HOMOLOGICAL ALGEBRA

Contents

Date: ?

So Okada has suggested a number of corrections.

Part 0. Announcements

0. What does the Homological algebra do?

Homological algebra is a general tool useful in various areas of mathematics. One tries to apply it to constructions that morally should contain more information then meets the eye. Homological algebra, when it applies, produces "derived" versions of the construction ("the higher cohomology"), which contain the "hidden" information. In a number of areas, the fact that that with addition of homological algebra one is not missing the less obvious information allows a development of superior techniques of calculation.

The goal of this course is to understand the usefulness of homological ideas in applications. and as usual, to use this process as an excuse to visit various interesting topics in mathematics.

0.1. Some examples of applications.

Some basic applications :

(1) **Algebraic topology.** It can loosely be described as a "systematic way of counting holes in manifolds". While we can agree that a circle has a 1-dimensional hole (in the sense of "a hole that can be made by a one dimensional object") and a sphere has a 2-dimensional hole, algebraic topology finds that the surface of a pretzel has one 2-dimensional hole and four 1-dimensional holes. These "holes" or "cycles" turn out to be essential in problems in geometry and analysis.

(2) **Cohomology of sheaves.** It deals with an omnipresent problem of relating local and global information on a manifold.

(3) **Subtle objects.** A classical example is that when one passes from smooth spaces to singular spaces the correct analogue of some standard objects become certain *complexes* of objects, i.e., the homological algebra generalizations of standard objects. Examples: *dualizing sheaves* in algebraic geometry and *perverse sheaves* in topology, are not really sheaves but complexes of sheaves. So, in general calculations on singular spaces usually require homological algebra just to start.

(4) **Subtle spaces.** In order to organize interesting objects such as all curves or all vector bundles on a given curve, into a mathematically meaningful space, one requires an extension of a notion of a space. In contemporary physics the basic objects of string theory – the D-branes – are expected to be highly sophisticated constructs of homological algebra.

Some special examples:

- **Dual of a module over a ring.** The dual V^* of a real vector space V is the space of linear maps from V to real numbers. If one tries to do the same for a module M over a ring k (say the ring of integers), it does not work as well since M^* can often be zero. However in the "derived" world the construction works as well as for vector spaces.

- **Algebraic analysis of linear differential equations** is based on the observation that any map between two spaces allows you to move a system of linear equations on one of the spaces to the other. These operations become most useful after passing to their derived versions.
- Group cohomology, Galois cohomology, Lie algebra cohomology,...
- Deformation theory.
- etc.

0.2. **Topics.**

- Algebraic topology.
- Duality of abelian groups.
- Derived functors, Ext and Tor.
- Solutions of linear differential equations with singularities.
- Sheaves and cohomology of sheaves.
- Derived categories.

Possible advanced topics: (1) Differential graded algebras, (2) n-categories. (3) Extended notions of a space: stacks and dg-schemes. (4) Homotopical algebra.

0.3. **The texts.**

(1) Weibel, Charles A., *An introduction to homological algebra*, Cambridge University Press, Cambridge [England] ; New York : Cambridge studies in advanced mathematics 38; ISBN/ISSN 0521435005.
(2) Gelfand, S. I., Manin Y. I., *Methods of homological algebra*, Springer, Berlin ; New York : ISBN/ISSN 3540547460 (Berlin) 0387547460 (New York).

These are very different books. Weibel's book deals with a more restricted subject, so it is less exciting but seems fairly pleasant to read once one knows what one wants from homological algebra.

Manin and Gelfand Jr. are top mathematicians and their book is full of exciting material from various areas, and it points towards hot developments. For that reason (and a laconic style of Russian mathematics), it is also more difficult.

Part 1. Intro

This part is an announcement for ideas that we will revisit in more detail.

1. Algebraic topology: Homology from triangulations

The *subject of algebraic topology* is measuring properties of shapes i.e., of topological spaces. In a simple example of a circle our "measurements" give the following observations: (0) it is connected, and (1) it has a hole. We will find a systematic approach (the *homology* of topological spaces) to finding such properties for more complicated topological spaces. This involves the following steps:

- In order to extract information about a topological space X we make a choice of additional data – a triangulation $\mathcal{T}$, i.e., we break X into oriented simplices (points, intervals, triangles,...).
- Some information on a triangulated space $(X, \mathcal{T})$ is then encoded into an algebraic construct, a *complex* $C_\bullet(X, \mathcal{T}; \Bbbk)$ of modules over a ring $\Bbbk$ (such as $\mathbb{Z}, \mathbb{Z}/n, \mathbb{Q}, \mathbb{R}, \mathbb{C}$). Here,
  1. modules C_i simply count the simplices of dimension i in the triangulation and
  2. the boundary maps between them encode the way the simplices are attached to form X.
- The modules C_i are not really interesting, they are large because they involve not only the interesting space X but also an auxiliary choice of a triangulation $\mathcal{T}$ of X. Now one distills is the interesting information. This is the *homology* $H_\bullet(X, \mathcal{T}; \Bbbk)$ of the complex $C_\bullet(X, \mathcal{T}; \Bbbk)$.
- It turns out that the homology is an *invariant* of the topological space X – though it was calculated using the extra information of a triangulation $\mathcal{T}$ it does not depend on the choice of $\mathcal{T}$ but only on the space X. So we may drop $\mathcal{T}$ from the notation and call it the

  Homology $H_\bullet(X; \Bbbk)$*. of the topological space* X *with coefficients in* $\Bbbk$

One obvious application of invariants I such as homology is that they may be used to distinguish objects, If for two topological spaces X, Y the invariants $I(X)$ and $I(Y)$ are different then the spaces are different, i.e., there is no homeomorphism between them. However, the use of invariants is much deeper. For instance one can integrate differential forms over homology classes.

We will start with the basic atoms of the theory, very simple topological spaces called *simplices*. We will see how to describe topological spaces in terms of simplices and then we will use such descriptions to calculate (co)homology of simplices. This will involve an algebraic idea of *complexes of abelian groups*.

1.1. Linear Simplices.

The idea of a *combinatorial topology* is to describe a given topological space as being glued from very simple topological spaces, appropriately called *simplices*. The gluing rules are combinatorial objects and so one can study the original topological space in terms of the gluing combinatorics.

Simplices are familiar objects: there is one in each dimension and 0-simplex is a point, 1-simplex is a closed interval, 2-simplex is a triangle, 3-simplex is a pyramid, etc. A systematic approach is given by the construction of

1.1.1. *Standard and linear simplices.* The standard n-simplex is $\sigma_n \subseteq \mathbb{R}^{n+1}$, the convex closure $\sigma_n = \ conv\{e_0, ..., e_n\}$ of the standard basis of $\mathbb{R}^{n+1}$. So, σ_n is in the first "quadrant" $x_i \geq 0$, and there it is given by the hyperplane $\sum\ x_i = 1$.

More generally, we say that a *linear i-simplex* in a real vector space is the convex closure $conv(v_0, ..., v_i)$ of a set $\mathcal{V} = \{v_0, ..., v_i\}$ of $i+1$ vectors which lie in an i-dimensional affine subspace but do not lie in any $(i-1)$-dimensional affine subspace.[1] We say that $\mathcal{V}$ is the set of *vertices* of the simplex $conv(\mathcal{V})$ and we often denote by $\sigma_{\mathcal{V}} \overset{\text{def}}{=} conv(\mathcal{V})$ the simplex with vertices $\mathcal{V}$. So, an i-simplex has $(i+1)$ vertices.

1.1.2. *Barycentric coordinates on a linear simplex $\sigma_{\mathcal{V}}$.* These are defined through the following lemma

Lemma. (a) Any point x in the simplex can be written as $x = \sum\ x_i v_i$ with $x_i \geq 0$ and $\sum\ x_i = 1$. Barycentric coordinates x_i are unique.

(b) One can recover vertices from a linear simplex $\sigma_{\mathcal{V}}$ as the the points with all but one coordinate zero.

(c) A bijection between vertices of two linear simplices extends canonically to a homeomorphism. For instance, an ordering of $\mathcal{V}$ gives a canonical identification $\sigma_{\mathcal{V}} \cong \ \sigma_{|\mathcal{V}|-1}$.

Proof. (a) In general, we can always translate a simplex into another one with $v_0 = 0$. Now, if $v_0 = 0$ then $v_1, ..., v_i$ have to be independent. So $x = \sum_0^i\ x_p v_p = \sum_1^i\ x_p v_p$ hence $x_p,\ p > 0$ are determined by x, and then so is $x_0 = 1 - \sum_1^i\ x_p$.

(b) is clear. The homeomorphism in (c) is specified by requiring that the coordinates are the same (for the given bijection of vertices).

1.1.3. *Facets, faces, interior.* The *facets* of the simplex $conv(\mathcal{V})$ are the simplices associated to subsets of the set of vertices – any subset $W \subseteq \mathcal{V}$ defines a facet of $conv(\mathcal{V})$ which is the simplex $conv(W)$. The facets are closed under intersections: $conv(W') \cap conv(W'') = conv(W' \cap W'')$.

The facets of codimension 1 are called *faces*. The *interior* $\sigma_{\mathcal{V}}^o$ of a linear simplex $conv(\mathcal{V})$ consists of the points with all $x_i > 0$.

[1] For instance $e_0, ..., e_n$ lie in the n-dimensional hyperplane $\sum\ x_i = 1$ but not in any $(n-1)$-dimensional affine subspace.

1.1.4. *Orientations of simplices.* We say that an ordering of the set of vertices $\mathcal{V}$ of a simplex $\sigma_{\mathcal{V}}$ gives an *orientation* of the simplex; and that two orderings give the same orientation if they differ by an even permutation of vertices. So, an orientation of a simplex is an orbit of the group of even permutation of vertices in the set of all orderings of vertices. We denote the set of orientations of an i-simplex σ by or_σ. Notice it has two elements for $i > 0$ and one for $i = 0$. We denote by $\alpha \mapsto \overline{\alpha}$ the operation of the change of triangulation of oriented simplices (for $i = 0$ it does not do anything).

Notice the parallel with the notion of orientation in a vector space given by a basis ordered up to even permutation or by a top form $dx^1 \wedge \cdots \wedge dx^n$ given by an ordering of coordinates up to even permutations. We will write an oriented simplex as $\sigma_{v_0 \cdots v_n}$ indicating that the orientation is given by the ordering $v_0 < \cdots < v_n$. So $\sigma_{abc} = \sigma_{bca} \neq \sigma_{bac}$. In lower dimensions this notion of orientation of simplices agrees with our intuition of orientation, say oriented simplex σ_{ab} means a segment with vertices a, b and an arrow from a to b, etc.

Standard simplices σ_n (and their facets) have *standard orientation* given by the ordering $e_0 < \cdots < e_n$, we write this ordering from the right to the left as $(e_n, ..., e_0)$.

1.2. **Topological simplices.** A *topological i-simplex* is a pair (S, ϕ) of a topological space S and a homeomorphism $\phi : \sigma_{\mathcal{V}} \to S$ with a linear i-simplex. For simplicity we usually omit ϕ from notation. Notice that the above notions of vertices, facets, coordinates, interior, orientation are defined for topological simplices via ϕ. For instance, facets of topological simplices are again topological simplices.

We sometimes denote the faces of a topological n-simplex S by S^i, $i \in \mathcal{V}$, where S^i is obtained by throwing out the vertex i.

1.3. **Triangulations.** The idea of *triangulation* is to present a given topological space as a combination of simple spaces – the simplices. Then we will extract the information on X from the way the simplices are patched together.

There is a number of versions of the idea of a triangulation:

- *simplicial triangulation* is a notion of a triangulation with certain properties **SC1-3** below that make it very easy to describe how simplices fit together to form the space X - everything is stated in terms of the set of vertices. The information about simplices and how they glue is encoded in a combinatorial object called the *simplicial complex* However, the price for the properties **SC1-3** is that in practice one needs a large number of simplices.
- A more loose notion of a *CW-complex* allows using few simplices, but makes the description of how they fit together more subtle. It is stated in terms i-cells in X, i.e., maps $\sigma_i \xrightarrow{\phi} X$ such that the restriction to the interior is a homeomorphism onto the image.
$$\sigma_i^o \xrightarrow{\cong} \phi(\sigma_i^o) \subseteq X.$$

1.3.1. *Simplicial triangulations.* A triangulation $\mathcal{T}$ of a topological space X is a covering $\mathcal{T}$ of X by topological simplices $\alpha \in \mathcal{T}$, such that

- (ST1) facet of simplices in $\mathcal{T}$ are again simplices in $\mathcal{T}$,
- (ST2) if $\alpha, \beta \in \mathcal{T}$ and $\alpha \subseteq \beta$ then α is a facet of β,
- (ST3) for any $\alpha, \beta \in \mathcal{T}$ the intersection $\alpha \cap \beta$ is $\emptyset$ or a simplex in $\mathcal{T}$.

We will denote by $\mathcal{T}_i$ the subset of i-simplices in $\mathcal{T}$.

Now observe that

Lemma. (a) A non-empty intersection of simplices α, β is a facet of both α, β.

(b) A simplex in $\mathcal{T}$ is determined by its vertices.

Proof. (a) follows from (ST3) and (ST2). (b) If two simplices α and β have the same set of vertices $\mathcal{V}$. Then $\alpha \cap \beta$ is $\neq \emptyset$ so it is a simplex $Y \in \mathcal{T}$ which is a facet of α and of β. However a facet that contains all vertices has to be the simplex itself.

1.3.2. *The idea of simplicial complexes.* This means that the way the simplices are attached will be completely described in terms of the combinatorics of the set of vertices $\mathcal{T}^0$. For that reason one can encode a simplicial triangulation as a combinatorial structure: a set $\mathcal{V}$ (set of all vertices in $\mathcal{T}$), endowed by a family $\mathcal{K}$ of subsets of $\mathcal{V}$ – the family of sets of vertices of all simplices in $\mathcal{T}$. We saw that for each simplex $Y \in \mathcal{T}$ the set of its vertices is a subset of $\mathcal{V}$, and the mutual position of two simplices in $\mathcal{T}$ is recorded in the intersection of the sets of their vertices).

This leads to an abstraction:

1.4. **Combinatorial Topology of Simplicial Complexes.** We will see how to describe some topological spaces in combinatorial terms. This will then be used to calculate their invariants purely algebraically using the combinatorics of the space rather then the space itself.

1.4.1. *Simplicial Complexes.* A *simplicial complex* is a set $\mathcal{V}$ together with a family $\mathcal{K}$ of finite non-empty subsets of $\mathcal{V}$ such that with any element $A \in \mathcal{K}$, family $\mathcal{K}$ also contains all subsets of A.

Lemma. (a) Any simplicial triangulation $\mathcal{T}$ defines a simplicial complex $\mathcal{K}(\mathcal{T})$.

(b) To any simplicial complex $\mathcal{K}$ we can associate a topological space $|\mathcal{K}|$ called its *realization.* It comes with a triangulation $\mathcal{T}$ such that $\mathcal{K}(\mathcal{T})$ is naturally identified with $\mathcal{K}$.

Proof. Procedure (a) has been described above. In (b) we start by associating to each finite set $A \in \mathcal{K}$ a topological simplex σ_A with vertices A (i.e., with vertices parameterized by A). This gives a topological space $\widetilde{\mathcal{K}} \stackrel{\text{def}}{=} \sqcup_{A \in \mathcal{K}}\ \sigma_A$, the disjoint union of all simplices

σ_A. Then the topological space $|\mathcal{K}|$ is obtained as a quotient $\widetilde{\mathcal{K}}/\sim$ of $\mathcal{K}$ by the equivalence relation $\sim$ on X given by $x\in\sigma_A$ and $y\in\sigma_B$ are equivalent if (i) x lies in the facet $\sigma_{A,A\cap B}$ of σ_A given by the subset $A\cap B\subseteq A$, (i) y lies in $\sigma_{B,A\cap B}\subseteq\sigma_B$ and (iii) the coordinates of x and y with respect to the set of vertices $A\cap B$ are the same (i.e., x and y are identified by the canonical identification of topological simplices $\sigma_{A,A\cap B}$ and $\sigma_{B,A\cap B}$ given by the obvious identification of the sets of vertices of these two simplices).

Notice that the canonical map $\pi:\widetilde{\mathcal{K}}\twoheadrightarrow|\mathcal{K}|$ is injective on each simplex $\sigma_A\subseteq\widetilde{\mathcal{K}}$ and gives a homeomorphism $\pi_A:\sigma_A\to\pi(\sigma_A)$. So, one can identify the image with σ_A and then σ_A's cover $|\mathcal{K}|$ and one can check that they form a triangulation $\mathcal{T}$ of $\mathcal{K}$.

Theorem. If we start with a triangulated topological space $(X,\mathcal{T})$ then the realization $|\mathcal{K}(\mathcal{T})|$ of the corresponding simplicial complex $\mathcal{K}(\mathcal{T})$ is canonically homeomorphic to X.

Proof. It is easy to construct a continuous map $\pi:|\mathcal{K}\to X$ for $\mathcal{K}=\mathcal{K}(\mathcal{T})$. Since $|\mathcal{K}|$ has a quotient topology from $\widetilde{\mathcal{K}}$ such map is the same as a continuous map $\widetilde{\pi}:\widetilde{\mathcal{K}}\to X$ such that $x\sim y\Rightarrow\widetilde{\pi}(x)=\widetilde{\pi}(y)$. Now, for any simplex $\alpha\in\mathcal{T}$ I will denote by $\mathcal{V}_\alpha$ its set of vertices. Then the simplices $\sigma_{\mathcal{V}_\alpha}\subseteq\widetilde{\mathcal{K}}$ and $\alpha\subseteq X$ can be canonically identified since the sets of vertices are the same. This gives a map $\widetilde{\pi}:\widetilde{\mathcal{K}}\to X$ – subsets $A\in\mathcal{K}$ are of the form $\mathcal{V}_\alpha$ for some $\alpha\in\mathcal{T}$ and then $\sigma_A=\sigma_{\mathcal{V}_\alpha}\xrightarrow{\cong}\alpha\subseteq X$. Since $\widetilde{\pi}$ is continuous on each σ_A it is continuous on the disjoint union $\widetilde{\mathcal{K}}$.

1.4.2. *Triangulations of spheres.* To describe a triangulation of S^1 we choose an orientation of S^1 and n distinct points $A_1,...,A_n$ that go in the direction of the orientation. The triangulation is given by 0-simplices $\mathcal{T}_0=\{A_1,...,A_n\}$ and and 1-simplices $\mathcal{T}_1=\{\sigma_{A_1A_2},...,\sigma A_nA_1\}$ (I denote by σ_{AB} or just AB the segment from A to B).

If $n=1$ this is not a simplicial complex since A_1A_1 is not really a 1-simplex by our definition – it is a circle hence not homeomorphic to σ_1. [(2)] $n=2$ still does not give a simplicial complex since the intersection $\sigma A_2A_1\cap\sigma_{A_1A_2}$ consists of two points so it is not a simplex. For $n\geq 3$ we do get a simplicial complex. The associated simplicial complex has vertices $\mathcal{V}=\{A_1,...,A_n\}$ and $\mathcal{K}=\ \{A_1,...,A_n;\{A_1A_2\},...,\{A_n,A_1\}\ \}$.

For any finite set $\mathcal{V}$ with n elements $\mathcal{K}=\{A\subseteq\mathcal{V};\ A\neq\emptyset\}$ is a simplicial complex. Its realization $|\mathcal{K}|$ is the simplex $\sigma_\mathcal{V}$ of dimension $|\mathcal{V}|$. However, if we remove the largest simplex: $\mathcal{L}=\{A\subseteq\mathcal{V};\ \mathcal{V}\neq A\neq\emptyset\}$ the realization is the boundary of $\sigma_\mathcal{V}$, i.e., a sphere of dimension $|\mathcal{V}|-1$.

For instance an obvious triangulation of S^2 is given by four points a,b,c,d on the sphere, six segments between pairs of points and four triangles in vertices in 3 out of 4 points, so $\mathcal{K}=\{a,b,c,d,\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$.

[2]However, it is a 1-cell and $\mathcal{T}=\{A_1,A_1A_1\}$ is an efficient CW-complex.

1.5. **Complex $C_*(X,\mathcal{T};\Bbbk)$.** Our first goal is to encode a triangulation $\mathcal{T}$ algebraically. In order to pass from topological spaces to linear algebra we make a choice of a *coefficient ring* $\Bbbk$ so that we calculate in the linear algebra of $\Bbbk$-modules. The set of simplices will be encoded as a basis of a $\Bbbk$-module $C_\bullet(X,\mathcal{T};\Bbbk)$ of $\Bbbk$-valued chains in X. The *boundary operator* $\partial : C_\bullet(X,\mathcal{T};\Bbbk) \to C_\bullet(X,\mathcal{T};\Bbbk)$ will encode the way the simplices in $\mathcal{T}$ are glued in X,

1.5.1. *Coefficients.* In order to pass from topological spaces to algebra we make a choice of a *coefficient ring* $\Bbbk$. $\Bbbk$ can be any abelian group but we usually choose it to be a commutative ring. The most interesting case is $\Bbbk = \mathbb{Z}$ but for now we will be happy with $\Bbbk$ a field such as $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

1.5.2. *Oriented triangulations.* An *oriented triangulation* Σ on a topological space X is a pair $(\mathcal{T}, o)$ of a triangulation $\mathcal{T}$ of X and a choice o_α of an orientation of each simplex $\alpha \in \mathcal{T}$. So one can say that Σ is a family of oriented simplices which give a triangulation once orientation is forgotten.

1.5.3. *Chains.* We will define the space of i-chains $C_i(X,\mathcal{T};\Bbbk)$ for any triangulation $\mathcal{T}$ of a space X. A choice of an orientation o for the triangulation $\mathcal{T}$ will then give a simpler way of thinking of groups C_i.

The space of i-chains for an oriented triangulation (X,Σ) is the free $\Bbbk$-module

$$C_i = C_i(X,\Sigma;\Bbbk) \stackrel{\text{def}}{=} \oplus_{\alpha\in\Sigma^i}\ k\alpha$$

with the basis given by the set of i-simplices Σ^i in the oriented triangulation Σ.

To define the space of i-chains for a triangulation $(X,\mathcal{T})$ we start with the free $\Bbbk$-module $\widetilde{C}_i(X,\mathcal{T};\Bbbk) = \ \oplus_{\alpha\in\mathcal{T}^i,\ o_\alpha\in \mathrm{or}_\alpha}\ k\alpha$ with the basis given by all i-simplices α with all possible choices of orientations o_α. Then $C_i(X,\mathcal{T};\Bbbk)$ is the quotient of $\widetilde{C}_i(X,\mathcal{T};\Bbbk)$ obtained by imposing $\overline{\sigma} = (-1)\cdot\sigma$ for oriented i-simplices $\sigma = (\alpha, o_\alpha)$ with $i > 0$.

We see that a choice of an orientation o for a triangulation $\mathcal{T}$ identifies group $C_i(X,\mathcal{T};\Bbbk)$ with the same construction $C_i(X,\mathcal{T},o;\Bbbk)$ for the oriented triangulation $(\mathcal{T}, o)$ since the composition $C_i(X,\mathcal{T},o;\Bbbk)\subseteq\widetilde{C}_i(X,\mathcal{T};\Bbbk)\twoheadrightarrow C_i(X,\mathcal{T};\Bbbk)$ is an isomorphism.

1.5.4. *Boundary operator $\partial : C_i \to C_{i-1}$.* We start with some examples of what a boundary should be for oriented simplices in lower dimension. A point has no boundary $\partial\sigma_a = 0$. For an oriented segment σ_{ab} the boundary is "target-source", i.e., $\partial_{\sigma_{ab}} = \sigma_b - \sigma_a$ (as in the fundamental theorem of calculus $\int_a^b\ f' = f(b) - f(a)$). For a triangle σ_{abc} with vertices a, b, c and the orientation given by ordering abc the boundary is a triangle with the induced orientation, i.e., $\sigma_{ab} + \sigma_{bc} + \sigma_{ac}$. If we rewrite it as $\partial\sigma_{abc} = \sigma_{ab} - \sigma_{cb} + \sigma_{ac}$ we get an idea of how to define the boundary for any oriented simplex by algebraic formula

$$\partial_i\sigma_{v_0\cdots v_i} = \sum_{0\le p\le i} (-1)^i\ \sigma_{v_0\cdots\widehat{v_i}\cdots v_i},$$

where $\widehat{v_i}$ means that we omit v_i. This is indeed a sum of all faces of $\sigma_{v_0\cdots v_i}$ with orientations given by the ordering $v_0\cdots\widehat{v_i}\cdots v_i$ and the sign $(-1)^i$.

Lemma. (a) The above formula for ∂_i gives a well defined $\mathbb{k}$-map $\partial_i : C_i(X,\mathcal{T};\mathbb{k}) \to C_{i-1}(X,\mathcal{T};\mathbb{k})$.

Proof.

(1) First one checks that the formula only depends on the orientation. For instance for two orderings xyz and zxy which give the same orientation one has $\partial\sigma_{zxy} = \sigma_{xy} - \sigma_{zy} + \sigma_{zx}$ and $\partial\sigma_{xyz} = \sigma_{yz} - \sigma_{xz} + \sigma_{xy}$ coincide.

Now we have defined a map from the the basis of $\widetilde{C}_i$ to C_{i-1}, i.e., a $\mathbb{k}$-linear map $\widetilde{C}_i \to C_{i-1}$.

(2) Next one needs to check hat the map descends to $C_i \to C_{i-1}$, i.e., that opposite orientations produce opposite results. For instance for two orderings xyz and yxz which give opposite orientations one has $\partial\sigma_{yxz} = \sigma_{xz} - \sigma_{yz} + \sigma_{yx}$ which is the opposite of $\partial\sigma_{xyz} = \sigma_{yz} - \sigma_{xz} + \sigma_{xy}$.

The two requirements together say that for any permutation τ of $0,...,i$ one has $\partial\sigma_{v_{\tau 0}\cdots v_{\tau i}} = \varepsilon_\tau \cdot \sigma_{v_0\cdots v_i}$ where ε_τ is the sign of the permutation τ. This statement it suffices to check when τ is one of the transpositions τ_p which exchange $p-1$ and p, $1 \le p \le i$.

1.5.5. *Remark.* The above formula for ∂ is for the complex associated to a triangulation $\mathcal{T}$. If one uses an oriented triangulation $\Sigma = (\mathcal{T}, o)$ then one can adjust the formula so then one needs no extra orientations of simplices in $\mathcal{T}$. The boundary operator $\partial_i : C_i \longrightarrow C_{i-1}$ sends an oriented i-simplex $Y \in \Sigma$ to the sum of its faces, with certain orientation and a certain sign. For a given face Z if the orientation from $\partial_i Y$ agrees with orientation on Z from Σ we do not need any adjustments, otherwise we change the orientation from $\partial_i Y$ to the one from Σ and change the sign.

1.5.6. $\partial^2 = 0$. A simplex σ of dimension $i > 0$ has a boundary (in the topological sense) which is a sphere of dimension $i-1$. Now this sphere has no boundary points! So, our topological intuition says that $del{\circ}\partial$ should "kill" simplices and therefore also everything that is glued from simplices. This agrees with algebra:

Lemma. $\partial_i\partial_{i-1} = 0$.

Remark. This observation is the origin of homological algebra. The above lemma was the inspiration to define the

1.5.7. *Algebraic notion of a complex.* A *complex of cochains* is a sequence of $\Bbbk$-modules and maps

$$\cdots \xrightarrow{\partial^{-2}} C^{-1} \xrightarrow{\partial^{-1}} C^{0}\cdots \xrightarrow{\partial^{0}} C^{1} \to\cdots,$$

such that the compositions of *coboundary operators* ∂^i are zero: $\partial^{i+1}\partial^i = 0,\ i \in \mathbb{Z}$. We often omit the index on the coboundary operator, so we can write the preceding requirement as $\partial^2 = 0$.

From a complex of cochains we get three sequences of $\Bbbk$-modules

- i-cocycles $Z^i \overset{\text{def}}{=} \mathrm{Ker}(\partial^i) \subseteq C^i$,
- i-coboundaries $B^i \overset{\text{def}}{=} Im(\partial^{i-1}) = Im(\partial^{i-1}) \subseteq C^i$,
- i-cohomologies $H^i \overset{\text{def}}{=} Z^i/B^i$,

Here we used $B^i \subseteq Z^i$ which follows from $\partial\partial = 0$.

A *complex of chains* is the same thing but with maps going down

$$\cdots \xleftarrow{\partial_{-1}} C_{-1} \xleftarrow{\partial_0} C_0\cdots \xleftarrow{\partial_0} C_1 \to\cdots.$$

In this case we lower the indices and we talk of i-cycles $Z_i \subseteq C_i$, i-boundaries $B_i \subseteq C_i$, and i-homologies $H_i \overset{\text{def}}{=} Z_i/B_i$.

The difference off two notions is only in terminology and notation. One can pass from a complex of cochains $(C^\bullet, d^b u)$ to a complex of chains $(C_\bullet, d_\bullet)$ by $C_i \overset{\text{def}}{=} C^{-i}$ and then $d_i : C_i \to C_{i-1}$ is defined as $d^{-i} : C^{-i} \to C_{-i+1}$. When we just say "complex" we usually mean "complex of cochains".

1.5.8. *Corollary.* $C_*(X, \mathcal{T}; \Bbbk)$ is a complex (of chains).

Proof. This is the above lemma 1.5.6.

1.5.9. *Homology groups of a topological space.* We have seen that any triangulation $\mathcal{T}$ of X associates to a topological space X the homology groups

$$H_i(X, \mathcal{T}; \Bbbk) \ \overset{\text{def}}{=}\ H_i[\big(C_\bullet(X, \mathcal{T}; \Bbbk), \partial\big)].$$

However, by the next theorem these groups are really invariants of X itself so we call them the homology groups of X and denote them by $H_i(X, \Bbbk)$.

Theorem. The homology groups $H_i(X, \mathcal{T}; \Bbbk)$ do not depend on the choice of a triangulation $\mathcal{T}$, in the sense that for any two triangulations of X there is a canonical isomorphism

$$\phi_{\mathcal{T}'',\mathcal{T}'} :\ H_i(X, \mathcal{T}'; \Bbbk) \xrightarrow{\cong} H_i(X, \mathcal{T}''; \Bbbk).$$

Proof. We say that a triangulation $\mathcal{S}$ is a refinement of a triangulation $\mathcal{T}$ if for each $\alpha \in \mathcal{T}$ the subset $\mathcal{S}_\alpha = \ \{\sigma \in \mathcal{S};\ \sigma \subseteq \alpha\}$ is a triangulation of α. Now the theorem follows from the following lemma

Lemma. (a) For a refinement $\mathcal{S}$ of $\mathcal{T}$ there is a canonical isomorphism $H_i(X,\mathcal{T};\Bbbk) \xrightarrow{\cong} H_i(X,\mathcal{S};\Bbbk)$ obtained by sending $\alpha \in \mathcal{T}_i$ with orientation u to $\sum_{\sigma\in\mathcal{S}_\alpha\cap\mathcal{S}_i} (\sigma, o|\sigma)$ where $o|\sigma$ is the orientation o restricted to σ.

(b) Any two triangulations $\mathcal{T}', \mathcal{T}''$ of X have a common refinement $\mathcal{T}$.

1.6. **Examples.**

1.6.1. S^n *for* $n = 1$ *or* 2.

1.6.2. *Torus* T^2. We can view T^2 as a quotient of a rectangle, this makes the drawing of triangles easier. There is a simple CW-triangulation where one divides the rectangle by a diagonal into two triangles. It gives a fast calculation of homology.

One can get a simplicial complex, for instance by dividing the rectangle into nine rectangles and each of these into two triangles. Then H_0 and H_2 are easy and the dimension of H_1 can be computed from the invariance of Euler characteristic under taking homology (Homework 2.2).

Of a particular interest is a basis of H_1 – one can see that it corresponds to two "main" circles on the torus. Classically such basis controls the indeterminacy of elliptic integrals. In the modern algebraic geometry one says that such basis produces the so called periods, the basic invariants of elliptic curves.

1.6.3. *Dependence on coefficients.* Integers are the universal coefficient ring, i.e., integral homology (with integer coefficients) has the most information. Passing to $\mathbb{Q}$ or $\mathbb{Z}/n\mathbb{Z}$ in general kills some information and therefore – if we view on the positive side – it leads to simpler computations giving some (partial) information.

1.6.4. *Example.* S^3 is the unit sphere $S\subseteq\mathbb{R}^4$ which we can think of as $\mathbb{C}^2$. Then $S = \{x \in \mathbb{R}^4;\ (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1\}$ can be written as $S = \{z \in \mathbb{C}^2;\ |z^1|^2 + |z^2|^2 = 1\}$. This point of view makes it obvious that the group $\mathbb{T} \cong S^1$ of unit complex numbers acts on S by $z\cdot(z^1, z^2) = (zz^1, zz^2)$. This is a free action (i.e., there are no stabilizers), and the quotient is homeomorphic to S^2. The quotient map $S \to S^2$ is called *Hopf map.* This is one basic example of a nontrivial fibration: all fibers are homeomorphic (to S^1) but the map is still quite nontrivial. We will revisit the Hopf map once we acquire the machinery of spectral sequences.

However let us consider the quotients S/μ_n where $\mu_n\subseteq\mathbb{T}$ is the group of all n^{th} roots of unity in $\mathbb{C}$. Then $H_*(S/\mu_n;\mathbb{R})$ is naturally identified with $H_*(S;\mathbb{R})$ and the same is true for homology with coefficients in $Z/m\mathbb{Z}$ as long as m is prime to n. However when m is not prime to n then $H_*(S/\mu_n;\mathbb{Z}/m\mathbb{Z})$ is more complicated then $H_*(S;\mathbb{R})$. All such complications (for all m's) are already stored in $H_*(S/\mu_n;\mathbb{Z})$.

One can check the above statements using simplicial triangulations, however it will be much easier to do it with the machinery of sheaves. It provides a systematic use of *maps* in calculating homology.

2. **Duality for modules over rings**

2.1. **Capturing a class of objects in terms of a smaller and better behaved subclass.** This is the basic idea of homological algebra. We will describe it here on the example of constructing a reasonable notion of a dual for modules over a given ring $\Bbbk$.

2.1.1. *The problem.* The construction of a dual vector space over a field equally makes sense for modules over any ring.[3] However this "naive" notion of duality is not very useful since it does not have the standard properties of the duality for vector spaces.

2.1.2. *The idea.* The resolution of the problem starts with the easy observation that the naive duality still works well on some modules – the free modules. The nontrivial idea is that any module can be captured (described) in terms of finitely generated free modules. This is achieved by the notion of a *resolution.* Now the correct notion of duality is obtained by applying the naive duality not directly to the module, but to its resolution, i.e., a description in terms of free modules. The effect is that all computations are done with free modules and therefore the new duality has all good properties that the naive duality had on free modules.

2.1.3. *Machinery involved in realizing the above program.* Replacing modules by resolutions is done by passing from modules to *complexes of modules.* It is in this larger world that we find the hidden parts of "naive" constructions. There are two steps:

(1) thinking of abelian groups as complexes in degree 0,
(2) "Identifying" some complexes, in particular a module should be identified with its resolution.

These steps mean that we change twice the realms (categories) in which calculate :

$$\mathfrak{m}(\Bbbk) \xrightarrow{(1)} \mathcal{C}^*(\mathfrak{m}(\Bbbk)) \xrightarrow{(2)} D(\mathfrak{m}(\Bbbk)).$$

We start in the category of $\Bbbk$-modules $\mathfrak{m}(\Bbbk)$ and expand to the category of complexes of $\Bbbk$-modules $\mathcal{C}^*(\mathfrak{m}(\Bbbk))$, and then we pass to a more subtle *derived category* $D(\mathfrak{m}(\Bbbk))$ of $\Bbbk$-modules. (Objects are still complexes but there are more morphisms.)

Step (1) allows one to think of any $\Bbbk$-module in terms of particularly nice modules (say, the free modules). Step (2) introduces the optimal setting $D(\mathfrak{m}(\Bbbk))$, which makes it precise what I mean by "identifying some complexes".

2.2. **Rings.** Some of the classes of interesting rings $\Bbbk$

- fields such as $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ or the finite fields $\mathbb{F}_q$ with q elements,
- $\mathbb{Z}$ (related to number theory, i.e., to everything),
- smooth functions $C^\infty(M)$ on a manifold M (related to differential geometry),

[3] One just has to pay attention to the difference between left and right modules.

- polynomial functions $\mathcal{O}(\mathbb{A}^n) = \mathbb{C}[x_1, ..., x_n]$ (related to algebraic geometry),
- Differential operators D_M on M (related to *linear* differential equations).

By a module we mean a left module unless specified differently. Let $\mathfrak{m}(\Bbbk) = \mathfrak{m}^l(\Bbbk)$ be the category of left $\Bbbk$-modules and $\mathfrak{m}^r(\Bbbk)$ the right $\Bbbk$-modules.

2.3. **Duality and biduality of $\Bbbk$-modules.** We start with some linear algebra over a ring $\Bbbk$ – the properties of the "naive" duality d.

We will denote by ${}_{\Bbbk}\Bbbk$ the set $\Bbbk$ viewed as a left $\Bbbk$-module via the left multiplication, by $\Bbbk_{\Bbbk}$ the set $\Bbbk$ viewed as aright $\Bbbk$-module via the right multiplication, and by by ${}_{\Bbbk}\Bbbk_{\Bbbk}$ the set $\Bbbk$ viewed as a $(\Bbbk, \Bbbk)$-bimodule.

We will usually state and prove claims for left modules $\mathfrak{m}(\Bbbk) = \mathfrak{m}^l(\Bbbk)$, the analogous statements for right $\Bbbk$-modules are then obvious since $\mathfrak{m}^r(\Bbbk) = \mathfrak{m}^l(\mathfrak{k}^o)$ for the opposite ring $\Bbbk^o$.

2.3.1. d^l *and* d^r. The dual of a left $\Bbbk$-module M is the space $d(M) = M^*$ of linear functionals

$$M^* \overset{\text{def}}{=} \text{Hom}_{\Bbbk}(M, \Bbbk) \overset{\text{def}}{=} \{f : M \to \Bbbk;\, f(c{\cdot}m) = c{\cdot}f(m)\ \&\ f(m'+m'') = f(m')+f(m''),\ c \in \Bbbk,\ m, m', m'' \in M\}.$$

Duality operation d is defined similarly for right module N, here one asks that $f(n{\cdot}c) = f(n){\cdot}n$. When we deal with left and right modules we may denote by d^l and d^r the duality operations on left and right modules.

The duality construction is a *functor*, i.e., it is defined not only on $\Bbbk$-modules but also on maps of $\Bbbk$-modules; the dual of $f : M_1 \to M_2$ is the adjoint map $d(f) = f^* : M_2^* \to M_1^*$, $f^*(\nu)\ m = \ \langle \nu, fm \rangle,\ m \in M_1, \nu \in M_2^*$.

Lemma. $\mathfrak{m}^l(\Bbbk) \xrightarrow{d^l} \mathfrak{m}^r(\Bbbk)$ and $\mathfrak{m}^r(\Bbbk) \xrightarrow{d^r} \mathfrak{m}^l(\Bbbk)$.

2.3.2. *Biduality maps* ι_M.

Lemma. (a) For $M \in \mathfrak{m}^l(\Bbbk)$, the canonical map $\iota_M : M \to (M^*)^*$ is well defined by $\iota_M(m)\ (\lambda) = \ \langle \lambda, m \rangle,\ m \in M,\ \lambda \in M^*$.

(b) If $\Bbbk$ is a field and $M \in \mathfrak{m}_{fd}(\Bbbk)$ (i.e., M is a finite dimensional vector space over $\Bbbk$), the biduality map ι_M is an isomorphism.

Remarks. We call ι_M the *biduality map* for M. Claim (b) is an essential part of our experience with duality. Our main goal in this section is to force this to be true for modules over any ring $\Bbbk$.

2.3.3. *Duality for free modules.* Biduality is not always isomorphism (see 2.4), and now we distinguish a class of modules for which this is true. We start with $M = \Bbbk$:

Lemma. For the module ${}_{\mathbb{k}}\mathbb{k} \in \mathfrak{m}^l(\mathbb{k})$:

(a) The map that assigns to $a \in \mathbb{k}_{\mathbb{k}}$ the operator of right multiplication $R_a: {}_{\mathbb{k}}\mathbb{k} \to {}_{\mathbb{k}}\mathbb{k}$, $x \stackrel{\text{def}}{=} x\cdot a$, gives an isomorphism of right $\mathbb{k}$-modules $\mathbb{k}_{\mathbb{k}} \xrightarrow{R} ({}_{\mathbb{k}}\mathbb{k})^*$.

(b) $\iota_{{}_{\mathbb{k}}\mathbb{k}}$ is an isomorphism.

When this is pushed a little further, we get a nice class of modules for which $M \xrightarrow{\cong} (M^*)^*$:

Proposition. ι_M is an isomorphism for any finitely generated free $\mathbb{k}$-module.

Proof. It follows from the lemma 2.3.3b, and from

2.3.4. *Sublemma.* For two $\mathbb{k}$- modules P, Q;

- (a) $(P \oplus Q)^* \cong P^* \oplus Q^*$,
- (b) map $\iota_{P\oplus Q}$ is an isomorphism iff both ι_P and ι_Q are isomorphisms.

2.4. **What is the dual of the abelian group $\mathbb{Z}_n$?** We start by noticing that the duality operation $M \mapsto M^*$ is not very good for arbitrary modules M of any ring $\mathbb{k}$. Even when $\mathbb{k}$ is a field, biduality is an isomorphism only for the finite dimensional vector spaces. Therefore, for general $\mathbb{k}$ the duality can have "best" properties only on the subcategory $\mathfrak{m}_{fg}(\mathbb{k})$ of finitely generated $\mathbb{k}$-modules.

A more serious problem is encountered in the example when $\mathbb{k} = \mathbb{Z}$ the ring of integers and $M = \mathbb{Z}_n$ is a torsion module. For $\mathbb{k} = \mathbb{Z}$, category of $\mathbb{Z}$-modules is just the category of abelian groups: $\mathfrak{m}(\mathbb{Z}) = \mathcal{A}b$. So we have the notion of a dual of an abelian group $M^* \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{A}b}(M, \mathbb{Z})$. However, for $M = \mathbb{Z}_n \stackrel{\text{def}}{=} \mathbb{Z}/n\mathbb{Z}$ one has $M^* = 0$, so duality loses all information.

On this example we will develop our strategy of describing modules in terms of a subclass of free modul;es which behaves well under duality.

2.4.1. *The passage from $\mathbb{Z}_n$ to its resolution $P^\bullet$.* We know that biduality works for the abelian group $M = \mathbb{Z}$ (by 2.3.3), and $\mathbb{Z}_n$ is clearly intimately related to $\mathbb{Z}$. The quotient map $\mathbb{Z} \xrightarrow{q} \mathbb{Z}_n$ relates $\mathbb{Z}_n$ to $\mathbb{Z}$, however it does not tell the whole story – the difference between $\mathbb{Z}_n$ and $\mathbb{Z}$ is in the kernel $\mathrm{Ker}(q) = n\mathbb{Z}$. However, the inclusion $n\mathbb{Z} \subseteq \mathbb{Z}$ captures the definition of $\mathbb{Z}_n$ as $\mathbb{Z}/n\mathbb{Z}$, and since the abelian group $n\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ by $\mathbb{Z} \ni x \mapsto nx \in n\mathbb{Z}$, we will replace $n\mathbb{Z}$ by $\mathbb{Z}$ in this map. Then it becomes the multiplication map $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$.

Now we can think of $\mathbb{Z}_n$ as encoded in the map $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$. For a more complicated $\mathbb{k}$-module such encoding will be more complicated, the proper setting will turn out to require to think of $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$ as a complex $P^\bullet = \ (\cdots \to 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0 \to \cdots)$, with $\mathbb{Z}'s$ in degrees -1 and 0.

So we have passed from $\mathbb{Z}_n$ to a complex $P\bullet$. Now we need to know how to dualize it.

2.4.2. *Duality operation on complexes.* The dual of a complex of $\Bbbk$-modules $C^\bullet = (\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots)$ is the complex $dC^\bullet$ obtained by applying d to modules and maps. Since d is contravariant (i.e., it changes directions), we will also have to change the indexing. So, $(dC^\bullet)^n \overset{\text{def}}{=} d(C^{-n})$ and $d^n_{dC^\bullet}$ is the adjoint of $d^{-n-1}_{C^\bullet}$.

In order to calculate $dP^\bullet$ we will need

2.4.3. *Sublemma.* (a) $\Bbbk$-linear maps between left modules $\Bbbk^r$ and $\Bbbk^s$ can be described in terms of right multiplication by matrices. Precisely, if we denote for $A \in M_{r,s}(k)$ by R_A the right multiplication operator $\Bbbk^r \ni x \mapsto xA \in \Bbbk^s$ on row-vectors, then $M_{rs} \xrightarrow{R} \mathrm{Hom}_\Bbbk(\Bbbk^r, \Bbbk^s)$ is an isomorphism.

(b) The adjoint of R_A is the left multiplication $L_{A^{tr}}$ with the transpose of A (acting on column-vectors).

2.4.4. *Biduality is an isomorphism on complexes over the subcategory of complexes over* $\mathfrak{m}_{fg,free}(\Bbbk) \subseteq \mathfrak{m}(\Bbbk)$. Let $\mathcal{P} \overset{\text{def}}{=} \mathfrak{m}_{fg,free}(\Bbbk)$ be the category of all free finitely generated $\Bbbk$-modules.

Lemma. The biduality map $\iota_{C^\bullet}$ is an isomorphism for any complex $C^\bullet$ in $\mathcal{C}^*(\mathcal{P})$.

Proof. Observe that $(ddC^\bullet)^n = dd(C^n)$, and define the map $\iota_{C^\bullet} : C^\bullet \to ddC^\bullet$ as the collection of maps $\iota_{C^n} : C^n \to ddC^n$. If all C^n are in $\mathcal{P}$ then all maps ι_{C^n} are isomorphisms and hence so is $\iota_{C^\bullet}$.

2.4.5. *The derived dual Ld.* On $\Bbbk$-modules we define the *left derived duality operation Ld* by

$$Ld(M) \overset{\text{def}}{=} dP^\bullet$$

for any resolution $P^\bullet$ of M by free modules.

Let us see what this means for $\Bbbk = \mathbb{Z}$ and $M = \mathbb{Z}_n$. When we identify $d\mathbb{Z}$ with $\mathbb{Z}$ then the adjoint of the map $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$ is again $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$ (see 2.4.3). From the point of view of complexes this says that $dP^\bullet$ is the complex $\cdots \to 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0 \to \cdots$, but this time $\mathbb{Z}$'s are in degrees 0 and 1 ! So $dP^\bullet \cong P^\bullet[-1]$ where one denotes by $C^\bullet[n]$ the shift of the complex $C^\bullet$ by n places to the left.

Now observe hat it is natural to identify any module N with a complex (again denoted N), which has N in degree 0 and all other terms zero (hence all maps are zero). So, since we have also identified $\mathbb{Z}_n$ with $P^\bullet$, we should identify the smart dual $Ld(\mathbb{Z}_n) \overset{\text{def}}{=} dP^\bullet \cong P^\bullet[-1]$ with $\mathbb{Z}_n[-1]$. So,

$$Ld(\mathbb{Z}_n) = \mathbb{Z}_n[-1] = \text{the shift of } \mathbb{Z}_n \text{ by one to the right}.$$

This is the complex which has $\mathbb{Z}_n$ in degree 1 and all other terms zero.

2.4.6. *Conclusion.* The smart dual of $\mathbb{Z}_n$ is not a module but a complex in degrees ≥ 0. The fact that $H^0[Ld(\mathbb{Z}_n)] = 0$ corresponds to the fact that the naive definition of the dual gives $d(\mathbb{Z}_n) = 0$. So, the naive definition does not see the hidden part of the dual which is $H^1[Ld(\mathbb{Z}_n)] = \mathbb{Z}_n$.

Observe that since the computation of the derived dual is in the setting of complexes of free finitely generated modules, the biduality works (by 2.4.4), so the canonical map $\mathbb{Z}_n \xrightarrow{L\iota_{\mathbb{Z}_n}} (Ld)(Ld)(\mathbb{Z}_n)$ is an isomorphism.

In 2.5 and 2.7 we will make precise some ideas used in the calculation of $Ld(\mathbb{Z}_n)$. We will concentrate on formulations which will allow us to apply these ideas in many situations.

2.5. **Resolutions.** Here we repeat for any module M what we have been able to do for $\mathbb{Z}_n$. The precise, formal, solution of our wish to describe a module M in terms of maps between some (hopefully nicer) modules P^n. is the notion of a resolution of M.

2.5.1. *Exact complexes and short exact sequence.* A complex of $\Bbbk$-modules $C^\bullet = (\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots)$ is said to be *exact* if all of its cohomologies vanish, i.e., inclusion $B^n \subseteq Z^n$ is equality. (Exact complexes are also called exact sequences.)

Examples. Consider the meaning of exactness for complexes with few terms :

(1) If in the complex $C^\bullet$ all terms $C^n,\ n \neq 0$ are zero. Then $C^\bullet$ is exact iff $C^0 = 0$.
(2) If $C^n = 0$ for $n \neq -1, 0$, then $C^\bullet$ is exact iff $d^{-1} : C^{-1} \to C^0$ is an isomorphism.
(3) If $C^n = 0$ for $n \neq -1, 0, 1$, then $C^\bullet$ is exact iff $d^{-1} : C^{-1} \to C^0$ is injective, $d^0 : C^0 \to C^1$ is surjective, and in C^0 one has $\mathrm{Ker}(d^0) = Im(d^{-1})$. So all exact complexes with three terms (and such complexes are also called *short* exact sequences), are of the following form: module C^0 has a submodule C^{-1} and the quotient $C^1 = C^0/C^{-1}$.

2.5.2. *Resolutions.* A *left resolution* of a module M is an exact complex

$$\cdots \to \underset{-2}{P^{-2}} \to \underset{-1}{P^{-1}} \to \underset{0}{P^0} \xrightarrow{q} \underset{1}{M} \to 0 \to \cdots$$

(the numbers beneath are the positions in the complex). For example $\cdots \to 0 \to \underset{-1}{\mathbb{Z}} \xrightarrow{n} \underset{0}{\mathbb{Z}} \xrightarrow{q} \underset{1}{\mathbb{Z}_n} \to 0 \to \cdots$ is a resolution of $\mathbb{Z}_n$.

The notion of a resolution has a nice meaning in the world of complexes. For this we will think of both the module M and of its resolution as complexes. We start by finishing the construction of the world of complexes:

2.5.3. *Category of complexes.* A morphisms of complexes $f : A^\bullet \longrightarrow B^\bullet$ is a system of maps f^n of the corresponding terms in complexes, which "preserves" the differential in the sense that in the diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A^{-2} & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & \cdots \\
\downarrow & & {\scriptstyle f^{-2}}\downarrow & & {\scriptstyle f^{-1}}\downarrow & & {\scriptstyle f^{0}}\downarrow & & {\scriptstyle f^{1}}\downarrow & & \downarrow \\
\cdots & \longrightarrow & B^{-2} & \longrightarrow & B^{-1} & \longrightarrow & B^0 & \longrightarrow & B^1 & \longrightarrow & \cdots
\end{array}$$

all squares commute in the sense that two possible ways of following arrows give the same result: $f^n \circ d_A^{n-1} = d_B^{n-1} \circ f^{n-1}$, for all n; i.e., $f \circ d = d \circ f$.

Now we have a category of complexes of $\Bbbk$-modules $\mathcal{C}^\bullet[\mathfrak{m}(\Bbbk)]$: objects are complexes and morphisms are maps of complexes.

Lemma. (a) This is a category.

(b) Constructions Z^n, B^n, H^n are functors from $\mathcal{C}^\bullet(\mathfrak{m}(\Bbbk))$ to $\mathfrak{m}(\Bbbk)$.

2.5.4. *Modules as complexes.* To each module M we can associate a very simple complex $M^\#$ which is M in degree zero and 0 in other degrees (so all maps are zero). (However, we will usually denote $M^\#$ just by M again.)

Lemma. This gives a functor

$$\mathfrak{m}(\Bbbk) \ \longrightarrow \mathcal{C}^*(\mathfrak{m}(\Bbbk)), \quad M \mapsto M^\#$$

which is fully-faithful, i.e., $\mathfrak{m}(\Bbbk)$ is a full subcategory of $\mathcal{C}^*(\mathfrak{m}(\Bbbk))$.

2.5.5. *Resolutions as maps of complexes.* We will also use the terminology *resolution* for the equivalent data of a complex $P^\bullet = (\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow 0 \longrightarrow \cdots)$, together with the map $q : P_0 \rightarrow M$.

However, we can now think of resolutions completely in terms of complexes by viewing the map q as a morphism of complexes

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & {\scriptstyle q}\downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

It remains to encode the exactness of $\cdots \rightarrow P^{-1} \rightarrow P^0 \rightarrow M \rightarrow 0 \rightarrow \cdots$ in terms of complexes. For this we introduce the notion of

2.5.6. *Quasi-isomorphisms.* We say that a map of complexes $f : A^\bullet \longrightarrow B^\bullet$ is a quasi-isomorphism ("qis") if the induced maps of cohomology groups $H^n(f) : \ H^n(A^\bullet) \longrightarrow H^n(B^\bullet)$, $n \in \mathbb{Z}$, are all isomorphisms.

Lemma. A left resolution of M is the same as a quasi-isomorphism of complexes $P^\bullet \to M^\#$ such that $P^i = 0$ for $i > 0$.

Proof. If $(P^\bullet, q)$ is a resolution of M then the only non-zero cohomology group of $P^\bullet$ is $H^0(P^\bullet) \cong M$, the same is true for $M^\#$. Moreover, the morphism of complexes $P^\bullet \to M^\#$ is given by $q : P^0 \to M$ which induces isomorphism of $H^0(P^\bullet) = P^0/dP^{-1}$ onto $M = H^0(M\#)$.

Remark. Now it is clear how to define a right resolution – as a quasi-isomorphism of complexes $M^\# \to I^\bullet$ such that $I^i = 0$ for $i < 0$.

2.6. **Free resolutions.** We consider subcategories of $\Bbbk$-modules $\mathcal{P} = \mathcal{P}^l \overset{\text{def}}{=} \mathfrak{m}_{fg,free}(\Bbbk) \subseteq \mathfrak{m}^l{}_{fg}(\Bbbk) \subseteq \mathfrak{m}^l(\Bbbk)$ consisting of free modules and of finitely generated modules. The intersection is $\mathcal{F}ree_{fg}{}^l(\Bbbk)$. Now we can say that a free resolution of M is a resolution $P^\bullet$ such that all P^i are free $\Bbbk$-modules, etc.

Lemma. Any module M has a free resolution.

Proof. (1) *There is a free module F and a surjective map $F \twoheadrightarrow M$* ("a free cover of M"). For this we choose any set $\mathcal{G} \subseteq M$ of generators of M (for instance $\mathcal{G} = M$), and let F be the free $\Bbbk$-module with the basis $\mathcal{G}$.

(2) Let $P^0 \xrightarrow{q} M$ be the map $F \to M$ from (1). If q has no kernel, we are done – we choose $P^k = 0,\ k < 0$. Otherwise we use again (1) to choose a free cover $P^{-1} \to \mathrm{Ker}(q)$, then ∂^{-1} is the composition $P^{-1} \to \mathrm{Ker}(q) \subseteq P^0$. Etc.

Lemma. If the ring $\Bbbk$ is noetherian any finitely generated module M has a resolution by free finitely generated modules.

Proof. (1) If $\Bbbk$-module M is finitely generated a free finitely generated module F and a surjective map $F \twoheadrightarrow M$. This is as before, except that we can now choose a set $\mathcal{G} \subseteq M$ of generators of M, to be a finite set.

To repeat the step (2), we need the new modules to be covered, such as $\mathrm{Ker}(q) \subseteq P^0$, $\mathrm{Ker}(\partial^{-1}) \subseteq P^{-1}, \ldots$ are finitely generated. That's what the noetherian assumption means: we say that a ring $\Bbbk$ is noetherian if any submodule of a finitely generated module is finitely generated.

2.6.1. *Projective modules.* We say that a $\Bbbk$-module P is projective if it is a summand of a free $\Bbbk$-module. So, free modules are projective and we get a larger class $\mathcal{P}roj(\Bbbk) \supseteq \mathcal{F}ree(\Bbbk)$. However it has the same good properties of free modules (so projective resolutions are as good as free resolutions!).

For instance: if P is a summand of a finitely generated free $\Bbbk$-module then the biduality map ι_P is again an isomorphism by sublemma 2.3.4.

2.6.2. *Homological dimension of a ring and finiteness of resolutions.* The homological dimension of $\Bbbk$ is the infimum of all n such that any $\Bbbk$-module has a projective resolution of length $\leq n$ (i.e., $P^i = 0,\ i < -n$).

Examples. Any field has dimension 0, $\mathbb{Z}$ has dimension one and $\mathbb{C}[x_1, ..., x_n]$ has dimension n. However the ring $\mathcal{O}(Y) = \mathbb{C}\oplus xy\mathbb{C}[x,y]\subseteq \mathbb{C}[x,y]$ of functions on the crossing $Y = \{xy = 0\}\subseteq\mathbb{A}^2$ has infinite dimension (because Y has a singularity).

2.7. **Derived category of $\Bbbk$-modules.** Making the definition of the derived version of duality:

$$Ld(M) \overset{def}{=} d(P^\bullet) \textit{ for any free resolution } P^\bullet \textit{ of } M;$$

completely correct, depends on resolving two problems:

(1) existence of a free resolution $P^\bullet$ of M,
(2) independence of the choice of a free resolution $P^\bullet$.

The first one has already been dealt with. For the second one recall that a resolution is a quasi-isomorphism $P^\bullet \to M\#$. Our problem would disappear if this quasi-isomorphism were an isomorphism since we would be replacing $M\#$ with an isomorphic object. So our problem will be resolved if we can find a setting in which all quasi-isomorphisms in $\mathcal{C}^\bullet(\mathfrak{m}(\mathbb{K}))$ become isomorphisms. Such setting exists, the so called *derived category* of $\Bbbk$-modules $D(\mathfrak{m}(\Bbbk))$.

The passage from $\mathcal{C}^\bullet(\mathfrak{m}(\Bbbk))$ to $D(\mathfrak{m}(\Bbbk))$ requires inverting all quasi-isomorphisms in $\mathcal{C}^\bullet(\mathfrak{m}(\Bbbk))$. This can be done either by (i) universal abstract construction of inverting morphisms in a category, or (ii) using some convenient subcategory of $\mathfrak{m}(\Bbbk)$. We will eventually do both since both ideas are useful in applications.

For the approach (i) we will first recall the solution of an analogous problem in rings rather then categories:

2.7.1. *Localization of rings.* Localization of a ring A with respect to a subset $S\subseteq A$ is the ring A_S obtained by inverting all elements of S. More precisely, localization of a ring A with respect to $S\subseteq A$ is a pair of a ring A_S and a map of rings $A \xrightarrow{i} A_S$ such that $i(S)\subseteq (A_S)^* \overset{\text{def}}{=}$ the set of invertible elements of A_S. There may be many such pairs, and so we have to be still more precise, it is the universal such pair (i.e., the best such pair), in the sense that for each pair $(B, A \xrightarrow{k} B)$ such that $k(S)\subseteq B^*$, there is a unique map of rings $A_S \xrightarrow{\iota} B$ such that $k = \iota\circ i$.

Theorem. At least if A is commutative the localization of $S\subseteq A$ exists (and can be described).

2.7.2. *Localization of categories.* The localization of a category $\mathcal{A}$ with respect to a class of morphisms $\mathcal{S}\subseteq Mor(\mathcal{A})$ is the (universal!) functor, i.e., morphism of categories, $\mathcal{A} \xrightarrow{i} \mathcal{A}_{\mathcal{S}}$ such that the images of all morphisms in $\mathcal{S}$ are isomorphisms in $\mathcal{A}_{\mathcal{S}}$ (i.e., have inverses in $\mathcal{A}_{\mathcal{S}}$). Again, localization exists and can be described under some conditions.

2.7.3. *Maps in the localized category.* To make this less abstract I will sketch how one goes about constructing $\mathcal{A}_{\mathcal{S}}$. However we will return to this more precisely.

Observe that a map in $\mathcal{A}$, say, $\alpha \in \mathrm{Hom}_{\mathcal{A}}(a,b)$ gives a map in $\mathcal{A}_{\mathcal{S}}$, the map is $i(\alpha) \in \mathrm{Hom}_{\mathcal{A}_{\mathcal{S}}}[i(a),i(b)]$. Moreover a wrong direction map $\sigma \in \mathrm{Hom}_{\mathcal{A}}(b,a)$ which lies in $\mathcal{S}$ will also give a map from $i(a)$ to $i(b)$ in $\mathcal{A}_{\mathcal{S}}$, the map is $i(\sigma)^{-1} \in \mathrm{Hom}_{\mathcal{A}_{\mathcal{S}}}[i(a),i(b)]$. Since these are the only kinds of maps that we are asking to have in $\mathcal{A}_{\mathcal{S}}$, it is natural that all maps in $\mathcal{A}_{\mathcal{S}}$ should be generated from these two kinds of maps by using composition of maps.

This leads to the following idea: we will have $\mathrm{Ob}(\mathcal{A}_{\mathcal{S}}) = Ob(\mathcal{A})$ and i will be identity on objects. For $a,b \in Ob(\mathcal{A})$ the morphisms from a to b in $\mathcal{A}_{\mathcal{S}}$ will come from diagrams in $\mathcal{A}$

$$a \longrightarrow x \longleftarrow p \longrightarrow \cdots q \longrightarrow y \longleftarrow b,$$

where all backwards maps are in $\mathcal{S}$. The precise meaning of this is that $\mathrm{Hom}_{\mathcal{A}_{\mathcal{S}}}(a,b)$ will consists of equivalence classes of diagrams as above for a certain equivalence relation (which we still need to describe!).

2.7.4. *Derived category of modules and complexes of free modules.* According to the above definition $D(\mathfrak{m}(\Bbbk))$ is a very abstract construction. Fortunately it will turn out that there is a simple description of $D(\mathfrak{m}(\Bbbk))$ in terms of *homotopy* in the category of complexes over the subcategory of free modules. (This is the the approach (ii) above.)

2.7.5. *Do we really want the derived category?* The historical origin of the idea is as we have introduced it: it is a good setting for doing calculations with complexes. However, the derived category $D(\mathcal{A})$ of a category $\mathcal{A}$ (say $\mathcal{A} = \mathfrak{m}(\Bbbk)$ as above), may be more "real" than the simple category $\mathcal{A}$ we started with. One indication is that there are pairs of very different categories $\mathcal{A}$ and $\mathcal{B}$ such that their derived categories $D(\mathcal{A})$ and $D(\mathcal{B})$ are canonically equivalent. For instance $\mathcal{A}$ and $\mathcal{B}$ could be the categories of graded modules for the symmetric algebra $S(V)$ and the exterior algebra $\overset{\bullet}{\wedge} V^*$ for dual vector spaces V and V^*. This turns out to be important, but there are more exciting examples: the relation between linear differential equations and their solutions, mirror symmetry.

2.7.6. *Bounded categories of complexes.* We say that a complex $C^\bullet$ is bounded from above if $C^n = 0,\ n >> 0$. The categories of such complexes is denoted $\mathcal{C}^-(\mathcal{A})$ and $D^-(\mathcal{A})$ (meaning that the complexes are allowed to stretch in the negative direction) Similarly one has $\mathcal{C}^+(\mathcal{A})$ and $D^+(\mathcal{A})$. We say that a complex $C^\bullet$ is bounded (or finite) if $C^n = 0$ for all but a finitely many $n \in \mathbb{Z}$, this gives $\mathcal{C}^b(\mathcal{A})$ and $D^b(\mathcal{A})$.

2.8. **Derived versions of constructions.** After introducing the heroes of the story we again explain, somewhat more precisely, how passage to complexes produces a derived version Ld of a 'naive' construction d.

2.8.1. *Improving objects $M \in \mathcal{A}$.* Let $\mathcal{A} = \mathfrak{m}(\Bbbk)$ and $M \in \mathcal{A}$. We improve M by replacing it with a complex $P^\bullet$ of free (or say, projective) modules. This can be schematically described as

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\boldsymbol{\alpha}"] \arrow[d, "\beta"'] & \text{Projective resolutions} \arrow[d, "\delta"'] \\ \mathcal{C}^\bullet(\mathcal{A}) \arrow[r, "\boldsymbol{\alpha}'"] & \mathcal{C}^\bullet(\mathcal{P}roj(\mathcal{A})); \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\beta"'] & (\cdots \to P^{-1} \to P^0 \xrightarrow{q} M \to 0 \to \cdots) \arrow[d, "\delta"'] \\ (\cdots \to 0 \to \underset{0}{M} \to 0 \to \cdots) \arrow[r, "\gamma"] & (\cdots \to P^{-1} \to P^0 \to 0 \to \cdots). \end{tikzcd}$$

Notice that vertical arrows a natural constructions (i.e., functors), while horizontal arrows require some choices.

The composition of $\boldsymbol{\alpha}$ and δ is a description of M in terms of complexes of projective modules. The other route $\boldsymbol{\alpha}' \circ \beta$ indicates a more formal formulation of the same idea – we first view modules as complexes via β and then α' means describing complexes in $\mathcal{A}$ in terms of quasi-isomorphic complexes in $\mathcal{P}roj(\mathcal{A})$.

2.8.2. *Any (additive) functor $\mathcal{D} : \mathcal{A} \to \mathcal{B}$ extends to complexes.* Let $\mathcal{A} = \mathfrak{m}(\Bbbk)$ and $\mathcal{B} = \mathfrak{m}(\Bbbk')$ be categories of modules over two rings, and let $\mathcal{D}$ be a way to construct from a module for $\Bbbk$ a module for $\Bbbk'$, i.e., a functor $\mathcal{D} : \mathcal{A} \to \mathcal{B}$. It extends to a functor from $\mathcal{A}$-complexes to $\mathcal{B}$-complexes $\mathcal{D}^\bullet : \mathcal{C}^\bullet(\mathcal{A}) \to \mathcal{C}^\bullet(\mathcal{B})$, that assigns to each $\mathcal{A}$-complex $A^\bullet = (\cdots \to A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots)$ a $\mathcal{B}$-complex

$$\mathcal{D}^\bullet(A^\bullet) = (\cdots \to \mathcal{D}(A^{-1}) \xrightarrow{\mathcal{D}(d^{-1})} \mathcal{D}(A^0) \xrightarrow{\mathcal{D}(d^0)} \mathcal{D}(A^1) \xrightarrow{\mathcal{D}(d^1)} \cdots).$$

(As we know, if $\mathcal{D}$ is contravariant – for instance if $\mathcal{D}$ is the duality $\mathcal{D}_\Bbbk$ – the formula for $\mathcal{D}^\bullet A^\bullet$ would involve switching n and $-n$).

The main point is that $\mathcal{D}^\bullet \mathcal{A}^\bullet$ really is a complex: since $\mathcal{D}$ is a functor it preserves compositions of morphisms, hence $\mathcal{D}(d^n) \circ \mathcal{D}(d^{n-1}) = \mathcal{D}(d^n \circ d^{n-1}) = \mathcal{D}(0) = 0$. Asking that $\mathcal{D}$ is additive i.e., $\mathcal{D}(A' \oplus A'') = \mathcal{D}(A') \oplus \mathcal{D}(A'')$, $A', A'' \in \mathcal{A}$, is needed for the last step: $\mathcal{D}(0) = 0$.

2.8.3. *Left derived version $L\mathcal{D}$ of $\mathcal{D}$.* It really means that we do not apply $\mathcal{D}$ directly to M but to its improved version $P^\bullet$:

$$\begin{tikzcd} M \arrow[r, "\in"] & \mathcal{A} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \text{Projective resolutions} \arrow[d, "\delta"'] & & \\ & \mathcal{C}^\bullet(\mathcal{A}) \arrow[r, "\gamma"] & \mathcal{C}^\bullet(\mathcal{P}roj(\mathcal{A})) \arrow[r, "\mathcal{D}^\bullet"] & \mathcal{C}^\bullet(\mathcal{B}) & \mathcal{D}^\bullet(\mathcal{P}^\bullet) \overset{\text{def}}{=} L\mathcal{D}(M). \arrow[l, "\ni"'] \end{tikzcd}$$

2.8.4. *Left and right derived functors.* In order to say that $L\mathcal{D}$ is really an improvement of $\mathcal{D}$, we need to know that $H^0[L\mathcal{D}(M)] = \mathcal{D}(M)$, then $L\mathcal{D}(M)$ contains the information on $\mathcal{D}(M)$ and also a "hidden part" given by higher cohomologies $H^i[L\mathcal{D}(M)],\ i > 0$.

This is going to be true precisely if $\mathcal{D}$ has a property called right exactness (duality $\mathcal{D}$ is right exact!). There are important functors which are not right exact but have a "dual" property of left exactness, they will require a "dual" strategy: a right resolution of M:

$$\cdots \to M \to I^0 \to I^1 \to \cdots$$

by injective modules. We'll be back to that.

2.9. **A geometric example: duality for the ring of polynomials.** The commutative ring $\mathbb{k} = \mathbb{C}[x_1, ..., x_n]$ is the algebra of functions on the n-dimensional affine space $\mathbb{A}^n \overset{\text{def}}{=} \mathbb{C}^n$. Natural examples of $\mathbb{k}$-modules have geometric meaning.

2.9.1. *Affine algebraic varieties.* We say that an affine algebraic variety is a subset Y of some $\mathbb{A}^n$ which is given by polynomial conditions: $Y = \{z = (z_1, ..., z_n) \in \mathbb{C}^n;\ f_1(z) = \cdots = f_c(z) = 0\}$. The set $\mathcal{I}_Y$ of functions that vanish on Y is an ideal in $\mathbb{k}$ (i.e., a $\mathbb{k}$-submodule of the $\mathbb{k}$-module $\mathbb{k}$). We define the ring $\mathcal{O}(Y)$ of polynomial functions on Y as the all restrictions $f|Y$ of polynomials $f \in \mathbb{k}$ to Y. So $\mathcal{O}(Y) = \mathbb{k}/\mathcal{I}_Y$ is also a module for $\mathbb{k} = \mathcal{O}(\mathbb{A}^n)$.

2.9.2. *Duality.* We will consider the $\mathbb{k}$-module $\mathcal{O}(Y)$ where Y is the origin in $\mathbb{A}^n$. Then $\mathcal{I}_Y = \sum\ x_i{\cdot}k$ and therefore $\mathcal{O}(Y) = \mathbb{k}/\sum\ x_i{\cdot}k$ is isomorphic to $\mathbb{C}$ as a ring ($\mathbb{C}$-valued functions on a point!). However it is more interesting as a $\mathbb{k}$-module.

2.9.3. $n = 1$. Here $\mathbb{C}[x]$ and $\mathcal{I}_Y = x\mathbb{C}[x]$, so we have a resolution $\cdots \to 0 \to \mathbb{C}[x] \xrightarrow{x} \mathbb{C}[x] \xrightarrow{q} \mathcal{O}(Y) \to 0 \to \cdots$ and the computation of the dual of $\mathcal{O}(Y)$ is the same as in the case of $\mathbb{Z}_n$. One finds that $\mathbb{D}[\mathcal{O}(Y)] \cong\ \mathcal{O}(Y)[-1]$.

2.9.4. $n = 2$. Then $\mathcal{O}(\mathbb{A}^2) = \mathbb{C}[x,y]$ and $\mathcal{O}(Y) = \mathbb{C}[x,y]/\langle x,y\rangle = \mathbb{C}[x,y]/(x\mathbb{C}[x,y] + y\mathbb{C}[x,y]) = \mathbb{k}/(x\mathbb{k} + y\mathbb{k})$. The kernel of the covering $P^0 = \mathbb{k} \xrightarrow{q} \mathcal{O}(Y)$ is $x\mathbb{k} + y\mathbb{k}$. We can cover it turn with $P^0 = \mathbb{k}\oplus\mathbb{k} \xrightarrow{\alpha} x\mathbb{k} + y\mathbb{k}, \alpha(f,g) = x\alpha + y\beta$. This covering still contains surplus: $\operatorname{Ker}(\alpha) =\ \{(-yh, xh);\ h \in \mathbb{k}\}$. However, this is a free module so the next covering $P^{-2} = \mathbb{k} \xrightarrow{\beta} \operatorname{Ker}(\alpha)\subseteq P^{-1},\ \beta(h)(-yh, xh)$. This gives a resolution

$$\cdots \to 0 \to \mathbb{C}[x,y] \xrightarrow{\beta} \mathbb{C}[x,y]\oplus\mathbb{C}[x,y] \xrightarrow{\alpha} \mathbb{C}[x,y] \xrightarrow{q} \mathcal{O}(Y) \to 0 \to \cdots.$$

As a complex this resolution is

$$P^\bullet = [\cdots \to 0 \to \underset{-2}{\mathbb{C}[x,y]} \xrightarrow{\beta=(-y,x)} \underset{-1}{\mathbb{C}[x,y]\oplus\mathbb{C}[x,y]} \xrightarrow{\alpha=(x,y)} \underset{0}{\mathbb{C}[x,y]} \to 0 \to \cdots].$$

Therefore, computing the adjoints by lemma 2.4.3 gives

$$L\mathbb{D}[\mathcal{O}(Y)] = \mathbb{D}P^\bullet = [\cdots \to 0 \to \underset{0}{\mathbb{C}[x,y]} \xrightarrow{\alpha^*=(x,y)} \underset{1}{\mathbb{C}[x,y]\oplus\mathbb{C}[x,y]} \xrightarrow{\beta^*=(-y,x)} \underset{2}{\mathbb{C}[x,y]} \to 0 \to \cdots].$$

The cohomology of this complex is easy to compute (nothing new!), it gives

Lemma. $L\mathbb{D}[\mathcal{O}(Y)] \cong \mathcal{O}(Y)[-2]$.

2.9.5. *The general* n. The resolutions above for $n = 1, 2$ are example of the Koszul complex which we will meet later.

For the ring $\mathbb{C}[x_1, ..., x_n] = \mathcal{O}(\mathbb{A}^n)$ of functions on the n-dimensional affine space, the derived dual of the $\mathbb{C}[x_1, ..., x_n]$-module $\mathcal{O}(origin)$ is the same module $\mathcal{O}(origin)$ shifted to the degree n. So the shift is clearly the codimension of the origin in $\mathbb{A}^n$, and it is equal to n because $\dim(\mathbb{A}^n) = n$.

2.9.6. *Geometric nature of integers.* In view of the similarity of computations for $\mathcal{O}(\mathbb{A}^1) = \mathbb{C}[x]$ and for $\mathbb{Z}$, we may expect that $\mathbb{Z}$ is also a ring of functions on some geometric object, and that its dimension is one. So $\mathbb{Z}$ should correspond to some geometric object which we will denote $\mathrm{Spec}(\mathbb{Z})$, and $\mathrm{Spec}(\mathbb{Z})$ is some sort of a curve.

2.10. **Comments.**

2.10.1. *The parallel of resolutions and triangulations.* The idea is the same – explain complicated objects in terms of combining simple ones. As this can be done in several ways, in the end one has to check that whatever we produced is independent of choices. (This we leave for later.)

3. **Sheaves**

Sheaves are a machinery which addresses an essential problem – the relation between local and global information – so they appear throughout mathematics.

3.1. **Definition.** We start with a familiar example:

3.1.1. *Smooth functions on* $\mathbb{R}$. Let X be $\mathbb{R}$ or any smooth manifold. The notion of smooth functions on X gives the following data:

- for each open $U \subseteq X$ an algebra $C^\infty(U)$ (the smooth functions on U),
- for each inclusion of open subsets $V \subseteq U \subseteq X$ a map of algebras $C^\infty(U) \xrightarrow{\rho_V^U} C^\infty(V)$ (the restriction map);

and these data have the following properties

(1) (transitivity of restriction) $\rho_V^U \circ \rho_V^U = \rho_W^U$ for $W \subseteq V \subseteq U$,
(2) (gluing) if the functions $f_i \in C^\infty(U_i)$ on open subsets $U_i \subseteq X$, $i \in I$, are compatible in the sense that $f_i = f_j$ on the intersections $U_{ij} \stackrel{\text{def}}{=} U_i \cap U_j$, then they glue into a unique smooth function f on $U = \cup_{i \in I} U_i$.

So, smooth functions can be restricted and glued from compatible pieces. We formalize the idea of objects which can be restricted and glued together, into the notion of

3.1.2. *Sheaves on a topological space.* A *sheaf of sets* $\mathcal{S}$ on a topological space $(X, \mathcal{T})$ consists of the following data:

- for each open $U \subseteq X$ a set $\mathcal{S}(U)$,
- for each inclusion of open subsets $V \subseteq U \subseteq X$ a map $\mathcal{S}(U) \xrightarrow{\rho_V^U} \mathcal{S}(V)$ (called the restriction map);

and these data are required to satisfy the following properties

(1) (transitivity of restriction) $\rho_V^U \circ \rho_V^U = \rho_W^U$ for $W \subseteq V \subseteq U$,
(2) (gluing) Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of an open $U \subseteq X$. For a family of elements $f_i \in \mathcal{S}(U_i)$, $i \in I$, compatible in the sense that $\rho_{U_{ij}}^{U_i} f_i = \rho_{U_{ij}}^{U_i} f_j$ in $\mathcal{S}(U_{ij})$ for $i, j \in I$; there is a unique $f \in \mathcal{S}(U)$ such that on the intersections $\rho_{U_{ij}}^{U_i} f = f_i$ in $\mathcal{S}(U_i)$, $i \in I$.
(3) $\mathcal{S}(U) = \emptyset$.

We can equally define sheaves of abelian groups, rings, modules, etc – only the last, and least interesting requirement has to be modified, say in abelian groups we would ask that $\mathcal{S}(U)$ is the trivial group $\{0\}$.

3.1.3. *Examples.* (1) *Structure sheaves.* On a topological space X one has a sheaf of continuous functions C_X. If X is a smooth manifold there is a sheaf C_X^∞ of smooth functions, etc., there are holomorphic functions $\mathcal{H}_X$ on a complex manifold, "polynomial" functions $\mathcal{O}_X$ on an algebraic variety. What is important is that in each of these cases the topology on X and the sheaf contain all information on the structure of X.

(2) To a set S one can associated the *constant sheaf* S_X on any topological space X:

$$S_X(U) \textit{ is the set of } \text{locally } \textit{constant functions from } U \textit{ to } X.$$

(3) Constant functions do not form a sheaf, and neither do the functions with compact support. A given class $\mathcal{C}$ of objects forms a sheaf if it is defined by *local conditions*. For instance, being a (i) function with values in S, (ii) non-vanishing (i.e., invertible) function, (iii) solution of a given system $(*)$ of differential equations; these are all local conditions: they can be checked in a neighborhood of each point.

3.2. **Global sections functor** $\Gamma : \mathcal{S}heaves(X) \to \mathcal{S}ets$**.** Elements of $\mathcal{S}(U)$ are called the sections of a sheaf $\mathcal{S}$ on $U \subseteq X$ (this terminology is from classical geometry). By $\Gamma(X, \mathcal{S})$ we denote the set $\mathcal{S}(X)$ of global sections.

The construction $\mathcal{S} \mapsto \Gamma(\mathcal{S})$ means that we are looking at *global objects* in a given class $\mathcal{S}$ of objects. We will see that that the construction Γ comes with a hidden part, the cohomology $\mathcal{S} \mapsto H^\bullet(X, \mathcal{S})$ of sheaves on X.

Γ acquires different meaning when applied to different classes of sheaves. For instance for the constant sheaf pt_X, $\Gamma(X, \mathrm{pt}_X)$ is the set of connected components of X. On any smooth manifold X, $\Gamma(C^\infty) = C^\infty(X)$ is "huge" and there are no higher cohomologies ("nothing hidden"). The holomorphic setting is more subtle in this sense, on a compact connected complex manifold $\Gamma(X, \mathcal{H}_H)$ consists of only the constant functions and a lot of information may be stored in higher cohomology groups.

3.2.1. *Solutions of differential equations.* Solutions of a system $(*)$ of differential equation on X form a sheaf $\mathcal{S}ol_{(*)}$. If X is an interval I in $\mathcal{R}$ and $(*)$ is one equation of the form $y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_0(t) = 0$ with $a_i \in C^\infty(I)$ then any point $c \in I$ gives evaluation isomorphism of vector spaces $\mathcal{S}ol_{(*)}(I) \xrightarrow{E_c} \mathbb{C}^n$ by $E_c(y) = (y(c), ..., y^{(n-1)}(c))$ (solutions correspond to initial conditions!). The sheaf theoretic encoding of this property of the *initial value problem* is :

Lemma. $\mathcal{S}ol_{(*)}$ is a constant sheaf on X.

On the other hand let $(*)$ be the equation $zy' = \lambda y$ considered as equation in holomorphic functions on $X = \mathbb{C}^*$. The solutions are multiples of functions z^λ defined using a branch of logarithm. On any disc $D \subseteq X$, evaluation at a point $c \in D$ still gives $\mathcal{S}ol_{(*)}(D) \xrightarrow{\cong} \mathbb{C}$, so the local behavior of the is simple – it is a locally constant sheaf. However, $\Gamma(X, \mathcal{S}ol_{(*)}) = 0$

if $\lambda \notin \mathbb{Z}$ (any global solution would change by a factor of $e^{2\pi i\lambda}$ as we move once around the origin). So locally there is the expected amount but nothing globally.

3.3. **Projective line $\mathbb{P}^1$ over $\mathbb{C}$.** $\mathbb{P}^1 = \mathbb{C} \cup \infty$ can be covered by $U_1 = U = \mathbb{C}$ and $U_2 = V = \mathbb{P}^1 - \{0\}$. We think of $X = \mathbb{P}^1$ as a complex manifold by identifying U and V with $\mathbb{C}$ using coordinates u, v such that on $U \cap V$ one has $uv = 1$.

Lemma. $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$.

Proof. (1) *Proof using a cover.* A holomorphic function f on X restricts to $f|U = \sum_{n\geq 0} \alpha_n u^n$ and to $f|V = \sum_{n\geq 0} \beta_n v^n$. On $U \cap V = \mathbb{C}^*$, $\sum_{n\geq 0} \alpha_n u^n = \sum_{n\geq 0} \beta_n u^{-n}$, and therefore $\alpha_n = \beta_n = 0$ for $n \neq 0$.

(2) *Proof using maximum modulus principle.* The restriction of a holomorphic function f on X to $U = \mathbb{C}$ is a bounded holomorphic function (since X is compact), hence a constant.

3.4. **Čech cohomology of a sheaf $\mathcal{A}$ with respect to a cover $\mathcal{U}$.** Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X. We will use finite intersections $U_{i_0,\dots,i_p} \overset{\text{def}}{=} U_{i_0} \cap \cdots \cap U_{i_p}$.

3.4.1. *Calculations of global sections using a cover.* Motivated by the calculation of global sections in 3.3, to a sheaf $\mathcal{A}$ on X we associate

- Set $\mathcal{C}^0(\mathcal{U}, \mathcal{A}) \overset{\text{def}}{=} \prod_{i\in I} \mathcal{A}(U_i)$ whose elements are systems $f = (f_i)_{i\in I}$ with one section $f_i \in \mathcal{A}(U_i)$ for each open set U_i,
- $\mathcal{C}^1(\mathcal{U}, \mathcal{A}) \overset{\text{def}}{=} \prod_{(i,j)\in I^2} \mathcal{A}(U_{ij})$ whose elements are systems $g = (g_{ij})_{I^2}$ of sections g_{ij} on all intersections U_{ij}.

Now, if $\mathcal{A}$ is a sheaf of abelian groups we can reformulate the calculations of global sections of $\mathcal{A}$ in terms of the open cover $\mathcal{U}$

For this we encode the comparison of f_i's on intersections U_{ij} in terms of a map d : $\mathcal{C}^0(\mathcal{U}, \mathcal{A}) \xrightarrow{d} \mathcal{C}^1(\mathcal{U}, \mathcal{A})$ which sends $f = (f_i)_I \in \mathcal{C}^0$ to $df \in C^1$ with

$$(df)_{ij} = \rho^{U_j}_{U_{ij}} f_j - \rho^{U_i}_{U_{ij}} f_i.$$

More informally, $(df)_{ij} = f_j|U_{ij} - f_i|U_{ij}$.

Lemma. $\Gamma(\mathcal{A}) \xrightarrow{\cong} \mathrm{Ker}[C^0(\mathcal{U}, \mathcal{A}) \xrightarrow{d} C^1(\mathcal{U}, \mathcal{A})]$.

3.4.2. *Čech complex* $\mathcal{C}^\bullet(\mathcal{U},\mathcal{A})$. Emboldened, we try more of the same and define the abelian groups

$$\mathcal{C}^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \prod_{(i_0,\dots,i_n)\in I^n} \mathcal{A}(U_{i_0,\dots,i_n})$$

of systems of sections on multiple intersections, and relate them by the maps $\mathcal{C}^n(\mathcal{U},\mathcal{A}) \xrightarrow{d^n} \mathcal{C}^{n+1}(\mathcal{U},\mathcal{A})$ which sends $f = (f_{i_0,\dots,i_n})_{I^n} \in \mathcal{C}^n$ to $d^n\ f \in \mathcal{C}^{n+1}$ with

$$(d^n\ f)_{i_0,\dots,i_{n+1}} = \sum_{s=0}^{n+1} (-1)^s f_{i_0,\dots,i_{s-1},i_{s+1},\dots,i_{n+1}}.$$

Lemma. $(\mathcal{C}^\bullet(\mathcal{U},\mathcal{A}), d^\bullet)$ is a complex, i.e., $d.{\circ}d^{n-1} = 0$.

3.4.3. *Čech cohomology* $\check{H}^\bullet(X,\mathcal{U};\mathcal{A})$. It is defined as the cohomology of the Čech complex $\mathcal{C}^\bullet(\mathcal{U},\mathcal{A})$. We have already observed that

Lemma. $\check{H}^0(X,\mathcal{U};\mathcal{A}) = \Gamma(\mathcal{A})$.

3.4.4. *The "small Čech complex".* If the set I has a complete ordering, we can choose in $\mathcal{C}^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \prod_{(i_0,\dots,i_n)\in I^n} \mathcal{A}(U_{i_0,\dots,i_n})$ a subgroup $C^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \prod_{i_0<\cdots<i_n} \mathcal{A}(U_{i_0,\dots,i_n})$. This is what we will usually use in computations since it is smaller but it also computes the Čech cohomology:

Lemma. (a) $C^\bullet(\mathcal{U},\mathcal{A}) \subseteq \check{C}^\bullet(\mathcal{U},\mathcal{A})$ is a subcomplex (i.e., it is invariant under the differential).

(b) Map of complexes $C^\bullet(\mathcal{U},\mathcal{A}) \subseteq \mathcal{C}^\bullet(\mathcal{U},\mathcal{A})$ is a quasi-isomorphism.

Proof. (a) is clear. (b) is intuitively plausible since the extra data in $\mathcal{C}$ is a duplication of data in $C\bullet$, say $\mathcal{C}^1$ contains $\mathcal{S}(U_{ii}) = \mathcal{S}(U_i) \subseteq C^0$ and for $i < j$ it contains $\mathcal{S}(U_{ij})$ the second time under the name $\mathcal{S}(U_{ji})$.

3.5. **Vector bundles.** We recall the notion of a vector bundle, i.e., a vector space smeared over a topological space. We will be interested in calculating cohomology of sheaves associated to vector bundles.

3.5.1. *Vector bundle over space* X. In general one can extend many notions to the *relative setting* over some *base* X. For instance, a reasonable notion of a "vector space over a set X" is a collection $V = (V_x)_{x\in X}$ of vector spaces, one for each point of X. Then the total space $V = \sqcup_{x\in X} V_x$ maps to X and the fibers are vector spaces. If X is a topological space, we want the family of V_x to be "continuous in x". This leads to the notion of a vector bundle over a topological space.

Le $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$. The data for a $\mathbb{k}$-vector bundle of rank n over a topological space X consists of a map of topological spaces $\pi : V \to X$ and the vector space structures on fibers $V_x = \pi^{-1}x,\ x \in X$. These data should locally be isomorphic to to $X{\times}\Bbbk^n$ in the sense that each point has a neighborhood U such that there exists a homeomorphism $\phi : V|U \stackrel{\text{def}}{=} \pi^{-1}U \xrightarrow{\cong} U{\times}\Bbbk^n$ such that

$$\begin{array}{ccc} V|U & \xrightarrow{\phi} & U{\times}\Bbbk^n \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle pr_1}\downarrow \\ X & \xrightarrow{=} & X \end{array}$$

and that the corresponding maps of fibers $V_x \to \Bbbk^n,\ x \in U$, are isomorphisms of vector bundles.

Similarly one defines vector bundle over manifolds or over complex manifolds by requiring that π are local trivialization maps ϕ are smooth or holomorphic.

3.5.2. *Examples.*

(1) The smallest interesting example is the *Moebius strip*. Moebius strip is a line bundle over S^1 (it projects to the central curve S^1 and the fibers are real lines).
(2) *(Co)tangent bundles* On each manifold X there are the tangent and cotangent vector bundles TX, T^*X. In terms of local coordinates x_i at a, the fibers are $T_aX = \oplus\ \mathbb{R}\frac{\partial}{\partial x_i}$ and $T_a^*X = \ \oplus\ \mathbb{R}dx_i$.
(3) Any vector bundle can be obtained by gluing trivial vector bundles $V_i = U_i{\times}\Bbbk^n$ on an open cover $\mathcal{U} = (U_i)_{i\in I}$, The gluing data is given by transition functions

$$\phi_{ij} : U_{ij} \to GL_n(\Bbbk).$$

The corresponding vector bundle is the quotient

$$V \ = \ [\sqcup_{i\in I}\ U_i{\times}\Bbbk^n]/\sim$$

for the equivalence relation given by: $(u_i, z) \in U_i{\times}\Bbbk^n = V_i$ and $(u_j, w) \in U_j{\times}\Bbbk^n = V_j$ are equivalent iff they are related by the corresponding transition function, i.e., $u_i = u_j$ and $z = \phi_{ij}(u_j){\cdot}w$.

3.5.3. *Sheaf $\mathcal{V}$ associated to a vector bundle V.* Let $V \xrightarrow{\pi} M$ be a vector bundle over M. Define the *sections of the vector bundle* V over an open $U{\subseteq}X$, by

$$\mathcal{V}(U) \stackrel{\text{def}}{=} \{s : U \to V;\ \pi{\circ}s = id_U\}.$$

More precisely,

If V is obtained by gluing trivial vector bundles $V_i = U_i{\times}\mathbb{C}^n$ by transition functions ϕ_{ij}, then $\mathcal{V}(U)$ consists of all systems of $f_i \in \mathcal{H}(U_i \cap U, \mathbb{C}^n)$ such that on all intersections $U_{ij} \cap U$ one has $f_i = \phi_{ij} f_j$.

3.6. **Čech cohomology of line bundles on $\mathbb{P}^1$.**

Lemma. $\check{H}^\bullet(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \check{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$.

Proof. Since the cover we use $\mathcal{U} = \{U_1, U_2\}$ has two elements, $C^n = 0$ for $n > 1$. We know $\check{H}^0 = \Gamma(\mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$, so it remains to understand $\check{H}^1 = C^1/dC^0 = \mathcal{O}(U\cap V)/[\mathcal{O}(U)+\mathcal{O}(V)$, i.e., all Laurent series $\phi = \sum_{-\infty}^{+\infty} \gamma_n u^n$ that converge on $\mathbb{C}^*$, modulo the series $\sum_0^{+\infty} \lambda_n u^n$ and $\sum_0^{+\infty} \beta_n u^{-n}$, that converge on $\mathbb{C}$ and on $\mathbb{P}^1 - 0$. However, if a Laurent series $\phi = \sum_{-\infty}^{+\infty} \gamma_n u^n$ converges on $\mathbb{C}^*$,then Laurent series $\phi^+ = \sum_0^{+\infty} \gamma_n u^n$ converges on $\mathbb{C}$, and $\phi^- = \sum_{-\infty}^{-1} \gamma_n u^n$ converge on $\mathbb{C}^* \cup \infty$.

3.6.1. *Line bundles L_n on $\mathbb{P}^1$.* On $\mathbb{P}^1$ let L_n be the vector bundle obtained by gluing trivial vector bundles $U\times\mathbb{C}$, $V\times\mathbb{C}$ over $U\cap V$ by identifying $(u,\xi) \in U\times\mathbb{C}$ and $(v,\zeta) \in V\times\mathbb{C}$ if $uv = 1$ and $\zeta = u^n{\cdot}\xi$. So for $U_1 = U$ and $U_s = V$ one has $\phi_{12}(u) = u^n$, $U \in U\cap V\subseteq U$. Let $\mathcal{L}_n$ be the sheaf of holomorphic sections of L_n.

Lemma. (a) $\Gamma(\mathbb{P}^1, \mathcal{L}_n) = 0$ for $n < 0$ and for $n \geq 0$ the dimension is $n+1$ and

$$\Gamma(\mathbb{P}^1, \mathcal{L}_n) \cong \mathbb{C}_{\leq n}[u] \stackrel{\text{def}}{=} \text{the polynomials in } u \text{ of degree} \leq n$$

$$\cong \mathbb{C}_n[x,y] \stackrel{\text{def}}{=} \text{homogeneous polynomials in } x,y \text{ of degree } n.$$

(b) $\check{H}^1_{\mathcal{U}}(\mathbb{P}^1; \mathcal{L}_n) = 0$ for $n \geq -1$ and for $d \geq 1$, we have $\dim[\check{H}^1_{\mathcal{U}}(\mathbb{P}^1; \mathcal{L}_{-d})] = d-1$.

3.6.2. *Sheaves of meromorphic functions associated to divisors.* For distinct points $P_1, ..., P_n$ on $\mathbb{P}^1$, and integers D_i, define the sheaf $\mathcal{L} = \mathcal{O}(\sum D_i P_i)$ by $\mathcal{L}(U) \stackrel{\text{def}}{=}$ "all holomorphic functions f on $U - \{P_1, ..., P_n\}$, such that $\text{ord}_{P_i} f \geq -D_i$. Then

Lemma. $\mathcal{O}(\sum D_i P_i) \cong \mathcal{L}_{\sum D_i}$.

3.7. **Geometric representation theory.** Group $SL_2(\mathbb{C})$ acts on $\mathbb{C}^2$ and therefore on

- (i) polynomial functions $\mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x,y]$,
- (ii) each $\mathbb{C}_n[x,y]$;
- (iii) complex manifold $\mathbb{P}^1$ (the set of all lines in $\mathbb{C}^2$), and less obviously on
- (iv) each $\mathcal{L}_n$, hence also on
- (v) each $H^i(\mathbb{P}^1, \mathcal{L}_n)$.

In fact,

Lemma. $\mathbb{C}_n[x,y] = \Gamma(\mathbb{P}^1, \mathcal{L}_n), n = 0, 1, 2, ..$ is the list of

all irreducible finite dimensional holomorphic representations of SL_2.

By restricting the action to $SU(2)\subseteq SL(2,\mathbb{C})$ we find that this is also the list of all irreducible finite dimensional representations of $SU(2)$ on complex vector spaces.

3.7.1. *Borel-Weil-Bott theorem.* For each semisimple (reductive) complex group G there is a space $\mathcal{B}$ (the flag variety of G) such that all irreducible finite dimensional holomorphic representations of G are obtained as global sections of all line bundles on $\mathcal{B}$.

3.8. **Relation to topology.** Let $\Bbbk$ be any field. The cohomology of the constant sheaf $\Bbbk_X$ on a topological space X coincides with the *cohomology* $H^\bullet(X, \Bbbk)$ of X with coefficients in $\Bbbk$. The cohomology is defined as the dual of homology

$$H^i(X, \Bbbk) \overset{\text{def}}{=} H_i(X, \Bbbk)^*.$$

For instance,

Lemma. For $X = S^1$, $\check{H}^*_{\mathcal{U}}(X, \Bbbk_X)$ is dual to $H_*(X, \Bbbk)$.