HOMOLOGICAL ALGEBRA

Contents

Date: ?

10. **Bicomplexes and the extension of resolutions and derived functors to complexes**

For a right exact functor $F:\mathcal{A}\longrightarrow\mathcal{B}$ we have defined its left derived functor $LF:\mathcal{A}\longrightarrow K^{-}(\mathcal{B})$ (if $\mathcal{A}$ has enough projectives) by replacing objects with their projective resolutions. However, it is necessary to extend this construction to $LF:\ K^{-}(\mathcal{A})\longrightarrow K^{-}(\mathcal{B})$. For one thing, for calculational reasons we need the property $L(F\circ G)A=LF(LG(A))$, but this means that we have to apply LF to a complex.

We will need the notion of bicomplexes (roughly "complexes that stretch in a plane rather then on a line"), as a tool for finding projective resolutions of complexes.

10.1. **Filtered and graded objects.** A graded object A of $\mathcal{A}$ is a sequence of objects $A^n\in\mathcal{A},\ n\in\mathbb{Z}$. We think of it as a sum $A=\oplus_{\mathbb{Z}}\ A^n$ (and this is precise if the sum exists in $\mathcal{A}$).

An increasing (resp. decreasing) filtration on an object $a\in\mathcal{A}$ is an increasing (resp. decreasing) sequence of subobjects $a_n\hookrightarrow a,\ n\in\mathbb{Z}$. When we talk of a filtration F we denote a_n by F_na.

A filtration defines a graded object $Gr(a)$ with $Gr^n(a)\overset{\text{def}}{=}a_n/a_{n-1}$ (resp. $Gr^n(a)\overset{\text{def}}{=}a_n/a_{n+1}$). Also, a graded object $A=\ \oplus\ A^n$ has a canonical increasing filtration $A_n=\ \oplus_{i\le n}\ A^i$ (if the sums exist), and a canonical decreasing filtration $A_n=\ \oplus_{i\ge n}\ A^i$. (However, these grading and filtering are far from being inverse to each other.)

We will be interesting in decreasing filtrations F of complexes $(A=\oplus\ A^i,d)$. This is a sequence of subcomplexes $F_nA=\ \oplus\ F_n(A^i)$, i.e.,

(1) $\cdots F_{-1}(A^i)\subseteq F_0(A^i)\subseteq F_1(A^i)\supseteq\cdots\subseteq A^i$ is a filtration of A^i and
(2) $d(F_nA^i)\subseteq F_n(A^{i+1})$.

A filtration F of a complex $(A=\oplus\ A^i,d)$ gives a filtration F of its cohomology groups by

$$F_n[\mathrm{H}^i(A)]\overset{\text{def}}{=}Im[\mathrm{H}^i(F_nA)\longrightarrow\mathrm{H}^i(A)].$$

10.2. **Bicomplexes.**

10.2.1. *Bicomplexes.* A bicomplex is a bigraded object $B=\oplus_{p.q\in\mathbb{Z}}\ B^{p,q}$ with differentials $B^{p,q}\xrightarrow{d'}B^{p+1,q}$ and $B^{p,q}\xrightarrow{d''}B^{p,q+1}$, such that $d=d'+d''$ is also a differential. So we ask that i.e.,

$$0=d^2=(d'+d'')^2=(d')^2+d'd''+d''d'+(d'')^2=d'd''+d''d'.$$

We draw a bicomplex as a two dimensional object:

$$\begin{tikzcd}
\cdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \cdots \\
\cdots \arrow[r, "d'"] \arrow[u, "d''"] & B^{-1,2} \arrow[r, "d'"] \arrow[u, "d''"] & B^{0,2} \arrow[r, "d'"] \arrow[u, "d''"] & B^{1,2} \arrow[r, "d'"] \arrow[u, "d''"] & B^{2,2} \arrow[r, "d'"] \arrow[u, "d''"] & B^{3,2} \arrow[r, "d'"] \arrow[u, "d''"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[r, "d'"] \arrow[u, "d''"] & B^{-1,1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{0,1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{1,1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{2,1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{3,1} \arrow[r, "d'"] \arrow[u, "d''"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[r, "d'"] \arrow[u, "d''"] & B^{-1,0} \arrow[r, "d'"] \arrow[u, "d''"] & B^{0,0} \arrow[r, "d'"] \arrow[u, "d''"] & B^{1,0} \arrow[r, "d'"] \arrow[u, "d''"] & B^{2,0} \arrow[r, "d'"] \arrow[u, "d''"] & B^{3,0} \arrow[r, "d'"] \arrow[u, "d''"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[r, "d'"] \arrow[u, "d''"] & B^{-1,-1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{0,-1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{1,-1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{2,-1} \arrow[r, "d'"] \arrow[u, "d''"] & B^{3,-1} \arrow[r, "d'"] \arrow[u, "d''"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[r, "d'"] \arrow[u, "d''"] & \vdots \arrow[r, "d'"] \arrow[u, "d''"] & \vdots \arrow[r, "d'"] \arrow[u, "d''"] & \vdots \arrow[r, "d'"] \arrow[u, "d''"] & \vdots \arrow[r, "d'"] \arrow[u, "d''"] & \vdots \arrow[r, "d'"] \arrow[u, "d''"] & \cdots \arrow[u, "d''"] .
\end{tikzcd}$$

So, B^{pq} has horizontal position p and height q, and d' is a horizontal differential while d'' is a vertical differential.

10.2.2. *Remarks.* (1) Anti-commutativity relation $d'd'' + d''d' = 0$ can be interpreted as commutativity in the correct framework: the super-mathematics.

(2) If d' and d'' would happen to commute, we would have to correct one of these (say replace d'' by $(\check{d''})^{p,q} \stackrel{\text{def}}{=} (-1)^p (d'')^{p,q}$).

10.2.3. *The total complex* $Tot(B)$ *of a bicomplex and the cohomology of a bicomplex.* The total complex of a bicomplex is the complex $(Tot(B), d)$ with $Tot(B)^n \stackrel{\text{def}}{=} \oplus_{p+q=n} B^{p,q}$. The cohomology of B is by definition the cohomology of the complex $Tot(B)$.

10.2.4. *Decreasing filtrations* $'F$ *and* $''F$ *on a bicomplex and on the total complex.* The fact that the complex $Tot(B)$ has come from a bicomplex will be used to produce two decreasing filtrations on the complex $Tot(B)$. Actually, any complex A has a stupid decreasing filtration F where the subcomplex F_nA is obtained by erasing all terms A^k with $k < n$:

$$F_nA \stackrel{\text{def}}{=} \quad (\cdots \to 0 \to 0 \to A^n \to A^{n+1} \to A^{n+2} \to \cdots).$$

In turn, any bicomplex B has two decreasing filtrations $'F$ and $''F$. The sub-bicomplex $'F_iB$ of a bicomplex B is obtained by erasing the part of B which is on the left from the

i^{th} column, and symmetrically, $''F_jB$ is obtained by erasing beneath the j^{th} row. Say, the subbicomplex $'F_iB$ is given by

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow \\
\cdots & \xrightarrow{d'} & 0 & \xrightarrow{d'} & B^{i,2} & \xrightarrow{d'} & B^{i+1,2} & \xrightarrow{d'} & B^{i+2,2} & \xrightarrow{d'} & B^{i+3,2} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow \\
\cdots & \xrightarrow{d'} & 0 & \xrightarrow{d'} & B^{i,1} & \xrightarrow{d'} & B^{i+1,1} & \xrightarrow{d'} & B^{i+2,1} & \xrightarrow{d'} & B^{i+3,1} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow \\
\cdots & \xrightarrow{d'} & 0 & \xrightarrow{d'} & B^{i,0} & \xrightarrow{d'} & B^{i+1,0} & \xrightarrow{d'} & B^{i+2,0} & \xrightarrow{d'} & B^{i+3,0} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow \\
\cdots & \xrightarrow{d'} & 0 & \xrightarrow{d'} & B^{i,-1} & \xrightarrow{d'} & B^{i+1,-1} & \xrightarrow{d'} & B^{i+2,-1} & \xrightarrow{d'} & B^{i+3,-1} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow & & {\scriptstyle d''}\big\uparrow \\
\cdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \cdots .
\end{array}$$

This then induces filtrations on the total complex, say

$$[{}'F_iTot(B)]^n \stackrel{\text{def}}{=} Tot({}'F_iB)^n = \oplus_{p+q=n,\ p\geq i} B^{p,q} \subseteq Tot(B)^n \supseteq \oplus_{p+q=n,\ q\geq j} B^{p,q} = {}''F_j[Tot(B)^n].$$

Finally, ‘F and $''F$ induce filtrations on the cohomology

$${}'F_i\mathrm{H}^n(Tot\ B) \stackrel{\text{def}}{=} Im[\mathrm{H}^n(Tot\ {}'F_iB) \longrightarrow \mathrm{H}^n(Tot\ {}'F_iB)],$$

so the cohomology groups are extensions of pieces

$${}'Gr_i[\mathrm{H}^n(Tot\ B)] \stackrel{\text{def}}{=} \frac{{}'F_i\mathrm{H}^n(Tot\ B)}{{}'F_{i+1}\mathrm{H}^n(Tot\ B)}.$$

These pieces can be calculated by the method of spectral sequences (see 11).

10.3. **Partial cohomologies.** By taking the "horizontal" cohomology we obtain a bigraded object $'\mathrm{H}(B)$ with

$${}'\mathrm{H}(B)^{p.q} \stackrel{\text{def}}{=} \mathrm{H}^p(B^{\bullet,q}) = \frac{\mathrm{Ker}(B^{p,q} \xrightarrow{d'} B^{p+1.q})}{Im(B^{p-1,q} \xrightarrow{d'} B^{p.q})}.$$

The vertical differential d'' on B factors to a differential on $'\mathrm{H}(B)$ which we denote again by d'':

$${}'\mathrm{H}(B)^{p,q} \xrightarrow{d''} {}'\mathrm{H}(B)^{p,q+1}.$$

Next, we take the "vertical" cohomology of ${}'\mathrm{H}(B)$ (i.e., with respect to the new d''), and get a bigraded object ${}''\mathrm{H}({}'\mathrm{H}(B))$ with

$$''({}'\mathrm{H}(B))^{p,q} \overset{\text{def}}{=} \mathrm{H}^q({}'\mathrm{H}(B)^{p,\bullet}) = \frac{\mathrm{Ker}[{}'\mathrm{H}(B)^{p,q} \xrightarrow{d''} {}'\mathrm{H}(B)^{p.q+1}]}{Im[{}'\mathrm{H}(B)^{p,q-1} \xrightarrow{d''} {}'\mathrm{H}(B)^{p.q}]}.$$

One defines ${}''\mathrm{H}(B)$ and ${}'\mathrm{H}({}''\mathrm{H}(B))$ by switching the roles of the first and second coordinates.

10.3.1. *Remark.* Constructions ${}'\mathrm{H}(B)$ and ${}''\mathrm{H}(B)$ are upper bounds on the cohomology of the bicomplex, and ${}'\mathrm{H}({}''\mathrm{H}(B))$ and ${}''\mathrm{H}({}'\mathrm{H}(B))$ are even better upper bounds. The precise relation is given via the notion of spectral sequences (see 11).

10.4. **Resolutions of complexes.** An injective resolution of a complex $A \in C * \mathcal{A})$ is a quasi-isomorphism $A \longrightarrow I$ such that all I^n are injective objects of the abelian category $\mathcal{A}$. The next two theorems will state that injective resolutions of complexes exist and and are can be chosen compatible with short exact sequences of complexes.

10.4.1. *Theorem.* If $\mathcal{A}$ has enough injectives any $A \in C^+(\mathcal{A})$ has an injective resolution. More precisely,

(a) There is a bicomplex

$$\begin{tikzcd}
\cdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \cdots \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & I^{-1,2} \arrow[u, "d''"] \arrow[r, "d'"] & I^{0,2} \arrow[u, "d''"] \arrow[r, "d'"] & I^{1,2} \arrow[u, "d''"] \arrow[r, "d'"] & I^{2,2} \arrow[u, "d''"] \arrow[r, "d'"] & I^{3,2} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & I^{-1,1} \arrow[u, "d''"] \arrow[r, "d'"] & I^{0,1} \arrow[u, "d''"] \arrow[r, "d'"] & I^{1,1} \arrow[u, "d''"] \arrow[r, "d'"] & I^{2,1} \arrow[u, "d''"] \arrow[r, "d'"] & I^{3,1} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & I^{-1,0} \arrow[u, "d''"] \arrow[r, "d'"] & I^{0,0} \arrow[u, "d''"] \arrow[r, "d'"] & I^{1,0} \arrow[u, "d''"] \arrow[r, "d'"] & I^{2,0} \arrow[u, "d''"] \arrow[r, "d'"] & I^{3,0} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & A^{-1} \arrow[u, "d''"] \arrow[r, "d'"] & A^{0} \arrow[u, "d''"] \arrow[r, "d'"] & A^{1} \arrow[u, "d''"] \arrow[r, "d'"] & A^{2} \arrow[u, "d''"] \arrow[r, "d'"] & A^{3} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"]
\end{tikzcd}$$

such that the columns are injective resolutions of terms in the complex A.

(b) For any such bicomplex the canonical map $A \longrightarrow Tot(I)$ is an injective resolution of A.

10.4.2. *Theorem.* Let P and R be projective resolutions of objects A and C that appear in an exact sequence $0A \longrightarrow B \longrightarrow C \longrightarrow 0$. Then $Q = P \oplus R$ appears in a short exact sequence of projective resolutions

We start with the baby case of the theorem 10.4.2.

10.4.3. *Lemma.* Assume that $\mathcal{A}$ has enough injectives. A short exact sequence in $\mathcal{A}$ can always be lifted to a short exact sequence of injective resolutions. More precisely,

(a) Let I and K be injective resolutions of objects A and C that appear in an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$. Then all $J^n = I^n \oplus K^n$ appear in a short exact sequence of injective resolutions

$$\begin{tikzcd}
\vdots \arrow[r] & \vdots \arrow[r] & \vdots \arrow[r] & \vdots \arrow[r] & \vdots \\
0 \arrow[r] \arrow[u] & I^2 \arrow[r, "\alpha^2"] \arrow[u, "d_I^2"] & J^2 \arrow[r, "\beta^2"] \arrow[u, "d_J^2"] & K^2 \arrow[r] \arrow[u, "d_K^2"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & I^1 \arrow[r, "\alpha^1"] \arrow[u, "d_I^1"] & J^1 \arrow[r, "\beta^1"] \arrow[u, "d_J^1"] & K^1 \arrow[r] \arrow[u, "d_K^1"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & I^0 \arrow[r, "\alpha^0"] \arrow[u, "d_I^0"] & J^0 \arrow[r, "\beta^0"] \arrow[u, "d_J^0"] & K^0 \arrow[r] \arrow[u, "d_K^0"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & A \arrow[r, "\alpha"] \arrow[u, "\iota_A"] & B \arrow[r, "\beta"] \arrow[u, "\iota_B"] & C \arrow[r] \arrow[u, "\iota_C"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

with α^n the inclusion of the first summand and $\beta^n$the projection to the second summand.

(b) Any short exact sequence of injective resolutions

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & J \arrow[r] & K \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & A \arrow[r, "\alpha"] \arrow[u, "\iota_A"] & B \arrow[r, "\beta"] \arrow[u, "\iota_B"] & C \arrow[r] \arrow[u, "\iota_C"] & 0 \arrow[u]
\end{tikzcd}$$

necessarily splits on each level, i.e., $J^n \cong I^n \oplus K^n$. (However, complex J is not a sum of I and K.)

Proof. (a) We need to define ι_B so that the middle column is a resolution and the diagram commutes. Define $\iota_B : B \longrightarrow J^0 = I^0 \oplus K^0$ by

$$\iota_B(b) \stackrel{\text{def}}{=} \widetilde{\iota}_A(b) \oplus \iota_C(\beta b)$$

where $\widetilde{\iota_A} : B \longrightarrow I^0$ is any extension of $\iota_A : A \longrightarrow I^0$ (it exists since I^0 is injective). This choice ensures that the two squares that contain ι_B commute. Moreover, ι_B is injective since the kernel of the second component $\iota_C \circ \beta$ is $\mathrm{Ker}(\beta) = A$ and on $A \subseteq B$ ι_B is ι_A.

To continue in this way, we denote $\widetilde{B} = \mathrm{Coker}(\iota_B) = (I^0 \oplus K^0)/\iota_B(B)$ and notice that the second projection gives a surjection from $\widetilde{B}$ to $\widetilde{C} \overset{\text{def}}{=} K^0/\iota_C(C)$. Its kernel is the inverse of $\iota_C(C) \subseteq K^0$ under the second projection, taken modulo $\iota_B(B)$, i.e., $(I^0 \oplus \iota_C(C))/\iota_B(B) \cong I^0/\iota_A(A) \overset{\text{def}}{=} \widetilde{A}$. So we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I^1 \arrow[r, "\alpha^1"] & I^1 \oplus K^1 \arrow[r, "\beta^1"] & K^1 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & \widetilde{A} \arrow[r, "\alpha^1"] \arrow[u, "\overline{d^0_I}"] & \widetilde{B} \arrow[r, "\beta^1"] & \widetilde{C} \arrow[r] \arrow[u, "\overline{d^0_K}"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & I^0 \arrow[r, "\alpha^0"] \arrow[u, "q'"] & I^0 \oplus K^0 \arrow[r, "\beta^0"] \arrow[u, "q"] & K^0 \arrow[r] \arrow[u, "q''"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & A \arrow[r, "\alpha"] \arrow[u, "\iota_A"] & B \arrow[r, "\beta"] \arrow[u, "\iota_B"] & C \arrow[r] \arrow[u, "\iota_C"] & 0 \arrow[u]
\end{tikzcd}$$

in which $\overline{d^1_I}$ and $\overline{d^1_K}$ are factorizations of d^1_I and d^1_K through the the canonical quotient maps q' and q''. Maps $\overline{d^1_I}$ and $\overline{d^1_K}$ are embeddings, and we need to supply an embedding $\widetilde{B} \xrightarrow{\overline{d^1_J}} I^1 \oplus K^1$ which would give two more commuting squares, then d^1_J is defined as a composition of $\overline{d^1_J}$ and the quotient map q. However, this is precisely the problem we solved in the first step.

(b) Since I^n is injective, one can extend the identity map on I^n to $J^n \xrightarrow{\phi^n} I^n$, and gives a splitting i.e., a complement $\mathrm{Ker}(\phi^n)$ to I^n in J^n.

10.4.4. *Existence of injective resolutions of complexes: proof of the theorem 10.4.2.* .
(0) *About the proof.* We choose injective resolutions of coboundaries and cohomologies of A: $B^n(A) \longrightarrow J^n = J^{\bullet,n}$, $\mathrm{H}^n(A) \longrightarrow K^n = K^{\bullet,n}$. Then the n$^{\text{th}}$ column of I is $I^{\nu}, n = J^n \oplus K^n \oplus J^{n+1}$ and the horizontal differentials are the compositions

$$I^{p,q} = J^{p,q} \oplus K^{p,q} \oplus J^{p,q+1} \twoheadrightarrow J^{p,q+1} \subseteq J^{p,q+1} \oplus K^{p,q+1} \oplus J^{p,q+2} = I^{p,q+1}.$$

The vertical differential make the n^{th} column into a complex $I^{\bullet,n}$ such that

- $J^{\bullet,n} \subseteq I^{\bullet,n}$ is a subcomplex, and
- $K^{\bullet,n} \subseteq I^{\bullet,n}/J^{\bullet,n}$ is a subcomplex.

The fist task is to choose suitable injective resolutions of everything in site. We start by choosing injective resolutions of coboundaries and cohomologies

$$B^n(A) \to \mathcal{B}^n = \mathcal{B}^{n,\bullet}, \ \mathrm{H}^n(A) \to \mathcal{H}^n = \mathcal{H}^{n,\bullet}.$$

Now, cocycles are an extension of cohomologies and coboundaries, i.e., there is an exact sequence, and then A^n is an extension of $Z^n(A)$ and B^{n+1}, i.e., there are exact sequences

$$0 \to B^n(A) \to Z^n(A) \to \mathrm{H}^n(A) \to 0 \qquad \text{and} \qquad 0 \to Z^n(A) \to A^n \to B^{n+1}(A) \to 0.$$

By the preceding lemma 10.4.3 we can combine $\mathcal{H}$ and $\mathcal{B}$ to get injective resolutions of these short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{B}^n \arrow[r] & \mathcal{Z}^n \arrow[r] & \mathcal{H}^n \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & B^n(A) \arrow[r] \arrow[u, "\iota"] & Z^n(A) \arrow[r] \arrow[u, "\iota"] & \mathrm{H}^n(A) \arrow[r] \arrow[u, "\iota"] & 0 \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{Z}^n \arrow[r] & \mathcal{A}^n \arrow[r] & \mathcal{B}^{n+1} \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & Z^n(A) \arrow[r] \arrow[u, "\iota"] & A^n \arrow[r] \arrow[u, "\iota"] & B^{n+1}(A) \arrow[r] \arrow[u, "\iota"] & 0 \arrow[u]
\end{tikzcd}$$

Since $\mathcal{A}^n$ is an injective resolution of A^n, we can use them to build a bicomplex as in the the part (a) of the theorem, with the vertical differentials d'' the differentials in $\mathcal{A}^n$'s. Now we need the horizontal differentials $\mathcal{A}^n \xrightarrow{d'^n} \mathcal{A}^{n+1}$), these are the compositions

$$(\mathcal{A}^n \xrightarrow{d'^n} \mathcal{A}^{n+1}) \stackrel{\text{def}}{=} [\mathcal{A}^n \to \mathcal{B}^{n+1} \xrightarrow{\subseteq} \mathcal{Z}^{n+1} \xrightarrow{\subseteq} \mathcal{A}^{n+1}].$$

Since these are morphisms of complexes vertical and horizontal differentials commute, however this is easily corrected to $d'd'' + d''d' = 0$ by 10.2.2(2).

(b) Let I be a bicomplex from the part (a). A canonical map $A \xrightarrow{\iota_A} Tot(I)$ comes from $A^n \xrightarrow{\iota_{A^n}} I^{n,0} \subseteq Tort(I)^n$. Moreover, ι_A is a quasi isomorphism since all maps are quasi isomorphisms. This is easy to see directly and there is an elegant tool for such problems – the concept of spectral sequences (see 11).

10.4.5. *Existence of injective resolutions of exact sequences of complexes: proof of the theorem 10.4.2.* This is a combination of ideas in proofs of the lemma 10.4.3 and the theorem 10.4.1.

10.4.6. *Corollary.* If $\mathcal{A}$ has enough injectives any short exact sequence in $C^+(\mathcal{A})$ defines a distinguished triangle in $K(\mathcal{A})$.

Proof. By the theorem any short exact sequence in $C^+(\mathcal{A})$ is isomorphic in $K(\mathcal{A})$ to a short exact sequence of complexes with injective objects. Since such sequence splits on each level it defines a distinguished triangle in $K(\mathcal{A})$.

11. **Spectral sequences**

In general, spectral sequences[1] are associated to filtered complexes but we will be happy with the special case of spectral sequences associated to bicomplexes. The idea of a spectral sequence is to relate the cohomology of a complex with the cohomology of a simplified complex which may be more accessible.

11.1. **The notion of a spectral sequence.** A spectral sequence in an abelian category $\mathcal{A}$ is a sequence $(E_r, d_r, \iota_r),\ r \geq 0$. such that

(1) E_r is bigraded family of objects of $\mathcal{A}$, $E_r = \ (E_r^{p,q})_{r\geq 0}$
(2) $d_r : R_r \to E_r$ is a "differential", i.e., $d_r^2 = 0$, and it has type $(r, 1-r)$, i.e.,

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q+1-r}.$$

(3) ι_r is an isomorphism $E_{r+1} \overset{\cong}{\longrightarrow} \mathrm{H}(E_r, d_r)$, i.e.,

$$E_{r+1}^{p,q} \overset{\cong}{\longrightarrow} \mathrm{Ker}(E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q+1-r})/Im(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q}).$$

Remark. When dealing wilth spectral sequences one usually draws the $\mathbb{Z}^2$-grid in the plane and then one draws the directions of differentials: d_0 goes vertically by one, d_1 horizontaly by one, then d_2 goes two right and one down, etc.

11.1.1. *The limit of a spectral sequence.* The limit $E_\infty = \ \lim\ E)r$ can be defined for any spectral sequence but the most interesting case is when the spectral sequence stabilizes. We will say that the (p,q)-term stabilizes in the r^{th} term if for $s \geq r$ the differentials d_s from the (p,q)-term and into the (p,q)-term are zero. Then clearly

$$E_r^{p,q} \cong\ E_{r+1}^{p,q} \cong\ E_{r+2}^{p,q} \cong \cdots.$$

Then we say that

$$E_\infty^{p,q} \overset{\text{def}}{=} E_r^{p,q}.$$

We will say that the spectral sequence stabilizes (degenerates) in the r^{th} term if $d_s = 0,\ s \geq r$. Then E_∞ is by definition E_r.

11.1.2. *Stabilization criteria.*

[1]An approximate second hand quote:
"*Those were great times when only three of us*[2] *knew spectral sequences. We could prove everything and no one else could prove anything.*"

Lemma. (a) If there exists some $r_0 \geq 0$ and some $a, b \in \mathbb{Z}$ such that E_{r_0} is supported in the quadrant $\{(p,q) \in \mathbb{Z}^2;\ p \geq a \quad \text{and} \quad q \geq b\}$ then for each pair (p,q) the terms E_r^{pq} stabilize.

(b) If for some r all $p+q$ with $E_r^{pq} \neq 0$ are of the same parity, then $0 = d_r = d_{r+1} = \cdots$, i.e., sequence stabilizes at E_r.

Proof. First notice that if $E_r^{pq} =$ for some p, q, r then for $s \geq r$ one also has $E_s^{pq} = 0$ since E_s^{pq} is a subquotient of E_r^{pq}.

(a) $d_r : E_r^{pq} \to E_r^{p+r,q+1-r}$ is zero when $r \geq r_0$ and $q+1-r < b$, i.e., $r > q+1-b$. Also, $d_r : E_r^{p-r,q-1+r} \to E_r^{p,q}$ is zero when $r \geq r_0$ and $p - r < a$, i.e., $r > p - a$.

(b) Since d_s is of type $(s, 1-s)$ it changes the total parity $p+q$. So. always either the source or the target of d^s is zero.

Remarks. (0) The simplest way to prove that a spectral sequence degenerates at E_r is if one can see that for $s \geq r$ and any p, q one of objects $E_r^{p,q}$ or $E_r^{p+s,q+1-s}$ is zero.

(1) However, sometimes one needs to really study particular differential d_r to see that it is zero.

11.2. **Spectral sequence of a filtered complex.** Consider a complex $K \in C(\mathcal{A})$ with a decreasing filtration $F_\bullet$, so $F_pK^n \subseteq K^n$ and the differential $d = d_K$ takes F_pK^n to F_pK^{n+1}.

Then one can grade and obtain complexes $Gr_nK = F_pK/F_{p+1}K$ which one can think of as building blocks for K itself since K is an extension of these graded pieces. We will use a filtration of K to get information about cohomology H^*K. First observe that

11.2.1. *A filtration F of a complex K induces a filtration F on the cohomology.* Inclusion $F_pK \subseteq K$ is a map of complexes so it gives maps $H^n(F_pK) \to H^n(K)$, and therefore it defines subgroups

$$F_p\big(H^n(K)\big) \stackrel{\text{def}}{=} Im[H^n(F_pK) \to H^n(K)] \subseteq H^n(K),\ p \in \mathbb{Z}.$$

Notice that

$$H^n(K) = \frac{Z^n(K)}{B^n(K)} = \frac{\{a \in K^n;\ da = 0\}}{dK^{n-1}} \supseteq \frac{\{a \in F_pK^n;\ da = 0\} + dK^{n-1}}{dK^{n-1}} = F_p\big(H^n(K)\big).$$

Therefore the graded pieces are

$$Gr_p\big(H^n(K)\big) = \ .$$

11.2.2. *Theorem.* A decreasing filtration F of a complex K defines a spectral sequence E with

(1) $E_0^{p,q} = Gr_pH^n(K)$
(2) $E_1^{p,q} = H^{p+q}[Gr_p(K)]$
(3) $E_\infty^{p,q} = Gr_p[\mathrm{H}^{p+q}(K)]$

In general,

$$E_r^{p,q} \overset{\text{def}}{=} \frac{\{a \in F_p K^{p+q};\ da \in F_{p+r} K^{p+q+1}\}}{d\ F_{p+1-r} K^{p+q-1}\ +\ F_{p+1} K^{p+q}},$$

and each d_r is a factorization to E_r of the differential d on K.

11.2.3. *Remark.* (Usefulness of this formalism.) The idea is that passing from K to $H^*(K)$ kills some less important information. The decrease of information is clear since $H^n(K)$ is a subquotient of K^n so it is in some sense lesser then K^n.

Now, if we have a filtration of K it provides a way of killing information in many smaller and simpler steps. One first passes from K to $Gr(K) = E^0$ by grading and then from E_r to E_{r+1} by taking cohomology. This process converges to E_∞ which is not quite the cohomology $H^*(K)$ but it is close since $H^n(K)$ is a an extension of pieces $Gr_p[H^n(K)] = E_\infty^{p,n-p},\ p \in \mathbb{Z}$.

11.2.4. *Remark.* The basic information we get from here is that the "size" of $H^n(K)$ has an upper bound $\oplus_{p\in\mathbb{Z}} H^{p+q}[Gr_p(K)]$. The precise version is that: $H^n(K)$ is an extension of pieces E_∞^{pq} which are subquotients of groups $E_1^{pq} = H^{p+q}[Gr_p(K)]$.

Of course we get better information (a finer upper bound) if we can calculate more terms E_r, beyond E_1; and ideally we would like to calculate all E_r's in a given situation. This is actually "often" possible and usually in the situation when sequence stabilizes early. A number of deep theorems in mathematics takes form of degeneration of a spectral sequence in E_2 or E_3 term.

11.2.5. *Remark.* (The origin of this formalism.) One constructs the spectral sequence by starting with the idea of replacing K by its simplified version $E_0 = Gr(K)$. This is clearly a bigraded object since two indices appear in $Gr_p(K^n)$. so all computations will be in the realm of bigraded objects. It is very clear $E_0^{p\bullet}$ is a complex since it is ta quoteint of a complex F_pK by a subcomplex, in other words, the differential on K defines a differential d_0 on the subquotient $E_0 = Gr(K)$ of K. Therefore one can define E_1 as cohomology of (E_0, d_0). However, we again notice that E_1 is a suquotient of K and the differential on K defines a new differential d_1 on E_1. So, one can define E_2. After repeating this process a few times we start to expect that it will continue forever and we get a feeling for what any E_r, d_r will look like (this is the above formula for E_r). Then we verify this expectation by straightforward algebra. Finally, once we have the formulas for E_r's it is easy to we notice that E_r's converge to the graded cohomology of K.

11.2.6. *Proof.* The idea is that E_r is the "r^{th} apprximation" of cohomology of K : Here the meanining of "r^{th} apprximation" is that we take those a's in F_p that the differential moves to F_{p+r}, now, we expect that for large r the filtered piece F_{p+r} will be much smaller then F_p; so we are really asking that da be "r-smaller" then a; and this is an approximation to $da = 0$.

(**A.** $\boldsymbol{E_0}$) When $r=0$ we get

$$E_0^{p,q} \stackrel{\text{def}}{=} \frac{\{a\in F_pK^{p+q};\ da\in F_pK^{p+q+1}\}}{dF_{p+1}K^{p+q-1}\ +\ F_{p+1}K^{p+q}}\ =\ \frac{F_pK^{p+q}}{F_{p+1}K^{p+q}} =\ Gr_p(K^{p+q}),$$

since $dF_pK^{p+q}\subseteq F_pK^{p+q+1}$ and $dF_{p+1}K^{p+q-1}\subseteq F_{p+}K^{p+q}$.

(**B. Differential** d_r) Now define the differentials $d_r: E_r^{pq}\to E_r^{p+r,p+1-r}$ by sending the class

$$[a] \stackrel{\text{def}}{=} a+d\ F_{p+1-r}K^{p+q-1}\ +\ F_{p+1}K^{p+q}\in E_r^{pq}$$

to the class

$$[da]+d\ F_{(p+r)+1-r}K^{(p+r)+(q+1-r)-1}\ +\ F_{(p+r)+1}K^{(p+r)+(q+1-r)}$$
$$=da+\ d\ F_{p+1}K^{p+q}\ +\ F_{p+r+1}K^{p+q+1}.$$

To see that this is well defined first recall that classes $[a]\in E_r^{pq}$ are represented by elements $a\in F_pK^{p+q}$ such that $da\in F_{p+r}K^{p+q+1}$. The second condition, together with $d(da)=0\in F_{(p+r)+r}K^{p+q+1+1}$ implies that da really defines a class $[da]$ in $E_r^{p+r,q+1-r}$.

Next, the class $[da]\in E_r^{p+r,q+1-r}$ depend only on the class $[a\in E_r^{pq}]$ (and not on the choice of a), since the differential d in K to sends the denominator $d(F_{p+1-r}K^{p+q-1})+\ F_{p+1}K^{p+q}$ in E_r^{pq} to the denominator $d(F_{p+1}K^{p+q})+\ F_{p+r+1}K^{p+q+1}$.

(**C.** $H^*(E_r)=\ E_{r+1}$)

(**D.** $\boldsymbol{E_1}$) The description of E_1^{pq} as $H^{p+q}[Gr_p(K)]$ now follows since

- (i) We have identified bigraded objects $E_0^{pq}=(Gr_pK)^{p,q}$, (ii) this is really an identification of complexes since the differentials on both E_0^{pq} and $(Gr_pK)^{p,q}$ are induced from the differential d on K.
- (iii) So, the cohomology $H^*(Gr_pK)$ of $Gr_p(K)$ gets identified with the cohomology E_1 of (E_0,d_0).

To see this also directly,

$$E_1^{p,q} \stackrel{\text{def}}{=} \frac{\{a\in F_pK^{p+q};\ da\in F_{p+1}K^{p+q+1}\}}{d\ F_pK^{p+q-1}\ +\ F_{p+1}K^{p+q}}$$

The condition on $a\in F_pK^{p+q}=\ (F_pK)^{p+q}$ to define a class in E_1^{pq} is that da lies in $F_{p+1}K^{p+q+1}=(F_{p+1}K)^{p+q+1}$, i.e., that for the class

$$[a]=a+F_{p+1}K^{p+q}\in\ F_pK^{p+q}/F_{p+1}K^{p+q}=(Gr_pK)^{p+q}),$$

the differential $d[a]\stackrel{\text{def}}{=}[da]\in\ (Gr_pK)^{p+q+1})=F_pK^{p+q+1}/F_{p+1}K^{p+q+1}$ is zero, i.e., that $[a]$ defines a cohomology class in $H^{p+q}(Gr_pK)$.

(**E.** $\boldsymbol{E_\infty}$)

11.3. **Spectral sequences of bicomplexes.**

11.3.1. *Theorem.* To any bicomplex B one associates two ("symmetric") spectral sequences $'E$ and $''E$. The first one satisfies

(1) $'E_0^{p,q} = \; B^{p,q}$
(2) $'E_1^{p,q} = \; ''\mathrm{H}^{p,q}(B)$
(3) $'E_2^{p,q} = \; '\mathrm{H}^p[''\mathrm{H}^{\bullet,q}(B)$
(4) $'E_\infty^{p,q} = \; 'Gr_p[\mathrm{H}^{p+q}(Tot\ B)]$

11.3.2. *Remark.* The basic consequence is that the piece $'Gr_p[\mathrm{H}^n(Tot\ B)]$ of $\mathrm{H}^n(Tot\ B)$ is a subquotient of $'\mathrm{H}^{n-p}[''\mathrm{H}^{\bullet,q}(B)]$, hence in particular, of $''\mathrm{H}^{p,n-p}(B)$ (since $E_{r+1}^{i,j}$ is always a subquotient of $E_r^{i,j}$). This gives upper bounds on the dimension of $\mathrm{H}^n(Tot\ B)$.

11.3.3. *Remark.* We are fond of bicomplexes such that the first spectral sequence degenerates at E_2, then we recover the constituents of $\mathrm{H}^n(Tot\ B)$ from partial cohomology

$$Gr_\bullet[\mathrm{H}^n(Tot\ B)] \cong \; \oplus_{p+q=n}\ '\mathrm{H}^p[''\mathrm{H}^{\bullet,n-p}(B).$$

12. **Derived categories of abelian categories**

Historically the notion of triangulated categories has been discovered independently in

- Algebra: derived category of an abelian category $\mathcal{A}$ is a convenient setting for doing homological algebra – i.e., the calculus of complexes in $\mathcal{A}$ (and much more).
- Topology: the stable homotopy theory deals with the category whose objects are topological spaces (rather then complexes!), but the total structure is the same as for a derived category $D(\mathcal{A})$ (shift is giving by the operation of suspension of topological spaces, exact triangles come from the topological construction of the mapping cone C_f corresponding to a continuous map $f : X \to Y$ of topological spaces ...)[(3)]

Here we will only be concerned with the (more popular) appearance of derived categories in algebra.

12.1. **Derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$.** The objects of $D(\mathcal{A})$ are again the complexes in $\mathcal{A}$, however $\mathrm{Hom}_{D(\mathcal{A})}(A, B)$ is an equivalence class of diagrams in $K(\mathcal{A})$:

(1) Let $\widetilde{\mathrm{Hom}}_{D(\mathcal{A})}(A, B)$ be the class of all diagrams in $K(\mathcal{A})$ of the form

$$A \xrightarrow{\phi} X \xleftarrow{s} B, \qquad \text{with } s \text{ a quasi-isomorphism.}$$

[3]Topologists did not notice the Octahedron axiom. Also, here is still no published/readable presentation of the construction of derived category of topological spaces.

On the other hand it is still not known whether the Octahedron axiom is a consequence of other axioms. It requires existence of an exact triangle such that five squares commute. Existence of the traingle and commutativity of 4 squares is a consequence of the remaining axioms.

(2) Two diagrams $A \xrightarrow{\phi_i} X_i \xleftarrow{s_i} B,\ i=1,2;$ are equivalent iff they are quasi-isomorphic to a third diagram, in the sense that there are quasi-isomorphisms $X_i \xrightarrow{u_i} X,\ i=1,2;$ such that

$$\begin{array}{rcl} u_1\circ\phi_1 = & u_2\circ\phi_2 \\ u_1\circ s_1 = & u_2\circ s_2 \end{array}, \qquad i.e., \qquad \begin{tikzcd} A \arrow[r, "\phi_1"] \arrow[d, "="'] & X_1 \arrow[d, "u_1"'] & B \arrow[l, "s_1"'] \arrow[d, "="'] \\ A \arrow[r, "\phi"] & X & B \arrow[l, "s"'] \\ A \arrow[r, "\phi_2"] \arrow[u, "="] & X_2 \arrow[u, "u_2"] & B. \arrow[l, "s_2"'] \arrow[u, "="] \end{tikzcd}$$

12.1.1. *Symmetry.* It follows from the next lemma that one can equivalently represent morphisms by diagrams $A \xleftarrow{s} C \xrightarrow{\phi} B$ with s a quasi-isomorphism.

12.1.2. *Lemma.* If s is a quasi-isomorphism a diagram $A \xleftarrow{s} B$ in $K(\mathcal{A})$ can be canonically completed to a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "u"'] & B \arrow[d, "s"'] \\ A \arrow[r, "\phi"] & X \end{tikzcd}$$

with u a quasi-isomorphism. (Conversely, one can also complete u, ψ to s, ϕ.)

12.1.3. *Remarks.* (0) We can denote the map represented by the diagram $A \xrightarrow{\phi} X \xleftarrow{s} B$ by $[s,\phi]$, its intuitive meaning is that it the morphism $s^{-1}\circ\phi$ once s is inverted. Identifications of diagrams should correspond to equalities ${s_1}^{-1}\circ\phi_1 = {s_2}^{-1}\circ\phi_2$, to compare it with the requirement (2) rewrite it as first as $(u_1\circ s_1)^{-1}\circ(u_1\circ\phi_1) = (u_2\circ s_2)^{-1}\circ(u_2\circ\phi_2)$, and then as $(u_1\circ\phi_1)\circ(u_2\circ\phi_2)^{-1} = (u_1\circ s_1)\circ(u_2\circ s_2)^{-1}$; then (2) actually says that both sides are equal to the identity 1_X.

(1) The composition of (equivalence classes of) diagrams is based on lemma 12.1.2. Putting together two diagrams $[s,\phi]$ and $[u,\psi]$ gives $A \xrightarrow{\phi} X \xleftarrow{s} B \xrightarrow{\psi} Y \xleftarrow{u} C$. Now lemma 12.1.2 allows us to replace the inner part $X \xleftarrow{s} B \xrightarrow{\psi} Y$ by $X \xleftarrow{\psi'} Z \xrightarrow{s'} Y$ with s' a quasi-isomorphism. This gives a diagram $A \xrightarrow{\phi} X \xleftarrow{\psi'} Z \xrightarrow{s'} Y \xleftarrow{u} C$ and we define

$$\overset{\text{def}}{=}[u,\psi]\circ[s,\phi]\circ[s'\circ u, \psi'\circ\phi].$$

(2) The above procedure inverts quasi-isomorphisms in $K(\mathcal{A})$. We can actually invert quasi-isomorphisms directly in $C(\mathcal{A})$, however lemma 12.1.2 does not hold in $C(\mathcal{A})$, so

one is forced to take a more complicated definition of maps as coming from long diagrams

$$A \xrightarrow{\phi_0} X_0 \xleftarrow{s_0} B_0 \xrightarrow{\phi_1} X_0 \xleftarrow{s_1} B_1 \xrightarrow{\phi_2} X_2 \xleftarrow{s_2} \cdots \xrightarrow{\phi_{n-1}} X_{n-1} \xleftarrow{s_{n-1}} B_{n-1} \xrightarrow{\phi_n} X_n \xleftarrow{s_n} B_n = B$$

which are composed in the obvious way.

12.1.4. *Theorem.* $D(\mathcal{A})$ is a triangulated category if we define the exact triangles as images of exact triangles in $K(\mathcal{A})$.

One can also define triangulated subcategories $D^?(\mathcal{A})$ for $? \in \{+, b, -\}$, these are full subcategories of all complexes A such that $H^\bullet(A) \in C^?(\mathcal{A})$ (with zero differential). For $\mathcal{Z} \subseteq \mathcal{Z}$ one can also define a full subcategory $D^{\mathcal{Z}}(\mathcal{A})$ by again requiring a condition on cohomology: that $H^\bullet(A) \in C^{\mathcal{Z}}(\mathcal{A})$.

12.1.5. *The origin of exact triangles in $D(\mathcal{A})$.* They can be associated to either of the following:

(1) a map of complexes,
(2) a short exact sequence of complexes that splits on each level,
(3) if $\mathcal{A}$ has enough injectives, any short exact sequence of complexes in $C^+(\mathcal{A})$, and if $\mathcal{A}$ has enough projectives, any short exact sequence of complexes in $C^-(\mathcal{A})$.

12.1.6. *Cohomology functors $H^i : D(\mathcal{A}) \to \mathcal{A}$.* H^i is clearly defined on objects, on morphisms it is well defined by $\mathrm{H}^i([s,\phi]) \overset{\text{def}}{=} \mathrm{H}^i(s)^{-1} \circ \mathrm{H}^i(\phi)$.

12.2. **Truncations.** These are functors

$$D^{\le n}(\mathcal{A}) \xleftarrow{\tau_{\le n}} D(\mathcal{A}) \xrightarrow{\tau_{\le n}} D^{\ge n},$$

they come with canonical maps $\tau_{\le n}A \to A \to \tau_{\ge n}A$ defined by

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \longrightarrow & Z^n(A) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \quad (\tau_{\le n}(A)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & A^{n+2} & \longrightarrow & \cdots \quad (\tau_{\le n}(A)) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A^n/B^n(A) & \longrightarrow & A^{n+1} & \longrightarrow & A^{n+2} & \longrightarrow & \cdots \quad (\tau_{\ge n}(A))
\end{array}$$

12.2.1. *Lemma.* (a) $\tau_{\le n}$ is the left adjoint to the inclusion $D^{\le n}(\mathcal{A}) \subseteq D(\mathcal{A})$, and $\tau_{\ge n}$ is the right adjoint to the inclusion $D^{\ge n}(\mathcal{A}) \subseteq D(\mathcal{A})$.

(b) $\mathrm{H}^i(\tau_{\le n}A) = \left\{ \begin{smallmatrix} \mathrm{H}^i(A) & i \le n \\ 0 & i > n \end{smallmatrix} \right\}$, and $\mathrm{H}^i(\tau_{\ge n}A) = \left\{ \begin{smallmatrix} \mathrm{H}^i(A) & i \ge n \\ 0 & i < n \end{smallmatrix} \right\}$.

12.3. **Inclusion $\mathcal{A} \hookrightarrow D(\mathcal{A})$.**

Lemma. (a) By interpreting each $A \in \mathcal{A}$ as a complex concentrated in degree zero, one identifies $\mathcal{A}$ with a full subcategory of $C(\mathcal{A})$, $K(\mathcal{A})$ or $D(\mathcal{A})$.

(b) The inclusion of $\mathcal{A}$ into the full subcategory $D^0(\mathcal{A})$ (all complexes A with $H^i(A) = 0$ for $i \neq 0$) is an equivalence of categories. (The difference is that $D^0(\mathcal{A})$ is closed in $D(\mathcal{A})$ under isomorphisms and $\mathcal{A}$ is not).

Proof. In (a) we need to see that for A, B in $\mathcal{A}$ the canonical map $\mathrm{Hom}_{\mathcal{A}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{X}}(A, B$ is an isomorphism:

- when $\mathcal{X} = C(\mathcal{A})$ since any homotopy between two complexes concentrated in degree zero is clearly 0 (recall that $h^n : A^N \longrightarrow B^{n-1}$),
- when $\mathcal{X} = D(\mathcal{A})$ one shows that
  (1) any diagram in $K(\mathcal{A})$ of the form $A \xrightarrow{\phi} X \xleftarrow{s} B$ with s a quasi-isomorphism, is equivalent to a diagram in $\mathcal{A}$ $A \xrightarrow{\mathrm{H}^0(\phi)} \mathrm{H}^0(X) \xleftarrow{\mathrm{H}^0(s)} B$, hence to $A \xrightarrow{\mathrm{H}^0(s)^{-1}\mathrm{H}^0(\phi)} B \xleftarrow{1_B} B$, hence it comes from a map $A \xrightarrow{\mathrm{H}^0(s)^{-1}\mathrm{H}^0(\phi)} B$ in $\mathcal{A}$,
  (2) Two diagrams of the form $A \xrightarrow{\alpha_i} B \xleftarrow{1_B} B$, are equivalent iff $\alpha_1 = \alpha_2$.

Remark. Part (b) of the lemma describes $\mathcal{A}$ inside $D(\mathcal{A})$ (up to equivalence) by only using the functors H^i on $D(\mathcal{A})$.

12.4. **Homotopy description of the derived category.** The following provides a down to earth description of the derived category and gives us a way to calculate in the derived category.

12.4.1. *Theorem.* Let $\mathcal{I}_{\mathcal{A}}$ be the full subcategory of $\mathcal{A}$ consisting of all injective objects.

(a) $\mathcal{I}_{\mathcal{A}}$ is an abelian subcategory.

(b) If $\mathcal{A}$ has enough injectives the canonical functors

$$K^+(\mathcal{I}_{\mathcal{A}}) \xrightarrow{\sigma} D^+(\mathcal{I}_{\mathcal{A}}) \xrightarrow{\tau} D^+(\mathcal{A})$$

are equivalences of categories.

Proof. (A sketch.) One first observes that

- Quasi-isomorphism between complexes in $C^+(\mathcal{I})$ are always homotopical equivalences.

This tells us that quasi-isomorphisms in $K^+(\mathcal{I})$ are actually isomorphisms in $K^+(\mathcal{I})$. Since quasi-isomorphisms in $K^+(\mathcal{A})$ are already invertible the passage to $D^+(\mathcal{I})$ obviously gives an equivalence $K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{I})$.

We know that any complex A in $C^+(\mathcal{A})$ is quasi-isomorphic to its injective resolution I, and also any map $A' \longrightarrow A''$ is quasi-isomorphic to a map of injective resolutions $I' \longrightarrow I''$. This is the surjectivity of τ on objects and morphisms.

The remaining observation is that

- For $I, J \in D^+(\mathcal{I})$ map $\mathrm{Hom}_{D(\mathcal{I})}(I,J) \to \mathrm{Hom}_{D(\mathcal{A})}(I,J)$ is injective.

13. **Derived functors**

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.

13.1. **Derived functors $R^pF : \mathcal{A} \to \mathcal{A}$.** Suppose that $\mathcal{A}$ has enough injectives and set

$$R^pF(A) = \mathrm{H}^p(FI) \quad \text{for any injective resolution } I \text{ of } A.$$

We call these the (right) derived functors of F.

13.1.1. *Theorem.* (a) Functors R^p are well defined.

(b) $R^0F \cong F$.

(c) Any short exact sequence $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ in $\mathcal{A}$ defines a long exact sequence of derived functors

$$0 \to R^0F(A') \xrightarrow{R^0F(\alpha)} R^0F(A) \xrightarrow{RF^n(\beta)} R^0F(A'') \xrightarrow{\partial^0} \cdots \xrightarrow{\partial^{n-1}} R^nF(A') \xrightarrow{R^nF(\alpha)} R^nF(A) \xrightarrow{RF^n(\beta)} R^nF(A'')$$

Proof. (a) and (b) follow from **??**. (c) follows from the lemma 10.4.3. First we can choose an injective resolution $0 \to I' \xrightarrow{\alpha} I \xrightarrow{\beta} I'' \to 0$ of the short exact sequence $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$, and then we apply F to it. The sequence of complexes $0 \to F(I') \xrightarrow{\alpha} F(I) \xrightarrow{\beta} F(I'') \to 0$ is exact since the short exact sequence of resolutions splits levelwise (lemma 10.4.3b), and F is additive.

13.1.2. *Remark.* , even if F is not left exact the above construction produces a left exact functor R^F.

13.2. **Derived functors $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$.** We start with the definition of its right derived functor RF as the universal one among all extensions of $\mathcal{A} \xrightarrow{F} \mathcal{B}$ to $D^+(\mathcal{A}) \to D^+(\mathcal{B})$. Then we see that the "replacement by injective resolution" construction satisfies the universality property.

13.2.1. *Notion of the derived functor of F.* A functor between two triangulated categories is said to be a morphism of triangulated categories (a triangulated or ∂-functor) if it preserves all structure of these categories:

1. it is additive,
2. it preserves shifts
3. it preserves exact triangles

The right derived functor of an additive functor $F:\mathcal{A}\to\mathcal{B}$ is the universal one among all extensions of F to $D^+(\mathcal{A})\to D^+(\mathcal{B})$. Precisely, it consists of the following data

- A triangulated functor $RF: D^+(\mathcal{A})\to D^+(\mathcal{B})$,
- a morphism of functors $i_{\mathcal{B}}\circ F \xrightarrow{\xi} RF\circ i_{\mathcal{A}}$;

and these data should satisfy the universality property:

- for any other such pair $(\widetilde{RF},\widetilde{\xi})$, morphism $\widetilde{\xi}$ factors uniquely through ξ, i.e., there is a unique morphism of functors $RF \xrightarrow{\mu} \widetilde{RF}$ such that $\widetilde{\xi}=\mu\circ\xi$, i.e., $i_{\mathcal{B}}\circ F \xrightarrow{\widetilde{\xi}} \widetilde{R}F\circ i_{\mathcal{A}}$ is the composition $i_{\mathcal{B}}\circ F \xrightarrow{\xi} RF\circ i_{\mathcal{A}} \xrightarrow{\mu\circ 1_{\mathcal{A}}} \widetilde{R}F\circ i_{\mathcal{A}}$.

13.2.2. *Remark.* (0) For $A\in\mathcal{A}$, $i_{\mathcal{A}}A$ is A viewed as a complex, and ξ_A relates objects that should be the same if RF extends F: $RF(i_{\mathcal{A}}A)\xrightarrow{\xi_A} i_{\mathcal{B}}(FA)$. So, intuitively ξ is the "commutativity constraint" for the diagram,
$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ {\scriptstyle i_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle i_{\mathcal{B}}} \\ D^+(\mathcal{A}) & \xrightarrow{RF} & D^+(\mathcal{B}) \end{array}$$
, it takes care of the fact that the functors $i_{\mathcal{B}}\circ F$ and $RF\circ i_{\mathcal{A}}$ are not literally the same but only canonically isomorphic (though the universality property does not require ξ to be an isomorphism, in practice it will be an isomorphism).

(1) The problem is when does the universal extension exist? The simplest case is when F is exact, then then one can define RF simply as F acting on complexes. In general the most useful criterion is

13.2.3. *Theorem.* Suppose that $\mathcal{A}$ has enough injectives. Then for any additive functor $\mathcal{A}\xrightarrow{F}\mathcal{B}$:

(a) $RF: D^+(\mathcal{A})\to D(\mathcal{B})$ exists.

(b) For any complex $A\in D^+(\mathcal{A})$, there is a canonical isomorphism $(RF)A\cong\ F(I)$ for any injective resolution I of A.

(c) In particular, for $A\in\mathcal{A}$ the cohomologies of $(RF)A$ are the derived functors $(R^iF)A$ introduced above.

Proof. Recall from 12.4.1 that one has equivalences $K^+(\mathcal{I}_{\mathcal{A}})\xrightarrow{\sigma} D^+(\mathcal{I}_{\mathcal{A}})\xrightarrow{\tau} D^+(\mathcal{A})$, and from recall that an additive functor $F_{\mathcal{I}}\overset{\text{def}}{=}(\mathcal{I}_{\mathcal{A}}\subseteq\mathcal{A}\xrightarrow{F}\mathcal{B})$ has a canonical extension $K(F_{\mathcal{I}}): K(\mathcal{A})\to K(\mathcal{B})$. So we can define RF as a composition

$$RF\overset{\text{def}}{=}\ [D^+(\mathcal{A})\ \xrightarrow{(\tau\circ\sigma)^{-1}}\ K^+(\mathcal{I}_{\mathcal{A}})\xrightarrow{K(F_{\mathcal{I}})} K^+(\mathcal{B})\to D^+(\mathcal{B})].$$

Our construction satisfies the description of RF in (b). We have $(\tau\sigma)^{-1}I = I$ and a quasi-isomorphism $A \to I$. Therefore, $RF(A) \stackrel{\text{def}}{=} [K(F_{\mathcal{I}})\circ(\tau\circ\sigma)^{-1}]\ A \ = \ K(F_{\mathcal{I}})\ I \ = \ F(I)$. (c) follows from (b).

It remains to check that our RF is the universal extension.

RF preserves shifts by its definition. Any exact triangle in $D^{(}\mathcal{A})$ is isomorphic to a triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ that comes from an exact sequence $0 \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\gamma} C \to 0$ in $C(\mathcal{A})$ that splits on each level. Moreover, we can replace the exact sequence $0 \xrightarrow{\alpha} A' \xrightarrow{\beta} A \xrightarrow{\gamma} A'' \to 0$ with an isomorphic (in $D(\mathcal{A})$) short exact sequence of injective resolutions $0 \to I' \xrightarrow{a} I \xrightarrow{b} I'' \to 0$. Since it also splits on each level its F-image $0 \to F(I') \xrightarrow{F(a)} F(I) \xrightarrow{F(b)} F(I'') \to 0$ is an exact sequence in $C(\mathcal{B})$. Therefore it defines an exact triangle $F(I') \xrightarrow{F(a)} F(I) \xrightarrow{F(b)} F(I'') \xrightarrow{\widetilde{\gamma}} F(I')[1]$ in $D(\mathcal{A})$. To see that this is the triangle we observe that by definition $F(I') = RF(A')$ $F(I) = RF(A)$ $F(I'') = RF(A'')$, and also $F(a) = RF(\alpha)$ and $F(b) = RF(\beta)$. It remains to see that $\widetilde{\gamma} = RF(\gamma)$. For this recall that γ and $\widetilde{\gamma}$ are defined using splittings of the first and second row of

$$\begin{array}{ccccccccc}
0 & \xrightarrow{\alpha} & A' & \xrightarrow{\beta} & A & \xrightarrow{\gamma} & A'' & \longrightarrow & 0 \\
 & & {\scriptstyle\iota'}\downarrow & & {\scriptstyle\iota}\downarrow & & {\scriptstyle\iota''}\downarrow & & \\
0 & \longrightarrow & I' & \xrightarrow{a} & I & \xrightarrow{b} & I'' & \longrightarrow & 0.
\end{array}$$

So, we need to be able to choose the splitting of the second row compatible with the one in the first row. However, this is clearly possible by the construction of the second row.

13.3. **Usefulness of the derived category.** We list a few more reasons to rejoice in derived categories.

13.3.1. *Some historical reasons for the introduction of derived categories.* For a left exact functor F we have derived functors R^iF and and then concept of derived category allows us to glue them into one functor RF. Does this *gluing* make a difference?

(1) When we calculate cohomology from a complex A we loose some information. This is not a problem if our end goal is the calculation of this cohomology $H^*(A)$, however if this is just the first step, the next step may not be doable becuasu of the lost information. For this reason it is better not to discard the complex A but to keep all relevant information it contains. This is achieved by the setting of derived categories.
(2) The first example where loss of information is kept is the relation between derived functors of F, G and the composition $G\circ F$. It is easy to check from definitions that for two left exact functors F and G one has $R(G\circ F) \cong \ RG\circ RF$. If we instead work with R^iF, R^jG and $R^n(G\circ F)$ the above formula degenerates to a relation which is weak (some information gets lost) and complicated (uses language

of spectral sequences). So, seemingly more complicated construction RF is more natural and has better properties then a bunch of functors R^iF.

If $G:\mathcal{B}\to\mathcal{C}$ is right exact and $\mathcal{B}$ has enough projectives then one similarly has functors $L^iG:\mathcal{B}\to\mathcal{C}$ and they glue to $LG:D^-(\mathcal{B})\to D^-(\mathcal{C})$.

However, if one has a combination of two functors $\mathcal{A}\xrightarrow{F}\mathcal{B}\xrightarrow{G}\mathcal{C}$ and F is left exact while G is right exact, we have a functor $D^b(\mathcal{A})\xrightarrow{LG\circ RF}D^b(\mathcal{C})$ which is often very useful but has no obvious analogue as a family of functors from $\mathcal{A}$ to $\mathcal{C}$ since the composition $G\circ F$ need not be neither left nor right exact.[(4)]

(3) Some of essential objects and tools exist only on the level of derived categories: the dualizing sheaves in topology and algebraic geometry, the perverse sheaves in topology (and more recently also in algebraic geometry).

13.3.2. *Some unexpected gains.* It turns out that in practice many deep relations between abelian categories $\mathcal{A}$ and $\mathcal{B}$ become understandable only on the level of derived categories, for instance $D(\mathcal{A})$ and $D(\mathcal{B})$ are sometimes equivalent though the abelian categories $\mathcal{A}$ and $\mathcal{B}$ are very different.

This observation has revolutionarized several areas of mathematics and physics.

[4] One of the famous EGA books (Elements de Geometrie algebrique, the foundations of the contemporary language of algebraic geometry) deals just with the construction of the derived functor of the bifunctor $\Gamma(X,\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{B})$ which (there were no derived categories at the time), they had to do in a very explicit and involved way since Γ is left exact and tensoring is right.

Appendix A. **Manifolds**

Appendix A on manifolds contains some basic definitions and reformulation of manifolds in terms of structure sheaves. Out of all of this we will only need the notion of vector bundles.

A.1. **Real manifolds.**

A.1.1. *Charts, atlases, manifolds.* A homeomorphism $U \xrightarrow{\phi} V$ with $M \overset{open}{\supseteq} U$ and $V \overset{open}{\subseteq} \mathbb{R}^n$ for some n, is called a *local chart* on the topological space M. Two charts $(U_k \xrightarrow{\phi} V_k)$ on M $(k \in \{i, j\})$, are said to be *compatible* if (for $U_{ij} = U_i \cap U_j$), the *comparison function* (or *transition function*),

$$V_j \supseteq \phi_j(U_{ij}) \xrightarrow{\phi_{ij} \overset{\text{def}}{=} \phi \circ \phi_j{}^{-1}} \phi_i(U_{ij}) \subseteq V_i$$

is a C^∞-map between two open subsets of $\mathbb{R}^n$. An atlas on M is a family of compatible charts on M that cover M.

We say that any atlas defines on M a *structure of a manifold*, and two atlases define the same manifold structure if they are compatible, i.e., if their union is again an atlas.

So, "compatible" is an equivalence relation on atlases, and a structure of a manifold on a topological space M is precisely an equivalence class of compatible atlases on M. On the other hand, if $\mathcal{A}$ is an atlas on M the set $\widetilde{\mathcal{A}}$ of all charts on M that are compatible with the charts in $\mathcal{A}$ is a maximal atlas on M. So, any equivalence class of atlases contains the largest element and we can think of manifold structures on M as *maximal atlases* on M.[5]

A.1.2. *Once again.* A real manifold M of dimension n is a topological space M which is locally isomorphic to $\mathbb{R}^n$ in a smooth way and without contradictions. Here,

- *Locally isomorphic to* $\mathbb{R}^n$ means that we are given an open cover U_i, $i \in I$, of M, and for each $i \in I$ a topological identification (*homeomorphism*), $\phi : U_i \xrightarrow{\cong} V_i$ with V_i open in $\mathbb{R}^n$.
- *Smooth way without contradictions* means that for any $i, j \in I$ (and $U_{ij} = U_i \cap U_j$), the transition function

$$V_j \supseteq \phi_j(U_{ij}) \xrightarrow{\phi_{ij} \overset{\text{def}}{=} \phi \circ \phi_j{}^{-1}} \phi_i(U_{ij}) \subseteq V_i$$

  is a C^∞-map between two open subsets of $\mathbb{R}^n$.

[5]Later, we will find a nicer way to describe the manifold structure in terms of *ringed spaces*.

A.1.3. *The sheaf C^∞_M of smooth functions on a manifold M.* For any open $U\subseteq M$ we define $C^\infty(U,\mathbb{R})$ to consist of all functions $f: U\to\mathbb{R}$ such that for any chart $(U_i \xrightarrow{\phi} V_i)$ the function $f\circ\phi^{-1}:\ \phi_i(U\cap U_i)\to\mathbb{R}$ is C^∞ on the open subset $\phi_i(U\cap U_i)\subseteq V_i\subseteq\mathbb{R}^n$.

Because of the *no-contradiction* policy one does not have to check all charts, but only sufficiently many to cover U.

Lemma. (a) Though the definition of C^∞_M is complicated, locally we get just the usual smooth functions on $\mathbb{R}^n$. If U lies in some chart (U_i,ϕ,V_i) (i.e., in $U\subseteq U_i$), then ϕ_i gives identification $C^\infty(U)\cong C^\infty(\phi_i(U))$ of smooth fonctions on U with smooth functions on an open part of $\mathbb{R}^n$.

(b) C^∞_M is a sheaf of $\mathbb{R}$-algebras on M,, i.e.,

- (0) for each open $U\subseteq X$ $C^\infty(U)$ is an $\mathbb{R}$-algebra,
- (1) for each inclusion of open subsets $V\subseteq U\subseteq X$ the restriction map $C^\infty(U)\xrightarrow{\rho^U_V} C^\infty(V)$ is map of $\mathbb{R}$-algebras

and these data satisfy

- (Sh0) $\rho^U_U = id$
- (Sh1) (Transitivity of restriction) $\rho^U_V\circ\rho^U_V =\ \rho^U_W$ for $W\subseteq V\subseteq U$
- (Sh2) (Gluing) If $(W_j)_{j\in J}$ is an open cover of an open $U\subseteq M$[6], we ask that any family of compatible $f_j\in C^\infty(W_j),\ j\in J$, glues uniquely. This means that if all f_j agree on intersections in the sense that $\rho^{W_i}_{W_{ij}}f_i = \rho^{W_i}_{W_{ij}}f_j$ in $C^\infty(W_{ij})$ for any $i,j\in J$; then there is a unique $f\in C^\infty(U)$ such that $\rho^U_{W_j}f = f_j$ in $C^\infty(W_j),\ j\in J$.
- (Sh3) $C^\infty(\emptyset)$ is $\{0\}$.

Proof. (a) is clear from definitions. The notion of $\mathcal{F}$ is a sheaf", that appears in (b), is really a shorthand for "$\mathcal{F}$ is of local nature", i.e., "$\mathcal{F}$ is defined by some local property". Now C^∞_M is a sheaf because to check that a function f on U is smooth, one only has to check locally, i.e., one has to consider f on a small neighborhood of each point.

A.1.4. *Examples.* The following are real manifolds

(1) $M=\mathbb{R}^n$
(2) M an open subset of $\mathbb{R}^n$
(3) $M=S^1$ or $M=S^n$.
(4) $M=\mathbb{RP}^1$ or $M=\mathbb{RP}^n$.

A.1.5. *Category of real manifolds.* For two real manifolds M',M'' we define the set $\mathrm{Hom}(M',M'') = Map(M',M'')$ of *smooth maps* or *morphisms of manifolds* , to consist of all maps $F: M'\to M''$ which are smooth when checked in local charts.

[6]We denote $W_{ij}=W_i\cap W_j$ etc.!

This means that for each $x \in M'$ there are charts $M' \supseteq U_i \xrightarrow{\phi} V_i \subseteq \mathbb{R}^{m'}$ and $M'' \supseteq U''_j \xrightarrow{\phi} V''_j \subseteq \mathbb{R}^{m''}$, such that $x \in U'_i$ and $F(x) \in U''_j$, and the map

$$\begin{array}{ccc} U'_i \cap F^{-1}U''_j & \xrightarrow{F} & U''_j \\ {\scriptstyle \phi'_i}\big\downarrow & & {\scriptstyle \phi''_j}\big\downarrow \\ V'_i \supseteq \phi'_i(U'_i \cap F^{-1}U''_j) & \xrightarrow{F_{ij}} & \phi''_j(U''_j) \subseteq V''_j \end{array}$$

is a smooth map between open subsets of $\mathbb{R}^{m'}$ and $\mathbb{R}^{m''}$.

Again, no-contradiction policy implies that if the above is true for one pair of charts at x and $F(x)$, it is true for any pair of charts.

A.1.6. *Examples.*

(1) For any manifold M, $\mathrm{Hom}(M, \mathbb{R}^n) = C^\infty(M, \mathbb{R})^n$.
(2) A smooth map $F \in \mathrm{Hom}(M, N)$ defines for any pair of open subsets $U \subseteq M$ and $V \subseteq N$ the pull-back map $C^\infty_N(V) \xrightarrow{F^*} C^\infty_M(U)$, $g \mapsto F^*g = g \circ F|U$.

A.2. **Vector bundles.** The notion of a *vector bundle* is the relative version of the notion of a vector space, i.e., a "vector bundle on a space M" will mean a "vector space spread over M" with the certain "continuity" property. The basic examples will be the tangent bundle $TM \to M$ which organize all tangent spaces T_aM, $a \in M$, into one manifold TM. The appropriate level of organization (structure) on the union $TM = \cup_{a \in M} T_aM$ is described in the following notion:

A.2.1. *Vector bundles.*

(1) ($\mathcal{S}ets$.) If M is a <u>set</u> a vector bundle V over M consists of a map $V \xrightarrow{p} M$ and a structure of a vector space on each fiber $V_m = p^{-1}(m)$, $m \in M$.
(2) ($\mathcal{T}op$.) If M is a *topological space*, we also ask that V is a topological space, the map $V \xrightarrow{p} M$ is continuous and the vector space structure of the fibers does not change wildly in the sense that

each $m \in M$ has a neighborhood U such that
there exists a <u>homeomorphism</u> $\phi : V|U \to U \times \mathbb{R}^n$ such that

(a) ϕ maps each fiber to a fiber, i.e., the diagram

$$\begin{array}{ccc} V|U & \xrightarrow{\phi} & U \times \mathbb{R}^n \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle pr_U} \\ U & \xrightarrow{=} & U \end{array}$$

commutes,
(b) The restriction of ϕ to fibers is an isomorphism of vector spaces.

(3) (*Manifolds.*) If M is a *manifold*,[7] we ask that V is a manifold, the map $V \xrightarrow{p} M$ is a map of manifolds and the vector space structure on fibers changes smoothly in the sense that

each $m \in M$ has a neighborhood U such that there exists an isomorphism of manifolds $\phi : V|U \to U \times \mathbb{R}^n$, which preserves fibers and the restrictions of ϕ to fibers are isomorphisms of vector spaces.

A.2.2. *Lemma.* For a vector bundle V on M, any map of manifolds $f : N \to M$ can be used to pull-back the vector bundle V to a vector bundle

$$f^*V \stackrel{\text{def}}{=} \cup_{n \in N} \ V_{f(n)}$$

on N. So, by definition $(f^*V)_n = V_{f(n)}$, i.e., the fiber of f^*V at $n \in N$ is the same as the fiber of V at $f(n) \in M$.

A.3. **The (co)tangent bundles.**

A.3.1. *Cotangent spaces $T_a^*(M)$.* The cotangent space at a point $m \in M$ is defined by

$$T_m^*(M) \stackrel{\text{def}}{=} \mathfrak{m}_a/\mathfrak{m}_a^2 \quad \text{for} \ \ \mathfrak{m}_a \stackrel{\text{def}}{=} \{g \in C^\infty(M);\ g(a) = 0\}.$$

For any open $U \subseteq M$ and $f \in C^\infty(U)$, the differential at a of f is defined as the image

$$d_a f \stackrel{\text{def}}{=} (f - f(a)) + \mathfrak{m}_a^2 \in \ T_a^*(M)$$

of $f - f(a)$ in $T_a^* M$.

A.3.2. *Tangent spaces $T_a(M)$.* The *tangent vectors* at $a \in M$ are the "derivatives at a", i.e., all linear functionals in the tangent space

$$T_m(M) \stackrel{\text{def}}{=} \{\xi \in \mathrm{Hom}_{\mathbb{R}}[C^\infty(M), \mathbb{R}];\ \xi(fg) = \xi(f) \cdot g(a) + f(a) \cdot \xi(g)\}.$$

The *vector fields* on M are all "derivatives on M", i.e., all linear operators in

$$X(M) \stackrel{\text{def}}{=} \{\Xi \in \mathrm{Hom}_{\mathbb{R}}[C^\infty(M), C^\infty(M)];\ \Xi(fg) = \Xi(f) \cdot g + f \cdot \Xi(g)\}.$$

A vector field Ξ defines a tangent vector $\Xi_a \in T_a(M)$ at each point $a \in M$

$$\Xi_a(f) \stackrel{\text{def}}{=} (\Xi f)(a), \qquad f \in C^\infty(M).$$

A.3.3. *Local coordinates.* For any open $U \subseteq M$, we say that functions $x_1, ..., x_n \in C^\infty(U)$ form a coordinate system on U if $\phi = (x_1, ..., x_n) : U \to \mathbb{R}^n$ gives a chart, i.e.,

- $\phi(U)$ is open in $\mathbb{R}^n$,
- $\phi : U \to \phi(U)$ is a bijection, and
- the inverse function is a map of manifolds.

[7] Smooth or holomorphic.

Lemma. The last condition is equivalent to

For each $a \in U$, the differentials $d_a x_i$ form a basis of $T_a^* M$.

Proof. Implicit Function Theorem.

A.3.4. *Lemma.* For any manifold M,

$$TM \stackrel{\text{def}}{=} \cup_{a\in M}\ T_aM \quad \text{and} \quad T^*M \stackrel{\text{def}}{=} \cup_{a\in M}\ T_a^*M$$

are naturally vector bundles over the manifold M.

A.4. **Constructions of manifolds.**

A.4.1. *The differential of manifold maps.* A map of manifolds $f : M \to N$, produces for any open $V \subseteq N$ and $U = \subseteq M$ such that $f(U) \subseteq V$, the pull-back of functions

$$f^* : C_N^\infty(V) \to C_M^\infty(U), \quad \phi \mapsto f^*\phi \stackrel{\text{def}}{=} \phi \circ f|U.$$

For each $a \in M$, $f^* I_{f(a)}^N \subseteq I_a^M$, so we get a linear map

$$d_a^* f :\ T_{f(a)}^*(N) = I_{f(a)}^N/(I_{f(a)}^N)^2 \to I_a^M/(I_a^M)^2 = T_a^*(M), \quad d_a f(d_{f(a)}\phi) \stackrel{\text{def}}{=} d_a(\phi \circ f).$$

In other words,

$$d_a f(\ [\phi - \phi(f(a))] + (I_{f(a)}^N)^2\ )\ =\ [\phi \circ f - (\phi \circ f)(a))] + (I_a^M)^2.$$

In the opposite (covariant) direction one has the map called the differential of f

$$d_a f :\ T_a(M) \to T_{f(a)}(N),\ (d_a f \xi)\phi \stackrel{\text{def}}{=} \xi(f^*\phi)\ =\ \xi(\phi \circ f)$$

which is the adjoint of d_a^f. In terms of the local coordinates x_i around $a \in M$ and y_j around $f(a) \in N$,

$$(d_a f)\partial_{i,a}\ =\ \sum_j\ \partial_{i,a}(y_j \circ f) \cdot \partial_{j,f(a)}$$

and the matrix $(\partial_{i,a}(y_j \circ f))_{i,j}$ of $d_a f$ in the bases $\partial_{i,a}$, $\partial_{j,f(a)}$ is called the Jacobian of f at a.

A.4.2. *Theorem.* Let $f : M \to N$ be a map of manifolds which is of *constant rank* (i.e., all differentials $d_a f : T_a(M) \to T_{f(a)}(N)$ have the same rank). Then the fibers $M_b \stackrel{\text{def}}{=} f^{-1}b,\ b \in N$, are naturally manifolds.

This is again a consequence of the Implicit Function Theorem.

A.4.3. *Examples.* Let $f \in C^\infty(M)$ and $b \in \mathbb{R}$ be such that $d_a f \neq 0$ for any $a \in M_b$. Then M_b is a submanifold.

Proof. $d_a f \neq 0$ for any $a \in M_b$, so the same is true for a in some neighborhood U of M_b. Now, $M_b = f^{-1}b =\ (f|U)^{-1}b$ and on U the rank is 1.

A.5. **Complex manifolds.** A complex manifold M of dimension n is a topological space M which is locally isomorphic to $\mathbb{C}^n$ in a holomorphic way and without contradictions. Here,

- *Locally isomorphic to* $\mathbb{C}^n$ means that we are given an open cover $U_i,\ i\in I$, of M, and for each $i\in I$ a topological identification (*homeomorphism*), $\phi : U_i \xrightarrow{\cong} V_i$ with V_i open in $\mathbb{C}^n$.
- *In a holomorphic way* means that for any $i,j\in I$, the transition function ϕ_{ij} is a holomorphic map between two open subsets of $\mathbb{C}^n$.[8]
- *No contradictions* means that for any $i,j,k \in I$, the two identifications of $\phi_k(U_{ijk})\subseteq V_k$ and $\phi_i(U_{ijk})\subseteq V_i$, are the same: $\phi_{ij}\circ\phi_{jk} = \phi_{ik}$.

We call each $(U_i \xrightarrow{\phi} V_i)$ a local chart on the manifold. A collection $(U_i \xrightarrow{\phi} V_i)_{i\in I}$, of charts on a topological space is said to be compatible if it satisfies the conditions *smooth way* and *no-contradictions*. A collection of compatible charts that cover M is called an *atlas* on M. We say that any atlas defines on M a structure of a manifold, and two atlases define the same manifold structure if they are compatible, i.e., if their union is again an atlas.

So a structure of a manifold on a topological space M can be viewed is an equivalence class of compatible atlases on M. On the other hand, if $\mathcal{A}$ is an atlas on M the set $\widetilde{\mathcal{A}}$ of all charts on M that are compatible with the charts in $\mathcal{A}$ is a maximal atlas on M. So, any equivalence class of atlases contains the largest element.

A.5.1. *The sheaf $\mathcal{O}_M^{an}$ of holomorphic functions on a manifold M.* For any open $U\subseteq M$ we define $\mathcal{O}^{an}(U,\mathbb{R})$ to consist of all functions $f : U \to \mathbb{R}$ such that for any chart $(U_i \xrightarrow{\phi} V_i)$ the function $f\circ\phi^{-1}:\ \phi_i(U\cap U_i) \to \mathbb{R}$ is $\mathcal{O}^{an}$ on the open subset $\phi_i(U\cap U_i)\subseteq V_i\subseteq\mathbb{R}^n$.

Because of the *no-contradiction* policy one does not have to check all charts, but only sufficiently many to cover U.

Lemma. (a) If U lies in some chart U_i then ϕ gives identification $\mathcal{O}^{an}(U) \cong \mathcal{O}^{an}(\phi_i(U))$ of holomorphic fonctions on U with holomorphic functions on an open part of $\mathbb{C}^n$.

(b) $\mathcal{O}_M^{an}$ is a sheaf of $\mathbb{C}$-algebras on M,, i.e.,

- (0) for each open $U\subseteq X$ $\mathcal{O}^{an}(U)$ is a $\mathbb{C}$-algebra,
- for each inclusion of open subsets $V\subseteq U\subseteq X$ the restriction map $\mathcal{O}^{an}(U) \xrightarrow{\rho_V^U} \mathcal{O}^{an}(V)$ is map of $\mathbb{C}$-algebras

and these data satisfy

- (Sh0) $\rho_U^U = id$

[8]...

- (Sh1) (Transitivity of restriction) $\rho^U_V \circ \rho^U_V = \rho^U_W$ for $W \subseteq V \subseteq U$
- (Sh2) (Gluing) If $(W_j)_{j\in J}$ is an open cover of an open $U \subseteq M$ we ask that any family of compatible $f_j \in \mathcal{O}^{an}(W_j),\ j \in J$, glues uniquely.
- (Sh3) $\mathcal{O}^{an}(\emptyset)$ is $\{0\}$.

A.5.2. *Examples.*

(1) $M = \mathbb{C}^n$
(2) M an open subset of $\mathbb{C}^n$
(3) $M = \mathbb{CP}^1$ or $M = \mathbb{CP}^n$.

A.5.3. *Category of complex manifolds.* For two complex manifolds M', M'' we define the set $\mathrm{Hom}(M', M'') = Map(M', M'')$ of *holomorphic maps* or *morphisms of complex manifolds* to consist of all maps $F : M' \to M''$ which are holomorphic when checked in local charts.

A.5.4. *Examples.*

(1) For any manifold M, $\mathrm{Hom}(M, \mathbb{C}^n) = \mathcal{O}^{an}(M, \mathbb{C})^n$.
(2) A holomorphic map $F \in \mathrm{Hom}(M, N)$ defines for any pair of open subsets $U \subseteq M$ and $V \subseteq N$ the pull-back map $\mathcal{O}^{an}_N(V) \xrightarrow{F^*} \mathcal{O}^{an}_M(U),\ g \mapsto F^*g = g \circ F|U$.

A.6. **Manifolds as ringed spaces.** We will see that a geometric space (for instance a manifold of a certain type) can naturally be thought of as a topological space with a sheaf of rings.

A.6.1. *Ringed spaces.* A ringed space consists of a topological space X and a sheaf of rings $\mathcal{O}$ on X. Usually we call $\mathcal{O}$ the structure sheaf of X and we denote it $\mathcal{O}_X$.

A.6.2. *Real manifolds as ringed spaces.* As we have seen, any real manifold M defines a ringed space (M, C^∞_M). Actually,

Lemma. (a) For a manifold M one can recover the manifold structure on M from the sheaf of rings C^∞_M[9]

(b) Manifolds are the same as ringed spaces $(X, \mathcal{O}_X)$ that are locally isomorphic to $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$.[10]

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[9]The largest atlas for the manifold M consists of all data $M \overset{open}{\supseteq} U \xrightarrow[homeomorphism]{\phi} V \overset{open}{\subseteq} \mathbb{R}^n$, such that for any $g \in C^\infty(V)$ the pull-back $g \circ \phi$ is in $C^\infty_M(U)$.

[10]This means that X can be covered by open sets U such that

(1) there is a homeomorphism $\phi : U \xrightarrow{\cong} V$ with V open in some $\mathbb{R}^n$, with the property that
(2) for any U' open in U, the restriction of ϕ to $U' \to \pi(U') = V'$ identifies $\mathcal{O}_X(U')$ and $C^\infty_{\mathbb{R}^n}(V')$.

A.6.3. *Complex manifolds as ringed spaces.* The story is the same. Any complex manifold M defines a locally ringed space $(M, \mathcal{O}_M^{an})$. Actually, complex manifolds are the same as ringed spaces $(X, \mathcal{O}_X)$ that are locally isomorphic to $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}^{an})$.

A.6.4. *Terminology.* We will speak of a $\Bbbk$-manifold $(M, \mathcal{O}_M)$ where $\Bbbk$ is either $\mathbb{R}$ or $\mathbb{C}$, and we will mean the above notion of a real manifold with $\mathcal{O}_M = C_M^\infty$ if $\Bbbk = \mathbb{R}$, or the above notion of a complex manifold with $\mathcal{O}_M = \mathcal{O}_M^{an}$ if $\Bbbk = \mathbb{C}$.

A.6.5. *Use of sheaves.* Sheaves are more <u>fundamental</u> for $\mathbb{C}$-manifolds then for $\mathbb{R}$-manifolds because for an $\mathbb{R}$-manifold M, all information is contained in one ring $C^\infty(M)$, while for a $\mathbb{C}$-manifold the global functions need not contain enough information – for instance $\mathcal{O}^{an}(\mathbb{CP}^n) = \mathbb{C}$. This forces one to control all local function rather then just the global functions (i.e., the sheaf $\mathcal{O}_M$ rather then just $\mathcal{O}_M(M)$).

However, the general role of sheaves is that they control the relation between local and global objects, and this make them useful in many a context.

A.7. **Manifolds as locally ringed spaces.** We saw that geometric space can naturally be thought of as a ringed spaces, actually their geometric nature will be reflected in a special property of the corresponding ringed spaces – these are the *locally ringed spaces.*

A.7.1. *Stalks.* The stalk of the sheaf $\mathcal{O}$ at $a \in X$ is intuitively $\mathcal{O}(\mathcal{U})$ for a "very small neighborhood $\underline{U}$ of a". More precisely, if $a \in V \subseteq U$ then $\mathcal{O}(U)$ and $\mathcal{O}(V)$ are related by the restriction map $\mathcal{O}(U) \xrightarrow{\rho_V^U} \mathcal{O}(V)$, and the stalk at a is a certain limit of these restriction maps (called *inductive limit* or *colimit*), i.e.,

$$\mathcal{O}_a \stackrel{\text{def}}{=} \lim_{\substack{\rightarrow \\ U \ni a}} \mathcal{O}(U)$$

of $\mathcal{O}(U)$ over smaller and smaller neighborhoods U of a in X.

The elements of $\mathcal{O}_a$ are called the *germs* of $\mathcal{O}$-functions at a, and $\mathcal{O}_a$ can be described in en elementary way

(1) For any neighborhood U of a point a any $f \in \mathcal{O}(U)$ defines a germ $\underline{f}_a = \underline{(U, f)}_a \in \mathcal{O}_a$, and any germ is obtained in this way.
(2) Two germs $\underline{(U, f)}_a$ and $\underline{(V, g)}_a$ at a, are the same if there is neighborhood $W \subseteq U \cap V$ such that $f = g$ on W.

Then one defines the structure of a ring on $\mathcal{O}_a$ by

$$\underline{(U, f)}_a + \underline{(V, g)}_a \stackrel{\text{def}}{=} \underline{(U \cap V, f + g)}_a \quad \text{and} \quad \underline{(U, f)}_a \cdot \underline{(V, g)}_a \stackrel{\text{def}}{=} \underline{(U \cap V, f \cdot g)}_a.$$

A.7.2. *Local rings.* A commutative ring A is said to be a *local ring* if it has the *largest proper ideal.*

Examples.

(1) Any field is local, the largest ideal is 0.
(2) The ring of formal power series $\Bbbk[[X_1, ..., X_n]]$ over a field $\Bbbk$ is local, the largest ideal $\mathfrak{m}$ consists of series that vanish at 0 (i.e, the constant term is 0).
(3) $\mathbb{C}[x]$ is not at all local.

A commutative ring is local iff it has precisely one maximal ideal (then this is the largest ideal). Remember that maximal ideals correspond to the naive notion of "ordinary" points of a space. So, uniqueness of a maximal ideal in a ring A intuitively means that this ring corresponds to a *space with one ordinary point.*

A.7.3. *Locally ringed spaces.* We say that a ringed space $(X, \mathcal{O})$ is *locally ringed* if all $\mathcal{O}(U)$ are commutative rings and each stalk $\mathcal{O}_a$, $a \in X$, is a *local ring*, i.e., it has the *largest proper ideal.* This ideal is then denoted $\mathfrak{m}_a \subseteq \mathcal{O}_a$.

Example. The stalk of the sheaf of analytic functions $\mathcal{O}^{an}_{\mathbb{C}^n,0}$ consists of all formal series in n variables $f(Z_1, ..., Z_n) = \ sum_I \ f_I \cdot Z^I$ which converge on some ball around $0 \in \mathbb{C}^n$ (think of $\underline{(U, f)}_0$ as the expansion of f at 0). This is a local ring, and the largest ideal is

$$\mathfrak{m}_a \stackrel{\text{def}}{=} \mathcal{O}_a \cap \sum Z_i \cdot \mathbb{C}[[Z_1, ..., Z_n]] = \text{ all germs at } a \text{ of functions that vanish at } a.$$

Remark. Remember that a local ring intuitively corresponds to a space with one ordinary point. Therefore, it makes sense that the stalk $\mathcal{O}_{X,a}$ should be a local ring since $\mathcal{O}_{X,a}$ should only see one ordinary point – the point a.

A.7.4. *Manifolds as locally ringed spaces.* As we have seen, any manifold M (real or complex) defines a ringed space. Actually,

Lemma. The ringed space of any manifold M is a locally ringed space. The largest ideal $\mathfrak{m}_a$ of the stalk at a consists of germs of functions that vanish at a.

Proof. Let $\mathcal{O}$ be the structure sheaf (i.e., C^∞_M or $\mathcal{O}^{an}_M$) and let $\phi \in \mathcal{O}_a$ be the germ $\phi = \underline{(U, f)}_a$ of a function at a. If $\phi \notin \mathfrak{m}_a$, i.e., $f(a) \neq 0$ then the restriction of f to the neighborhood $V = f^{-1}\Bbbk^*$ (for $\Bbbk = \mathbb{R}$ or $\mathbb{C}$) of a, is invertible. Therefore ϕ is invertible (so ϕ can not lie in a proper ideal!).

Appendix B. **Categories: more**

Appendix B on categories, contains additional material which will not be needed in this course.

B.1. **Construction (description) of objects via representable functors.** Yoneda lemma bellow says that passing from an object $a \in \mathcal{A}$ to the corresponding functor $\mathrm{Hom}_{\mathcal{A}}(-,a)$ does not loose any information – a can be recovered from the functor $\mathrm{Hom}_{\mathcal{A}}(-,a)$.[11] This has the following applications:

(1) One can describe an object a by describing the corresponding functor $\mathrm{Hom}_{\mathcal{A}}(-,a)$. This turns out to be **the** *most natural* description of a.
(2) One can start with a functor $F : \mathcal{A}^o \to \mathcal{S}ets$ and ask whether it comes from some objects of a. (Then we say that a represents F and that F is representable).
(3) Functors $F : \mathcal{A}^o \to \mathcal{S}ets$ behave somewhat alike the objects of $\mathcal{A}$, and we can think of their totality as a natural enlargement of $\mathcal{A}$ (like one completes $\mathbb{Q}$ to $\mathbb{R}$).

B.1.1. *Category* $\widehat{\mathcal{A}}$. To a category $\mathcal{A}$ one can associate a category

$$\widehat{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{F}unct(\mathcal{A}^o, \mathcal{S}ets)$$

of contravariant functors from $\mathcal{A}$ to sets. Observe that each object $a \in \mathcal{A}$ defines a functor

$$\iota_a = \mathrm{Hom}_{\mathcal{A}}(-,a) \in \widehat{\mathcal{A}}.$$

The following statement essentially says that one can recover a form the functor $\mathrm{Hom}_{\mathcal{A}}(-,a)$, i.e., that this functor contains all information about a.

B.1.2. *Theorem.* (Yoneda lemma)
(a) Construction ι is a functor $\iota : \mathcal{A} \to \widehat{\mathcal{A}}$.

(b) For any functor $F \in \widehat{\mathcal{A}} = \mathcal{F}unct(\mathcal{A}^o, \mathcal{S}ets)$ and any $a \in \mathcal{A}$ there is a canonical identification

$$\mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, F) \cong F(a).$$

Proof. (b) Recall that a map of functors $\eta : \iota_a \to F$ (functors from $\mathcal{A}^o$ to $\mathcal{S}ets$), means for each $x \in \mathcal{A}$ one map of sets $\eta_x : \iota_a(x) = \mathrm{Hom}_{\mathcal{A}}(x,a) \to F(x)$, and this system of maps should be such that for each morphism $y \xrightarrow{\alpha} x$ in $\mathcal{A}$ (i.e., $x \xrightarrow{\alpha} y$ in $\mathcal{A}^o$), the following diagram commutes

$$\begin{array}{ccc} F(x) & \xrightarrow{F(\alpha)} & F(y) \\ {\scriptstyle\eta_x}\uparrow & & {\scriptstyle\eta_y}\uparrow \\ \iota_a(x) & \xrightarrow{\iota_a(\alpha)} & \iota_a(y) \end{array} \quad , \quad i.e., \quad F(\alpha)\circ\eta_x = \eta_y\circ\iota_a(\alpha).$$

Such η in particular gives $\eta_a : \iota_a \to F(a)$, and since $\iota_a = \mathrm{Hom}_{\mathcal{A}}(a,a) \ni 1_a$ we get an element $\overline{\eta} \stackrel{\text{def}}{=} \eta_a(1_a)$ of $F(a)$.

[11] This is the precise form of the *Interaction Principle* on the level of categories that we used to pass from varieties to spaces and stacks. The interactions of a with all objects of the same kind are encoded in the functor $\mathrm{Hom}_{\mathcal{A}}(-,a)$, so Yoneda says that if you know the interactions of a you know a.

In the opposite direction, a choice of $f \in F(a)$, gives for any $x \in \mathcal{A}$ the composition of functions

$$\widetilde{f}_x \stackrel{\text{def}}{=} [\iota_a(x) = \mathrm{Hom}_{\mathcal{A}}(x,a) = \mathrm{Hom}_{\mathcal{A}^o}(a,x) \xrightarrow{F}= \mathrm{Hom}_{\mathcal{S}ets}[F(a),F(x)] \xrightarrow{ev_f} F(x)].$$

Now one checks that

- (i) $\widetilde{f}$ is a map of functors $\iota_a \to F$, and
- (ii) procedures $\eta \mapsto \overline{\eta}$ and $f \mapsto \widetilde{f}$ are inverse functions between $\mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, F)$ and $F(a)$.

Corollary. (a) Yoneda functor $\iota : \mathcal{A} \to \widehat{\mathcal{A}}$ is a full embedding of categories, i.e., for any $a, b \in \mathcal{A}$ the map

$$\iota : \mathrm{Hom}_{\mathcal{A}}(a,b) \ \to \ \mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, \iota_b),$$

given by the functoriality of ι, is an isomorphism.

(b) Functor $\mathrm{Hom}_{\mathcal{A}}(-,a) = \iota_a$ determines a up to a unique isomorphism, i.e., if $\iota_a \cong \iota_b$ in $\widehat{\mathcal{A}}$ then $a \cong b$ in $\mathcal{A}$.

Proof. (a) follows the part (b) of the Yoneda lemma (take $F = \iota_b$). (b) follows from (a).

Remark. We say that a functor $F : \mathcal{B} \to \mathcal{C}$ is a full embedding of categories if for any $a, b \in \mathcal{B}$ the map $\mathrm{Hom}_{\mathcal{A}}(a,b) \xrightarrow{F_{a,b}} \mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, \iota_b)$ given by the functoriality of F, is an isomorphism. The meaning of this is we put $\mathcal{B}$ into a larger category which has objects from $\mathcal{B}$ and maybe also some new objects, but the old objects (from $\mathcal{B}$) relate to each other in $\mathcal{C}$ the same as they used to in $\mathcal{B}$. We also say that F makes $\mathcal{B}$ into a full subcategory of $\mathcal{C}$.

B.2. **Yoneda completion $\widehat{\mathcal{A}}$ of a category $\mathcal{A}$.** Yoneda lemma says that $\mathcal{A}$ lies in a larger category $\widehat{\mathcal{A}}$. The hope is that the category $\widehat{\mathcal{A}}$ may contain many *beauties* that should morally be in $\mathcal{A}$ (but are not). One example will be a way of treating inductive systems in $\widehat{\mathcal{A}}$. In particular we will see inductive systems of infinitesimal geometric objects that underlie the differential calculus.

B.2.1. *Distributions.* This Yoneda completion is a categorical analogue of one of the basic tricks in analysis:

since among functions one can not find beauties like the δ-functions, we extend the notion of of functions by adding distributions.

Remember that the distributions on an open $U \subseteq \mathbb{R}^n$ are the (nice) linear functionals on the vector space of of (nice) functions: $\mathcal{D}(U,\mathbb{C}) \subseteq C_c^\infty(U,\mathbb{C})^* = \mathrm{Hom}_{\mathbb{C}}[C_c^\infty(U),\mathbb{C}]$.

B.2.2. *Representable functors.* First we get a feeling for how objects of $\mathcal{A}$ are viewed inside $\widehat{\mathcal{A}}$, i.e., the relation between thinking of $a \in \mathcal{A}$ and the functor ι_a.

We will say that a functor $F \in \widehat{\mathcal{A}}$, i.e., $F : \mathcal{A}^o \longrightarrow \mathcal{S}ets$, is *representable* if there is some $a \in \mathcal{A}$ and an isomorphism of functors $\eta :\ \mathrm{Hom}_{\mathcal{A}}(-, a) \longrightarrow F$. Then we say that a represents F. This is the *basic categorical trick for describing an object* a *up to a canonical isomorphism:* :

instead of describing a *directly we describe a functor* F *isomorphic to* $\mathrm{Hom}_{\mathcal{A}}(-, a)$.

B.2.3. *Examples.* (1) *Products.* A product of a and b is an object that represents the functor

$$\mathcal{A} \ni x \mapsto \mathrm{Hom}(x, a) \times \mathrm{Hom}_{\mathcal{A}}(x, b) \in \ \mathcal{S}ets.$$

(2) In the category of $\Bbbk$-varieties, functor

$$X \mapsto \{(f_1, ..., f_n);\ f_i \in \mathcal{O}(X)\} \ = \ \mathcal{O}(X)^n$$

represents $\mathbb{A}^n$.

(3) In the category of schemes,

$$X \mapsto \{f \in \mathcal{O}(X);\ f^2 = 0\}$$

represents the double point scheme $\mathrm{Spec}(\mathbb{Z}[x]/x^2)$.

(4) If $\mathbb{A}^n = \oplus_1^n \ \Bbbk{\cdot}e_i$, then the set

$$\mathbb{A}^\infty \stackrel{\text{def}}{=} \cup_0^\infty \ \mathbb{A}^n = \oplus_1^\infty \ \Bbbk{\cdot}e_i$$

is an increasing union of $\Bbbk$-varieties. In analogy with (2), we see that the functor corresponding to this construction should be given by all infinite sequences of functions

$$X \mapsto \{(f_0, f_1, ..., f_n, ...);\ f_i \in \mathcal{O}(X)\} \ = \ Map(\mathbb{N}, \mathcal{O}(X)).$$

However, this functor is *not* representable in $\Bbbk$-varieties, i.e., $\mathbb{A}^\infty$ is not a $\Bbbk$-variety. We may expect that it lives in the larger world of schemes, but even this fails. So, its natural ambient is the category the Yoneda completion $\Bbbk$-$\widehat{Varieties}$ of the category $\Bbbk$-$Varieties$.

B.2.4. *Limits.* One can describe the completion of $\mathcal{A}$ to $\widehat{\mathcal{A}}$ as *adding to* $\mathcal{A}$ *all limits of* inductive systems *in* $\mathcal{A}$, just as one constructs $\mathbb{R}$ from $\mathbb{Q}$. The simplest kinds of inductive systems in $\mathcal{A}$ are the diagrams $\boldsymbol{a} = (a_0 \to a_1 \to \cdots)$ in $\mathcal{A}$[12] The limit $\varinjlim \boldsymbol{a}$ is roughly speaking the object that should naturally appear at the end: $(a_0 \to a_1 \to \cdots \to \varinjlim_n \ a_n)$. It need not exist in $\mathcal{A}$ at least it is easy to see that if $\mathcal{A} = \mathcal{S}ets$ then all inductive limits always exist!

A consequence of this good situation in the category $\mathcal{S}ets$ is that:

[12] Here *inductive* means that it stretches to the right, while for instance $(\cdots \longleftarrow b_n \longleftarrow b_1 \longleftarrow b_0)$ would be called a *projective system*.

even if $\lim_{\to\, n}\ a_n$ *does not exist in* $\mathcal{A}$*, it always exists in the larger category* $\widehat{\mathcal{A}}$ *.*

An inductive system $\boldsymbol{a}$ defines an object in $\mathcal{A}$ *if* the limit $\lim_{\to} a_n$ exists in $\mathcal{A}$, however it *always* defines a functor $\iota_{\boldsymbol{a}} = \varinjlim_n \iota_{a_n} \in \widehat{\mathcal{A}}$, by

$$\iota_{\boldsymbol{a}}(c) \overset{\text{def}}{=} \lim_{\to\, n}\ \iota_{a_n}(c) = \lim_{\to\, n}\ \mathrm{Hom}_{\mathcal{A}}(c, a_n) \in \mathcal{S}ets.$$

(This definition uses the existence of inductive limits in the category $\mathcal{S}ets$!)

This allows us to <u>*think*</u> of the functor $\iota_{\boldsymbol{a}}$ as the limit of the inductive system $\boldsymbol{a}$ that exists in the larger category $\widehat{\mathcal{A}}$. All together, we can think of any inductive system *as if* it were an object $\varinjlim a_i$ in $\mathcal{A}$ (since we can identify it with $\boldsymbol{a} \in \widehat{\mathcal{A}}$). For this reason an inductive system in $\mathcal{A}$ is called an *ind-object of* $\mathcal{A}$ (while it really gives an object of $\widehat{\mathcal{A}}$). [13]

Examples. The basic example of inductive system is an increasing union. Some infinite increasing unions of $\Bbbk$-schemes are not $\Bbbk$-schemes but they are objects of the category of $\Bbbk^{(\overset{\text{def}}{=})}\widehat{\Bbbk\text{-}\mathcal{S}chemes}$. The most obvious examples are $\mathbb{A}^\infty$ (above) which should be a $\Bbbk$-variety but it is not, and the formal neighborhood of a closed subscheme (bellow).

B.3. **Category of $\Bbbk$-spaces (Yoneda completion of the category of $\Bbbk$-schemes).** This will be our main example of a Yoneda completion of a category. For examples of non-representable functors, i.e., functors which are in $\widehat{\mathcal{A}}$ but not in $\mathcal{A}$.

This is a geometric example. The geometry we use here is the algebraic geometry. Its geometric objects are called schemes and they are obtained by gluing schemes of a somewhat special type, which are called affine schemes (like manifolds are all obtained by gluing open pieces of $\mathbb{R}^n$'s). We start with a brief review.

B.3.1. *Affine* $\Bbbk$*-schemes.* Fix a commutative ring $\Bbbk$.

An affine scheme S over $\Bbbk$ is determined by its algebra of functions $\mathcal{O}(S)$, which is a $\Bbbk$-algebra. Moreover, any commutative $\Bbbk$-algebra A is the algebra of functions on some $\Bbbk$-scheme – the scheme is called the spectrum of A and denoted $\mathrm{Spec}(A)$. So, affine $\Bbbk$-schemes are really the same as commutative $\Bbbk$-algebras, except that a map of affine schemes $X \xrightarrow{\phi} Y$ defines a map of functions $\mathcal{O}(Y) \xrightarrow{\phi^*} \mathcal{O}(X)$ in the opposite direction (the pull-back $\phi^*(f) = f\circ\phi$). The statement

information contained in two kinds of objects is the same but the directions reverse when one passes from geometry to algebra

is stated in categorical terms:

[13] Similarly one calls projective systems pro-objects of $\mathcal{A}$.

categories $\mathcal{A}ff\mathcal{S}ch_{\Bbbk}$ *and* $(\mathcal{C}om\mathcal{A}lg_{\Bbbk})^o$ *are equivalent.*

[14]

The basic strategy. Our intuition is often geometric. So, one starts by translating geometric ideas into precise statements in algebra. These are then proved in algebra. Once sufficiently many geometric statements are verified in algebra, one can build up on these and do more purely in geometry.

B.3.2. *Formal neighborhood of* $0 \in \mathbb{A}^1$. Consider the contravariant functor on $\Bbbk$-*Schemes*

$$\Bbbk\text{-}Schemes \ni X \mapsto F(X) = \{f \in \mathcal{O}(X);\ f \text{ is nilpotent}\} \in \mathcal{S}ets.$$

It is an increasing union of subfunctors

$$\Bbbk\text{-}Schemes \ni X \mapsto F_n(X) = \{f \in \mathcal{O}(X);\ f^{n+1} = 0\} \in \mathcal{S}ets.$$

Looking for geometric interpretation of these functor we start with the n^{th} infinitesimal neighborhood $IN^n_{\mathbb{A}^1_{\Bbbk}}(0)$ of the point 0 in the line $\mathbb{A}^1_{\Bbbk} = \mathrm{Spec}(\Bbbk[x])$. This is the $\Bbbk$-scheme defined by the algebra

$$\mathcal{O}\big(IN^n_{\mathbb{A}^1_{\Bbbk}}(0)\big) \overset{\text{def}}{=} \Bbbk[x]/x^{n+1}, \quad \text{i.e.,} \quad IN^n_{\mathbb{A}^1}(0) \overset{\text{def}}{=} \mathrm{Spec}(\Bbbk[x]/x^{n+1}).$$

For instance, $IN^0_{\mathbb{A}^1}(0) = \{0\}$ is a point while $IN^1_{\mathbb{A}^1}(0)$ is a *double point*, etc.

We see that the functor F_n is representable – it is represented by the scheme $IN^n_{\mathbb{A}^1_{\Bbbk}}(0)$. Therefore, one should think of the functor F as the increasing union of infinitesimal neighborhoods of $0 \in \mathbb{A}^1$. For that reason we call F the *formal neighborhood of* $0 \in \mathbb{A}^1$.

B.3.3. *Formal neighborhood of a closed subscheme.* In general if Y is a closed subscheme of a scheme X given by the ideal $I_Y = \{f \in \mathcal{O}(X);\ f|Y = 0\}$, one can again define the n^{th} infinitesimal neighborhood of Y in X as an affine scheme

$$IN^n_X(Y) \overset{\text{def}}{=} \mathrm{Spec}(\mathcal{O}(X)/I_Y^{n+1}),$$

and then one defines the formal neighborhood $FN_X(Y)$ as a $\Bbbk$-space which is the union of infinitesimal neighborhoods, i.e., as the functor

$$\Bbbk\text{-}Schemes \ni Z \mapsto \cup_n \ Map[Z, IN^n_X(Y)].$$

B.4. **Groupoids (groupoid categories).** We consider a special class of categories, the *groupoid categories* . We get a new respect for categories when we notice that this special case of categories, is a common generalization of both groups and equivalence relations.

B.4.1. A groupoid category is a category such that all morphisms are invertible (i.e., isomorphisms).

[14] One can simplify this kind of thinking and *define* the category of affine schemes over $\mathbb{C}$ as the the opposite of the category of commutative $\mathbb{C}$-algebras. The part that would be skipped in this approach is how one develops a geometric point of view on affine schemes defined in this way.

B.4.2. *Example: Group actions and groupoids.* An action of a group G on a set X, produces a category X_G with $Ob(X_G) = X$ and

$$\mathrm{Hom}_{X_G}(a,b) \overset{\text{def}}{=} \{(b,g,a);\ g \in G \quad \text{and} \quad b = \ ga\}.$$

Here $1_a = (a,1,a)$ and the composition is given by multiplication in G: $(c,h,b)\circ(b,g,a) \overset{\text{def}}{=} (c,hg,a)$. This is a groupoid category: $(b,g^{-1},a)\circ(b,g,a) \overset{\text{def}}{=} (a,1,a)$.

B.4.3. *Example: Equivalence relations.* Any equivalence relation $\cong$ on a set X defines a category $X_{\cong}$ with $Ob(X_{\cong}) = X$ and $\mathrm{Hom}_{X_{\cong}}(a,b)$ is a point $\{b,a)\}$ if $a \cong b$ and otherwise $\mathrm{Hom}_{X_{\cong}}(a,b) = \emptyset$. The composition is $(c,b)\circ(b,a) \overset{\text{def}}{=} (c,a)$ and $1_a = (a,a)$. This is a groupoid category: $(a,b)\circ(b,a) = 1_a$.

B.4.4. *Lemma.* Let $\mathcal{C}$ be a groupoid category.

(a) A groupoid category $\mathcal{C}$ gives: a set $\pi_0(\mathcal{C})$ of isomorphism classes of objects of $\mathcal{C}$, and

(b) for each object $a \in \mathcal{G}$ a group $\pi_1(\mathcal{C},a) \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(a,a)$.

(b) If $a,b \in \mathcal{C}$ are isomorphic then $\mathrm{Hom}_{\mathcal{C}}(a,b)$ is a *bitorsor* for $(\mathrm{Hom}_{\mathcal{C}}(a,a), \mathrm{Hom}_{\mathcal{C}}(b,b))$, i.e., a torsor for each of the groups $\mathrm{Hom}_{\mathcal{C}}(a,a)$ and $\mathrm{Hom}_{\mathcal{C}}(a,a)$, and the actions of the two groups commute.

(c) A groupoid category on one object is the same as a group.

B.4.5. *Examples.* (1) For the *action groupoid* associated to an action of G on X

$$\pi_0(X_G) = X/G \quad \text{and} \quad \pi_1(X_G,a) = \ G_a.$$

(2) If $X_{\cong}$ is the groupoid given by an equivalence relation $\cong$ on X then

$$\pi_0(X_{\cong}) = X/\cong \quad \text{and} \quad \pi_1(X_{\cong},a) = \ \{1\}.$$

B.4.6. *Remarks.* Passing from a groupoid category $\mathcal{C}$ to the set $\pi_0(\mathcal{C})$ of isomorphism classes in $\mathcal{C}$, the main information we forget is the *automorphism groups* $\mathrm{Hom}_{\mathcal{C}}(a,a) = \mathrm{Aut}_{\mathcal{C}}(a)$ of objects.

To see the importance of this loss, we will blame the formation of singularities in the invariant theory quotients on passing from a groupoid category to the set of isomorphism classes. Remember that when $G = \{\pm 1\}$ acting on $X = \mathbb{A}^2$, one can organize the set theoretic quotient X/G into algebraic variety $X//G$ which has one singular point – the image of $\mathbf{0} = (0,0$. Recall that $\mathbf{0}$ is the only point in X which has a non-trivial stabilizer, i.e., which has a non-trivial automorphism group $\mathrm{Aut}_{X_{\cong}}(\mathbf{0})$ when we encode the action of G on X as a category structure $X_{\cong}$ on X.

So, the hint we get from this example is:

One may be able to remove some singularities in sets of isomorphism classes by remembering the automorphisms, i.e., remembering the corresponding groupoid category rather then just the set of isomorphism classes of objects.

This is the principle behind the introduction of *stack quotients*.

B.5. **Categories and sets.** Some of the relations:

- All sets form a category $\mathcal{S}ets$ which can be viewed as the basic example of a category.
- Each set S defines a (small) category $\underline{S}$ with $Ob(\underline{S}) = S$ and for $a, b \in S$ $\mathrm{Hom}_{\underline{S}}(a, b)$ is $\{1_a\}$ if $a = b$ and it is empty otherwise. In the opposite (and more stupid) direction, each small category $\mathcal{C}$ gives a set $Ob(\mathcal{C})$ (we just forget the morphisms).
- The structure of a category can be viewed as a more advanced version of the structure of a set.

B.5.1. *Question.* If all sets form a structure more complicated then a set – a category $\mathcal{S}ets$, what do all categories form? (All categories form a more complicated structure, a 2-category. Moreover all n-categories form an $(n+1)$-category ...)

B.5.2. *Operations on categories.* The last remark suggests that what we can do with sets, we should be able to do with categories (though it may get more complicated).

For instance the product of sets lifts to a notion of a product of categories $\mathcal{A}$ and $\mathcal{B}$, The category $\mathcal{A}\times\mathcal{B}$ has $Ob(\mathcal{A}\times\mathcal{B}) = Ob(\mathcal{A})\times Ob(\mathcal{B})$ and $\mathrm{Hom}_{\mathcal{A}\times\mathcal{B}}[(a', b'), (a'', b'')] \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{A}}[a', a'']\times\mathrm{Hom}_{\mathcal{B}}[b', b'']$.

However, since we are dealing with a finer structure there are operations on categories that do not have analogues in sets. Say the dual (opposite) category of $\mathcal{A}$ is the category $\mathcal{A}^o$ with $Ob(\mathcal{A}^o) = Ob(\mathcal{A})$, but

$$\mathrm{Hom}_{\mathcal{A}^o}(a, b) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{A}}(b, a).$$

This is the formal meaning of the observation that reversing the arrows gives a "duality operation" for constructions in category theory. For instance, a projective system in $\mathcal{A}$ is the same as an inductive system in $\mathcal{A}^o$, a sum in $\mathcal{A}$ is the same as a product in $\mathcal{A}^o$, etc. This is useful: for any statement we prove for projective systems there is a "dual" statement for inductive systems which is automatically true.

B.6. **Higher categories.** Notice that

(1) The class $\mathcal{C}at$ of all categories has a category like structure with $Ob(\mathcal{C}at) \stackrel{\mathrm{def}}{=}$ categories, and (for any two categories $\mathcal{A}$ and $\mathcal{B}$) $\mathrm{Hom}_{\mathcal{C}at}(\mathcal{A}, \mathcal{B}) \stackrel{\mathrm{def}}{=} Funct(\mathcal{A}, \mathcal{B}) \stackrel{\mathrm{def}}{=}$

functors from $\mathcal{A}$ to $\mathcal{B}$. However, the class $\mathrm{Hom}_{\mathcal{C}at}(\mathcal{A},\mathcal{B})$ need not be a set unless the categories $\mathcal{A}$ and $\mathcal{B}$ are small.

(2) Moreover, for any two categories $\mathcal{A}$ and $\mathcal{B}$, $\mathrm{Hom}_{\mathcal{C}at}(\mathcal{A},\mathcal{B}) = Funct(\mathcal{A},\mathcal{B})$ is actually a category with $Ob(Funct(\mathcal{A},\mathcal{B}))$ = functors from $\mathcal{A}$ to $\mathcal{B}$, and for $F, G \in Funct(\mathcal{A},\mathcal{B})$, the morphisms $\mathrm{Hom}_{Funct(\mathcal{A},\mathcal{B})}(F,G) \stackrel{\text{def}}{=}$ natural transforms from F to G.

The two structures (1) and (2) (together with natural compatibility conditions between them) make $\mathcal{C}at$ into what is called a 2-category. We leave this notion vague as it is not central to what we do now. (We will see that complexes and topological spaces are also 2-categories.)

Geometrically, a category $\mathcal{A}$ defines a 1-dimensional simplicial complex $|\mathcal{A}|$ (the nerve of $\mathcal{A}$) where vertices=$Ob(\mathcal{A})$ and (directed) edges between vertices a and b are given by $\mathrm{Hom}_{\mathcal{C}}(a,b)$. A 2-category $\mathcal{B}$ defines a 2-dimensional topological object $|\mathcal{B}|$ (the nerve of $\mathcal{B}$). If say, $\mathcal{B} = \mathcal{C}at$ then vertices $Ob(\mathcal{C}at)$ = categories, edges between vertices $\mathcal{A}$ and $\mathcal{B}$ corresponds to all functors from $\mathcal{A}$ to $\mathcal{B}$, and for two functors $\mathcal{A} \xrightarrow{F,G} \mathcal{B}$ (i.e., two edges from the point $\mathcal{A}$ to the point $\mathcal{B}$), morphisms $F \xrightarrow{\eta} G$ correspond to (directed) 2-cells in $|\mathcal{C}at|$ whose boundary is the union of edges corresponding to F and G. So one dimensional topology controls the level of our thinking when we use categories, 2-dimensional when we use 2-categories and in this way one can continue to define more complicated frameworks for thinking of mathematics, modeled on more complicated topology: the nerve of an n-category is an n-dimensional topological object for $n = 0, 1, 2, ..., \infty$ (here "1-category" means just "category" and a 0-category is a set).