HOMOLOGICAL ALGEBRA

Contents

Date: ?

4. **Categories**

We will use the language of categories seriously on several levels. Some examples:

- *Abelian categories.* This is a basic setting for homological algebra. It explains why we can calculate with sheaves "the same" as with abelian groups.
- *Triangulated categories.* This is the optimal setting for homological algebra needed for more subtle calculations and constructions.
- *Study of sheaves.* Categories appear from the beggining since we are interested in sheaves with values in a certain category. The notion of a stalk of a sheaf, i.e., a restriction of a sheaf is an instance of a notion of a limit in a category.

4.0.1. *Why categories?* The notion of a category is misleadingly elementary. It formalizes the idea that we study *certain kind of objects* (i.e., objects endowed with some specified structures) and that it makes sense to *go* from one such object to another via something (a "morphism") that preserves the relevant structures. Since this is indeed what we usually do, the *language* of categories is convenient.

However, soon one finds that familiar notions and constructions[1] **categorify**, i.e., have analogues (and often more then one) in general categories[2] This enriched language of categories was recognized as fundamental for describing various complicated phenomena, and the study of *special kinds of categories* mushroomed somewhat similarly as the study of special classes of functions in analysis.

4.1. **Categories.** A *category* $\mathcal{C}$ consists of

(1) a class $Ob(\mathcal{C})$, its elements are called *objects* of $\mathcal{C}$,
(2) for any $a, b \in Ob(\mathcal{C})$ a set $\mathrm{Hom}_{\mathcal{C}}(a, b)$ whose elements are called *morphisms* ("maps") from a to b in $\mathcal{C}$,
(3) for any $a, b, c \in Ob(\mathcal{C})$ a function $\mathrm{Hom}_{\mathcal{C}}(b, c)\times\mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}}(a, c)$, called *composition*,
(4) for any $a \in Ob(\mathcal{C})$ an element $1_a \in \mathrm{Hom}_{\mathcal{C}}(a, a)$,

such that

- the composition is associative and
- 1_a is a neutral element for composition.

Instead of $a \in \mathrm{Ob}(\mathcal{C})$ we will usually just say that $a \in \mathcal{C}$.

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[1] Such as (i) empty set, (ii) union of sets, (iii) product of sets, (iv) abelian group, ...

[2] Respectively: (i) initial object, final object, zero object; (ii) sum of objects or more generally a direct (inductive) limit of objects; (iii) product of objects or more generally the inverse (projective) limit of objects; (iv) additive category, abelian category; ...

4.1.1. *Examples.*

(1) Categories of *sets with additional structures*: $\mathcal{S}ets$, $\mathcal{A}b$, $\mathfrak{m}(\Bbbk)$ for a ring $\Bbbk$ (denoted also $\mathcal{V}ect(\Bbbk)$ if $\Bbbk$ is a field), $\mathcal{G}roups$, $\mathcal{R}ings$, $\mathcal{T}op$, $\mathcal{O}rd\mathcal{S}ets \overset{\text{def}}{=}$ category of ordered sets, ...
(2) To a category $\mathcal{C}$ one attaches the opposite category $\mathcal{C}^o$ so that objects are the same but the "direction of arrows reverses":
$$\mathrm{Hom}_{\mathcal{C}^o}(a,b) \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(b,a).$$
(3) Any partially ordered set $(I, \leq)$ defines a category with $Ob = I$ and $\mathrm{Hom}(a,b)$ is a point if $a \leq b$ (call this point (a,b)), and $\emptyset$ otherwise.
(4) Sheaves of sets on a topological space X, Sheaves of abelian groups on X,...

4.2. **Objects.** The categorical thinking allows us to view some phenomena in various parts of mathematics in a uniform way, by observing that these are examples of notions that make sense in every category. The first example are some special classes of objects and maps, and some constructions of objects.

4.2.1. *Some special objects and maps.* We say that $i \in \mathcal{C}$ is an *initial* object if for any $a \in \mathcal{C}$ set $\mathrm{Hom}_{\mathcal{C}}(i,a)$ has one element. Also, $t \in \mathcal{C}$ is a *terminal* or *final* object if for any $a \in \mathcal{C}$ set $\mathrm{Hom}_{\mathcal{C}}(a,t)$ has one element. We say that $z \in \mathcal{C}$ is a *zero* object if it is both initial and terminal.

A map $\phi \in \mathrm{Hom}_{\mathcal{C}}(a,b)$ is said to be an *isomorphism* if it is invertible, i.e., if there is a $\psi \in \mathrm{Hom}_{\mathcal{C}}(b,a)$ such that ...

Examples. (1) In $\mathcal{S}ets$ empty set is the only initial object, terminal objects are precisely the one-pint sets, and so there are no zero objects. In $\mathcal{A}b$ initial, terminal and zero objects coincide – these are the one element groups, i.e., zero groups. In $\mathcal{R}ings$, $\mathbb{Z}$ is the initial object and there is no terminal object.

Lemma. Initial object in $\mathcal{C}$ (if it exists) is unique up to a canonical isomorphism, i.e., for any two initial objects i, j in $\mathcal{C}$ there is a canonical isomorphism. (The same for terminal and zero objects.)

4.2.2. *Which notion of "equality" is useful in categories?* We first deal with equality of objects. Here. we make a philosophical remark which is essential for navigation in a category. In a set two elements may be equal or different, however when we try to extend this idea to objects in a category, a new subtlety appears. Two objects in a category $\mathcal{C}$ can be :

- (i) the same,
- (ii) isomorphic,
- (iii) isomorphic by a canonical (given) isomorphism.

It turns out that (i) is too restrictive, (ii) is too lax and (iii) is the most useful – the correct analogue of equality of elements of a set. In practice this means that we will often be imprecise, we will often say that "$a = b$" and mean that "we have in mind a specific isomorphism $\phi : a \rightarrow b$".

Example. An example of this style of thinking are, say, the terminal objects in a category – if they exist they are all "equal" in the sense that any two are *canonically* isomorphic.

4.2.3. *Products of objects.* A *product* of objects a and b in $\mathcal{C}$ is a triple (Π, p, q) where $\Pi \in \mathcal{C}$ is an object while $p \in \mathrm{Hom}_{\mathcal{C}}(\Pi, a)$, $q \in \mathrm{Hom}_{\mathcal{C}}(\Pi, b)$ are maps such that for any $x \in \mathcal{C}$ the function

$$\mathrm{Hom}_{\mathcal{C}}(x, \Pi) \ni \phi \mapsto (p \circ \phi, q \circ \phi) \in \ \mathrm{Hom}_{\mathcal{C}}(x, a) \times \mathrm{Hom}_{\mathcal{C}}(x, b)$$

is a bijection.

Remarks. (0) If $\mathcal{C} = \mathcal{S}ets$ then the product of sets $\Pi = a \times b$ together with projections p, q satisfies this property. Actually, the above categorical notion of a product is just the abstract formulation of properties of a product of sets.

(1) From our experience with sets and groups etc, we expect that a product of a and b should be a specific object built from a and b. However, this is not what the categorical definition above says. For given a, b there may be many triples (Π, p, q) satisfying the product property. However it is easy to see that any two such $(\Pi_i, p_i, q_i), i = 1, 2;$ are related by a *canonical isomorphism* $\phi : \Pi_1 \rightarrow \Phi_2$ provided by the the defining property of the product. This is another example of 4.2.2.

(2) We often abuse the language and say that "Π is product of a and b". What we mean is that we remember the additional data p, q but are too lazy to mention these. Moreover, we may even denote Π by $a \times b$ suggesting (in general incorrectly) that the product can be constructed naturally from a and b – what we mean by notation $a \times b$ is *some* object Π supplied with maps p, q with properties as above (again an example of the idea 4.2.2).

(3) The product of a and b is an example of a standard construction of an *object defined by a universal property.* In this case the property is that a map into a product is the same as a pair of maps into a and b. We can also think of it as *object (co)representing a functor.* In our case Π corepresents a contravariant functor $\mathcal{C} \ni x \mapsto F(x) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{C}}(x, a) \times \mathrm{Hom}_{\mathcal{C}}(x, b) \in \mathcal{S}ets$, in the sense that the functor F is identified with the functor $\mathrm{Hom}_{\mathcal{C}}(-, \Pi)$ that one gets from Π.

In general, *an object defined by some universal property* $\mathcal{P}$ is

- (i) not really a single object but a system of various objects related by (compatible) isomorphisms,
- (ii) each of these objects does not come alone but is supplied with some additional data consisting of some morphisms (such as p, q above).

(4) A product of two objects a and b in a given category $\mathcal{C}$ need not exist! (Find one example!)

4.2.4. *Sums.* A *sum* of objects a and b in $\mathcal{C}$ is a triple (Σ, i, j) where $\Sigma \in \mathcal{C}$ is an object while $i \in \mathrm{Hom}_{\mathcal{C}}(a,\Sigma),\ j \in \mathrm{Hom}_{\mathcal{C}}(b,\Sigma)$ are maps such that for any $x \in \mathcal{C}$ the function

$$\mathrm{Hom}_{\mathcal{C}}(\Sigma, x) \ni \phi \mapsto (\phi\circ i, \phi\circ j) \in \ \mathrm{Hom}_{\mathcal{C}}(a,x)\times\mathrm{Hom}_{\mathcal{C}}(b,x)$$

is a bijection.

Example. In $\mathcal{S}ets$ the sums exist and the sum of a and b is the disjoint union $a \sqcup b$.

4.2.5. *Sums and products of families of objects.* This is the same as for two objects. A *product in* $\mathcal{C}$ of a family of objects $a_i \in \mathcal{C},\ i \in I$, is a pair $(P, (p_i)_{i\in I}))$ where $P \in \mathcal{C}$ and $p_i : P \to a_i$ are such that the map

$$\mathrm{Hom}_{\mathcal{C}}(x, P) \ni \phi \mapsto (p_i\circ\phi)_{i\in I} \in \ \Pi_{i\in I}\ \mathrm{Hom}_{\mathcal{C}}(x, a_i)$$

is a bijection. We use the notation $\prod_{i\in I}\ a_i$.

A sum of $a_i \in \mathcal{C},\ i \in I$ is a pair $(S, (j_i)_{i\in I}))$ where $j_i : a_i \to S$ gives a bijection

$$\mathrm{Hom}_{\mathcal{C}}(S, x) \ni \phi \mapsto (\phi\circ j_i)_{i\in I} \in \ \Pi_{i\in I}\ \mathrm{Hom}_{\mathcal{C}}(a_i, x).$$

The notation is $\sqcup_{i\in I}\ a_i$ or $\oplus_{i\in I}\ a_i$.

Lemma. For a ring $\Bbbk$ the category $\mathfrak{m}(\Bbbk)$ has sums and products.

(1) The product $\prod_{i\in I}\ M_i$ is (as a set) just the product of sets, so it consists of all families $m = (m_i)_{i\in I}$ with $m_i \in M_i,\ i \in I$.[3]
(2) The sum $\oplus_{i\in I}\ a_i$ happens to be the submodule of $\prod_{i\in I}\ M_i$ consisting of all finite families $m = (m_i)_{i\in I}$, i.e.families such that $m_i = 0$ for all but finitely many $i \in I$.

Remark. This is how we get familiar with categorical constructions: by checking what they mean in familiar categories.

4.3. **Limits.** Categorical thinking allows us to extend the idea of *limits* from analysis to many other settings.[4]

This is often indispensable. We will need it in order to be able to restrict a sheaf on a topological space to a given point (this is the notion of the *stalk* of a sheaf, see 4.3.8).

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[3] Such family is often written as a – possibly infinite – sum $\sum_{i\in I}\ m_i \stackrel{\text{def}}{=} (m_i)_{i\in I}$.

[4] This is one of the two most useful ideas in category theory. The other one is the notion of *adjoint functors*, see 4.6.

4.3.1. *Example.* In some instances it is clear what one should mean by a limit of a family of objects. Consider a sequence of *increasing subsets* $A_0 \subseteq A_1 \subseteq \cdots$ of a set A, we will say that its limit $\lim\limits_{\rightarrow} A_i$ is the subset $\cup_{i\geq 0} A_i$ of A. Similarly, the limit of a *decreasing sequence* of subsets $B_0 \supseteq B_1 \subseteq \cdots$ of A, will be the subset $\lim\limits_{\leftarrow} B_i \stackrel{\text{def}}{=} \cap_{i\geq 0} B_i$ of A.

Now we give precise meaning to two constructions corresponding to these two examples:

4.3.2. *Inductive limits.* An *inductive system* of objects of $\mathcal{C}$ over a partially ordered set $(I, \leq)$, consists of

(1) a family of objects $a_i \in \mathcal{C},\ i \in I$; and a
(2) system of maps $\phi_{ji} : a_i \longrightarrow a_j$ for all $i \leq j$ in I;

such that

- $\phi_{ii} = 1_{a_i},\ i \in I$ and
- $\phi_{kj} \circ \phi_{ji} = \phi_{ki}$ when $i \leq j \leq k$.

Its *inductive limit* is a pair $(a, (\rho_i)_{i\in I})$ of an object $a \in \mathcal{C}$ and a system of maps $\rho_i : a_i \longrightarrow a,\ i \in I$, such that

- (i) $\rho_j \circ \phi_{ji} = \rho_i$ for $i \leq j$, and moreover
- (ii) $(a, (\rho_i)_{i\in I})$ is universal with respect to this property in the sense that

for any $(a', (\rho'_i)_{i\in I})$ that satisfies $\rho'_j \circ \phi_{ji} = \rho'_i$ for $i \leq j$,
there is a unique map $\rho : a \longrightarrow a'$ such that $\rho'_i = \rho \circ \rho_i,\ i \in I$.

Informally, we write $\lim\limits_{\rightarrow\, I,\leq} a_i =\ a$.

4.3.3. *Example.* A system of increasing subsets $A_0 \subseteq A_1 \subseteq \cdots$ of A really form an inductive system in the category $\mathcal{S}ets$ (and over $\mathbb{N}$ with the standard order) and $\lim\limits_{\rightarrow} A_i = \cup_{i\geq 0} A_i$.

4.3.4. *Projective limits.* A *projective system* of objects of $\mathcal{C}$, over a partially ordered set $(I, \leq)$, consists of

(1) a family of objects $a_i \in \mathcal{C},\ i \in I$; and
(2) a system of maps $\phi_{ij} : a_j \longrightarrow a_i$ for all $i \leq j$ in I;

such that

- $\phi_{ii} = 1_{a_i},\ i \in I$ and
- $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ when $i \leq j \leq k$.

Its limit is a pair $(a, (\sigma_i)_{i\in I})$ of $a \in \mathcal{C}$ and maps $\sigma_i : a \longrightarrow a_i$ such that

- $\phi_{ij} \circ \sigma_j = \sigma_i$ for $i \leq j$, and

- $(a,(\sigma_i)_{i\in I})$ is universal in the sense that for any $(a',(\sigma_i')_{i\in I})$ that satisfies $\phi_{ij}\circ\sigma_j' = \sigma_i'$ for $i\le j$, there is a unique map $\sigma: a'\to a$ such that $\sigma_i' = \sigma_i\circ\sigma,\ i\in I$.

Informally, $\lim\limits_{\leftarrow\ I,\le} a_i = a$.

Example. A decreasing sequence of subsets $B_0\supseteq B_1\subseteq\cdots$ of A forms a projective system and $\lim\limits_{\leftarrow} B_i = \cap_{i\ge 0} B_i$ of A.

Remark. In these basic examples we have: "$\lim\limits_{\rightarrow}$ = growth", and "$\lim\limits_{\leftarrow}$ = decline", but in general it can also go the opposite way:

4.3.5. *Lemma.* (a) Let $(I,\le)$ be $\{1,2,3,...\}$ with the order $i\le j$ if i divides j. In $\mathcal{A}b$ let $A_i = \mathbb{Q}/\mathbb{Z}$ for all $i\in I$, and let ϕ_{ji} be the multiplication by j/i when i divides j. This is an inductive system and $\lim\limits_{\rightarrow} A_i = ?$.

(b) Let $(I,\le)$ be $\mathbb{N}=\{0,1,...\}$ with the standard order. In $\mathcal{R}ings$ let $\Bbbk_n = \mathbb{C}[x]/x^{n+1}$ and for $i\le j$ let ϕ_{ij} be the obvious quotient map. This is a projective system and $\lim\limits_{\leftarrow} \Bbbk_i = ?$.

4.3.6. *Limits are functorial.* Let $\mathcal{IS}_{(I,\le)}(\mathcal{C})$ be the category of inductive systems in the category $\mathcal{C}$ and over a partially ordered set $(I,\le)$. Objects are the inductive systems $(A_i)_{i\in I},(\phi_{ji})_{i\le j}$ and a map μ from $(A_i')_{i\in I},(\phi_{ji}')_{i\le j}$ to $(A_i'')_{i\in I},(\phi_{ji}'')_{i\le j}$ is a system of maps $\mu_i: A_i'\to A_i'',\ i\in I$, which intertwine the structure maps of the two inductive systems i.e., for $i\le j$ the diagram

$$\begin{array}{ccc} A_i' & \xrightarrow{\mu_i} & A_i'' \\ {\scriptstyle \phi_{ji}'}\downarrow & & {\scriptstyle \phi_{ji}''}\downarrow \\ A_j' & \xrightarrow{\mu_i} & A_j'' \end{array}$$

commutes.

Lemma. If limits of both systems exist then a map μ of systems defines a map $\lim\limits_{\rightarrow} A_i' \xrightarrow{\lim_{\rightarrow}\mu_i} \lim\limits_{\rightarrow} A_i''$.

Proof. By definition of $\lim\limits_{\rightarrow}$.

4.3.7. *Limits in sets, abelian groups, modules and such.* Next we will see that in the category $\mathcal{S}ets$ one has inductive and projective limits (i.e., each inductive or projective system has a limit):

Lemma. Let $(I,\leq)$ be a partially ordered set such that for any $i,j\in I$ there is some $k\in I$ such that $i\leq k\geq j$.

(a) (**Construction of projective limits of sets**.) Let $(A_i)_{i\in I}$ and maps $(\phi_{ij}:A_j\to A_i)_{i\leq j}$ be a projective system of sets. Then

$\lim\limits_{\leftarrow} A_i$ *is the subset of* $\prod_{i\in I}\ A_i$ *consisting of all families* $a=(a_i)_{i\in I}$, *such that* $\phi_{ij}a_j=a_i$ *for* $i\leq j$.

b) (**Construction of inductive limits of sets**.) Let the family of sets $(A_i)_{i\in I}$ and maps $(\phi_{ji}:A_i\to A_j)_{i\leq j}$ be an inductive system of sets. Then

(1) The relation $\sim$ defined on the disjoint union $\sqcup_{i\in I}\ A_i \stackrel{\text{def}}{=} \cup_{i\in I}\ A_i\times\{i\}$ by
- $(a,i)\sim(b,j)$ (for $a\in A_i,\ b\in A_j$), if there is some $k\geq i,j$ such that "$a=b$ in A_k", i.e., if $\phi_{ki}a=\phi_{kj}b$,

is an equivalence relation.

(2) $\lim\limits_{\rightarrow} A_i$ is the quotient $[\sqcup_{i\in I}\ A_i]/\sim$ of the disjoint union by the above equivalence relation.

Remarks. (1) The above lemma gives simple descriptions of limits in sets: $\lim\limits_{\leftarrow} A_i$ is a certain subset of the product $\prod\ A_i$, and $\lim\limits_{\rightarrow} A_i$ is a certain quotient of sum (i.e., disjoint union) $\sqcup_{i\in I}\ A_i$.

(2) Even better, $\lim\limits_{\rightarrow} A_i$ can be described in English:

- for $i\in I$, any $a\in A_i$ defines an element $\overline{a}$ of $\lim\limits_{\rightarrow} A_i$,
- all elements of $\lim\limits_{\rightarrow} A_i$ arise in this way, and
- for $a\in A_i$ and $b\in A_j$ one has $\overline{a}=\overline{b}$ iff for some $k\in I$ with $i\leq k\geq j$ one has $a=b$ in A_k.

Proof. This is just a simple retelling of the lemma, here $\overline{a}$ is the image of (a,i) in the quotient $[\sqcup_{i\in I}\ A_i]/\sim$.

(2) The same existence and description of limits works in many categories such as abelian groups, $\Bbbk$-modules, $\mathcal{G}roups$, etc (these are all categories where objects are sets with some additional structure).

For instance for an inductive system of abelian groups A_i over $(I,\leq)$, the inductive limit $\lim\limits_{\rightarrow} A_i$ always exists, it can be described as in (1), but one has to also explain what is the group structure (addition) on the set $[\sqcup_{i\in I}\ A_i]/\sim$. This is clear – if $a\in A_i$ and $b\in A_j$ then $\overline{a}+\overline{b}=\overline{\phi_{ki}a+\phi_{kj}b}$ for any k with $i\leq k\geq j$.

(3) Moreover, the lemma generalizes to limits in an arbitrary category (see lemma 4.3.12 bellow). The formulation is slightly longer since we have to explicitly ask for the existence of some standard constructions which are obvious in the categories mentioned in (2).

(4) In general, limits need not exist. For instance the category of finite sets clearly does not have infinite sums or products, hence can not have limits.[5]

4.3.8. *Stalks of a sheaf.* We want to restrict a sheaf $\mathcal{F}$ on a topological space X to a point $a \in X$. The restriction $\mathcal{F}|a$ is a sheaf on a point, so it just one set $\mathcal{F}_a \stackrel{\text{def}}{=} (\mathcal{F}|a)(\{a\})$ called the stalk of $\mathcal{F}$ at a. It will give us one of fundamental intuitions about sheaves.

What should $\mathcal{F}_a$ be? It has to be related to all $\mathcal{F}(U)$ where $U \subseteq X$ is is open and contains a, and $\mathcal{F}(U)$ should be closer to $\mathcal{F}_a$ when U is a smaller neighborhood. A formal way to say this is that

- (i) the set $\mathcal{N}_a$ of neighborhoods of a in X is partially ordered by $U \leq V$ if $V \subseteq U$,
- (ii) the values of $\mathcal{F}$ on neighborhoods $(\mathcal{F}(U))_{U \in \mathcal{N}_a}$ form an inductive system,
- (iii) we define the stalk by $\mathcal{F}_a \stackrel{\text{def}}{=} \varinjlim_{U \in \mathcal{N}_a} \mathcal{F}(U)$.

The basic examples are given by

Lemma. (a) The stalk of a constant sheaf of sets $S_{\mathbb{R}^n}$ at any point is canonically identified with the set S.

(b) The stalk of a the sheaf $\mathcal{H}_{\mathbb{C}}$ of holomorphic functions at the origin is canonically identified with the ring of convergent power series. ("Convergent" means that the series converges on *some* disc around the origin.)

4.3.9. *Sums and products as limits.* Now we that sums and products are special cases of limits:

Lemma. Let $a_i \in \mathcal{C},\ i \in I$. If we supply I with the discrete partial order (i.e., $i \leq j$ iff $i = j$), then $\varinjlim a_i$ is the same as $\oplus_{i \in I} a_i$, and $\varprojlim a_i$ is the same as $\prod_{i \in I} a_i$.

Remark. Observe that I does not have the property which we occasionally assumed above, i.e., that for any $i, j \in I$ there is $l \in I$ with $i \leq l$ and $j \leq l$.

4.3.10. *Fibered products as limits.* For a diagram $a_1 \xrightarrow{p} b \xleftarrow{q} a_2$ in $\mathcal{C}$, we say the fibered product of a_1 and a_2 over the base b (denoted $a_1 \times_b a_2$), is defined as the projective limit of the projective system $a_i,\ i \in I$ where

- (i) $I = \{1, 2, k\}$ and $a_k = b$,
- (ii) the only nontrivial inequalities in I are $1 \geq k \leq 2$.

[5] But it can (and does) have *finite limits*, i.e., limits over finite ordered sets I !

Lemma. (a) In $\mathcal{S}ets$ fibered products exist and the fibered product of $A \xrightarrow{p} S \xleftarrow{q} B$ is the set $A \underset{S}{\times} B = \{...\}$.

(b) If $A \xrightarrow{\subseteq} S \xleftarrow{\supseteq} B$ are inclusions of subsets of S the fibered product of sets $A \underset{S}{\times} B$ is just the intersection $A \cap B$.

4.3.11. *Existence and construction of limits.* The following lemma gives a criterion for existence of limits in a category $\mathcal{C}$, and a way to describe them in terms of simpler constructions. The particular case when $\mathcal{C}$ is the category of sets is the lemma 4.3.7 – it is less abstract since we assume some familiar properties of sets. The general proof is "the same".

4.3.12. *Lemma.* (a) If a category $\mathcal{C}$ has

(1) products of families of objects and
(2) any pair of maps $\mu, \nu \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ has an *equalizer* (i.e., a map $e \xrightarrow{\sigma} a$ universal among all maps ϕ into a such that $\mu \circ \phi = \nu \circ \phi$),

then $\mathcal{C}$ has projective limits, and these can be described in terms of products and equalizers.

(b) Dually, if a category $\mathcal{C}$ has

(1) sums of families of objects and
(2) any pair of maps $\mu, \nu \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ has a *coequalizer* (i.e., a map $b \xrightarrow{\sigma} c$ universal among all maps ϕ from b such that $\phi \circ \mu = \phi \circ \nu$),

then $\mathcal{C}$ has inductive limits, and these can be described in terms of sums and coequalizers.

4.4. **Functors.** The analogue on the level of categories of a function between two sets is a *functor between two categories.*

A functor F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of

(1) for each object $a \in \mathcal{A}$ an object $F(a) \in \mathcal{B}$,
(2) for each map $\alpha \in \mathrm{Hom}_{\mathcal{A}}(a', a'')$ in $\mathcal{A}$ a map $F(\alpha) \in \mathrm{Hom}_{\mathcal{B}}(Fa', Fa'')$

such that

- (i) F preserves compositions and units, i.e., $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$ and
- (ii) $F(1_a) = 1_{Fa}$.

Remark. A *functor* means a *natural construction,* i.e., a construction of objects of $\mathcal{B}$ from objects in $\mathcal{A}$ which is sufficiently natural so that it extends to morphisms in the two categories.

Examples. (1) A map of rings $\Bbbk \xrightarrow{\phi} l$ gives

- a *pull-back* (or *inverse image*) functor $\phi^* : \mathfrak{m}(l) \to \mathfrak{m}(\Bbbk)$ where $\phi^* N$ is N as an abelian group, but now it is considered as module for $\Bbbk$ via ϕ.
- a *push-forward* (or *direct image*) functor $\phi_* : \mathfrak{m}(\Bbbk) \to \mathfrak{m}(l)$ where $\phi_* M \stackrel{\text{def}}{=} l \otimes_\Bbbk M$. This is also called *change of coefficients* from $\Bbbk$ to l.

To see that these are functors, we need to define them also on maps. So, a map $\beta : N' \to N''$ in $\mathfrak{m}(l)$ gives a map $\phi^*(\beta) : \phi^*(N') \to \phi^*(N'')$ in $\mathfrak{m}(\Bbbk)$ which as a function between sets is really just $\beta : N' \to N''$. On the other hand, $\alpha : M' \to M''$ in $\mathfrak{m}(\Bbbk)$ gives $\phi_*(\alpha) : \phi_*(M') \to \phi_*(M'')$ in $\mathfrak{m}(l)$, this is just the map $1_l \otimes \alpha : \ l \otimes_\Bbbk M' \to l \otimes_\Bbbk M'', \ c \otimes x \mapsto c \otimes \alpha(x)$.

(2) Here we see a general feature:

functors often come in pairs ("adjoint pairs of functors") and usually one of them is stupid and the other one an interesting construction.

(3) For any category $\mathcal{A}$ there is the identity functor $1_\mathcal{A} : \mathcal{A} \to \mathcal{A}$. Two functors $\mathcal{A} \xrightarrow{F} \mathcal{B}$ and $\mathcal{B} \xrightarrow{G} \mathcal{C}$ can be composed to a functor $\mathcal{A} \xrightarrow{G \circ F} \mathcal{C}$.

(3) An object $a \in \mathcal{A}$ defines two functors,

$$\mathrm{Hom}_\mathcal{A}(a, -) : \ \mathcal{A} \to \mathcal{S}ets \quad \text{and} \quad \mathrm{Hom}_\mathcal{A}(-, a) : \ \mathcal{A}^o \to \mathcal{S}ets.$$

Moreover, $\mathrm{Hom}_\mathcal{A}(-,-)$ is a functor from $\mathcal{A}^o \times \mathcal{A}$ to sets![6]

(4) For a ring $\Bbbk$, tensoring is a functor $- \otimes_\Bbbk - : \ \mathfrak{m}^r(\Bbbk) \times \mathfrak{m}^l(\Bbbk) \to \mathcal{A}b$.

4.4.1. *Contravariant functors.* We say that a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is given by assigning to any $a \in \mathcal{A}$ some $F(a) \in \mathcal{B}$, and for each map $\alpha \in \mathrm{Hom}_\mathcal{A}(a', a'')$ in $\mathcal{A}$ a map $F(\alpha) \in \mathrm{Hom}_\mathcal{B}(Fa'', Fa')$ – notice that we have changed the direction of the map so now we have to require $F(\beta \circ \alpha) = F(\alpha) \circ F(\beta)$ (and $F(1_a) = 1_{Fa}$).

This is just a way of talking, not a new notion since a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is the same as a functor F from $\mathcal{A}$ to $\mathcal{B}^o$ (or a functor F from $\mathcal{A}^o$ to $\mathcal{B}$).

4.4.2. *Notions of subcategory as categorifications of the notion of a "subset".* The categorical analogue of the notion of a subset of a set splits into two notions of a (full) subcategory of a category.

A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ is given by a subclass $Ob(\mathcal{C}') \subseteq Ob(\mathcal{C})$ and for any $a, b \in Ob(\mathcal{C}')$ a subset $\mathrm{Hom}_{\mathcal{C}'}(a,b) \subseteq \mathrm{Hom}_\mathcal{C}(a,b)$ such that $\mathrm{Hom}_{\mathcal{C}'}(a,a) \ni 1_a, \ a \in \mathcal{C}'$, and the sets $\mathrm{Hom}_{\mathcal{C}'}(a,b), \ a, b \in \mathcal{C}'$ are closed under the composition in $\mathcal{C}$.

A *full subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ is a subcategory $\mathcal{C}'$ such that for any $a, b \in \mathcal{C}'$ one has $\mathrm{Hom}_{\mathcal{C}'}(a,b) = \mathrm{Hom}_\mathcal{C}(a,b)$. Notice that choosing a full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is the same as choosing a subclass $Ob(\mathcal{C}') \subseteq Ob(\mathcal{C})$.

[6] Exercise. Define the product $\mathcal{A} \times \mathcal{B}$ of categories $\mathcal{A}$ and $\mathcal{B}$.

Examples. (1) $\mathcal{F}ree(\Bbbk)\subseteq\mathfrak{m}(\Bbbk)$. (2) Category $\mathcal{C}$ defines subcategory $\mathcal{C}^*$ where objects are the same and morphisms are the isomorphisms from $\mathcal{C}$.

4.4.3. *Some categorifications of notions injection, surjection, bijection.* The following properties of a functor $F:\mathcal{A}\hookrightarrow\mathcal{B}$ are categorical analogues of *injectivity* property of a function:

- F is *faithful* (also called an *embedding of categories*), if all maps $\mathrm{Hom}_{\mathcal{A}}(a',a'')\to\mathrm{Hom}_{\mathcal{B}}(Fa',Fa'')$, $a',a''\in\mathcal{A}$ are injective.
- F is called *fully faithful* (or a *full embedding*) if all maps $\mathrm{Hom}_{\mathcal{A}}(a',a'')\to\mathrm{Hom}_{\mathcal{B}}(Fa',Fa'')$ are bijections.

An analogue of surjectivity:

- F is said to be *essentially surjective*, if it is surjective on isomorphism classes of objects, i.e., any $b\in\mathcal{B}$ is isomorphic to Fa for some $a\in\mathcal{A}$.

An analogue of a bijection:

- F is said to be an *equivalence of categories* if it is essentially surjective and fully faithful.

Here we took a point of view that a bijection is a function which is both injective and surjective.

4.4.4. *Examples.* For $\phi:\Bbbk\to l$ consider $\phi^*:\mathfrak{m}(l)\to\mathfrak{m}(\Bbbk)$. For instance if ϕ is (i) the canonical map $\mathbb{Z}\to\Bbbk$ or (ii) the inclusion $\mathbb{R}\subseteq\mathbb{C}$ then ϕ^* is one of the forgetful functors (i) $\mathfrak{m}(\Bbbk)\to\mathcal{A}b$ or (ii) $\mathcal{V}ect_{\mathbb{C}}\to\mathcal{V}ect_{\mathbb{R}}$.

ϕ^* is always faithful but in our examples it is neither full nor essentially surjective.

Example. Let $\Bbbk$ be a field and $\mathcal{V}_{\Bbbk}$ the category such that $Ob(\mathcal{V}_k)=\ \mathbb{N}=\{0,1,...\}$ and $\mathrm{Hom}(n,m)=M_{mn}$, the matrices with m rows and n columns. Then $\mathcal{V}_{\Bbbk}$ is equivalent to the category $\mathcal{V}ect_{\Bbbk}^{fd}$ of finite dimensional vector spaces by the functor $\mathcal{V}_{\Bbbk}\xrightarrow{\iota}\mathcal{V}ect_{\Bbbk}^{fd}$, here $\iota(n)=\Bbbk^n$ and for a matrix $\alpha\in M_{mn}$, $\iota_\alpha:\ \Bbbk^m\to\Bbbk^n$ is the multiplication by α.

Notice that the categories $\mathcal{V}_{\Bbbk}$ and $\mathcal{V}ect_{\Bbbk}^{fd}$ are in some sense very different (say only the first one is small), however, their content is the same (the linear algebra). One of these categories is more convenient for computation and the other for thinking. Historically, equivalence ι is roughly the observation that one can do linear algebra without always choosing coordinates (i.e., a basis).

4.5. **Natural transformations of functors ("morphisms of functors").** A natural transformation η of a functor $F:\mathcal{A}\to\mathcal{B}$ into a functor $G:\mathcal{A}\to\mathcal{B}$ consists of maps

$\eta_a \in \mathrm{Hom}_{\mathcal{B}}(Fa, Ga),\ a\in\mathcal{A}$ such that for any map $\alpha : a' \to a''$ in $\mathcal{A}$ the following diagram commutes

$$\begin{array}{ccc} F(a') & \xrightarrow{F(\alpha)} & F(a'') \\ {\scriptstyle\eta_{a'}}\downarrow & & {\scriptstyle\eta_{a''}}\downarrow \\ G(a') & \xrightarrow{G(\alpha)} & G(a'') \end{array}, \quad i.e., \quad \eta_{a''}\circ F(\alpha) = G(\alpha)\circ\eta_{a'}.$$

So, η relates values of functors on objects in a way compatible with the values of functors on maps. In practice, any "natural" choice of maps η_a will have the compatibility property.

4.5.1. *Example.* For the functors $\phi_* M = l\otimes_{\Bbbk} M$ and $\phi^* N = N$ from 4.4(1), there are canonical morphisms of functors

$$\alpha :\ \phi_*\circ\phi^* \to 1_{\mathfrak{m}(l)}, \qquad \phi_*\circ\phi^*(N) = l\otimes_{\Bbbk} N \xrightarrow{\alpha_N} N = 1_{\mathfrak{m}(l)}(N)$$

is the action of l on N and

$$\beta :\ 1_{\mathfrak{m}(\Bbbk)} \to \phi^*\circ\phi_*, \qquad \phi^*\circ\phi_*(M) = l\otimes_{\Bbbk} M \xleftarrow{\beta_M} M = 1_{\mathfrak{m}(\mathbb{M})}(M)$$

is the map $m\mapsto 1_l\otimes m$.

For any functor $F : \mathcal{A}\to\mathcal{B}$ one has $1_F : F\to F$ with $(1_F)_a = 1_{Fa} : Fa \to Fa$. For three functors F, G, H from $\mathcal{A}$ to $\mathcal{B}$ one can compose morphisms $\mu : F\to G$ and $\nu : G\to H$ to $\nu\circ\mu : F\to H$

4.5.2. *Lemma.* For two categories $\mathcal{A},\mathcal{B}$, the functors from $\mathcal{A}$ to $\mathcal{B}$ form a category $Funct(\mathcal{A},\mathcal{B})$.

Proof. For $F, G : \mathcal{A}\to\mathcal{B}$ one defines $\mathrm{Hom}(F,G)$ as the set of natural transforms from F to G, then all the structure is routine.

4.6. **Adjoint functors.** This is often the most useful categorical idea.

4.6.1. *Useful definition.* An adjoint pair of functors is a pair of functors $(\mathcal{A}\xrightarrow{F}\mathcal{B}, \mathcal{B}\xrightarrow{G}\mathcal{A})$ together with "natural identifications"

$$\zeta_{a,b} :\ \mathrm{Hom}_{\mathcal{B}}(Fa, b)\xrightarrow{\cong}\mathrm{Hom}_{\mathcal{A}}(a, Gb),\ a\in\mathcal{A},\ b\in\mathcal{B}.$$

Here,

(1) "natural" means behaving naturally in a and b, and by this we mean that ζ is a natural transform of functors $\zeta :\ \mathrm{Hom}_{\mathcal{B}}(F-,-)\to\mathrm{Hom}_{\mathcal{A}}(-,G-)$ from $\mathcal{A}^o\times\mathcal{B}$ to sets.
(2) "identification" means that each function $\zeta_{a,b}$ is a bijection.

We say that F is the left adjoint of G and that that G is the left adjoint of F (in the identity of homomorphisms F appears on the left in Hom and G on the right).

4.6.2. *Lemma.* Functors (ϕ_*, ϕ^*) from 4.4(1) form an adjoint pair, i.e., there is a canonical identification

$$\mathrm{Hom}_{\mathfrak{m}(l)}(\phi_* M, N) \xrightarrow{\eta_{M,N}} \mathrm{Hom}_{\mathfrak{m}(\Bbbk)}(M, \phi^* N), \qquad M \in \mathfrak{m}(\Bbbk),\ N \in \mathfrak{m}(l).$$

If $l\otimes_{\Bbbk} M \xrightarrow{\sigma} N,\ M \xrightarrow{\tau} N$, then $\mathrm{Hom}_{\mathfrak{m}(l)}(l\otimes_{\Bbbk} M, N) \xrightarrow{\eta_{M,N}} \mathrm{Hom}_{\mathfrak{m}(\Bbbk)}(M, N)$ by

$$\eta(\sigma)(m) = \ \sigma(1\otimes m) \quad \text{and} \quad \eta^{-1}(\tau)(c\otimes m) = c\tau(m), \quad m \in M,\ c \in l.$$

4.6.3. *Remark.* As in this example, often an adjoint pair appears in the following way: there is an obvious functor A (so obvious that we usually do not pay it any attention), but it has an adjoint B which is an interesting construction. The point is that this "interesting construction" B is intimately tied to the original "stupid" functor A, hence the properties of B can be deduced from the properties of the original simpler construction A. In fact B is produced from A in an explicit way as the following lemma shows.

4.6.4. What is the relation between morphisms of functors $\phi_*\circ\phi^* \xrightarrow{\alpha} 1_{\mathfrak{m}(l)}$ and $\phi^*\circ\phi_* \xleftarrow{\beta} 1_{\mathfrak{m}(\Bbbk)}$, from 4.5.1, and the isomorphism of functors $\mathrm{Hom}_{\mathfrak{m}(l)}(\phi_* -, -) \xrightarrow{\eta} \mathrm{Hom}_{\mathfrak{m}(\Bbbk)}(-, \phi^* -)$ from 4.6.2 ? They are really the same thing, i.e., two equivalent ways to describe adjointness.

4.6.5. *Lemma.* (Existence of the right adjoint.)
(a) If F has a right adjoint F then for each $b \in \mathcal{B}$ the functor

$$\mathrm{Hom}_{\mathcal{B}}(F-, b) : \mathcal{A} \to \mathcal{S}ets,\ a \mapsto \mathrm{Hom}_{\mathcal{B}}(Fa, b)$$

is representable (see **??**).

(b) Suppose that for each $b \in \mathcal{B}$ the functor $\mathrm{Hom}_{\mathcal{B}}(F-, b) : \mathcal{A} \to \mathcal{S}ets$ is representable. For each $b \in B$ choose a representing object $Gb \in \mathcal{A}$, then G is a functor from $\mathcal{B}$ to $\mathcal{A}$ and it s the right adjoint of F.

(c) The right adjoint of F, if it exists, is unique up to a canonical isomorphism.

Of course the symmetric claims hold for left adjoints.

4.6.6. *Left adjoints of forgetful functors.* We say that a functor $\mathcal{F}$ is forgetful if it consists in dropping part of the structure of an object. Bellow we will denote its left adjoint by $\mathcal{G}$. Standard construction (that add to the structure of an object), are often adjoints of forgetful functors

(1) If $\mathcal{F} : \mathfrak{m}(\Bbbk) \to \mathcal{S}ets$ then $\mathcal{G}$ sends set S to the the free $\Bbbk$-module $\Bbbk[S] = \ \oplus_{s\in S}\ \Bbbk\cdot s$ with a basis S.
(2) Let $\Bbbk$ be a commutative ring. For $\mathcal{F} : \Bbbk - \mathcal{C}om\mathcal{A}lg \to \mathcal{S}et$ from commutative $\Bbbk$-algebras to sets, $\mathcal{G}$ sends a set S to the polynomial ring $\Bbbk[x_s,\ s \in S]$ where variables are given by all elements of S.

(3) If $\mathcal{F} : \Bbbk - \mathcal{C}om\mathcal{A}lg \to \mathfrak{m}(\Bbbk)$, then for a $\Bbbk$-module M, $\mathcal{G}(M)$ is the symmetric algebra $S(M)$. (To get exterior algebras in the same way one needs the notion of super algebras.)

(4) For the functor $\mathcal{F} : \Bbbk - \mathcal{A}lg \to \mathfrak{m}(\Bbbk)$ from $\Bbbk$-algebras to $\Bbbk$-modules, $\mathcal{G}(M)$ is the tensor algebra $S(M)$.

(5) Forgetful functor $\mathcal{F} : \mathcal{T}op\mathcal{S}ets$ has a left adjoint $\mathcal{D}$ that sends a set S to S with the discrete topology, and also the right adjoint C such that $C(S)$ is S with the topology such that only S and ϕ are open.

4.6.7. *Functors between categories of modules.* (1) For $\phi : \Bbbk \to l$, one has adjoint triple $(\phi_*, \phi^*, \phi_\star)$ with $\phi_\star(M) \overset{\text{def}}{=} \text{Hom}_\Bbbk(l, M)$, i.e., (ϕ_*, ϕ^*) and $(\phi^*, \phi_\star)$ are adjoint pairs. So ϕ^* has both a left and a right adjoint and they are very different.

(2) $(\Bbbk, l)$-bimodule X gives $X_* : \mathfrak{m}(l) \to \mathfrak{m}(\Bbbk)$, with $X_*(N) \overset{\text{def}}{=} X \otimes_l N$, what is its right adjoint?

4.6.8. *More categorifications of "injection", "surjection", "bijection".*

5. **Abelian categories**

An abelian category is a category $\mathcal{A}$ which has the formal properties of the category $\mathcal{A}b$, i.e., we can do in $\mathcal{A}$ all computations that one can do in $\mathcal{A}b$.

5.1. **Additive categories.** Category $\mathcal{A}$ is additive if

- (A0) For any $a, b \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(a, b)$ has a structure of abelian group such that then compositions are bilinear.
- (A1) $\mathcal{A}$ has a zero object,
- (A2) $\mathcal{A}$ has sums of two objects,
- (A3) $\mathcal{A}$ has products of two objects,

5.1.1. *Lemma.* (a) Under the conditions (A0),(A1) one has (A3) $\Leftrightarrow$ (A4).

(b) In an additive category $a \oplus b$ is canonically the same as $a \times b$,

For additive categories $\mathcal{A}, \mathcal{B}$ a functor $F : \mathcal{A} \to \mathcal{B}$ is additive if the maps $\mathrm{Hom}_{\mathcal{A}}(a', a'') \to \mathrm{Hom}_{\mathcal{B}}(Fa', Fa'')$ are always morphisms of abelian groups.

5.1.2. *Examples.* (1) $\mathfrak{m}(\Bbbk)$, (2) $\mathcal{F}ree(\Bbbk)$, (3) $\mathcal{F}ilt\mathcal{V}ect_{\Bbbk} \stackrel{\text{def}}{=}$ filtered vector spaces over $\Bbbk$.

5.2. **(Co)kernels and (co)images.** In module categories a map has kernel, cokernel and image. To incorporate these notions into our project of defining abelian categories we will find their abstract formulations.

5.2.1. *Kernels: Intuition.* Our intuition is based on the category of type $\mathfrak{m}(\Bbbk)$. For a map of $\Bbbk$-modules $M \xrightarrow{\alpha} N$

- the kernel $\mathrm{Ker}(\alpha)$ is a subobject of M,
- the restriction of α to it is zero,
- and this is the largest subobject with this property

5.2.2. *Categorical formulation.* Based on this, our general definition (in an additive category $\mathcal{A}$), of "k is a kernel of the map $a \xrightarrow{\alpha} b$", is

- we have a map $k \xrightarrow{\sigma} M$ from k to M,
- if we follow this map by α the composition is zero,
- map $k \xrightarrow{\sigma} M$ is universal among all such maps, in the sense that
  - · all maps into a, $x \xrightarrow{\tau} a$, which are killed by α,
  - · factor uniquely through k (i.e., through $k \xrightarrow{\sigma} a$).

  So, all maps from x to a which are killed by α are obtained from σ (by composing it with some map $x \to k$). This is the "universality" property of the kernel.

5.2.3. *Reformulation in terms of representability of a functor.* A compact way to restate the above definition is:

- The kernel of $a \xrightarrow{\alpha} b$ is any object that represents the functor

$$\mathcal{A} \ni x \mapsto {}_{\alpha}\mathrm{Hom}_{\mathcal{A}}(x,a) \stackrel{\mathrm{def}}{=} \{\gamma \in \mathrm{Hom}_{\mathcal{A}}(x,a);\ \alpha\circ\gamma = 0\ \}.$$

One should check that this is the same as the original definition.

We denote the kernel by $\mathrm{Ker}(\alpha)$, but as usual, remember that

- this is not one specific object – it is only determined up to a canonical isomorphism,
- it is not only an object but a pair of an object and a map into a

.

5.2.4. *Cokernels.* In $\mathfrak{m}(\Bbbk)$ the cokernel of $M \xrightarrow{\alpha} N$ is $N/\alpha(M)$. So N maps into it, composition with α kills it, and the cokernel is universal among all such objects. When stated in categorical terms we see that we are interested in the functor

$$x \mapsto \mathrm{Hom}_{\mathcal{A}}(b,x)_{\alpha} \stackrel{\mathrm{def}}{=} \{\tau \in \mathrm{Hom}_{\mathcal{A}}(b,x);\ \tau\circ\alpha = 0\ \},$$

and the formal definition is symmetric to the definition of a kernel:

- The cokernel of f is any object that represents the functor $\mathcal{A} \ni x \mapsto \mathrm{Hom}_{\mathcal{A}}(b,x)_{\alpha}$.

So this object $\mathrm{Coker}(\alpha)$ is supplied with a map $b \to \mathrm{Coker}(\alpha)$ which is universal among maps from b that kill α.

5.2.5. *Images and coimages.* In order to define the image of α we need to use kernels and cokernels. In $\mathfrak{m}(\Bbbk)$, $Im(\alpha)$ is a subobject of N which is the kernel of $N \to \alpha(M)$. We will see that the categorical translation obviously has a symmetrical version which we call coimage. Back in $\mathfrak{m}(\Bbbk)$ the coimage is $M/\mathrm{Ker}(\alpha)$, hence there is a canonical map $Coim(\alpha) = M/\mathrm{Ker}(\alpha) \to Im(\alpha)$, and it is an isomorphism. This observation will be the final ingredient in the definition of abelian categories. Now we define

- Assume that α has cokernel $b \to \mathrm{Coker}(\alpha)$, the image of α is $Im(\alpha) \stackrel{\mathrm{def}}{=} \mathrm{Ker}[b \to \mathrm{Coker}(\alpha)]$ (if it exists).
- Assume that α has kernel $\mathrm{Ker}(\alpha) \to a$, the coimage of α is $Coim(\alpha) \stackrel{\mathrm{def}}{=} \mathrm{Coker}[\mathrm{Ker}(\alpha) \to a]$. (if it exists).

5.2.6. *Lemma.* If α has image and coimage, there is a canonical map $Coim(\alpha) \to Im(\alpha)$, and it appears in a canonical factorization of α into a composition

$$a \to Coim(\alpha) \to Im(\alpha) \to b.$$

5.2.7. *Examples.* (1) In $\mathfrak{m}(\Bbbk)$ the categorical notions of a (co)kernel and image have the usual meaning, and coimages coincide with images.

(2) In $\mathcal{F}ree(\Bbbk)$ kernels and cokernels need not exist.

(3) In $\mathcal{FV}\stackrel{\text{def}}{=}\mathcal{F}ilt\mathcal{V}ect_{\Bbbk}$ for $\phi\in\mathrm{Hom}_{\mathcal{FV}}(M_*,N_*)$ (i.e., $\phi:M\to N$ such that $\phi(M_k)\subseteq N_k,\ k\in\mathbb{Z}$), one has

- $\mathrm{Ker}_{\mathcal{FV}}(\phi)=\ \mathrm{Ker}_{\mathcal{V}ect}(\phi)$ with the induced filtration $\mathrm{Ker}_{\mathcal{FV}}(\phi)_n=\ \mathrm{Ker}_{\mathcal{V}ect}(\phi)\cap M_n$,
- $\mathrm{Coker}_{\mathcal{FV}}(\phi)\ =\ N/\phi(M)$ with the induced filtration $\mathrm{Coker}_{\mathcal{FV}}(\phi)_n\ =$ image of N_n in $N/\phi(M)=\ [N_n+\phi(M)]/\phi(M)\cong\ N_n/\phi(M)\cap N_n$.
- $Coim_{\mathcal{FV}}(\phi)\ =\ M/\mathrm{Ker}(\phi)$ with the induced filtration $Coim_{\mathcal{FV}}(\phi)_n\ =$ image of M_n in $M/\mathrm{Ker}(\phi)=\ M_n+\mathrm{Ker}(\phi)/\mathrm{Ker}(\phi)\cong=\ M_n/M_n\cap\mathrm{Ker}(\phi)$,
- $Im_{\mathcal{FV}}(\phi)=\ Im_{\mathcal{V}ect}(\phi)\subseteq N$, with the induced filtration $Im_{\mathcal{FV}}(\phi)_n=\ Im_{\mathcal{V}ect}(\phi)\cap N_n$.

Observe that the canonical map $Coim_{\mathcal{FV}}(\phi)\to Im_{\mathcal{FV}}(\phi)$ is an isomorphism of vector spaces $M/\mathrm{Ker}(\phi)\to Im_{\mathcal{V}ect}(\phi)$, however the two spaces have filtrations induced from filtrations on M and N respectively, and these need not coincide.

For instance one may have M and N be two filtrations on the same space V, if $M_k\subseteq N_k$ then $\phi=1_V$ is a map of filtered spaces $M\to N$ and $\mathrm{Ker}=0\mathrm{Coker}$ so that $Coim_{\mathcal{FV}}(\phi)=M$ and $Im_{\mathcal{FV}}(\phi)=N$ and the map $Coim_{\mathcal{FV}}(\phi)\to Im_{\mathcal{FV}}(\phi)$ is the same as ϕ, but ϕ is an isomorphism iff the filtrations coincide: $M_k=N_k$.

5.3. **Abelian categories.** Category $\mathcal{A}$ is abelian if

- (A0-3) It is additive,
- It has kernels and cokernels (hence in particular it has images and coimages!),
- The canonical maps $Coim(\phi)\to Im(\phi)$ are isomorphisms

5.3.1. *Examples.* Some of the following are abelian categories: (1) $\mathfrak{m}(\Bbbk)$ including $\mathcal{A}b=\mathfrak{m}(\mathbb{Z})$. (2) $\mathfrak{m}_{fg}(\Bbbk)$ if $\Bbbk$ is noetherian. (3) $\mathcal{F}ree(\Bbbk)\subseteq\mathcal{P}roj(\Bbbk)\subseteq\mathfrak{m}(\Bbbk)$. (4) $\mathcal{C}^\bullet(\mathcal{A})$. (5) Filtered vector spaces.

5.4. **Abelian categories and categories of modules.**

5.4.1. *Exact sequences in abelian categories.* Once we have the notion of kernel and cokernel (hence also of image), we can carry over from module categories $\mathfrak{m}(\Bbbk)$ to general abelian categories our homological train of thought. For instance we say that

- a map $i:a\to b$ makes a into a subobject of b if $\mathrm{Ker}(i)=0$ (we denote it $a\hookrightarrow b$ or even informally by $a\subseteq b$, one also says that i is a monomorphism or informally that it is an inclusion),
- a map $q:b\to c$ makes c into a quotient of b if $\mathrm{Coker}(q)=0$ (we denote it $b\twoheadrightarrow c$ and say that q is an epimorphism or informally that q is surjective),

- the quotient of b by a subobject $a \xrightarrow{i} b$ is $b/a \stackrel{\text{def}}{=} \text{Coker}(i)$,
- a complex in $\mathcal{A}$ is a sequence of maps $\cdots A^n \xrightarrow{d^n} A^{n+1} \to \cdots$ such that $d^{n+1} \circ d^n = 0$, its cocycles, coboundaries and cohomologies are defined by $B^n = Im(d^n)$ is a subobject of $Z^n = \text{Ker}(d^n)$ and $H^n = Z^n/B^n$;
- sequence of maps $a \xrightarrow{\mu} b \xrightarrow{\nu} c$ is exact (at b) if $\nu \circ \mu = 0$ and the canonical map $Im(\mu) \to \text{Ker}(\nu)$ is an isomorphism.

Now with all these definitions we are in a familiar world, i.e., they work as we expect. For instance, sequence $0 \to a' \xrightarrow{\alpha} a \xrightarrow{\beta} a'' \to 0$ is exact iff a' is a subobject of a and a'' is the quotient of a by a', and if this is true then

$$\text{Ker}(\alpha) = 0, \text{Ker}(\beta) = a', \text{Coker}(\alpha) = a'', \text{Coker}(\beta) = 0, \text{Im}(\alpha) = a', \text{Im}(\beta) = a''.$$

The difference between general abelian categories and module categories is that while in a module category $\mathfrak{m}(\Bbbk)$ our arguments often use the fact that $\Bbbk$-modules are after all abelian groups and sets (so we can think in terms of their elements), the reasoning valid in any abelian category has to be done more formally (via composing maps and factoring maps through intermediate objects). However, this is mostly appearances – if we try to use set theoretic arguments we will not go wrong:

5.4.2. *Theorem.* [Mitchell] Any abelian category is equivalent to a full subcategory of some category of modules $\mathfrak{m}(\Bbbk)$.

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- The kernel of $a \xrightarrow{\alpha} b$ is any object that represents the functor

$$\mathcal{A} \ni x \mapsto {}_{\alpha}\mathrm{Hom}_{\mathcal{A}}(x,a) \stackrel{\text{def}}{=} \{\gamma \in \mathrm{Hom}_{\mathcal{A}}(x,a);\ \alpha\circ\gamma = 0\ \}.$$

One should check that this is the same as the original definition.

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6.2.4. *Cokernels.* In $\mathfrak{m}(\Bbbk)$ the cokernel of $M \xrightarrow{\alpha} N$ is $N/\alpha(M)$. So N maps into it, composition with α kills it, and the cokernel is universal among all such objects. When stated in categorical terms we see that we are interested in the functor

$$x \mapsto \mathrm{Hom}_{\mathcal{A}}(b,x)_{\alpha} \stackrel{\text{def}}{=} \{\tau \in \mathrm{Hom}_{\mathcal{A}}(b,x);\ \tau\circ\alpha = 0\ \},$$

and the formal definition is symmetric to the definition of a kernel:

- The cokernel of f is any object that represents the functor $\mathcal{A} \ni x \mapsto \mathrm{Hom}_{\mathcal{A}}(b,x)_{\alpha}$.

So this object $\mathrm{Coker}(\alpha)$ is supplied with a map $b \to \mathrm{Coker}(\alpha)$ which is universal among maps from b that kill α.

6.2.5. *Images and coimages.* In order to define the image of α we need to use kernels and cokernels. In $\mathfrak{m}(\Bbbk)$, $Im(\alpha)$ is a subobject of N which is the kernel of $N \to \alpha(M)$. We will see that the categorical translation obviously has a symmetrical version which we call coimage. Back in $\mathfrak{m}(\Bbbk)$ the coimage is $M/\mathrm{Ker}(\alpha)$, hence there is a canonical map $Coim(\alpha) = M/\mathrm{Ker}(\alpha) \to Im(\alpha)$, and it is an isomorphism. This observation will be the final ingredient in the definition of abelian categories. Now we define

- Assume that α has cokernel $b \to \mathrm{Coker}(\alpha)$, the image of α is $Im(\alpha) \stackrel{\text{def}}{=} \mathrm{Ker}[b \to \mathrm{Coker}(\alpha)]$ (if it exists).
- Assume that α has kernel $\mathrm{Ker}(\alpha) \to a$, the coimage of α is $Coim(\alpha) \stackrel{\text{def}}{=} \mathrm{Coker}[\mathrm{Ker}(\alpha) \to a]$. (if it exists).

6.2.6. *Lemma.* If α has image and coimage, there is a canonical map $Coim(\alpha) \to Im(\alpha)$, and it appears in a canonical factorization of α into a composition

$$a \to Coim(\alpha) \to Im(\alpha) \to b.$$

6.2.7. *Examples.* (1) In $\mathfrak{m}(\Bbbk)$ the categorical notions of a (co)kernel and image have the usual meaning, and coimages coincide with images.

(2) In $\mathcal{F}ree(\Bbbk)$ kernels and cokernels need not exist.

(3) In $\mathcal{FV} \stackrel{\text{def}}{=} \mathcal{F}ilt\mathcal{V}ect_{\Bbbk}$ for $\phi \in \mathrm{Hom}_{\mathcal{FV}}(M_*, N_*)$ (i.e., $\phi : M \to N$ such that $\phi(M_k) \subseteq N_k,\ k \in \mathbb{Z}$), one has

- $\mathrm{Ker}_{\mathcal{FV}}(\phi) = \mathrm{Ker}_{\mathcal{V}ect}(\phi)$ with the induced filtration $\mathrm{Ker}_{\mathcal{FV}}(\phi)_n = \mathrm{Ker}_{\mathcal{V}ect}(\phi) \cap M_n$,
- $\mathrm{Coker}_{\mathcal{FV}}(\phi) = N/\phi(M)$ with the induced filtration $\mathrm{Coker}_{\mathcal{FV}}(\phi)_n =$ image of N_n in $N/\phi(M) = [N_n + \phi(M)]/\phi(M) \cong N_n/\phi(M) \cap N_n$.
- $Coim_{\mathcal{FV}}(\phi) = M/\mathrm{Ker}(\phi)$ with the induced filtration $Coim_{\mathcal{FV}}(\phi)_n =$ image of M_n in $M/\mathrm{Ker}(\phi) = M_n + \mathrm{Ker}(\phi)/\mathrm{Ker}(\phi) \cong = M_n/M_n \cap \mathrm{Ker}(\phi)$,
- $Im_{\mathcal{FV}}(\phi) = Im_{\mathcal{V}ect}(\phi) \subseteq N$, with the induced filtration $Im_{\mathcal{FV}}(\phi)_n = Im_{\mathcal{V}ect}(\phi) \cap N_n$.

Observe that the canonical map $Coim_{\mathcal{FV}}(\phi) \to Im_{\mathcal{FV}}(\phi)$ is an isomorphism of vector spaces $M/\mathrm{Ker}(\phi) \to Im_{\mathcal{V}ect}(\phi)$, however the two spaces have filtrations induced from filtrations on M and N respectively, and these need not coincide.

For instance one may have M and N be two filtrations on the same space V, if $M_k \subseteq N_k$ then $\phi = 1_V$ is a map of filtered spaces $M \to N$ and $\mathrm{Ker} = 0 \mathrm{Coker}$ so that $Coim_{\mathcal{FV}}(\phi) = M$ and $Im_{\mathcal{FV}}(\phi) = N$ and the map $Coim_{\mathcal{FV}}(\phi) \to Im_{\mathcal{FV}}(\phi)$ is the same as ϕ, but ϕ is an isomorphism iff the filtrations coincide: $M_k = N_k$.

6.3. **Abelian categories.** Category $\mathcal{A}$ is abelian if

- (A0-3) It is additive,
- It has kernels and cokernels (hence in particular it has images and coimages!),
- The canonical maps $Coim(\phi) \to Im(\phi)$ are isomorphisms

6.3.1. *Examples.* Some of the following are abelian categories: (1) $\mathfrak{m}(\Bbbk)$ including $\mathcal{A}b = \mathfrak{m}(\mathbb{Z})$. (2) $\mathfrak{m}_{fg}(\Bbbk)$ if $\Bbbk$ is noetherian. (3) $\mathcal{F}ree(\Bbbk) \subseteq \mathcal{P}roj(\Bbbk) \subseteq \mathfrak{m}(\Bbbk)$. (4) $\mathcal{C}^\bullet(\mathcal{A})$. (5) Filtered vector spaces.

6.4. **Abelian categories and categories of modules.**

6.4.1. *Exact sequences in abelian categories.* Once we have the notion of kernel and cokernel (hence also of image), we can carry over from module categories $\mathfrak{m}(\Bbbk)$ to general abelian categories our homological train of thought. For instance we say that

- a map $i : a \to b$ makes a into a subobject of b if $\mathrm{Ker}(i) = 0$ (we denote it $a \hookrightarrow b$ or even informally by $a \subseteq b$, one also says that i is a monomorphism or informally that it is an inclusion),
- a map $q : b \to c$ makes c into a quotient of b if $\mathrm{Coker}(q) = 0$ (we denote it $b \twoheadrightarrow c$ and say that q is an epimorphism or informally that q is surjective),
- the quotient of b by a subobject $a \xrightarrow{i} b$ is $b/a \stackrel{\text{def}}{=} \mathrm{Coker}(i)$,
- a complex in $\mathcal{A}$ is a sequence of maps $\cdots A^n \xrightarrow{d^n} A^{n+1} \to \cdots$ such that $d^{n+1} \circ d^n = 0$, its cocycles, coboundaries and cohomologies are defined by $B^n = Im(d^n)$ is a subobject of $Z^n = \mathrm{Ker}(d^n)$ and $H^n = Z^n/B^n$;
- sequence of maps $a \xrightarrow{\mu} b \xrightarrow{\nu} c$ is exact (at b) if $\nu \circ \mu = 0$ and the canonical map $Im(\mu) \to \mathrm{Ker}(\nu)$ is an isomorphism.

Now with all these definitions we are in a familiar world, i.e., they work as we expect. For instance, sequence $0 \to a' \xrightarrow{\alpha} a \xrightarrow{\beta} a'' \to 0$ is exact iff a' is a subobject of a and a'' is the quotient of a by a', and if this is true then

$$\mathrm{Ker}(\alpha) = 0, \mathrm{Ker}(\beta) = a', \mathrm{Coker}(\alpha) = a'', \mathrm{Coker}(\beta) = 0, \mathrm{Im}(\alpha) = a', \mathrm{Im}(\beta) = a''.$$

The difference between general abelian categories and module categories is that while in a module category $\mathfrak{m}(\Bbbk)$ our arguments often use the fact that $\Bbbk$-modules are after all abelian groups and sets (so we can think in terms of their elements), the reasoning valid in any abelian category has to be done more formally (via composing maps and factoring maps through intermediate objects). However, this is mostly appearances – if we try to use set theoretic arguments we will not go wrong:

6.4.2. *Theorem.* [Mitchell] Any abelian category is equivalent to a full subcategory of some category of modules $\mathfrak{m}(\Bbbk)$.

7. **Exactness of functors and the derived functors**

7.1. **Exactness of functors.** As we have observed, any functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories extends to a functor between the corresponding categories of complexes $\mathcal{C}^\bullet(F) : \mathcal{C}^\bullet(\mathcal{A}) \to \mathcal{C}^\bullet(\mathcal{B})$. We would like next to extend it to a functor between derived categories $D(F) : D(\mathcal{A}) \to D(\mathcal{B})$ – this is what one calls the derived version of F. We may denote it again by F, or by $D(F)$, or use some other notation which reflects on the way we produce the extension.

This extension is often obtained using exactness properties of F, i.e., it will depend on how much does F preserves exact sequences. Say, if F is exact then $D(F)$ is obvious: it is just the functor F applied to complexes. If F is right-exact $D(F)$ is the "left derived functor LF" obtained by replacing objects with projective resolutions. If F is left-exact, $D(F)$ is the "right derived functor RF" obtained by replacing objects with injective resolutions (see 7.5).

7.1.1. *Exactness of a sequence of maps.* A sequence of maps in an abelian category $M_a \xrightarrow{\alpha_a} M_{a+1} \to \cdots \to M_{b-1} \xrightarrow{\alpha_{b-1}} M_b$ is said to be exact at M_i (for some i with $a < i < b$) if $Im(\alpha_{i-1}) = \text{Ker}(\alpha_i)$. The sequence is said to be exact if it is exact at all $M_i,\ a < i < b$. (The sequence may possibly be infinite in one or both directions.)

7.1.2. *Exact functors.* Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. (One can think of the case where $\mathcal{A} = \mathfrak{m}(\Bbbk)$ and $\mathcal{B} = \mathfrak{m}(l)$ since the general case works the same.)

We will say that F is exact if it preserves short exact sequences, i.e., for any SES $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ in $\mathcal{A}$, its F-image in $\mathcal{B}$ is exact, i.e., the sequence $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ is a SES in $\mathcal{B}$.

Example. The pull-back functors ϕ^* from 4.4 are exact since they do not change the structure of abelian groups and the exactness for modules only involves the level of abelian groups.

In practice few interesting functors are exact so we have to relax the notion of exactness:

7.2. **Left exact functors.** We say, that F is left exact if for any SES its F-image $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ is exact except possibly in the A''-term, i.e., $F(\beta)$ need not be surjective.

7.2.1. *Lemma.* The property of left exactness is the same as asking that F preserves exactness of sequences of the form $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C''$, i.e., if $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A''$ is exact in $\mathcal{A}$, its F-image $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'')$ is exact in $\mathcal{B}$.

Proof. The property in the lemma seems a priori stronger because it produces the same conclusion in a larger number of cases. However it follows from the definition of left exactness by a diagram chase which uses left exactness in two places.

7.2.2. *Example: Invariants are left exact.* For a group G let $Rep_{\mathbb{k}}(G)$ be the category of all representations of G over a field $\mathbb{k}$. A representations of G is a pair (V,π) of a vector space V and a map of groups $G \xrightarrow{\pi} GL(V)$. We often denote $\pi(g)v$ by gv, and we omit V or π from the notation for representations. Representations of G are the same as modules for the algebra $\mathbb{k}[G] \overset{\text{def}}{=} \oplus_{g\in G}\ \mathbb{k}g$ (multiplication is obvious). So short exact sequences, etc., make sense in $Rep_{\mathbb{k}}(G)$.

Lemma. The functor of invariants, $I : Rep_{\mathbb{k}}(G) \to \mathcal{V}ect_{\mathbb{k}}$, by $I(V,\pi) \overset{\text{def}}{=} V^G \overset{\text{def}}{=} \{v \in V,\ gv = v,\ g \in G\}$ is left exact.

Proof is easy. It is more interesting to see how exactness fails at the right end.

Counterexample. For $G = \mathbb{Z}$, a representation is the same as a vector space V with an invertible linear operator A $(= \pi(1))$. Therefore, $I(V,A)$ is the 1-eigenspace V_1 of A. Short exact sequences in $Rep_{\mathbb{k}}(\mathbb{Z})$ are all isomorphic to the ones of the form $0 \to (V', A|V') \to (V,A) \to (V'', \overline{A}) \to 0$, i.e., one has a vector space V with an invertible linear operator A, an A-invariant subspace V' (we restrict A to it), and the quotient space $V'' = V/V'$ (we factor A to it). So the exactness on the right means that any $w \in V/V'$ such that $\overline{A}w = w$ comes from some v in V such that $Av = v$.

If $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ with $A = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $V' = \mathbb{C}e_1$ then $0 \to I(V', A|V') \to I(V,A) \to I(V'', \overline{A}) \to 0$, is just $0 \to \mathbb{C}e_1 \xrightarrow{id} \mathbb{C}e_1 \xrightarrow{0} V'' \to 0$, and the exactness fails on the right.

The principle "Invariants are left exact". It applies to many other situations. Also, since

$$I(V) = \operatorname{Hom}_G(\mathbb{k}, V),$$

it is a special case of the next lemma. The meaning of the last equality is:

- $\mathbb{k}$ denotes the trivial one dimensional representation of G on the vector space $\mathbb{k}$.
- Moreover, the equality notation $I(V) = \operatorname{Hom}_G(\mathbb{k}, V)$ is only a remainder of a more precise statement: there is a canonical isomorphism of functors $I \xrightarrow{\eta} \operatorname{Hom}_{Rep_{\mathbb{k}}(G)}(\mathbb{k}, -)$ from $Rep_{\mathbb{k}}(G)$ to $\mathcal{V}ect_{\mathbb{k}}$.
- The map η_V sends a G-fixed vector $w \in I(V)$ to a linear map $\eta_V(w) :\ \mathbb{k} \to V$, given by multiplying w with scalars: $\mathbb{k} \ni c \mapsto c{\cdot}v \in V$. One easily checks that η_V is an isomorphism of vector spaces.

7.2.3. *Lemma.* Let $\mathcal{A}$ be an abelian category, for any $a \in \mathcal{A}$, the functor

$$\mathrm{Hom}_{\mathcal{A}}(a,-): \ \mathcal{A} \to \mathcal{A}b,$$

is left exact!

Proof. For an exact sequence $0 \to b' \xrightarrow{\alpha} b \xrightarrow{\beta} b'' \to 0$ we consider the corresponding sequence $\mathrm{Hom}_{\mathcal{A}}(a,b') \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathcal{A}}(a,b) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathcal{A}}(a,b'')$.

(1) α_* is injective. if $a \xrightarrow{\mu} b'$ and $0 = \alpha_*(\mu) \overset{\text{def}}{=} \alpha\circ\mu$, then μ factors through the kernel $\mathrm{Ker}(\alpha)$ (by the definition of the kernel). However, $\mathrm{Ker}(\alpha) = 0$ (by definition of a short exact sequence), hence $\mu = 0$.

(2) $\mathrm{Ker}(\beta^*) = Im(\alpha_*)$. First, $\beta_*\circ\alpha_* = (\beta\circ\alpha)_* = 0_* = 0$, hence $Im(\alpha_*) \subseteq \mathrm{Ker}(\beta^*)$. If $a \xrightarrow{\nu} b$ and $0 = \beta_*(\nu)$, i.e., $0 = \beta\circ\nu$, then ν factors through the kernel $\mathrm{Ker}(\beta)$. But $\mathrm{Ker}(\beta) = a'$ and the factorization now means that ν is in $Im(\alpha_*)$.

7.2.4. *Counterexample.* Let $\mathcal{A} = \mathcal{A}b$ and apply $\mathrm{Hom}(a,-)$ for $a = \mathbb{Z}/2\mathbb{Z}$ to $0 \to 2\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \to 0$. Then $id_{\mathbb{Z}/2\mathbb{Z}}$ does not lift to a map from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$. So β_* need not be surjective.

7.3. **Right exact functors.** F is right exact if it satisfies one of two equivalent properties

(1) F-image $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ of a SES is exact except possibly in the A'-term, i.e., $F(\alpha)$ may fail to be injective.
(2) F preserves exactness of sequences of the form $C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$.

Version (1) is easier to check and (2) is easier to apply.

7.3.1. *Lemma.* Tensoring is right exact in each argument, i.e., for any left $\Bbbk$-module M the functor $M \underset{\Bbbk}{\otimes} - : \mathfrak{m}^r(\Bbbk) \to \mathcal{A}b$ is right exact, and so is $-\underset{\Bbbk}{\otimes} N : \mathfrak{m}(\Bbbk) \to \mathcal{A}b$ for any right $\Bbbk$-module N.

7.3.2. *Contravariant case.* Let us state the definition of right exactness also in the case that F is a contravariant from $\mathcal{A}$ to $\mathcal{B}$. The choice of terminology is such that one requires that the functor $F : \mathcal{A} \to \mathcal{B}^o$ is exact, this boils down to asking (again) that exactness is preserved except possibly at A' (the left end of the original sequence. So we need one of the following equivalent properties

(1) For a SES $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ in $\mathcal{A}$, its F-image $F(0) \to F(A'') \xrightarrow{F(\beta)} F(A) \xrightarrow{F(\alpha)} F(A') \to F(0)$ is exact except possibly at $F(A')$, i.e., $F(\alpha)$ may fail to be surjective.
(2) F preserves exactness of sequences of the form $C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$.

The most important example is

7.3.3. *Lemma.* For any $M \in \mathcal{A}$, the contravariant functor $G_M = \operatorname{Hom}_{\mathcal{A}}(-, M)$ is right exact.

7.4. **Projectives and the existence of projective resolutions.** Let $\mathcal{A}$ be an abelian category.

7.4.1. *Projectives.* We say that $p \in \mathcal{A}$ is a projective object if the functor $\operatorname{Hom}_{\mathcal{A}}(p, -) : \mathcal{A} \to \mathcal{A}b$ is exact. Since $\operatorname{Hom}_{\mathcal{A}}(p, -)$ is known to be always left exact, what we need is that for any short exact sequence $0 \to a \xrightarrow{\alpha} b \xrightarrow{\beta} c \to 0$ map $\operatorname{Hom}(p, b) \to \operatorname{Hom}(p, c)$ is surjective. In other words, if c is a quotient of b then any map from p to the quotient $p \xrightarrow{\gamma} c$ lifts to a map to b, i.e., there is a map $p \underset{\tilde{\gamma}}{\to} b$ such that $\gamma = \beta \circ \tilde{\gamma}$ for the quotient map $b \xrightarrow{\beta} c$.

7.4.2. *Lemma.* $\oplus_{i \in I}\, p_i$ is projective iff all summands p_i are projective.

This definition of projectivity generalizes our earlier definition in module categories since

7.4.3. *Lemma.* For a $\Bbbk$-module P, functor $\operatorname{Hom}_{\mathfrak{m}(\Bbbk)}(P, -) : \mathfrak{m}(\Bbbk) \to \mathcal{A}b$ is exact iff P is a summand of a free module.

We say that abelian category $\mathcal{A}$ has enough projectives if any object is a quotient of a projective object.

7.4.4. *Corollary.* Module categories have enough projectives.

The importance of "enough projectives" comes from

7.4.5. *Lemma.* For an abelian category $\mathcal{A}$ the following is equivalent

(1) Any object of $\mathcal{A}$ has a projective resolution (i.e., a left resolution consisting of projective objects).
(2) $\mathcal{A}$ has enough projectives.

7.5. **Injectives and the existence of injective resolutions.** Dually, we say that $i \in \mathcal{A}$ is an injective object if the functor $\operatorname{Hom}_{\mathcal{A}}(-, i) : \mathcal{A} \to \mathcal{A}b^o$ is exact.

Again, since $\operatorname{Hom}_{\mathcal{A}}(-, i)$ is always right exact, we need for any short exact sequence $0 \to a \xrightarrow{\alpha} b \xrightarrow{\beta} c \to 0$ that the map $\operatorname{Hom}(b, p) \xrightarrow{\alpha^*} \operatorname{Hom}(a, p)$, $\alpha^*(\phi) = \phi \circ \alpha$; be surjective. This means that if a is a subobject of b then any map $a \xrightarrow{\gamma} i$ from a subobject a to i extends to a map from b to i, i.e., there is a map $b \xrightarrow{\tilde{\gamma}} i$ such that $\gamma = \tilde{\gamma} \circ \alpha$. So, an object i is injective if each map from a subobject $a' \hookrightarrow a$ to i, extends to the whole object a.

7.5.1. *Example.* $\mathbb{Z}$ is projective in $\mathcal{A}b$ but it is not injective in $\mathcal{A}b$: $\mathbb{Z}\subseteq\frac{1}{n}\mathbb{Z}$ and the map $1_{\mathbb{Z}}:\mathbb{Z}\to\mathbb{Z}$ does not extend to $\frac{1}{n}\mathbb{Z}\to\mathbb{Z}$.

7.5.2. *Lemma.* Product $\prod_{i\in I} J_i$ is injective iff all factors J_i are injective.

7.5.3. *Lemma.* A $\mathbb{Z}$-module I is injective iff I is divisible, i.e., for any $a\in I$ and $n\in\{1,2,3,...\}$ there is some $\tilde{a}\in I$ such that $a=n{\cdot}\tilde{a}$. (i.e., multiplication $n:I\to I$ with $n\in\{1,2,3,...\}$ is surjective.)

The proof will use the Zorn lemma which is an essential part of any strict definition of set theory:

- Let $(I,\leq)$ be a (non-empty) partially ordered set such that any chain J in I (i.e., any totally ordered subset) is dominated by some element of I (i.e., there is some $i\in I$ such that $i\geq j,\ j\in J$). Then I has a maximal element.

Proof. For any $a\in I$ and $n>0$ we can consider $\frac{1}{n}\mathbb{Z}\supseteq\mathbb{Z}\xrightarrow{\alpha}I$ with $\alpha(1)=a$. If I is injective then α extends to $\widetilde{\alpha}:\frac{1}{n}\mathbb{Z}\to I$ and $a=n\widetilde{\alpha}(\frac{1}{n})$.

Conversely, assume that I is divisible and let $A\supseteq B\xrightarrow{\beta}I$. Consider the set $\mathcal{E}$ of all pairs (C,γ) with $B\subseteq C\subseteq A$ and $\gamma:C\to I$ an extension of β. It is partially ordered with $(C,\gamma)\leq(C',\gamma')$ if $C\subseteq C'$ and γ' extends γ. From Zorn lemma and the following observations it follows that $\mathcal{E}$ has an element (C,γ) with $C=A$:

(1) For any totally ordered subset $\mathcal{E}'\subseteq\mathcal{E}$ there is an element $(C,\gamma)\in\mathcal{E}$ which dominates all elements of $\mathcal{E}'$ (this is clear: take $C=\cup_{(C',\gamma')\in\mathcal{E}'}C'$ and γ is then obvious).
(2) If $(C,\gamma)\in\mathcal{E}$ and $C\neq A$ then (C,γ) is not maximal:
   - choose $a\in A$ which is not in C and let $\widetilde{C}=C+\mathbb{Z}{\cdot}a$ and $C\cap\mathbb{Z}{\cdot}a=\mathbb{Z}{\cdot}na$ with $n\geq 0$. If $n=0$ then $\widetilde{C}=C\oplus\mathbb{Z}{\cdot}a$ and one can extend γ to C by zero on $\mathbb{Z}{\cdot}a$. If $n>0$ then $\gamma(na)\in I$ is n-divisible, i.e., $\gamma(na)=nx$ for some $x\in I$. Then one can extend γ to $\widetilde{C}$ by $\widetilde{\gamma}(a)=x$ (first define a map on $C\oplus\mathbb{Z}{\cdot}a$, and then descend it to the quotient $\widetilde{C}$).

7.5.4. We say that abelian category $\mathcal{A}$ has enough injectives if any object is a subobject of an injective object.

7.5.5. *Lemma.* For an abelian category $\mathcal{A}$ the following is equivalent

(1) Any object of $\mathcal{A}$ has an injective resolution (i.e., a right resolution consisting of injective objects).
(2) $\mathcal{A}$ has enough injectives.

7.5.6. *Lemma.* For any abelian group M denote $\widehat{M} = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$. Then the canonical map $M \xrightarrow{\rho} \widehat{\widehat{M}}$ is injective.

(b) Category of abelian groups has enough injectives.

Proof. (a) For $m \in M,\ \chi \in \widehat{M},\ \rho(m)\ (\chi) \stackrel{\text{def}}{=} \chi(m)$. So, $\rho(m) = 0$ means that m is killed by each $\chi \widehat{M}$ ("each character of M"). If $m \neq 0$ then $\mathbb{Z}{\cdot}m$ is isomorphic to $\mathbb{Z}$ or to one of $\mathbb{Z}/n\mathbb{Z}$, in each case we can find a $\mathbb{Z}{\cdot}m \xrightarrow{\chi_0} \mathbb{Q}/\mathbb{Z}$ which is $\neq 0$ on the generator m. Since $\mathbb{Q}/\mathbb{Z}$ is injective we can extend χ_0 to M.

(b) To M we associate a huge injective abelian group $I_M = \prod_{x \in \widehat{M}} \mathbb{Q}/\mathbb{Z}{\cdot}x = (\mathbb{Q}/\mathbb{Z})^{\widehat{M}}$, its elements are $\widehat{M}$-families $c = (c_\chi)_{\chi \in \widehat{M}}$ of elements of $\mathbb{Q}/\mathbb{Z}$ (we denote such family also as a (possibly infinite) formal sum $\sum_{\chi \in \widehat{M}} c_\chi{\cdot}\chi$). By part (a), canonical map *io* is injective

$$M \xrightarrow{\iota} I_M, \quad \iota(m) = (\chi(m))_{\chi \in \widehat{M}} = \sum_{\chi \in \widehat{M}} \chi(m){\cdot}\chi, \quad m \in M.$$

7.5.7. *Lemma.* Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Its right adjoint preserves injectivity and its left adjoint preserves projectivity.

Proof. For the right adjoint G and an injective $b \in \mathcal{B}$, functor $\mathrm{Hom}_{\mathcal{A}}[-, Gb] \cong \mathrm{Hom}[F-, b]$ is exact.

7.5.8. *Corollary.* For a map of rings $\Bbbk \xrightarrow{\phi} l$ functor $\phi_* : \mathfrak{m}(\Bbbk) \to \mathfrak{m}(l),\ \ \phi_*(M) = l \otimes_{\Bbbk} M$ preserves projectivity and $\phi_\star : \mathfrak{m}(\Bbbk) \to \mathfrak{m}(l),\ \ \phi_\star(M) = \mathrm{Hom}_{\Bbbk}(l, M)$ preserves injectivity.

Proof. These are the two adjoints of the forgetful functor ϕ^*.

7.5.9. *Theorem.* Module categories $\mathfrak{m}(\Bbbk)$ have enough injectives.

Proof. The problem will be reduced to the case $\Bbbk = \mathbb{Z}$ via the canonical map of rings $\mathbb{Z} \xrightarrow{\phi} \Bbbk$. Any $\Bbbk$-module M gives a $\mathbb{Z}$-module $\phi^* M$, and by lemma 7.5.6 there is an embedding $M \xrightarrow{\iota} I_M$ into an injective abelian group. Moreover, by corollary 7.5.8 $\phi_\star I_M$ is an injective $\Bbbk$-module. So it suffices to have an embedding $M \hookrightarrow \phi_\star I_M$.

The adjoint pair $(\phi^*, \phi_\star)$ gives a map of $\Bbbk$-modules $M \xrightarrow{\zeta} \phi_\star \phi^* M = \mathrm{Hom}_{\mathbb{Z}}(\Bbbk, M)$, by $\zeta(m)\ c = cm,\ m \in M,\ c \in \Bbbk$. It remains to check that both maps in the composition $M \xrightarrow{\zeta} \phi_\star(M) \xrightarrow{\phi_\star(\iota)} \phi_\star(I_M)$ are injective. For ζ it is obvious since $\zeta(m)\ 1_{\Bbbk} = m$, and for $\phi_\star$ we recall that $\mathrm{Hom}_{\mathbb{Z}}(\Bbbk, -)$ is left exact, hence takes injective maps to injective maps.

7.5.10. *Examples.* (1) An injective resolution of the $\mathbb{Z}$-module $\mathbb{Z}$: $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/Z \to 0$.

(2) Injective resolutions are often big, hence more difficult to use in specific calculations then say, the free resolutions. We will need them mostly for the functor $\Gamma(X,-)$ of global sections of sheaves, and the functors $\mathrm{Hom}_{\mathcal{A}}(a,-)$.

7.6. **Exactness and the derived functors.** This is a preliminary motivation for the precise construction of derived functors in the next chapters.

7.6.1. *Left derived functor RF of a right exact functor F.* We observe that if F is right exact then the correct way to extend it to a functor on the derived level is the construction $LF(M)\overset{\text{def}}{=} F(P^\bullet)$, i.e., replacement of the object by a projective resolution. "Correct" means here that LF is really more then F – it contains the information of F in its zero$^{\text{th}}$ cohomology, i.e., $L^0F \cong F$ for $L^iF(M)\overset{\text{def}}{=}H^i[LF(M)]$. Letter L reminds us that we use a left resolution.

7.6.2. *Lemma.* If the functor $F:\mathcal{A}\to\mathcal{B}$ is right exact, there is a canonical isomorphism of functors $H^0(LF)\cong F$.

Proof. Let $\cdots\to P^{-2}\to P^{-1}\xrightarrow{d^{-1}} P^0\xrightarrow{q} M\to 0$ be a projective resolution of M.
(A) *The case when F is covariant.* Then $LF(M)= F[\cdots\to P^{-2}\to P^{-1}\xrightarrow{d^{-1}} P^0\to 0\to \cdots]$ equals

$$[\cdots\to F(P^{-2})\to F(P^{-1})\xrightarrow{F(d^{-1})} F(P^0)\to 0\to\cdots],$$

so $H^0[LF(M)] = F(P^0)/F(d^{-1})F(P^{-1})$.

If we apply F to the exact sequence $P^{-1}\xrightarrow{d^{-1}} P^0\to M\xrightarrow{q} 0$, the right exactness gives an exact sequence $F(P^{-1})\xrightarrow{F(d^{-1})} F(P^0)\xrightarrow{F(q)} F(M)\to 0$. Therefore, $F(q)$ factors to a canonical map $F(P^0)/F(d^{-1})F(P^{-1})\to F(M)$ which is an isomorphism.
(B) *The case when F is contravariant.* This is similar, $LF(M) = F(\cdots\to P^{-2}\to P^{-1}\xrightarrow{d^{-1}} P^0\to 0\to\cdots)$ equals

$$\cdots\to 0\to F(P^{-0})\xrightarrow{d^{-1}} F(P^{-1})\to F(P^{-2})\to 0\to\cdots,$$

and we get $H^0[LF(M)] = \mathrm{Ker}[F(P^{-0})\xrightarrow{d^{-1}} F(P^{-1})]$. However, applying F to the exact sequence $P^{-1}\xrightarrow{d^{-1}} P^0\to M\to 0$ gives an exact sequence $0\to F(M)\to F(P^0)\xrightarrow{F(d^{-1})} F(P^{-1})$. So the canonical map $F(M)\to \mathrm{Ker}[F(P^{-0})\xrightarrow{d^{-1}} F(P^{-1})]$ is an isomorphism.

7.6.3. *Remark.* As we see the argument is categorical and would not simplify if we only considered module categories.

7.6.4. *Example.* Recall the functor $i^o:\mathfrak{m}(D^1_{\mathbb{A}})\to\mathfrak{m}(D_{\mathbb{A}^0})$, $i^oM = M/xM$. It is right exact by 7.3.1 since $M/xM\cong M\otimes_{\Bbbk[x]}\Bbbk[x]/x\Bbbk[x]$, and therefore $H^0[Li^o(M)]\cong i^o(M)$ by the preceding lemma.

7.6.5. *Right derived functor RF of a left exact functor F.* Obviously, we want to define for any left exact functor $F : \mathcal{A} \to \mathcal{B}$ a right derived functor RF by replacing an object by its injective resolution. Then, as above $H^0(RF) \cong F$.

8. **Abelian category of sheaves of abelian groups**

For a topological space X we will denote by $\mathcal{S}h(X) = \mathcal{S}heaves(X, \mathcal{A}b)$ the category of sheaves of abelian groups on X. Since a sheaf of abelian groups is something like an abelian group smeared over X we hope to $\mathcal{S}h(X)$ is again an abelian category. When attempting to construct cokernels, the first idea does not quite work – it produces something like a sheaf but without the gluing property. This forces us to

- (i) generalize the notion of sheaves to a weaker notion of a presheaf,
- (ii) find a canonical procedure that improves a presheaf to a sheaf.

(We will also see that a another example that requires the same strategy is the pull-back operation on sheaves.)

Now it is easy to check that we indeed have an abelian category. What allows us to compute in this abelian category is the lucky break that one can understand kernels, cokernels, images and exact sequences just by looking at the stalks of sheaves.

8.1. **Categories of sheaves.**

A presheaf of sets $\mathcal{S}$ on a topological space $(X, \mathcal{T})$ consists of the following data:

- for each open $U \subseteq X$ a set $\mathcal{S}(U)$,
- for each inclusion of open subsets $V \subseteq U \subseteq X$ a map $\mathcal{S}(U) \xrightarrow{\rho^U_V} \mathcal{S}(V)$ (called the restriction map);

and these data are required to satisfy

- (Sh0)(Transitivity of restriction) $\rho^U_V \circ \rho^U_V = \rho^U_W$ for $W \subseteq V \subseteq U$

A sheaf of sets on a topological space $(X, \mathcal{T})$ is a presheaf $\mathcal{S}$ which also satisfies

- (Sh1) (Gluing) Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of an open $U \subseteq X$ (We denote $U_{ij} = U_i \cap U_j$ etc.). We ask that any family of compatible sections $f_i \in \mathcal{S}(U_i)$, $i \in I$, glues uniquely. This means that if sections f_i agree on intersections in the sense that $\rho^{U_i}_{U_{ij}} f_i = \rho^{U_i}_{U_{ij}} f_j$ in $\mathcal{S}(U_{ij})$ for any $i, j \in I$; then there is a unique $f \in \mathcal{S}(U)$ such that $\rho^U_{U_i} f = f_i$ in $\mathcal{S}(U_i)$, $i \in I$.
- $\mathcal{S}(\emptyset)$ is a *point*.

8.1.1. *Remarks.* (1) Presheaves of sets on X form a category $pre\mathcal{S}heaves(X, \mathcal{S}ets)$ when $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ consists of all systems $\phi = (\phi_U)_{U \subseteq X \ open}$ of maps $\phi_U : \mathcal{A}(U) \to \mathcal{B}(U)$ which are compatible with restrictions, i.e., for $V \subseteq U$

$$\begin{array}{ccc} \mathcal{A}(U) & \xrightarrow{\phi_U} & \mathcal{B}(U) \\ \rho^U_V \big\downarrow & & \rho^U_V \big\downarrow \\ \mathcal{A}(V) & \xrightarrow{\phi_V} & \mathcal{B}(V) \end{array}.$$

(One reads the diagram above as : "the diagram ... commutes".) The sheaves form a full subcategory $pre\mathcal{S}heaves(X, \mathcal{S}ets)$ of $\mathcal{S}heaves(X, \mathcal{S}ets)$.

(2) We can equally define categories of sheaves of abelian groups, rings, modules, etc. For a sheaf of abelian groups we ask that all $\mathcal{A}(U)$ are abelian groups, all restriction morphisms are maps of abelian groups, and we modify the least interesting requirement (Sh2): $\mathcal{S}(\phi)$ is the trivial group $\{0\}$. In general, for a category $\mathcal{A}$ one can define categories $pre\mathcal{S}heaves(X, \mathcal{A})$ and $\mathcal{S}heaves(X, \mathcal{A})$ similarly (the value on $\emptyset$ should be the final object of $\mathcal{A}$).

8.2. **Sheafification of presheaves.** We will use the wish to pull-back sheaves as a motivation for a procedure that improves presheaves to sheaves.

8.2.1. *Functoriality of sheaves.* Recall that for any map of topological spaces $X \xrightarrow{\pi} Y$ one wants a pull-back functor $\mathcal{S}heaves(Y) \xrightarrow{\pi^{-1}} \mathcal{S}heaves(X)$. As we have seen in the definition of a stalk of a sheaf (pull ,back to a point), the natural formula is

$$\underline{\pi^{-1}}(\mathcal{N})\ (U) \stackrel{\text{def}}{=} \lim_{\substack{\to \\ V \supseteq \pi(U)}} \ \mathcal{N}(V),$$

where limit is over open $V \subseteq Y$ that contain $\pi(U)$, and we say that $V' \leq V''$ if V'' better approximates $\pi(U)$, i.e., if $V'' \subseteq V'$.

8.2.2. *Lemma.* This gives a functor of presheaves $pre\mathcal{S}heaves(X) \xrightarrow{\underline{\pi^{-1}}} pre\mathcal{S}heaves(Y)$.

Proof. For $U' \subseteq U$ open, $\underline{\pi^{-1}}\mathcal{N}(U') = \lim_{\to\ V \supseteq \pi(U')} \mathcal{N}(V)$ and $\underline{\pi^{-1}}\mathcal{N}(U) = \lim_{\to\ V \supseteq \pi(U)} \mathcal{N}(V)$ are limits of inductive systems of $\mathcal{N}(V)$'s, and the second system is a subsystem of the first one, this gives a canonical map $\underline{\pi^{-1}}\mathcal{N}(U) \to \underline{\pi^{-1}}\mathcal{N}(U')$.

8.2.3. *Remarks.* Even if $\mathcal{N}$ is a sheaf, $\underline{\pi^{-1}}(\mathcal{N})$ need not be sheaf.

For that let $Y = pt$ and let $\mathcal{N} = S_Y$ be the constant sheaf of sets on Y given by a set S. So, $S_Y(\emptyset) = \emptyset$ and $S_Y(Y) = S$. Then $\underline{\pi^{-1}}(S_Y)\ (U) = \begin{cases} \emptyset & if\ U = \emptyset, \\ S & U \neq \emptyset \end{cases}$. We can say: $\underline{\pi^{-1}}(S_Y)\ (U) =$ constant functions from U to S. However, we have noticed that constant functions do not give a sheaf, so we need to correct the procedure $\underline{\pi^{-1}}$ to get sheaves from sheaves. For that remember that for the presheaf of constant functions there is a related sheaf S_X of *locally constant* functions.

Our problem is that the presheaf of constant functions is defined by a global condition (constancy) and we need to change it to a local condition (local constancy) to make it into a sheaf. So we need the procedure of

8.2.4. *Sheafification.* This is a way to improve any presheaf of sets $\mathcal{S}$ into a sheaf of sets $\widetilde{\mathcal{S}}$. We will imitate the way we passed from constant functions to locally constant functions. More precisely, we will obtained the sections of the sheaf $\tilde{\mathcal{S}}$ associated to the presheaf $\mathcal{S}$ in two steps:

(1) we glue systems of local sections s_i which are compatible in the weak sense that they are *locally* the same, and
(2) we identify two results of such gluing if the local sections in the two families are *locally* the same.

Formally these two steps are performed by replacing $\mathcal{S}(U)$ with the set $\widetilde{\mathcal{S}}(U)$, defined as the set of all equivalence classes of systems $(U_i, s_i)_{i\in I}$ where

(1) Let $\widehat{\mathcal{S}}(U)$ be the class of all systems $(U_i, s_i)_{i\in I}$ such that
   - $(U_i)_{i\in I}$ is an open cover of U and s_i is a section of $\mathcal{S}$ on U_i,
   - sections s_i are *weakly compatible* in the sense that they are locally the same, i.e., for any $i', i'' \in I$ sections $s_{i'}$ and $s_{i''}$ are the same near any point $x \in U_{i'i''}$. (Precisely, this means that there is neighborhood W such that $s_{i'}|W = s_{i''}|W$.)

(2) We say that two systems $(U_i, s_i)_{i\in I}$ and $(V_j, t_j)_{j\in J}$ are $\equiv$, iff for any $i \in I,\ j \in J$ sections s_i and t_j are weakly equivalent (i.e., for each $x \in U_i \cap V_j$, there is an open set W with $x \in W\subseteq U_i \cap V_j$ such that "$s_i = t_j$ on W" in the sense of restrictions being the same).

8.2.5. *Remark.* The relation $\equiv$ on $\widehat{\mathcal{S}}(U)$ really says that $(U_i, s_i)_{i\in I} \equiv (V_j, t_j)_{j\in J}$ iff the disjoint union $(U_i, s_i)_{i\in I} \sqcup (V_j, t_j)_{j\in J}$ is again in $\widehat{\mathcal{S}}(U)$.

8.2.6. *Lemma.* (a) $\equiv$ is an equivalence relation.

(b) $\widetilde{\mathcal{S}}(U)$ is a presheaf and there is a canonical map of presheaves $\mathcal{S} \xrightarrow{q} \widetilde{\mathcal{S}}$.

(c) $\widetilde{\mathcal{S}}$ is a sheaf.

Proof. (a) is obvious.

(b) The restriction of a system $(U_i, s_i)_{i\in I}$ to $V\subseteq U$ is the system $(U_i \cap V, s_i|U_i \cap V)_{i\in I}$. The weak compatibility of restrictions $s_i|U \cap V$ follows from the weak compatibility of sections s_i. Finally, restriction is compatible with $\equiv$, i.e., if $(U'_i, s'_i)_{i\in I}$ and $(U''_j, s''_j)_{j\in J}$ are $\equiv$, then so are $(U'_i \cap V, s'_i|U'_i \cap V)_{i\in I}$ and $(U''_j \cap V, s''_j|U''_j \cap V)_{j\in J}$.

The map $\mathcal{S}(U)\to\widetilde{\mathcal{S}}(U)$ is given by interpreting a section $s\in\mathcal{S}(U)$ as a (small) system: open cover of $(U_i)_{i\in\{0\}}$ is given by $U_0=U$ and $s_0=s$.

(c') Compatible systems of sections of the presheaf $\widetilde{\mathcal{S}}$ glue. Let $V^j,\ j\in J$, be an open cover of an open $V\subseteq X$, and for each $j\in J$ let $\sigma^j=[(U_i^j,s_i^j)_{i\in\ I_j}]$ be a section of $\widetilde{\mathcal{S}}$ on V_j. So, σ^j is an equivalence class of the system $(U_i^j,s_i^j)_{i\in\ I_j}$ consisting of an open cover $U_i^j,\ i\in\ I_j$, of V_j and weakly compatible sections $s_j^i\in\mathcal{S}(U_j^i)$.

Now, if for any $j,k\in J$ sections $\sigma^j=\ [(U_p^j,s_p^j)_{p\in\ I_j}]$ and $\sigma^k=\ [(U_q^k,s_q^k)_{q\in\ I_k}]$ of $\widetilde{\mathcal{S}}$ on V^j and V^k, agree on the intersection V^{jk}. This means that for any j,k $\sigma^j|V^{jk}=\sigma^k|V^{jk}$, i.e.,

$$(U_p^j\cap V^{jk},s_p^j|U_p^j\cap V^{jk})_{p\in\ I_j}\quad\equiv\quad(U_q^k\cap V^{jk},s_q^k|U_q^k\cap V^{jk})_{q\in\ I_k}.$$

This in turn means that for $j,k\in J$ and any $p\in I_j,\ q\in I_k$, sections s_p^j and s_q^k are weakly compatible. Since all sections $s_p^j,\ j\in J,\ p\in I_j$ are weakly compatible, the disjoint union of all systems $(U_i^j,s_i^j)_{i\in\ I_j},\ j\in J$ is a system in $\widehat{\mathcal{S}}(V)$. Its equivalence class σ is a section of $\widetilde{\mathcal{S}}$ on V, and clearly $\sigma|V^j=\sigma^j$.

(c") Compatible systems of sections of the presheaf $\widetilde{\mathcal{S}}$ glue uniquely. If $\tau\in\widetilde{\mathcal{S}}(V)$ is the class of a system $(U_i,s^i)_{i\in I}$ and $\tau|V^j=\sigma^j$ then σ's are compatible with all s_p^j's, hence $(U_i,s^i)_{i\in I}\equiv\sqcup_{j\in J}(U_i^j,s_i^j)_{i\in\ I_j}$, hence $\tau=\sigma$.

8.2.7. *Sheafification as a left adjoint of the forgetful functor.* As usual, we have not invented something new: it was already there, hidden in the more obvious forgetful functor

8.2.8. *Lemma.* Sheafification functor $pre\mathcal{S}heaves\ni\mathcal{S}\mapsto\widetilde{\mathcal{S}}\in\mathcal{S}heaves$, is the left adjoint of the inclusion $\mathcal{S}heaves\subseteq pre\mathcal{S}heaves$, i.e, for any presheaf $\mathcal{S}$ and any sheaf $\mathcal{F}$ there is a natural identification

$$\mathrm{Hom}_{\mathcal{S}heaves}(\widetilde{\mathcal{S}},\mathcal{F})\xrightarrow{\cong}\mathrm{Hom}_{pre\mathcal{S}heaves}(\mathcal{S},\mathcal{F}).$$

Explicitly, the bijection is given by $(\iota_{\mathcal{S}})_*\alpha=\ \alpha\circ\iota_{\mathcal{S}}$, i.e., $(\widetilde{\mathcal{S}}\xrightarrow{\alpha}\mathcal{F})\mapsto(\mathcal{S}\xrightarrow{\iota_{\mathcal{S}}}\widetilde{\mathcal{S}}\xrightarrow{\alpha}\mathcal{F})$.

8.3. **Inverse and direct images of sheaves.**

8.3.1. *Pull back of sheaves (finally!)* Now we can define for any map of topological spaces $X\xrightarrow{\pi}Y$ a pull-back functor

$$\mathcal{S}heaves(Y)\xrightarrow{\pi^{-1}}\mathcal{S}heaves(X),\qquad\pi^{-1}\mathcal{N}\overset{\text{def}}{=}\widetilde{\underline{\pi^{-1}}\mathcal{N}}.$$

8.3.2. *Examples.* (a) A point $a\in X$ can be viewed as a map $\{a\}\xrightarrow{\rho}X$. Then $\rho^{-1}\mathcal{S}$ is the stalk $\mathcal{S}_a$.

(b) Let $a:X\to\mathrm{pt}$, for any set S one has $S_X=a^{-1}S$.

8.3.3. *Direct image of sheaves.* Besides the pull-back of sheaves which we defined in 8.3.1, there is also a much simpler procedure of the push-forward of sheaves:

8.3.4. *Lemma.* (Direct image of sheaves.) Let $X \xrightarrow{\pi} Y$ be a map of topological spaces. For a sheaf $\mathcal{M}$ on X, formula

$$\pi_*(\mathcal{M})\ (V) \stackrel{\text{def}}{=} \mathcal{M}(\pi^{-1}V),$$

defines a sheaf $\pi_*\mathcal{M}$ on Y, and this gives a functor $\mathcal{S}heaves(X) \xrightarrow{\pi_*} \mathcal{S}heaves(Y)$.

8.3.5. *Adjunction between the direct and inverse image operations.* The two basic operations on sheaves are related by adjunction:

Lemma. For sheaves $\mathcal{A}$ on X and $\mathcal{B}$ on Y one has a natural identification

$$\mathrm{Hom}(\pi^{-1}\mathcal{B}, \mathcal{A}) \cong \mathrm{Hom}(\mathcal{B}, \pi_*\mathcal{A}).$$

Proof. We want to compare $\beta \in \mathrm{Hom}(\mathcal{B}, \pi_*\mathcal{A})$ with α in

$$\mathrm{Hom}_{\mathcal{S}h(X)}(\pi^{-1}\mathcal{B}, \mathcal{A}) = \mathrm{Hom}_{\mathcal{S}h(X)}(\widetilde{\underline{\pi^{-1}}\mathcal{B}}, \mathcal{A}) \cong \mathrm{Hom}_{pre\mathcal{S}h(X)}(\underline{\pi^{-1}}\mathcal{B}, \mathcal{A}).$$

α is a system of maps

$$\lim_{\to\ V\supseteq\pi(U)} \mathcal{B}(V) = \underline{\pi^{-1}}\mathcal{B}(U) \xrightarrow{\alpha_U} \mathcal{A}(U), \quad \text{for } U \text{ open in } X,$$

and β is a system of maps

$$\mathcal{B}(V) \xrightarrow{\beta_V} \mathcal{A}(\pi^{-1}V), \quad \text{for } V \text{ open in } Y.$$

Clearly, any β gives some α since

$$\lim_{\to\ V\supseteq\pi(U)} \mathcal{B}(V) \xrightarrow{\lim_{\to} \beta_V} \lim_{\to\ V\supseteq\pi(U)} \mathcal{A}(\pi^{-1}V) \to \mathcal{A}(U),$$

the second map comes from the restrictions $\mathcal{A}(\pi^{-1}V) \to \mathcal{A}(U)$ defined since $V\supseteq\pi(U)$ implies $\pi^{-1}V\supseteq U$.

For the opposite direction, any α gives for each V open in Y, a map $\lim_{\to\ W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W) = \underline{\pi^{-1}}\mathcal{B}(\pi^{-1}V) \xrightarrow{\alpha_{\pi^{-1}V}} \mathcal{A}(\pi^{-1}V)$. Since $\mathcal{B}(V)$ is one of the terms in the inductive system we have a canonical map $\mathcal{B}(V) \to \lim_{\to\ W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W)$, and the composition with the first map $\mathcal{B}(V) \to \lim_{\to\ W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W) \xrightarrow{\alpha_{\pi^{-1}V}} \mathcal{A}(\pi^{-1}V)$, is the wanted map β_V.

8.3.6. *Lemma.* (a) If $X \xrightarrow{\pi} Y \xrightarrow{\tau} Z$ then

$$\tau_*(\pi_*\mathcal{A}) \cong (\tau\circ\pi)_*\mathcal{A} \quad \text{and} \quad \tau_*(\pi_*\mathcal{A}) \cong (\tau\circ\pi)_*\mathcal{A}.$$

(b) $(1_X)_*\mathcal{A} \cong \mathcal{A} \cong (1_X)^{-1}\mathcal{A}$.

Proof. The statements involving direct image are very simple and the claims for inverse image follow by adjunction.

8.3.7. *Corollary.* (Pull-back preserves the stalks) For $a \in X$ one has $(\pi^{-1}\mathcal{N})_a \cong \mathcal{N}_{\pi(a)}$.

This shows that the pull-back operation which was difficult to define is actually very simple in its effect on sheaves.

8.4. **Stalks.** Part (a) of the following lemma is the recollection of the description of inductive limits of abelian groups from the remark 4.3.7.

8.4.1. *Lemma.* (Inductive limits of abelian groups.) (a) For an inductive system of abelian groups (or sets) A_i over $(I, \leq)$, inductive limit $\lim\limits_{\to} A_i$ can be described by

- for $i \in I$ any $a \in A_i$ defines an element $\overline{a}$ of $\lim\limits_{\to} A_i$,
- all elements of $\lim\limits_{\to} A_i$ arise in this way, and
- for $a \in A_i$ and $b \in A_j$ one has $\overline{a} = \overline{b}$ iff for some $k \in I$ with $i \leq k \geq j$ one has $a = b$ in A_k.

(b) For a subset $K \subseteq I$ one has a canonical map $\lim\limits_{\to\, i\in K} A_i \longrightarrow \lim\limits_{\to\, i\in I} A_i$.

Proof. In general (b) is clear from the definition of $\lim\limits_{\to}$, and for abelian groups also from (a).

8.4.2. *Stalks of (pre)sheaves.* Remember that the stalk of a presheaf $\mathcal{A}$b at a point x is $\mathcal{A}_x \stackrel{\text{def}}{=} \lim\limits_{\to} \mathcal{A}(U)$, the limit over (diminishing) neighborhoods u of x. This means that

- any $s \in \mathcal{A}(U)$ with $U \ni x$ defines an element s_x of the stalk,
- all elements of $\mathcal{A}_x$ arise in this way, and
- For $s' \in \mathcal{A}(U')$ and $s'' \in \mathcal{A}(U'')$ one has $s'_x = s''_x$ iff for some neighborhood W of x in $U' \cap U''$ one has $s' = s''$ on W.

8.4.3. *Lemma.* For a presheaf $\mathcal{S}$, the canonical map $\mathcal{S} \to \widetilde{\mathcal{S}}$ is an isomorphism on stalks.

Proof. We consider a point $a \in X$ as a map $\text{pt} = \{a\} \xrightarrow{i} X$, so that $\mathcal{A}_x = i^{-1}\mathcal{A}$. For a sheaf B on the point

$$\begin{aligned}\text{Hom}_{\mathcal{S}h(pt)}(i^{-1}\widetilde{\mathcal{S}}, \mathcal{B}) &\cong \text{Hom}_{\mathcal{S}h(X)}(\widetilde{\mathcal{S}}, i_*\mathcal{B}) \cong \text{Hom}_{pre\mathcal{S}h(X)}(\mathcal{S}, i_*\mathcal{B}) \\ &\cong \text{Hom}_{pre\mathcal{S}h(pt)}(i^{-1}\mathcal{S}, \mathcal{B}) = \text{Hom}_{\mathcal{S}h(pt)}(i^{-1}\mathcal{S}, \mathcal{B}).\end{aligned}$$

8.4.4. *Germs of sections and stalks of maps.* For any neighborhood U of a point x we have a canonical map $\mathcal{S}(U) \to \lim_{\to\, V\ni x} \mathcal{S}(V) \stackrel{\text{def}}{=} \mathcal{S}_x$ (see lemma 8.4.1), and we denote the image of a section $s \in \Gamma(U,\mathcal{S})$ in the stalk $\mathcal{S}_x$ by s_x, and we call it the germ of the section at x. The germs of two sections are the same at x if the sections are the same on some (possibly very small) neighborhood of x (this is again by the lemma 8.4.1).

A map of sheaves $\phi : \mathcal{A} \to \mathcal{B}$ defines for each $x \in M$ a map of stalks $\mathcal{A}_x \to \mathcal{B}_x$ which we call ϕ_x. It comes from a map of inductive systems given by ϕ, i.e., from the system of maps $\phi_U : \mathcal{A}(U) \to \mathcal{B}(U),\ U \ni x$ (see 4.3.6) ; and on germs it is given by $\phi_x(a_x) = [\phi_U(a)]_x,\ a \in \mathcal{A}(U)$.

For instance, let $\mathcal{A} = \mathcal{H}_{\mathbb{C}}$ be the sheaf of holomorphic functions on $\mathbb{C}$. Remember that the stalk at $a \in \mathbb{C}$ can be identified with all convergent power series in $z - a$. Then the germ of a holomorphic function $f \in \mathcal{H}_{\mathbb{C}}(U)$ at a can be thought of as the power series expansion of f at a. An example of a map of sheaves $\mathcal{H}_{\mathcal{C}} \xrightarrow{\Phi} \mathcal{H}_{\mathbb{C}}$ is the multiplication by an entire function $\phi \in \mathcal{H}_{\mathbb{C}}(\mathbb{C})$, its stalk at a is the multiplication of the the power series at a by the power series expansion of ϕ at a.

8.4.5. The following lemma from homework shows how much the study of sheaves reduces to the study of their stalks.

Lemma. (a) Maps of sheaves $\phi, \psi : \mathcal{A} \to \mathcal{B}$ are the same iff the maps on stalks are the same, i.e., $\phi_x = \psi_x$ for each $x \in M$.

(b) Map of sheaves $\phi : \mathcal{A} \to \mathcal{B}$ is an isomorphism iff ϕ_x is an isomorphism for each $x \in M$.

8.4.6. *Sheafifications via the etale space of a presheaf.* We will construct the sheafification of a presheaf $\mathcal{S}$ (once again) in an "elegant" way, using the etale space $\overset{\bullet}{\mathcal{S}}$ of the presheaf. It is based on the following example of sheaves

Example. Let $Y \xrightarrow{p} X$ be a continuous map. For any open $U \subseteq X$ the elements of

$$\Sigma(U) \stackrel{\text{def}}{=} \{s : U \to Y,\ s \text{ is continuous and } p \circ s = 1_u\}$$

are called the (continuous) sections of p over U. Σ is a sheaf of sets.

To apply this construction we need a space $\overset{\bullet}{\mathcal{S}}$ that maps to X:

- Let $\overset{\bullet}{\mathcal{S}}$ be the union of all stalks $\mathcal{S}_m,\ m \in X$.
- Let $p : \overset{\bullet}{\mathcal{S}} \to X$ be the map such that the fiber at m is the stalk at m.
- For any pair (U, s) with U open in X and $s \in \mathcal{S}(U)$, define a section $\tilde{s}$ of p over U by

$$\tilde{s}(x) \stackrel{\text{def}}{=} s_x \in \mathcal{S}_x \subset \overset{\bullet}{\mathcal{S}}, \qquad x \in U.$$

Lemma. (a) If for two sections $s_i \in \mathcal{S}(U_i),\ i = 1, 2;$ of $\mathcal{S}$, the corresponding sections $\tilde{s}_1$ and $\tilde{s}_2$ of p agree at a point then they agree on some neighborhood of of this point (i.e., if $\tilde{s}_1(x) = \tilde{s}_2(x)$ for some $x \in U_{12} \overset{\text{def}}{=} U_1 \cap U_2$, then there is a neighborhood W of x such that $\tilde{s}_1 = \tilde{s}_2$ on W).

(b) All the sets $\tilde{s}(U)$ (for $U \subseteq X$ open and $s \in \mathcal{S}(U)$), form a basis of a topology on $\dot{\mathcal{S}}$. Map $p : \dot{\mathcal{S}} \to M$ is continuous.

(c) Let $\tilde{\mathcal{S}}(U)$ denote the set of continuous sections of p over U. Then $\tilde{\mathcal{S}}$ is a sheaf and there is a canonical map of presheaves $\iota : \mathcal{S} \to \tilde{\mathcal{S}}$.

Remark. Moreover, p is "etale" meaning "locally an isomorphism", i.e., for each point $\sigma \in \dot{\mathcal{S}}$ there are neighborhoods $\sigma \in W \subseteq \dot{\mathcal{S}}$ and $p(\sigma) \subseteq U \subseteq X$ such that $p|W$ is a homeomorphism $W \xrightarrow{\cong} U$.

Lemma. The new $\widetilde{\mathcal{S}}$ and the old $\widetilde{\mathcal{S}}$ (from 8.2.4) are the same sheaves (and the same holds for the canonical maps $\iota : \mathcal{S} \to \widetilde{\mathcal{S}}$).

Proof. Sections of p over $U \subseteq X$ are the same as the equivalence classes of systems $\widehat{\mathcal{S}}/\equiv$ defined in 8.2.4.

8.5. **Abelian category structure.** Let us fix a map of sheaves $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ since the non-trivial part is the construction of (co)kernels. Consider the example where the space is the circle $X = \{z \in \mathbb{C},\ |z| = 1\}$ and $\mathcal{A} = \mathcal{B}$ is the sheaf $\mathcal{C}^\infty_X$ of smooth functions on X, and the map α is the differentiation $\partial = \frac{\partial}{\partial\theta}$ with respect to the angle θ. For $U \subseteq X$ open, $\mathrm{Ker}(\partial_U) :\ \mathcal{C}^\infty_X(U) \to \mathcal{C}^\infty_X(U)$ consists of locally constant functions and the cokernel $\mathcal{C}^\infty_X(U)/\partial_U \mathcal{C}^\infty_X(U)$ is

- zero if $U \neq X$ (then any smooth function on U is the derivative of its indefinite integral defined by using the exponential chart $z = e^{i\theta}$ which identifies U with an open subset of $\mathbb{R}$),
- one dimensional if $U = X$ – for $g \in C^\infty(X)$ one has $\int_X \ \partial g = 0$ so say constant functions on X are not derivatives (and for functions with integral zero the first argument applies).

So by taking kernels at each level we got a sheaf but by taking cokernels we got a presheaf which is not a sheaf (local sections are zero but there are global non-zero sections, so the object is not controlled by its local properties).

8.5.1. *Subsheaves.* For (pre)sheaves $\mathcal{S}$ and $\mathcal{S}'$ we say that $\mathcal{S}'$ is a sub(pre)sheaf of $\mathcal{S}$ if $\mathcal{S}'(U) \subseteq \mathcal{S}(U)$ and the restriction maps for $\mathcal{S}'$, $\mathcal{S}'(U) \xrightarrow{\rho'} \mathcal{S}'(V)$ are restrictions of the restriction maps for $\mathcal{S}$, $\mathcal{S}(U) \xrightarrow{\rho} \mathcal{S}(V)$.

8.5.2. *Lemma.* (Kernels.) Any map $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ has a kernel and $\mathrm{Ker}(\alpha)(U) = \mathrm{Ker}[\mathcal{A}(U) \xrightarrow{\phi(U)} \mathcal{B}(U)]$ is a subsheaf of $\mathcal{A}$.

Proof. First, $\mathcal{K}(U) \stackrel{\text{def}}{=} \mathrm{Ker}[\mathcal{A}(U) \xrightarrow{\phi(U)} \mathcal{B}(U)]$ is a sheaf, and then a map $\mathcal{C} \xrightarrow{\mu} \mathcal{A}$ is killed by α iff it factors through the subsheaf $\mathcal{K}$ of $\mathcal{A}$.

Lemma. (Cokernels.) Any map $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ defines a presheaf $C(U) \stackrel{\text{def}}{=} \mathcal{B}(U)/\alpha_U(\mathcal{A}(U))$, the associated sheaf $\mathcal{C}$ is the cokernel of α.

Proof. For a sheaf $\mathcal{S}$ one has

$$\mathrm{Hom}_{\mathcal{S}heaves}(\mathcal{B},\mathcal{S})_\alpha \cong \mathrm{Hom}_{pre\mathcal{S}heaves}(C,\mathcal{S}) \cong \mathrm{Hom}_{\mathcal{S}heaves}(\mathcal{C},\mathcal{S}).$$

The second identification is the adjunction. For the first one, a map $\mathcal{B} \xrightarrow{\phi} \mathcal{S}$ is killed by α, i.e., $0 = \phi \circ \alpha$, if for each U one has $0 = (\phi \circ \alpha)_U \mathcal{A}(U) = \phi_U(\alpha_U \mathcal{A}(U))$; but then it gives a map $C \xrightarrow{\overline{\phi}} \mathcal{S}$, with $\overline{\phi}_U : C(U) = \mathcal{B}(U)/\alpha_U \mathcal{A}(U) \to \mathcal{S}(U)$ the factorization of ϕ_U. The opposite direction is really obvious, any $\psi : C \to \mathcal{S}$ can be composed with the canonical map $\mathcal{B} \to C$ (i.e., $\mathcal{B}(U) \to \mathcal{B}(U)/\alpha_U \mathcal{A}(U)$) to give map $\mathcal{B} \to \mathcal{S}$ which is clearly killed by α.

8.5.3. *Lemma.* (Images.) Consider a map $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$.

(a) It defines a presheaf $I(U) \stackrel{\text{def}}{=} \alpha_U(\mathcal{A}(U)) \subseteq \mathcal{B}(U)$ which is a subpresheaf of $\mathcal{B}$. The associated sheaf $\mathcal{I}$ is the image of α.

(b) It defines a presheaf $c(U) \stackrel{\text{def}}{=} \mathcal{A}(U)/\mathrm{Ker}(\alpha_U)$, the associated sheaf is the coimage of α.

(c) The canonical map $Coim(\alpha) \to Im(\alpha)$ is isomorphism.

Proof. (a) $Im(\alpha) \stackrel{\text{def}}{=} \mathrm{Ker}[\mathcal{B} \to \mathrm{Coker}(\alpha)]$ is a subsheaf of $\mathcal{B}$ and $b \in \mathcal{B}(U)$ is a section of $Im(\alpha)$ iff it becomes zero in $\mathrm{Coker}(\alpha)$. But a section $b + \alpha_U \mathcal{A}(U)$ of C on U is zero in $\mathcal{B}$ iff it is locally zero in C, i.e., there is a cover U_i of U such that $b|U_i \in \alpha_{U_i}\mathcal{A}(U_i)$. But this is the same as saying that b is locally in the subpresheaf I of $\mathcal{B}$, i.e., the same as asking that b is in the corresponding presheaf $\mathcal{I}$ of $\mathcal{B}$.

(b) The coimage of α is by definition $Coim(\alpha) \stackrel{\text{def}}{=} \mathrm{Coker}[\mathrm{Ker}(\alpha) \to \mathcal{A}]$, i.e., the sheaf associated to the presheaf $U \mapsto \mathcal{A}(U)/\mathrm{Ker}(\alpha)(U) = c(U)$.

(c) The map of sheaves $Coim(\alpha) \to Im(\alpha)$ is associated to the canonical map of presheaves $c \to I$, however already the map of presheaves is an isomorphism: $c(U) = \mathcal{A}(U)/\mathrm{Ker}(\alpha)(U) \cong \alpha_U \stackrel{\text{def}}{=} \mathcal{A}(U) = I(U)$.

8.5.4. *Stalks of kernels, cokernels and images; exact sequences of sheaves.*

8.5.5. *Lemma.* For a map of sheaves $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ and $x \in X$

- (a) $\mathrm{Ker}(\mathcal{A} \xrightarrow{\alpha} \mathcal{B})_x = \mathrm{Ker}(\alpha_x : \mathcal{A}_x \to \mathcal{B}_x)$,
- (b) $\mathrm{Coker}(\mathcal{A} \xrightarrow{\alpha} \mathcal{B})_x = \mathrm{Coker}(\alpha_x : \mathcal{A}_x \to \mathcal{B}_x)$,
- (c) $Im(\mathcal{A} \xrightarrow{\alpha} \mathcal{B})_x = Im(\alpha_x : \mathcal{A}_x \to \mathcal{B}_x)$.

Proof. (a) Let $x \in U$ and $a \in \mathcal{A}(U)$. The germ a_x is killed by α_x if $0 = \alpha_x(a_x) \stackrel{\text{def}}{=} (\alpha_U(a))_x$, i.e., iff $\alpha_U(a) = 0$ on some neighborhood U' of x in U. But this is the same as saying that $0 = \alpha_U(a)|U' = \alpha_{U'}(a|U')$, i.e., asking that some restriction of a to a smaller neighborhood of x is a section of the subsheaf $\mathrm{Ker}(\alpha)$. And this in turn, is the same as saying that the germ a_x lies in the stalk of $\mathrm{Ker}(\alpha)$.

(b) Map $\mathcal{B} \xrightarrow{q} \mathrm{Coker}(\alpha)$ is killed by composing with α, so the map of stalks $\mathcal{B}_x \xrightarrow{q_x} \mathrm{Coker}(\alpha)_x$ is killed by composing with α_x.

To see that q_x is surjective consider some element of the stalk $\mathrm{Coker}(\alpha)_x$. It comes from a section of a presheaf $U \mapsto \mathcal{B}(U)/\alpha_U \mathcal{A}(U)$, so it is of the form $[b + \alpha_U(\mathcal{A}(U))]_x$ for some section $b \in \mathcal{B}(U)$ on some neighborhood U of x. Therefore it is the image $\alpha_x(b_x)$ of an element b_x of $\mathcal{B}_x$.

To see that q_x is injective, observe that a stalk $b_x \in \mathcal{B}_x$ (of some section $b\mathcal{B}(U)$), is killed by q_x iff its image $\alpha_x(b_x) = [b + \alpha_U(\mathcal{A}(U))]_x$ is zero in $\mathrm{Coker}(\alpha)$, i.e., iff there is a smaller neighborhood $U' \subseteq U$ such that the restriction $[b + \alpha_U(\mathcal{A}(U))]|U' = b|U' + \alpha_{U'}(\mathcal{A}(U'))$ is zero, i.e., $b|U'$ is in $\alpha_{U'}\mathcal{A}(U')$. But the existence of such U' is the same as saying that b_x is in the image of α_x.

(c) follows from (a) and (b) by following how images are defined in terms of kernels and cokernels.

8.5.6. *Corollary.* A sequence of sheaves is exact iff at each point the corresponding sequence of stalks of sheaves is exact.

9. **Homotopy category of complexes**

On the way to identifying any two quasi-isomorphic complexes, i.e., to inverting all quasi-isomorphisms, in the first step we will invert a special kind of isomorphisms – the homotopy equivalences.

This is achieved by passing from the category of complexes $C(\mathcal{A})$ to the so called "homotopy category of complexes" $K(\mathcal{A})$. Category $K(\mathcal{A})$ is no more an abelian category but it has a similar if less familiar structure of a "triangulated category". The abelian structure of the category $C(\mathcal{A})$ provides the notion of short exact sequences. This is essential since one can think of putting B into a short exact sequences $0 \to A \to B \to C \to 0$, as describing the complex B in terms of simpler complexes A and C; and indeed it turns out that the cohomology groups of B are certain combinations of cohomology groups of A and B. Since $K(\mathcal{A})$ is not abelian we are forced to find an analogue of short exact sequences which works in $K(\mathcal{A})$, this is the notion of "distinguished triangles", also called "exact triangles". The properties of exact triangles in $K(\mathcal{A})$ formalize into the concept of a "triangulated category" which turns out to be the best standard framework for homological algebra.

In particular, the passage to $(\mathcal{A})$ will solve the remaining foundational problem in the definition of derived functors:

- identify any two projective resolutions of one object

(so the derived functors will be well defined since we remove the dependence on the choice of a projective resolution).

9.1. **Category $C(\mathcal{A})$ of complexes in $\mathcal{A}$.** We observe some of the properties of the category $C(\mathcal{A})$.

9.1.1. *Structures.*

- *Shift functors.* For any integer n define a shift functor $[n]: C(\mathcal{A}) \to C(\mathcal{A})$ by $(A[n])^p \stackrel{\text{def}}{=} A^{n+p}$, and the differential $(A[n])^p \to A[n])^{p+1}$ given as $A^{p+n} \xrightarrow{(-1)^n d_A^{p+n}} A^{p+1+n}$.
- *Functors $H^i : C(\mathcal{A}) \to \mathcal{A}$.*
- *Special class of morphisms related to cohomology functors: the quasi-isomorphisms.*
- *Triangles.* These are diagrams of the form $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$.
- *Subcategories $C^b(\mathcal{A})$ etc.* If ? is one of the symbols $b, -, +$ we define a full subcategory $C^?(\mathcal{A})$ of $C^?(\mathcal{A})$, consisting respectively of bounded complexes (i.e. $A^n = 0$ for $|n| >> 0$), complexes bounded from bellow: $A^n = 0$ for $n << 0$ (hence allowed to stretch in the + direction), complexes bounded from above (so they may stretch in the − direction). Moreover, for a subset $\mathcal{Z}\subseteq\mathbb{Z}$ we can define $C^{\mathcal{Z}}(\mathcal{A})$ as a full

subcategory consisting of all complexes A with $A^n = 0$ for $n \notin \mathcal{Z}$. In particular one has $C^{\leq 0}(\mathcal{A}) \overset{\text{def}}{=} C^{\{\cdots,-2,-1,0\}}$ and $C^{\geq 0}(\mathcal{A}) \overset{\text{def}}{=} C^{\{0,1,2,\cdots\}}$, and $C^{\{0\}}(\mathcal{A})$ is equivalent to $\mathcal{A}$.

9.1.2. *Properties.* The next two lemmas give basic properties of the above structures on the category $C(\mathcal{A})$.

9.1.3. *Lemma.* $C(\mathcal{A})$ is an abelian category and a sequence of complexes is exact iff it is exact on each level!

Proof. For a map of complexes $A \xrightarrow{\alpha} B$ we can define $K^n = \text{Ker}(A^n \xrightarrow{\alpha^n} B^n)$ and $C^n = A^n/\alpha^n(B^n)$. This gives complexes since d_A induces a differential d_K on K and d_B a differential d_C on C. Moreover, it is easy to check that in category $C(\mathcal{A})$ one has $K = \text{Ker}(\alpha)$ and $C = \text{Coker}(\alpha)$. Now one finds that $Im(\alpha)^n = Im(\alpha^n) = \alpha^n(A^n)$ and $Coim(\alpha)^n = Coim(\alpha^n) = A^n/\text{Ker}(\alpha^n)$, so the canonical map $Coim \to Im$ is given by isomorphisms $A^n/\text{Ker}(\alpha^n) \xrightarrow{\cong} \alpha^n(A^n)$. Exactness claim follows.

9.1.4. *Lemma.* A short exact sequence of complexes $0 \to A \to B \to C \to 0$ gives a long exact sequence of cohomologies.

$$\cdots \xrightarrow{\partial^{n-1}} \mathrm{H}^n(A) \xrightarrow{\mathrm{H}^n(\alpha)} \mathrm{H}^n(B) \xrightarrow{\mathrm{H}^n(\beta)} \mathrm{H}^n(C) \xrightarrow{\partial^n} \mathrm{H}^{n+1}(A) \xrightarrow{\mathrm{H}^{n+1}(\alpha)} \mathrm{H}^{n+1}(B) \xrightarrow{\mathrm{H}^{n+1}(\beta)} \cdots$$

Proof. We need to construct for a class $\gamma \in \mathrm{H}^n(C)$ a class $\partial\gamma \in \mathrm{H}^{n+1}$. So if $\gamma = [c]$ is the class of a cocycle c, we need

(1) From a cocycle $c \in Z^n(C)$ a cocycle $a \in Z^{n+1}$.
(2) Independence of $[a]$ on the choice of c or any other auxiliary choices.
(3) The sequence of cohomology groups is exact.

Recall that a sequence of complexes. $0 \to A \to B \to C \to 0$ is a short exact sequence if for each integer n the sequence $0 \to A^n \to B^n \to C^n \to 0$ is exact.

The following calculation is in the set-theoretic language appropriate for module categories but can be rephrased in the language of abelian categories (and also the result for module categories implies the result for abelian categories since any abelian category is equivalent to a full subcategory of a module category).

(1) Since β^n is surjective, $c = \beta^n b$ for $b \in B^n$, Now $db = \alpha^{n+1}a$ for some $a \in A^{n+1}$ since $\beta^{n+1}(db) = d\beta^{n+1}b = dc = 0$. Moreover, a is a cocycle since $\alpha^{n+2}(d^{n+1}a) = d^{n+1}(\alpha^{n+1}a) = d^{n+1}(d^n b) = 0$.

(2) So we want to associate to $\gamma = [c]$ the class $\alpha = [a] \in \mathrm{H}^{n+1}(A)$. For that $[a]$ should be independent of the choices of c, B and a. So let $[c] = [c']$ and $c' = \beta^n b'$ with $b' \in B^n$, and $db' = \alpha^{n+1}a'$ for some $a' \in A^{n+1}$.

Since $[c]=[c']$ one has $c'=c+dz$ for some $z\in C^{n-1}$. Choose $y\in B^{n-1}$ so that $z=\beta^{n-1}y$, then

$$\beta^n b' = c' = c+dz = \beta^n b + d(\beta^{n-1}y) = \beta^n b + \beta^n dy = \beta^n(b+dy).$$

The exactness at B now shows that $b' = b+dy+\alpha^n x$ for some $x\in A^n$. So,

$$\alpha^{n+1}a' = db' = db + d\alpha^n x = \alpha^{n+1}a + \alpha^n(dx) = \alpha^{n+1}(a+dx).$$

Exactness at A implies that actually $a'=a+dx$.

(3) I omit the easier part: the compositions of any two maps are zero.

Exactness at $H^n(B)$. Let $b\in Z^n(B)$, then $\mathrm{H}^n(\beta)[b]=[\beta^n b]$ is zero iff $\beta^n b = dz$ for some $z\in C^{n-1}$. Let us lift this z to some $y\in B^{n-1}$, i.e., $z=\beta^{n-1}y$. Then $\beta^n(b-dy)=dz-dz=0$, hence $b-dy=\alpha^n a$ for some $a\in A^n$. Now a is a cocycle since $\alpha^n(da)=d(b-dy)=0$, and $[b]=[b-dy]=\mathrm{H}^n(\alpha)[a]$.

Exactness at $H^n(A)$. Let $a\in Z^n(A)$ be such that $\mathrm{H}^n(\alpha)[a]=[\alpha^n a]$ is zero, i.e., $\alpha^n a = db$ for some $b\in B^{n-1}$. Then $c=\beta b$ is a cocycle since $dc=\beta^n(db)=\ \beta^n\alpha^n a=0$; and by the definition of the connecting morphisms (in (1)), $[a]=\delta^{n-1}[c]$.

Exactness at $H^n(C)$. Let $c\in Z^n(C)$ be such that $\partial^n[c]=0$. Remember that this means that $c=\beta b$ and $db=\alpha a$ with $[a]=0$, i.e., $a=dx$ with $x\in A^{n-1}$. But then $db=\alpha(dx)=d(\alpha x)$, so $b-\alpha x$ is a cocycle, and then $c=\beta(b)=\beta(b-\alpha(x))$ implies that $[c]=\ \mathrm{H}^n(\beta)[b-\alpha(x)]$.

9.2. **Mapping cones.** The idea is that the cone of a map of complexes $A\xrightarrow{\alpha}B$ is a complex $\boldsymbol{C}_\alpha$ which measures how far α is from being an isomorphism. However, the main role of the mapping cone here is that it partially reformulates the short exact sequences of complexes. It will turn out that the following data are the same:

(1) a mapping cone and its associated triangle
(2) a short exact sequence of complexes which splits in each degree.

This is the content of the next three lemmas. These results, together with the preceding lemmas 9.1.4 and 9.1.3, all have analogues for the homotopy category of complex $K(\mathcal{A})$ which we meet in 9.3. These analogues (theorem 9.4.1), will be more complicated and will give rise to a notion of a triangulated category.

9.2.1. *Lemma.* (a) A map $A\xrightarrow{\alpha}B$ defines a complex C called the cone of α by

- $\boldsymbol{C}^n = B^n\oplus A^{n+1}$,
- $d^n_{\boldsymbol{C}}:\ B^n\oplus A^{n+1}\to B^{n+1}\oplus A^{n+2}$ combines the differentials in A and B and the map α by:

$$d(b^n\oplus a^{n+1})\stackrel{\text{def}}{=}(d^n_B b^n + \alpha^{n+1}a^{n+1})\oplus\ -d^{n+1}_A a^{n+1}$$

(b) The cone appears in a canonical triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\dot{\alpha}} \boldsymbol{C} \xrightarrow{\ddot{\alpha}} A[1], \quad \textit{with} \quad \dot{\alpha}^n(b^n) = b^n \oplus 0 \quad \text{and} \quad \ddot{\alpha}^n(b^n \oplus a^{n+1}) \overset{\text{def}}{=} a^{n+1}.$$

Proof. (a) One has $d_{\boldsymbol{C}} = (d_B + \alpha) \oplus - d_A$, hence

$$(d_{\boldsymbol{C}})^2 = d_{\boldsymbol{C}} \circ [(d_B + \alpha) \oplus - d_A] = (d_B^2 + d_B \circ \alpha + \alpha \circ - d_A) \oplus (-d_A)^2 = 0,$$

in more details

$$d^{n+1} d^n (b^n \oplus a^{n+1}) = d^{n+1}[(d_B^n b^n + \alpha^{n+1} a^{n+1}) \oplus - d_A^{n+1} a^{n+1}]$$
$$= (d_B^{n+1} d_B^n b^n + d^{n+1} \alpha^{n+1} a^{n+1} + \alpha^{n+1}(-d_A^{n+1} a^{n+1})) \oplus - d_A^{n+2} d_A^{n+1} a^{n+1} = 0.$$

(b) The claim is just that $\dot{\alpha}$ and $\ddot{\alpha}$ are maps of complexes. Clearly, $(d_{\boldsymbol{C}} \circ \dot{\alpha})(b^n) = d_{\boldsymbol{C}}(b^n \oplus 0) = d_B b^b \oplus 0 = \dot{\alpha}(d_B b^n)$ and

$$(d^n_{A[1]} \circ \ddot{\alpha})(b^n \oplus a^{n+1}) = (d^n_{A[1]} a^{n+1}) = -d_A^{n+1} a^{n+1}) = \ddot{\alpha}(d_B b^n + \alpha a^{n+1}) \oplus - d_A^{n+1} a^{n+1})$$
$$= (\ddot{\alpha} \circ d_{\boldsymbol{C}})(b^n \oplus a^{n+1}).$$

9.2.2. *Cone triangles and short exact sequences of complexes.* We will see that the triangles that arise from cones are reformulations of some short exact sequences of complexes.

Lemma. (a) For any map $A \xrightarrow{\alpha} B$ of complexes, the corresponding distinguished triangle $A \xrightarrow{\alpha} B \xrightarrow{\dot{\alpha}} \boldsymbol{C}_\alpha \xrightarrow{\ddot{\alpha}} A[1]$ contains an exact sequence of complexes

$$0 \to B \xrightarrow{\dot{\alpha}} \boldsymbol{C}_\alpha \xrightarrow{\ddot{\alpha}} A[1] \to 0.$$

Moreover, this exact sequence of complexes splits canonically in each degree, i..e, short exact sequences $0 \to B^n \xrightarrow{\dot{\alpha}} (\boldsymbol{C}_\alpha)^n \xrightarrow{\ddot{\alpha}} (A[1])^n \to 0$ in $\boldsymbol{\mathcal{A}}$ have canonical splittings, i.e., identifications

$$(\boldsymbol{C}_\alpha)^n \xrightarrow{\cong} B^n \oplus (A[1])^n.$$

(b) Actually, one can recover the whole triangle from the short exact sequence and its splittings.

Proof. In (a), exactness is clear since $\dot{\alpha}$ is the inclusion of the first factor and $\ddot{\alpha}$ is the projection to the second factor. The splitting statement is the claim that the image of $B^n \hookrightarrow C^n_\alpha$ has a complement (which is then automatically isomorphic to $(A[1])^n$), but the image is $B^n \oplus 0$ and the complement is just $0 \oplus A^{n+1}$.

(b) Identification $(\boldsymbol{C}_\alpha)^n \xrightarrow{\cong} B^n \oplus (A[1])^n$ involves two maps $(A[1])^n \overset{\sigma^n}{\hookrightarrow} (\boldsymbol{C}_\alpha)^n \overset{\tau^n}{\twoheadrightarrow} B^n$ (actually σ^n determines τ^n and vice versa). Now we can recover the map $A^{n+1} \xrightarrow{\alpha^{n+1}} B^{n+1}$ as a part of the differential $(\boldsymbol{C}_\alpha)^n \xrightarrow{d^n_C} (\boldsymbol{C}_\alpha)^{n+1}$. Precisely, α^{n+1} is the composition

$$(A[1])^n \overset{\sigma^n}{\hookrightarrow} (\boldsymbol{C}_\alpha)^n \xrightarrow{d^n_C} (\boldsymbol{C}_\alpha)^{n+1} \overset{\tau^{n+1}}{\twoheadrightarrow} B^{n+1}.$$

For this one just recalls that d_C^n is $B^n \oplus A^{n+1} \xrightarrow{(d_B+\alpha)\oplus -d_A} B^{n+1}\oplus A^{n+2}$, i.e., $d_C^n(b^n\oplus a^{n+1}) \stackrel{\text{def}}{=} (d_B^n b^n + \alpha^{n+1}a^{n+1})\oplus\ -d_A^{n+1}a^{n+1}$.

9.2.3. *From a short exact sequences to a cone triangle.* Conversely, we will see that any short exact sequence that splits on each level, produces a cone triangle. However, the two procedures of passing between cone triangles and short exact sequence that splits on each level, turn out not to be inverse to each other. This gets resolved in 9.6.1 by adopting a "correct" categorical setting – the homotopy category of complexes.

Lemma. A short exact sequence of complexes

$$0 \to P \xrightarrow{\phi} Q \xrightarrow{\psi} R \to 0$$

and a splitting $Q^n \cong\ P^n\oplus R^n$ for each $n\in\mathbb{Z}$, define canonically a distinguished triangle

$$R[-1] \xrightarrow{\alpha} P \xrightarrow{\phi=\dot{\alpha}} Q \xrightarrow{\psi=\ddot{\alpha}} (R[-1])[1].$$

(b) Moreover, this triangle is isomorphic in $C(\mathcal{A})$ to a cone triangle.

(c) Explicitly, if the splitting is given by maps $R^n \stackrel{\sigma^n}{\hookrightarrow} Q^n \cong \stackrel{\tau^n}{\twoheadrightarrow} P^n$, then the map $R^n =\ (R[-1])^{n+1} \xrightarrow{\alpha^{n+1}} P^{n+1}$ is a component of the differential $Q^n \xrightarrow{d_Q^n} Q^{n+1}$, i.e., it is the composition

$$\alpha^{n+1}\ =\ (R^n \stackrel{\sigma^n}{\hookrightarrow} P^n\oplus R^n\ \cong\ Q^n\ \xrightarrow{d_Q^n}\ Q^{n+1}\oplus P^{n+1}\ \cong\ P^{n+1}\oplus R^{n+1} \stackrel{\tau^{n+1}}{\twoheadrightarrow}\ P^{n+1}).$$

Proof. (a) We just need to know that $\chi=\alpha[1] : R\to P[1]$ is a map of complexes. Since $\chi^n =\ \tau^{n+1}\circ d_Q^n\circ\sigma^n(r^n)$ one can decompose the action of d_Q on the image of σ^n into the P and R components, by

$$d_Q\sigma^n(r^n)) =\ \tau^{n+1}d_Q\sigma^n(r^n)]\ +\ \sigma^{n+1}d_R(r^n), \qquad r^n\in R^n.$$

Therefore, the differential on Q is given by $(p^n\in P^n,\ r^n\in R^n)$

$$d_Q(\phi^n(p^n)+\sigma^n(r^n)) =\ d_Q(\phi^n(p^n))+d_Q\sigma^n(r^n)) =\ \phi^{n+1}(d_P\ p^n))+d_Q\sigma^n(r^n))$$

$= [\phi^{n+1}(d_P\ p^n))+\tau^{n+1}d_Q\sigma^n(r^n)]\ +\ \sigma^{n+1}d_R(r^n) =\ [\phi^{n+1}(d_P\ p^n))+\chi^n(r^n)]\ +\ \sigma^{n+1}d_R(r^n)$.

Now,

$0 =\ d_Q^2(\phi^n(p^n)+\sigma^n(r^n)) =\ [\phi^{n+2}(d_P^2\ p^n))+d_Q\chi^n(r^n)+\chi^{n+1}\sigma^{n+1}d_R(r^n)]\ +\ \sigma^{n+2}d_R^2(r^n)$

shows that $\chi\circ d_r =\ -d_P\circ\chi =\ d_{P[1]}\circ\chi$.

(b) is clear: Q is isomorphic to the cone of α since $Q^n=\tau^n(P^n)\oplus\sigma^n(R^n)\cong\ P^n\oplus R^n = P^n\oplus(R[-1])^{n+1}$, and via these identification the differential in Q is precisely the cone differential

$$d_Q(p^n\oplus r^n) =\ (d_P\ p^n+\chi^n\ r^n)\oplus d_R\ r^n =\ (d_P\ p^n+\alpha^n\ r^n)\oplus\ -d_{R[-1]}\ r^n.$$

9.3. **The homotopy category $K(\mathcal{A})$ of complexes in $\mathcal{A}$.** We say that two maps of complexes $A \xrightarrow{\alpha,\beta} B$ are homotopic (we denote this $\alpha \mod \beta$), if there is a sequence h of maps $h^n: A^n \to B^{n-1}$, such that

$$\beta - \alpha = dh + hd, \quad i.e., \quad \beta^n - \alpha^n = d_B^{n-1}h^n + h^{n+1}d_A^n.$$

We say that h is a homotopy from α to β.

A map of complexes $A \xrightarrow{\alpha} B$ is said to be a homotopical equivalence if there is a map β in the opposite direction such that $\beta\circ\alpha \equiv 1_A$ and $\alpha\circ\beta \equiv 1_B$.

9.3.1. *Lemma.* (a) Homotopic maps are the same on cohomology.

(b) Homotopical equivalences are quasi-isomorphisms.

(c) A complex A is homotopy equivalent to the zero object iff $1_A = hd + dh$. Then the complex A is acyclic, i.e., $\mathrm{H}^*(A) = 0$.

(d) $\alpha \equiv \beta$ implies $\mu\circ\alpha \equiv \mu\circ\beta$ and $\alpha\circ\nu \equiv \beta\circ\nu$.

Proof. (a) Denote for $a \in Z^n(A)$ by $[a]$ the corresponding cohomology class in $\mathrm{H}^n(A)$. Then $\mathrm{H}(b)[a] - \mathrm{H}(\alpha)[a] = [d_B^{n-1}h^n(a) + h^{n+1}d_A^n(a)] = [d_B^{n-1}(h^n a)] = 0$.

(b) If $\beta\circ\alpha \equiv 1_A$ and $\alpha\circ\beta \equiv 1_B$ then $\mathrm{H}(\beta)\circ\mathrm{H}(\alpha) = \mathrm{H}(\beta\circ\alpha) = \mathrm{H}(1_A) = 1_{\mathrm{H}(A)}$ etc.

(c) The map $0 \xrightarrow{\alpha} A$ with $\alpha = 0$ is a homotopy equivalence if there is a map $A \xrightarrow{\beta} 0$ (then necessarily $\beta = 0$) such that $\beta\circ\alpha \equiv 1_0$ and $\alpha\circ\beta \equiv 1_A$. Since the LHS is always zero, the only condition is that $0 \equiv 1_A$.

(d) If $\beta - \alpha = d_B\circ h + h\circ d_A$ then for $X \xrightarrow{\nu} A \xrightarrow{\alpha} B \xrightarrow{\mu} Y$ one has $\mu\circ\beta - \mu\circ\alpha = d_C\circ(\mu\circ h) + (\mu\circ h)\circ d_A$ etc.

9.3.2. *Homotopy category $K(\mathcal{A})$.* The objects are again just the complexes but the maps are the homotopy classes $[\phi]$ of maps of complexes ϕ

$$\mathrm{Hom}_{K(\mathcal{A})}(A,B) \stackrel{\text{def}}{=} \mathrm{Hom}_{C(\mathcal{A})}(A,B)/\equiv .$$

Now identity on A in $K(\mathcal{A})$ is $[1_A]$ and the composition is defined by $[\beta]\circ[\alpha] \stackrel{\text{def}}{=} [\beta\circ\alpha]$, this makes sense by the part (d) of the lemma.

9.3.3. *Remarks.* (1) Observe that for a homotopy equivalence $\alpha: A \to B$ the corresponding map in $K(\mathcal{A})$, $[\alpha]: A \to B$ is an isomorphism. So we have accomplished a part of our long term goal – we have inverted some quasi-isomorphisms: the homotopy equivalences!

(2) More precisely, we know what are isomorphisms in $K(\mathcal{A})$. The homotopy class $[\alpha]$ of a map of complexes α, is an isomorphism in $K(\mathcal{A})$ iff α is a homotopy equivalence!

9.4. **The triangulated structure of $K(\mathcal{A})$.** $K(\mathcal{A})$ is not an abelian category but there are features that allow us to make similar computations:

- It is an additive category.
- It has shift functors $[n]$.
- It has a class $\mathcal{E}$ of "distinguished triangles" (or "exact triangles"), defined as all triangles isomorphic (in $K(\mathcal{A})$ to a cone of a map of complexes.
- It has cohomology functors $\mathrm{H}^i : K(\mathcal{A}) \to \mathcal{A}$.

9.4.1. *Theorem.* Distinguished triangles have the following properties

- (T0) The class of distinguished triangles is closed under isomorphisms.
- (T1) Any map α appears as the first map in some distinguished triangle.
- (T2) For any object A, triangle $A \xrightarrow{1_A} A \xrightarrow{0} 0 \to A[1]$ is distinguished.
- (T3) (Rotation) If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ is distinguished, so is

$$B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1] \xrightarrow{-\alpha[1]} B[1].$$

- (T4) Any diagram with distinguished rows

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & A[1] \\ {\scriptstyle\mu}\downarrow & & {\scriptstyle\nu}\downarrow & & & & \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & A'[1] \end{array}$$

can be completed to a morphism of triangles

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & A[1] \\ {\scriptstyle\mu}\downarrow & & {\scriptstyle\nu}\downarrow & & {\scriptstyle\eta}\downarrow & & {\scriptstyle\mu[1]}\downarrow \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & A'[1]. \end{array}$$

- (T5) (Octahedral axiom) If maps $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ and the composition $A \xrightarrow{\gamma=\beta\circ\alpha} C$, appear in distinguished triangles
  (1) $A \xrightarrow{\alpha} B \xrightarrow{\alpha'} C_1 \xrightarrow{\alpha''} A[1]$,
  (2) $B \xrightarrow{\beta} C \xrightarrow{\beta'} A_1 \xrightarrow{\beta''} B[1]$,
  (3) $A \xrightarrow{\gamma} C \xrightarrow{\gamma'} B_1 \xrightarrow{\gamma''} C[1]$;
  then there is a distinguished triangle

$$C_1 \xrightarrow{\phi} B_1 \xrightarrow{\psi} A_1 \xrightarrow{\chi} C_1[1]$$

that fits into the commutative diagram

$$
\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\alpha'} & C_1 & \xrightarrow{\alpha''} & A[1] \\
{\scriptstyle =}\big\downarrow & & {\scriptstyle \beta}\big\downarrow & & {\scriptstyle \phi}\big\downarrow & & {\scriptstyle =}\big\downarrow \\
A & \xrightarrow{\gamma} & C & \xrightarrow{\gamma'} & B_1 & \xrightarrow{\gamma''} & A[1] \\
{\scriptstyle \beta}\big\downarrow & & {\scriptstyle =}\big\downarrow & & {\scriptstyle \psi}\big\downarrow & & {\scriptstyle \beta[1]}\big\downarrow \\
B & \xrightarrow{\beta} & C & \xrightarrow{\beta'} & C_1 & \xrightarrow{\beta''} & B[1] \\
 & & & & {\scriptstyle \chi}\big\downarrow & & {\scriptstyle =}\big\downarrow \\
 & & & & C_1[1] & \xleftarrow{\alpha'[1]} & B_1[1]
\end{array}
$$

9.4.2. *Remarks.* (1) Octahedral axiom (T5) is the most complicated and the least used part.

(2) In (T4), the map γ is not unique nor is there a canonical choice. This is a source of some subtleties in using triangulated categories.

Proof. (T0) is a part of the definition of $\mathcal{E}$.

(T1) Any map in $K(\mathcal{A})$, $\alpha \in \mathrm{Hom}_{K(\mathcal{A})}(A, B)$ is a homotopy class $[\alpha]$ of some map of complexes $\alpha \in \mathrm{Hom}_{C(\mathcal{A})}(A, B)$. The cone of α gives a triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} \boldsymbol{C} \xrightarrow{\gamma} A[1]$ in $C(\mathcal{A})$ such that its image in $K(\mathcal{A})$ is a distinguished triangle $A \xrightarrow{[\alpha]} B \xrightarrow{[\beta]} \boldsymbol{C} \xrightarrow{[\gamma]} A[1]$ in $K(\mathcal{A})$ which starts with $[\alpha]$.

(T2) means that the cone $\boldsymbol{C}_{1_A} = C$ of the identity map on A is isomorphic in $K(\mathcal{A})$ to the zero complex, i.e., that the cone $\boldsymbol{C}_{1_A}$ is homotopically equivalent to zero. The homotopy $h^n : C^n \to C^{n-1}$ is simply identity on the common summand and zero on its complement

$$A^n \oplus A^{n+1} \to A^{n-1} \oplus A^n, \quad h^n(a^n \oplus a^{n+1}) \overset{\text{def}}{=} 0 \oplus a^n.$$

For that we calculate

$$(d_C^{n-1} h^n + h^{n+1} d_C^n)(a^n \oplus a^{n+1}) = \ d_C^{n-1}(0 \oplus a^n) \ + \ h^{n+1}[(d_A a^n + 1_A a^{n+1}) \oplus - d_A^{n+1} a^{n+1})]$$

$$= \ (1_A a^n \oplus - d_A a^n) \ + \ 0 \oplus (d_A a^n + 1_A a^{n+1}) = \ a^n \oplus a^{n+1}.$$

(T4) The obvious strategy is to lift the first diagram in (T4) to the level of $C(\mathcal{A})$. First, one can replace the rows with isomorphic ones which are cone triangles $A \xrightarrow{\alpha_0} B \xrightarrow{\overset{\bullet}{\alpha}_0} \boldsymbol{C}_{\alpha_0} \xrightarrow{\overset{\bullet\bullet}{\alpha}_0} A[1]$ and $A' \xrightarrow{\alpha_0'} B' \xrightarrow{\overset{\bullet}{\alpha}_0'} \boldsymbol{C}_{\alpha_0'} \xrightarrow{\overset{\bullet\bullet}{\alpha}_0'} A'[1]$ for maps α_0, α_0' in $C(\mathcal{A})$ (these maps are some representatives of homotopy classes α, α'). So, the diagram takes form (for any representatives μ_0, ν_0 of μ, ν)

$$\begin{array}{ccccccc}
A & \xrightarrow{[\alpha_0]} & B & \xrightarrow{[\overset{\bullet}{\alpha}_0]} & \boldsymbol{C}_{\alpha_0} & \xrightarrow{[\overset{\bullet\bullet}{\alpha}_0]} & A[1] \\
{\scriptstyle [\mu_0]}\downarrow & & {\scriptstyle [\nu_0]}\downarrow & & & & \\
A' & \xrightarrow{[\alpha'_0]} & B' & \xrightarrow{[\overset{\bullet}{\alpha}{}'_0]} & \boldsymbol{C}_{\alpha'_0} & \xrightarrow{[\overset{\bullet\bullet}{\alpha}{}'_0]} & A'[1]
\end{array}$$

It would be nice to lift the diagram completely into $C(\mathcal{A})$ in the sense that we look for representatives μ_0, ν_0 of μ, ν so that the diagram in $C(\mathcal{A})$ still commutes, then a representative η_0 of η would simply come from the functoriality ("naturality") of the cone construction. However we will make piece with the homotopic nature of the diagram and incorporate the homotopy corrections. Homotopical commutativity $[\nu_0]\circ[\alpha_0] = [\alpha'_0]\circ[\mu_0]$ means that one has maps $h^n : B^n \to (\boldsymbol{C}_{\alpha'_0})^{n-1}$ such that

$$\nu_0\circ\alpha_0 - \alpha'_0\circ\mu_0 = dh + hd.$$

Now, we construct a map $\boldsymbol{C}_{\alpha_0} \xrightarrow{\eta_0} \boldsymbol{C}_{\alpha'_0}$ such that the diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha_0} & B & \xrightarrow{\beta_0} & \boldsymbol{C}_{\alpha_0} & \xrightarrow{\gamma_0} & [A1] \\
 & & {\scriptstyle \nu_0}\downarrow & & {\scriptstyle \eta_0}\downarrow & & {\scriptstyle \mu_0}\downarrow \\
A' & \xrightarrow{\alpha'_0} & B' & \xrightarrow{\beta'_0} & \boldsymbol{C}_{\alpha'_0} & \xrightarrow{\gamma'_0} & A'[1]
\end{array}$$

commutes in $C(\mathcal{A})$, by

$$\eta_0 : B^n \oplus A^{n+1} \to (B')^n \oplus (A')^{n+1}, \qquad b^n \oplus a^{n+1} \mapsto (\nu_0 b^n + h^{n+1} a^{n+1}) \oplus \mu_0 a^{n+1}.$$

(T3) says that if one applies rotation to any cone triangle in $C(\mathcal{A})$, $A \xrightarrow{\alpha} B \xrightarrow{\beta} \boldsymbol{C}_\alpha \xrightarrow{\gamma} A[1]$, the resulting triangle

$$B \xrightarrow{\beta} \boldsymbol{C}_\alpha \xrightarrow{\gamma} A[1] \xrightarrow{-\alpha[1]} B[1]$$

is isomorphic in $K(\mathcal{A})$ to the cone triangle

$$B \xrightarrow{\beta} \boldsymbol{C}_\alpha \xrightarrow{\phi} \boldsymbol{C}_\beta \xrightarrow{\mu} B[1].$$

This requires a homotopy equivalence $A \xrightarrow{\zeta} \boldsymbol{C}_\beta$ such that the following diagram commutes in $K(\mathcal{A})$:

$$\begin{array}{ccccccc}
B & \xrightarrow{\beta} & \boldsymbol{C}_\alpha & \xrightarrow{\gamma} & A[1] & \xrightarrow{-\alpha[1]} & B[1] \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle \zeta}\downarrow & & {\scriptstyle =}\downarrow \\
B & \xrightarrow{\beta} & \boldsymbol{C}_\alpha & \xrightarrow{\phi} & \boldsymbol{C}_\beta & \xrightarrow{\mu} & B[1].
\end{array}$$

We define the map

$$A^{n+1} = (A[1])^n \xrightarrow{\zeta^n} (\boldsymbol{C}_\beta)^n = (B^n \oplus A^{n+1}) \oplus B^{n+1}, \quad a^{n+1} \mapsto 0 \oplus a^{n+1} \oplus -\alpha(a^{n+1});$$

and in the opposite direction $\xi(b^n \oplus a^{n+1} \oplus b^{n+1}) \stackrel{\text{def}}{=} a^{n+1}$. It suffices to check that

(1) ζ and ξ are maps of complexes,
(2) $\xi \circ \zeta = 1_{A[1]}$,
(3) $\zeta \circ \xi \equiv 1_{\boldsymbol{C}_\beta}$,
(4) $\zeta \circ \gamma = \phi$
(5) $\mu \circ \zeta = -\alpha[1]$

In (3), the homotopy h such that $1_{\boldsymbol{C}_\beta} - \zeta \circ \xi = dh + hd$, is the map $h^n : (\boldsymbol{C}_\beta)^n \to (\boldsymbol{C}_\beta)^{n-1}$ given by $B^n \oplus A^{n+1} \oplus B^{n+1} \ni b^n \oplus a^{n+1} \oplus b^{n+1} \mapsto 0 \oplus 0 \oplus b^n \in B^{n-1} \oplus A^n \oplus B^n$.

(T5) is a description of a certain "complicated" relation between cones of maps and compositions of maps.

9.4.3. *Triangulated categories.* These are additive categories with a functor [1] (called shift) and a class of distinguished triangles $\mathcal{E}$, that satisfy the conditions $(T0 - T5)$.

So, $K(\mathcal{A})$ is our first triangulated category.

9.5. **Long exact sequence of cohomologies.**

9.5.1. *Lemma.* Any distinguished triangle $X \to Y \to Z \to X[1]$ gives a long exact sequence of cohomologies

$$\cdots \to \mathrm{H}^i(X) \to \mathrm{H}^i(Y) \to \mathrm{H}^i(Z) \to \mathrm{H}^{i+1}(X) \to \cdots .$$

Proof. Up to isomorphism in $K(\mathcal{A})$, we can replace the triangle $X \to Y \to Z \to X[1]$ with the homotopy image of a cone triangle $X \xrightarrow{\alpha} Y \xrightarrow{\dot{\alpha}} \boldsymbol{C}_\alpha \xrightarrow{\ddot{\alpha}} X[1]$ in $C(\mathcal{A})$. We have observed that this triangle contains a short exact sequence of complexes $0 \xrightarrow{\alpha} Y \xrightarrow{\dot{\alpha}} \boldsymbol{C}_\alpha \xrightarrow{\ddot{\alpha}} X[1] \to 0$, and the short exact sequence of complexes does indeed provide a long exact sequence of cohomologies.

The proof was based on the relation of

9.6. **Exact (distinguished) triangles and short exact sequences of complexes.** By definition distinguished triangles in $K(\mathcal{A})$ come from maps $A \xrightarrow{\alpha} B$ in $C(\mathcal{A})$. We will just restate it as:

- distinguished triangles in $K(\mathcal{A})$ come from short exact sequences in $C(\mathcal{A})$ that split on each level.

9.6.1. *Lemma.* (a) Any short exact sequence of complexes in $C(\mathcal{A})$

$$0 \to P \to Q \to R \to 0$$

which splits on each level, defines a distinguished triangle in $K(\mathcal{A})$ of the form

$$P \to Q \to R \xrightarrow{-\chi} P[1].$$

(b) Any distinguished triangle in $K(\mathcal{A})$ is isomorphic to one that comes from a short exact sequence of complexes that splits on each level.

(c) Explicitly, in (a) the map χ in $C(\mathcal{A})$ comes from a choice of splittings $Q^n \cong P^n \oplus R^n$. The map $R^n \xrightarrow{\chi^n} P^{n+1}$ is the component of the differential $Q^n \xrightarrow{d_Q^n} Q^{n+1}$, i.e.,

$$\chi^n = (R^n \hookrightarrow P^n \oplus R^n \cong Q^n \xrightarrow{d_Q^n} Q^{n+1} P^{n+1} \cong P^{n+1} \oplus R^{n+1} \twoheadrightarrow P^{n+1}).$$

Proof. We know that the short exact sequence associated to a cone of a map splits canonically on each level (lemm 9.2.2). In the opposite direction, by lemma 9.2.3 any short exact sequence of complexes

$$0 \to P \xrightarrow{\phi} Q \xrightarrow{\psi} R \to 0$$

with a splitting $Q^n \cong P^n \oplus R^n$ on each level, defines a canonical cone triangle in $C(\mathcal{A})$

$$R[-1] \xrightarrow{\alpha} P \xrightarrow{\phi = \dot{\alpha}} Q \xrightarrow{\psi = \ddot{\alpha}} (R[-1])[1],$$

hence an exact triangle in $K(\mathcal{A})$.

Since we are in $K(\mathcal{A})$ we can rotate this triangle backwards using the property (T3) to get an exact triangle $P \to R \to Q \to P[1]$. Now the two procedures of going between short exact sequences in $C(\mathcal{A})$ and exact triangles in $K(\mathcal{A})$ are "inverse to each other".

The formulas in (c) come from lemma 9.2.3c.

9.7. **Extension of additive functors to homotopy categories.** The following lemma is quite obvious:

9.7.1. *Lemma.* (a) There is a canonical functor $C(\mathcal{A}) \to K(\mathcal{A})$ which sends each complex A to itself and each map of complexes ϕ to its homotopy class $[\phi]$.

(b) Any additive functor between abelian categories $\mathcal{A} \xrightarrow{F} \mathcal{B}$ extends to a functor $C(\mathcal{A}) \xrightarrow{C(F)} C(\mathcal{B})$, here $[C(F)A]^n = F(A^n)$ and the differential $d^n_{C(F)}$ is $F(d^n)$.

(c) Moreover, $C(\mathcal{A}) \xrightarrow{C(F)} C(\mathcal{B})$, factors to a functor $K(\mathcal{A}) \xrightarrow{K(F)} K(\mathcal{B})$, i.e., there is a unique functor $K(F)$ such that

$$\begin{array}{ccc} C(\mathcal{A}) & \xrightarrow{C(F)} & C(\mathcal{B}) \\ \downarrow & & \downarrow \\ K(\mathcal{A}) & \xrightarrow{K(F)} & K(\mathcal{B}). \end{array}$$

It is the same as $C(F)$ on objects and the action of $K(F)$ on homotopy classes of maps of complexes comes from the action of $C(F)$ on maps of complexes.

9.8. **Projective resolutions and homotopy.**

9.8.1. *Lemma.* If we have two complexes

$$\cdots \to P^{-n} \to \cdots \to P^0 \to a \to 0 \to \cdots$$

$$\cdots \to B^{-n} \to \cdots \to B^0 \to b \to 0 \to \cdots$$

such that all P^k are projective and the second complex is exact, then any map $\alpha : a \to b$ lifts to a map $P \xrightarrow{\phi} B$, i.e.,

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P^{-n} & \longrightarrow & \cdots & \longrightarrow & P^0 & \longrightarrow & a & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & {\scriptstyle\phi^{-n}}\downarrow & & & & {\scriptstyle\phi^0}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle=}\downarrow & & \\ \cdots & \longrightarrow & B^{-n} & \longrightarrow & \cdots & \longrightarrow & B^0 & \longrightarrow & b & \longrightarrow & 0 & \longrightarrow & \cdots. \end{array}$$

(b) Any two such lifts are homotopic.

Proof. (a) Since $\varepsilon_B : B^0 \to B$ is surjective and P^0 is projective, the map $P^0 \xrightarrow{\alpha\circ\varepsilon_A} B$ factors thru ε_B, i.e., it lifts to $P^0 \xrightarrow{phi^0} B^0$.

Actually, $\phi^0 \circ d_A{}^{-1}$ goes to $\mathrm{Ker}(\varepsilon_B)$ since

$$\varepsilon_B(\phi^0 d_A{}^{-1}) = (\alpha\circ\varepsilon_A)d_A = \alpha\circ 0 = 0.$$

Exactness of the second complex, shows that $d_B{}^{-1}$ gives a surjective map $d_B{}^{-1} : B^{-1} \to \mathrm{Ker}(\varepsilon_B)$. So, since P^{-1} is projective $\phi^0 \circ d_A{}^{-1}$ factors through $d_B{}^{-1}$, giving a map $\phi^{-1} : P^{-1} \to B^{-1}$, such that $\phi^0 \circ d_A{}^{-1} = d_B{}^{-1} \circ \phi^{-1}$.

In this way we construct all ϕ^n inductively.

(b) If we have another solution ψ then

$$\varepsilon_B \phi^0 = \alpha\circ\varepsilon_A = \varepsilon_B \psi^0$$

gives $\varepsilon_B(\phi^0 - \psi^0) = 0$, hence

$$(\phi^0 - \psi^0)P^0 \subseteq \mathrm{Ker}(\varepsilon_B) = d^{-1}{}_B(B^{-1}).$$

Now, since P^0 is projective, map $\phi^0 - \psi^0 : P^0 \to Im(d_B{}^{-1})$ lifts to $h^0 : P^0 \to B^{-1}$, i.e.,

$$\phi^0 - \psi^0 = d_B{}^{-1} \circ h^0.$$

One continuous similarly

$$d_B{}^{-1} \circ (\phi^{-1} - \psi^{-1}) = (\phi^0 - \psi^0) \circ d_A{}^{-1} = d_B{}^{-1} \circ h^0,$$

hence

$$Im[(\phi^{-1} - \psi^{-1}] - d_B{}^{-1} \circ h^0 \subseteq \mathrm{Ker}(D_B{}^{-1}) = Im(d_B^{-2}).$$

So,

$$\phi^{-1} - \psi^{-1} = h^{-1} \circ d_A^{-2}$$

for some map $h^{-1} : P^{-1} \to B^{-2}$. Etc.

9.8.2. *Corollary.* (a) If P and Q are projective resolutions of objects a and b in $\mathcal{A}$, then any map $a \to b$ lifts uniquely to a map $P \to Q$.

(b) Any two projective resolutions of the same object of $\mathcal{A}$ are canonically isomorphic in $K(\mathcal{A})$.

9.9. Derived functors $LF : \mathcal{A} \to K^-(\mathcal{B})$ and $RG : \mathcal{A} \to K^+(\mathcal{B})$.

9.9.1. *Lemma.* If $\mathcal{A}$ has enough projectives:

(1) There is a canonical projective resolution functor $\mathcal{P} : \mathcal{A} \to K^-(\mathcal{A})$.
(2) For any additive right exact functor $F : \mathcal{A} \to \mathcal{B}$ its left derived functor $LF : \mathcal{A} \to K^{\leq 0}(\mathcal{B})$ is well defined by replacing objects with their projective resolutions

$$LF(A) \stackrel{\text{def}}{=} F(\mathcal{P}(A))$$

and its zero cohomology is just the original functor F:

$$\mathrm{H}^0[(LF)(A) \cong F(A).$$

Proof. (1) is clear from the corollary and then (2) follows.