mathcal{G}$ sends set S to the the free $\Bbbk$-module $\Bbbk[S] = \oplus_{s\in S} \Bbbk{\cdot}s$ with a basis S.
(2) Let $\Bbbk$ be a commutative ring. For $\mathcal{F} : \Bbbk - \mathcal{C}om\mathcal{A}lg \to \mathcal{S}et$ from commutative $\Bbbk$-algebras to sets, $\mathcal{G}$ sends a set S to the polynomial ring $\Bbbk[x_s,\ s \in S]$ where variables are given by all elements of S.
(3) If $\mathcal{F} : \Bbbk - \mathcal{C}om\mathcal{A}lg \to \mathfrak{m}(\Bbbk)$, then for a $\Bbbk$-module M, $\mathcal{G}(M)$ is the symmetric algebra $S(M)$. (To get exterior algebras in the same way one needs the notion of super algebras.)
(4) For the functor $\mathcal{F} : \Bbbk - \mathcal{A}lg \to \mathfrak{m}(\Bbbk)$ from $\Bbbk$-algebras to $\Bbbk$-modules, $\mathcal{G}(M)$ is the tensor algebra $S(M)$.
(5) Forgetful functor $\mathcal{F} : \mathcal{T}op\mathcal{S}et$ has a left adjoint $\mathcal{D}$ that sends a set S to S with the discrete topology, and also the right adjoint C such that $C(S)$ is S with the topology such that only S and ϕ are open.

Question. Any (A, B)-bimodule X gives a functor $X_* : \mathfrak{m}(B) \to \mathfrak{m}(A)$, with $X_*(N) \overset{\text{def}}{=} X \otimes_B N$. What is its right adjoint?

3.8.7. *Equivalences of categories and adjoints.* The above definition of equivalences of categories is the categorical upgrade of "bijection=injection+surjection". Now we will see that it is also a lift of "f is bijection iff it has inverse function". This view on equivalence of categories is manifestly symmetric.

Theorem. (a) A functor $F : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories iff it has a right adjoint such that the unit and counit of adjunction (see 3.7.1) are isomorphisms.

(b) If a functor $F : \mathcal{A} \to \mathcal{B}$ is (fully) faithful then $\mathcal{A}$ is equivalent to a (full) subcategory of $\mathcal{B}$.

Proof. (a) If F is an equivalence then for $b \in \mathcal{B}$ we can choose some $b' \in \mathcal{A}$ and an isomorphism $\alpha : F(b') \underset{\cong}{\to} b$. Then we construct a right adjoint G on objects by $G(b) = b'$ and on maps via bijections

$$\mathrm{Hom}_{\mathcal{B}}(b_1, b_2) \cong \mathrm{Hom}_{\mathcal{B}}[F(b_1'), F(b_2')] \underset{\cong}{\leftarrow} \mathrm{Hom}_{\mathcal{A}}(b_1', b_2').$$

Now one checks that G is functor, a right adjoint of F and that the (co)units are isomorphisms.

In the opposite direction, for an adjoint pair (F, G) with invertible (co)units, it is easy to see that F is fully faithful and essentially surjective. $\square$