HOMOLOGICAL ALGEBRA
WITH THE EXAMPLE OF D-MODULES

Contents

Date: ?

So Okada has suggested a number of corrections.

Part 0. **Announcements**

0. **What does the Homological algebra do?**

Homological algebra is a general tool useful in various areas of mathematics. One tries to apply it to constructions that morally should contain more information then meets the eye. The homological algebra, if it applies, produces "derived" versions of the construction ("the higher cohomology"), which contain the "hidden" information.

The goal is to understand the usefulness of homological ideas in applications and to use this process as an excuse to visit various interesting topics in mathematics.

0.1. **Some examples of applications.** **Algebraic analysis of linear differential equations**

Algebraic analysis of linear differential equations is based on the observation that any map between two spaces allows you to move a system of linear equations on one of the spaces to the other. These operations become most useful after passing to their derived versions.

Algebraic topology. It can loosely be described as a "systematic way of counting holes in manifolds". While we can agree that a circle has a 1-dimensional hole (in the sense of "a hole that can be made by a one dimensional object") and a sphere has a 2-dimensional hole, algebraic topology finds that the surface of a pretzel has one 2-dimensional hole and four 1-dimensional holes. These "holes" or "cycles" turn out to be essential in problems in geometry and analysis.

Dual of a module over a ring. The dual V^* of a real vector space V is the space of linear maps from V to real numbers. If one tries to do the same for a module M over a ring k (say the ring of integers), it does not work as well since M^* can often be zero. However in the "derived" world the construction works as well as for vector spaces.

Cohomology of sheaves. It deals with an omnipresent problem of relating local and global information on a manifold.

Subtle spaces. In order to organize interesting objects such as all curves or all vector bundles on a given curve, into a mathematically meaningful space, one requires an extension of a notion of a space. In contemporary physics the basic objects of string theory – the D-branes – are expected to be highly sophisticated constructs of homological algebra.

0.2. **Topics.**

- Algebraic topology.
- Duality of abelian groups.
- Derived functors, Ext and Tor.
- Solutions of linear differential equations with singularities.
- Sheaves and cohomology of sheaves.
- Derived categories.

Possible advanced topics: (1) Differential graded algebras, (2) n-categories. (3) Extended notions of a space: stacks and dg-schemes. (4) Homotopical algebra.

0.3. **The texts.**

(1) Weibel, Charles A., *An introduction to homological algebra*, Cambridge University Press, Cambridge [England] ; New York : Cambridge studies in advanced mathematics 38; ISBN/ISSN 0521435005.
(2) Gelfand, S. I., Manin Y. I., *Methods of homological algebra*, Springer, Berlin ; New York : ISBN/ISSN 3540547460 (Berlin) 0387547460 (New York).

These are very different books. Manin and Gelfand Jr. are top mathematicians and their book is full of exciting material from various areas, and it points towards hot developments. For that reason (and a laconic al style of Russian mathematics) it is also more difficult. (Moreover it is said that the English translation has a huge number of confusing typos.)

Weibel's book deals with a more restricted subject, so it is less exciting but seems fairly pleasant to read once one knows what one wants from homological algebra.

In any case I will hand out copies of chapters from both books, and other material sufficient for the course.

Part 1. Intro

This part is an announcement for easily believable ideas we will revisit in more detail. The idea is for me to pose some problems and to solve these problems with a little help.

1. Algebraic topology: Homology from triangulations

The goal is to measure properties of shapes i.e., of topological spaces. In a simple example of a circle our measurements give the following observations: (0) it is connected, and (1) it has a hole. We will find a systematic approach (homology of a topological space) to finding such properties. This involves the following steps:

- In order to extract information about a topological space X we make a choice of additional data – a triangulation $\mathcal{T}$, i.e., we break X into oriented simplices (points, intervals, triangles,...).
- Some information on $(X,\mathcal{T})$ is encoded into an algebraic construct, a complex $C_\bullet(X,\mathcal{T};\Bbbk)$ of vector spaces over a field $\Bbbk$ (such as $\mathbb{Q},\mathbb{R},\mathbb{C}$): vector spaces C_i count simplices of dimension i in the triangulation and the boundary maps between them encode the way the simplices are attached to form X.
- Distill from this the interesting information: the homology $H_\bullet(X,\mathcal{T};\Bbbk)$ of the complex $C_\bullet(X,\mathcal{T};\Bbbk)$ is called the "homology of the topological space X with coefficients in $\Bbbk$ (calculated using triangulation $\mathcal{T}$)". This is a smaller object then $C_\bullet(X,\mathcal{T};\Bbbk)$ – the passage to homology of the complex selects the interesting information. encoded in smaller vector spaces H_i.
- The end result, i.e., the homology $H_\bullet(X,\mathcal{T};\Bbbk)$, does not depend on the auxiliary choice of the triangulation $\mathcal{T}$ so we denote it $H_\bullet(X;\Bbbk)$.

1.1. **Linear simplices.**

Calculation of homology of topological spaces is based on a geometric idea of a simplex: 0-simplex is a point, 1-simplex is a closed interval, 2-simplex is a triangle, 3-simplex is a pyramid, etc.

1.1.1. *Standard simplices.* The standard n-simplex is $\sigma_n \subseteq \mathbb{R}^{n+1}$, the convex closure $\sigma_n = conv\{e_0,...,e_n\}$ of the standard basis of $\mathbb{R}^{n+1}$. So, σ_n is in the first "quadrant" $x_i \geq 0$, and there it is given by the hyperplane $\sum\ x_i = 1$.

More generally, we say that a linear i-simplex in $\mathbb{R}^n$ is the convex closure $conv(v_0,...,v_i)$ of a set of $i+1$ vectors which lie in an i-dimensional affine subspace but do not lie in any $(i-1)$-dimensional affine subspace.

For instance $e_0,...,e_n$ lie in the n-dimensional hyperplane $\sum\ x_i = 1$ but not in any $(n-1)$-dimensional affine subspace.

1.1.2. *Facets.* We call $\mathcal{V} = \{v_0, ..., v_i\}$ the vertices of the simplex $conv(\mathcal{V})$, so an i-simplex has $(i+1)$ vertices. The facets of the simplex $conv(\mathcal{V})$ are the simplices associated to subsets of the set of vertices – any subset $W \subseteq \mathcal{V}$ defines a facet of $conv(\mathcal{V})$ which is the simplex $conv(W)$. The facets are closed under intersections: $conv(W') \cap conv(W'') = conv(W' \cap W'')$.

The facets of codimension 1 are called faces.

The facet of σ_n corresponding to a subset $I \subseteq \{0, ..., n\}$ is $\sigma_n(I) = conv\{e_i,\ i \in I\}$ Observe that the ordering of $I = \ \{i_0 < \cdots i_k\}$ gives a canonical identification $\sigma_n(I) \cong \ \sigma_k$.

1.1.3. *Orientations of simplices.* We say that an ordering of the set of vertices $\mathcal{V}$ of a simplex $conv(\mathcal{V})$ gives an orientation of the simplex; and that two orderings give the same orientation if they differ by an even permutation of vertices.

Standard simplices σ_n (and their facets) have standard orientation given by the ordering $e_0 < \cdots < e_n$, we write this ordering from the right to the left as $(e_n, ..., e_0)$.

1.1.4. *Barycentric coordinates on the simplex* $conv(v_0, ..., v_i)$. Any point x in the simplex can be written as $x = \sum\ x_i v_i$ with $x_i \geq 0$ and $\sum\ x_i = 1$. Barycentric coordinates x_i are unique.

Proof. If $v_0 = 0$ then $v_1, ..., v_i$ have to be independent. So $x = \sum_0^i\ x_p v_p = \sum_1^i\ x_p v_p$ hence $x_p,\ p > 0$ are determined by x, and then so is $x_0 = 1 - \sum_1^i\ x_p$. In general, we can always translate a simplex into another one with $v_0 = 0$.

In particular one can recover the vertices from the simplex $conv(v_0, ..., v_i)$ as the the points with all but one coordinate zero.

The interior $conv^\circ(\mathcal{V})$ of a linear simplex $conv(\mathcal{V})$ consists of the points with all $x_i > 0$.

1.2. **Topological simplices.** A topological i-simplex Y is a topological space Y with a homeomorphism $\phi :\ \sigma_i \xrightarrow{\cong} Y$. We say that the orientation of σ_n gives an orientation of Y.

We say that the facet of Y corresponding to a subset $I \subseteq \{0, ..., n\}$ is $\phi(\sigma_n(I))$, it has a canonical structure of a topological simplex.

The faces of Y are the facets Y^i corresponding to throwing out the i^{th} vertex, i.e., the image of $conv(e_0, ..., e_{i-1}, e_{i+1}, ..., e_n)$.

1.3. **Triangulations.** The idea of triangulation is to present a given topological space as a combination of simple spaces – the simplices. Then we will extract the information on X from the way the simplices are patched together.

There are some choices to be made in our approach. The notion of a simplicial complex is a notion of a triangulation with certain properties **SC1-3** that make it very easy to describe how simplices fit together to form the space X - everything is stated in terms of the set of vertices. However, the price for the properties **SC1-3** is that in practice one needs a large number of simplices.

A more loose notion of a CW-complex allows using few simplices, but makes the description of how they fit together more difficult. It is stated in terms i-cells in X, i.e., maps $\sigma_i \xrightarrow{\phi} X$ such that the restriction to the interior $\sigma_n^\circ \rightarrow \phi(\sigma_n^\circ)$ are homeomorphisms.

1.4. **Simplicial complexes.** A triangulation of a topological space X is a covering $\mathfrak{X} = \{X_i, i \in I\}$ of X by topological simplices X_i, such that (for topological simplices viewed as subsets of X)

- (SC1) any facet of any simplex X_i is one of the simplices in $\mathfrak{X}$,
- (SC2) if $X_i \subseteq X_j$ then X_i is a facet of X_j,
- (SC3) the intersection $X_i \cap X_j$ is $\emptyset$ or a simplex in $\mathfrak{X}$ (so it is one of spaces X_i, X_j or one of the facets of X_i or X_j).

Now a simplex is determined by its vertices. Suppose that two simplices X_i and X_j have the same set $\mathcal{V}$ of vertices. Then $X_i \cap X_j$ is $\neq \emptyset$ so it is a simplex Y. Now Y lies in both X_i and X_j, hence it is a facet of X_i and of X_j. However a facet that contains all vertices has to be the simplex itself.

This means that the way the simplices are attached will be completely described in terms of the combinatorics of the set of vertices $\mathcal{T}^0$. For that reason one can think of a simplicial complex as a combinatorial structure: a set $\mathcal{V}$ (set of vertices of $\mathcal{T}$), endowed by a family $\mathcal{F}$ of subsets of $\mathcal{V}$ (these subsets correspond to simplices in $\mathcal{T}$ – for each simplex Y the set of its vertices is a subset of $\mathcal{V}$, and the mutual position of two simplices in X is recorded in the intersection of the sets of their vertices).

1.4.1. *Triangulations of* S^1. They are obtained by choosing an orientation of S^1 and n distinct points $A_1, ..., A_n$ that go in the direction of the orientation. The triangulation is given by 0-simplices $\mathcal{T}^0 = \{A_1, ..., A_n\}$ and and 1-simplices $\mathcal{T}^1 = \{A_1A_0, ..., A_1A_n\}$ (I denote by BA the segment from A to B).

If $n = 1$ this is not a simplicial complex since A_1A_1 is not really a 1-simplex by our definition - it is a circle hence not homeomorphic to σ_1. However, it is a 1-cell and $\mathcal{T} = \{A_1, A_1A_1\}$ is an efficient CW-complex.

$n = 2$ still does not give a simplicial complex since $A_2A_1 \cap A_1A_2$ is two points , hence not a 0-simplex. For $n \geq 3$ we do get a simplicial complex.

1.5. **Complex** $C_*(X, \mathcal{T}; \Bbbk)$**.** Calculation of homology of X includes a choice of "coefficients group $\Bbbk$ on X". $\Bbbk$ can be any abelian group but we usually choose it to be a commutative ring. The most interesting case is $\Bbbk = \mathbb{Z}$ but for now we will be happy with $\Bbbk$ a field such as $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

The space of i-chains for $(X,\mathcal{T})$ is the free $\Bbbk$-module

$$C_i = C_i(X,\mathcal{T};\Bbbk) \overset{\text{def}}{=} \ \oplus_{\alpha\in\mathcal{T}^i}\ k\alpha$$

with the basis given by the set of i-simplices $\mathcal{T}^i$ in the triangulation $\mathcal{T}$.

The boundary operator $\partial_i : C_i \to C_{i-1}$ sends an i-simplex Y to the sum of its faces, $\partial_i Y = \sum_0^i\ Y^k$. Remember that any facet of a simplex $Y \in \mathcal{T}$ is some simplex in $\mathcal{T}$, hence a face Y^k is some $Z \in \mathcal{T}$ – but this is only an equality of sets, so we are forgetting the orientations. So to interpret $\partial_i Y = \sum_0^i\ Y^k$ as en element of C_{i-1} we replace Y^i with Z if the orientations of Y^k and Z are the same and we replace Y^i with $-Z$ otherwise.

1.5.1. *Notion of a complex.* A complex of cochains is a sequence of $\Bbbk$-modules and maps

$$\cdots \xrightarrow{\partial^{-2}} C^{-1} \xrightarrow{\partial^{-1}} C^0 \cdots \xrightarrow{\partial^0} C^1 \to \cdots,$$

such that the compositions of coboundary operators ∂^i are zero: $\partial^{i+1}\partial^i = 0,\ i \in \mathbb{Z}$. We often omit the index on the coboundary operator, so we can write the preceding requirement as $\partial\circ\partial = 0$.

From a complex of cochains we get three sequences of $\Bbbk$-modules

- i-cocycles $Z^i \overset{\text{def}}{=} \mathrm{Ker}(\partial^i) \subseteq C^i$,
- i-coboundaries $B^i \overset{\text{def}}{=} Im(\partial^{i-1}) = Im(\partial^{i-1}) \subseteq C^i$,
- i-cohomologies $H^i \overset{\text{def}}{=} Z^i/B^i$,

Here we used $B^i \subseteq Z^i$ which follows from $\partial\partial = 0$.

A complex of chains is the same thing but with maps going down

$$\cdots \xleftarrow{\partial_{-1}} C_{-1} \xleftarrow{\partial_0} C_0 \cdots \xleftarrow{\partial_0} C_1 \to \cdots.$$

The difference is only the terminology and the notation. In this case we lower the indices and we talk of i-cycles $Z_i \subseteq C_i$, i-boundaries $B_i \subseteq C_i$, and i-homologies $H_i \overset{\text{def}}{=} Z_i/B_i$. When we just say "complex" we usually mean "complex of cochains".

1.5.2. *Lemma.* $C_*(X,\mathcal{T};\Bbbk)$ is a complex (of chains).

Proof. One first considers $X = \sigma_n$ with the obvious triangulation $\mathcal{T}_n$ by all facets of σ_n. One has $\mathcal{T}^n = \{\sigma_n\}$ hence $C_n = \Bbbk\cdot\sigma_n$. Now one checks that $del_{n-1}(\partial_n\ \sigma_n) = 0$.

The case of any $(X,\mathcal{T})$ follows. Any simplex $Y \in \mathcal{T}^n$ comes with a homomorphism $\sigma_n \xrightarrow{\phi} Y \subseteq X$, and this produces maps $\phi_i : C_\bullet(\sigma_n,\mathcal{T}_n;\Bbbk) \to C_\bullet(\sigma_n,\mathcal{T}_n;\Bbbk)$ such that $\partial_i^X \circ f_i = f_{i+1}\circ\partial_i^{\sigma_n}$. Therefore $\partial\partial\sigma_n = 0$ implies $\partial\partial Y = 0$.

1.6. **Examples.**

1.6.1. S^n *for* $n = 1$ *or* 2.

1.6.2. *Torus T^2.* We can view T^2 as a quotient of a rectangle, this makes the drawing of triangles easier. There is a simple CW-triangulation where one divides the rectangle by a diagonal into two triangles. It gives a fast calculation of homology.

One can get a simplicial complex, for instance by dividing the rectangle into nine rectangles and each of these into two triangles. Then H_0 and H_2 are easy and the dimension of H_1 can be computed from the invariance of Euler characteristic under taking homology (Homework 2.2).

Of a particular interest is a basis of H_1 – one can see that it corresponds to two "main" circles on the torus. Classically such basis controls the indeterminacy of elliptic integrals. In the modern algebraic geometry one says that such basis produces the so called periods, the basic invariants of elliptic curves.

2. **Duality for modules over rings**

It is easy to extend the construction of a dual vector space to duality for modules over any ring. However this "naive" notion of duality is not very useful since it does not have the standard properties of the duality for vector spaces. The correct notion of duality involves going from modules to a larger world of complexes where we find the hidden part of the dual of a module. This involves:

(1) thinking of abelian groups as complexes in degree 0,
(2) "Identifying" some complexes, in particular a module should be identified with its resolution.

These steps mean that we change twice the realms (categories) in which calculate

$$\mathfrak{m}(\Bbbk) \xrightarrow{(1)} \mathcal{C}^*(\mathfrak{m}(\Bbbk)) \xrightarrow{(2)} D(\mathfrak{m}(\Bbbk)),$$

from the category of $\Bbbk$-modules $\mathfrak{m}(\Bbbk)$ to the category of complexes of $\Bbbk$-modules $\mathcal{C}^*(\mathfrak{m}(\Bbbk))$ and to the derived category $D(\mathfrak{m}(\Bbbk))$ of $\Bbbk$-modules. Step (1) allows one to think of any $\Bbbk$-module in terms of particularly nice modules – the free modules. Step (2) leads to the setting $D(\mathfrak{m}(\Bbbk))$ which makes it precise what I mean by "identifying some complexes".

2.1. **Rings.**

Some of the classes of interesting rings $\Bbbk$

- fields such as $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ or the finite fields $\mathbb{F}_q$ with q elements,
- $\mathbb{Z}$ (related to number theory, i.e., everything),
- polynomial functions $\mathcal{O}(\mathbb{A}^n) = \mathbb{C}[x_1, ..., x_n]$ (related to geometry),
- Differential operators $D_{\mathbb{A}^n}$ (related to *linear* differential equations).

By a module we mean a left module unless specified differently. Let $\mathfrak{m}(\Bbbk) = \mathfrak{m}^l(\Bbbk)$ be the category of left $\Bbbk$-modules and $\mathfrak{m}^r(\Bbbk)$ the right $\Bbbk$-modules.

2.2. **Duality and biduality of $\Bbbk$-modules.**

The dual of a left $\Bbbk$-module M is the space $\mathbb{D}(M) = M^*$ of linear functionals

$$M^* \overset{\text{def}}{=} \mathrm{Hom}_\Bbbk(M, \Bbbk) \overset{\text{def}}{=} \{f : M \to \Bbbk;\, f(c{\cdot}m) = c{\cdot}f(m)\ \&\ f(m'+m'') = f(m')+f(m''),\ c \in \Bbbk,\ m, m', m'' \in M$$

For a right module N one asks that $f(n{\cdot}c) = f(n){\cdot}n$. This construction is a functor, i.e., it is defined not only on $\Bbbk$-modules but also on maps of $\Bbbk$-modules; the dual of $f : M_1 \to M_2$ is the adjoint map $\mathbb{D}(f) = f^* : M_2^* \to M_1^*,\ f^*(\nu)\ m = \langle \nu, fm \rangle,\ m \in M_1, \nu \in M_2^*$.

2.2.1. *Lemma.* $\mathfrak{m}^l(\Bbbk) \xrightarrow{\mathbb{D}} \mathfrak{m}^r(\Bbbk)$ and $\mathfrak{m}^r(\Bbbk) \xrightarrow{\mathbb{D}} \mathfrak{m}^l(\Bbbk)$.

2.2.2. *Lemma.* For $M \in \mathfrak{m}^l(\Bbbk)$, the canonical map $\iota_M : M \to (M^*)^*$ is well defined by $\iota_M(m)\ (\lambda) = \langle \lambda, m \rangle,\ m \in M,\ \lambda \in M^*$.

2.2.3. *Lemma.* If $\Bbbk$ is a field and $M \in \mathfrak{m}_{fd}(\Bbbk)$ (i.e., M is a finite dimensional vector space over $\Bbbk$), then ι_M is an isomorphism.

We would like to have the same situation for each $\Bbbk$-module M. At least it works for $M = \Bbbk$ (but not always, see 2.3):

2.2.4. *Lemma.* For a module $M = \Bbbk \in \mathfrak{m}^l(\Bbbk)$:

(a) The map that assigns to $a \in \Bbbk$ the operator of right multiplication $R_a x \stackrel{\text{def}}{=} x{\cdot}a$, gives an isomorphism of right $\Bbbk$-modules $\Bbbk \xrightarrow{R} \Bbbk^*$.

(b) $\iota_\Bbbk$ is an isomorphism.

When this is pushed a little further, we get a nice class of modules for which $M \xrightarrow{\cong} (M^*)^*$:

2.2.5. *Lemma.* ι_M is an isomorphism for any finitely generated free $\Bbbk$-module.

Proof. It follows from the lemma 2.2.4b, and from

2.2.6. *Sublemma.* For two $\Bbbk$- modules P, Q; (a) $(P \oplus Q)^* \cong P^* \oplus Q^*$, (b) map $\iota_{P\oplus Q}$ is an isomorphism iff both ι_P and ι_Q are isomorphisms.

2.3. **What is the dual of the abelian group $\mathbb{Z}_n$?** We will notice that the duality operation $M \mapsto M^*$ is not very good for arbitrary modules M of a ring $\Bbbk$. To improve it we will be forced to

- (i) pass to finitely generated free $\Bbbk$-modules (here duality behaves as for finite dimensional vector spaces), and
- (ii) from modules to complexes of modules (this is what it takes to be able to describe any finitely generated module in terms of free finitely generated modules).

We will be guided through this by an example with $\Bbbk$ the ring of integers and $M = \mathbb{Z}_n$ a torsion module. For $\Bbbk = \mathbb{Z}$, category of $\mathbb{Z}$-modules is just the category of abelian groups: $\mathfrak{m}(\mathbb{Z}) = \mathcal{A}b$. So we have the notion of a dual of an abelian group $M^* \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{A}b}(M, \mathbb{Z})$. For $M = \mathbb{Z}_n \stackrel{\text{def}}{=} \mathbb{Z}/n\mathbb{Z}$ one has $M^* = 0$, so duality loses all information.

2.3.1. *The passage from $\mathbb{Z}_n$ to its resolution $P^\bullet$.* We know that biduality works for the abelian group $M = \mathbb{Z}$ (by 2.2.4), and $\mathbb{Z}_n$ is clearly intimately related to $\mathbb{Z}$. The quotient map $\mathbb{Z} \xrightarrow{q} \mathbb{Z}_n$ relates $\mathbb{Z}_n$ to $\mathbb{Z}$, however it does not tell the whole story – the difference between $\mathbb{Z}_n$ and $\mathbb{Z}$ is in the kernel $\mathrm{Ker}(q) = n\mathbb{Z}$. However, the inclusion $n\mathbb{Z} \subseteq \mathbb{Z}$ captures the definition of $\mathbb{Z}_n$ as $\mathbb{Z}/n\mathbb{Z}$, and since the abelian group $n\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ by $\mathbb{Z} \ni x \mapsto nx \in n\mathbb{Z}$, we will replace $n\mathbb{Z}$ by $\mathbb{Z}$ in this map. Then it becomes the multiplication map $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$.

We will think of $\mathbb{Z}_n$ as encoded in the map $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$. For a more complicated $\Bbbk$-module such encoding will be more complicated, the proper setting will turn out to require to think of $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$ as a complex $P^\bullet = (\cdots \to 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0 \to \cdots)$, with $\mathbb{Z}'s$ in degrees -1 and 0.

So we have passed from $\mathbb{Z}_n$ ti a complex $P\bullet$. Now we need to know how to dualize it.

2.3.2. *Duality operation on complexes.* The dual of a complex of $\Bbbk$-modules $C^\bullet = (\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots)$ is the complex $\mathbb{D}C^\bullet$ obtained by applying $\mathbb{D}$ to modules and maps. Since $\mathbb{D}$ is contravariant (i.e., it changes directions), we will also have to change the indexing. So, $(\mathbb{D}C^\bullet)^n \stackrel{\text{def}}{=} \mathbb{D}(C^{-n})$ and $d^n_{\mathbb{D}C^\bullet}$ is the adjoint of $d^{-n-1}_{C^\bullet}$.

In order to calculate $\mathbb{D}P^\bullet$ we will need

2.3.3. *Sublemma.* (a) $\Bbbk$-linear maps between $\Bbbk^r$ and $\Bbbk^s$ can be described in terms of matrices. If we denote for $A \in M_{r,s}(k)$ by R_A the right multiplication operator $\Bbbk^r \ni x \mapsto xA \in \Bbbk^s$ on row-vectors, then $M_{rs} \xrightarrow{R} \mathrm{Hom}_\Bbbk(\Bbbk^r, \Bbbk^s)$ is an isomorphism.

(b) The adjoint of R_A is the left multiplication $L_{A^{tr}}$ with the transpose of A (on column-vectors).

2.3.4. *Biduality is true on $Free_{fg}(\Bbbk)$-complexes.* Let $Free_{fg}(\Bbbk)$ be the category of all free finitely generated $\Bbbk$-modules.

2.3.5. *Lemma.* The biduality map $\iota_{C^\bullet}$ is an isomorphism for any complex $C^\bullet$ in $\mathcal{C}^*(Free_{fg})$.

Proof. Observe that $(\mathbb{D}\mathbb{D}C^\bullet)^n = \mathbb{D}\mathbb{D}(C^n)$, and define the map $\iota_{C^\bullet} : C^\bullet \to \mathbb{D}\mathbb{D}C^\bullet$ as the collection of maps $\iota_{C^n} : C^n \to \mathbb{D}\mathbb{D}C^n$. If all C^n are in $Free_{fg}(\Bbbk)$ then all maps ι_{C^n} are isomorphisms and hence so is $\iota_{C^\bullet}$.

2.3.6. *The derived dual $L\mathbb{D}(\mathbb{Z}_n) \stackrel{def}{=} \mathbb{D}(P^\bullet)$.* We can identify $\mathbb{D}\mathbb{Z}$ with $\mathbb{Z}$ and then the adjoint of $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$ is again $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$ (see 2.3.3). So, $\mathbb{D}P^\bullet$ is the complex $\cdots \to 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0 \to \cdots$, but this time $\mathbb{Z}$'s are in degrees 0 and 1 ! So $\mathbb{D}P^\bullet \cong P^\bullet[-1]$ where one denotes by $C^\bullet[n]$ the shift of the complex $C^\bullet$ by n places to the left.

Finally, it is natural to identify any N with a complex which has N in degree 0 and all other terms zero (hence all maps are zero). So, since we have also identified $\mathbb{Z}_n$ with $P^\bullet$, we should identify the smart dual $L\mathbb{D}(\mathbb{Z}_n) \stackrel{\text{def}}{=} \mathbb{D}P^\bullet \cong P^\bullet[-1]$ with $\mathbb{Z}_n[-1]$. So,

$$L\mathbb{D}(\mathbb{Z}_n) = \mathbb{Z}_n[-1], \text{ the shift of } \mathbb{Z}_n \text{ by one to the right },$$

i.e., the complex which has $\mathbb{Z}_n$ in degree 1 and all other terms zero.

2.3.7. *Conclusion.* The smart dual of $\mathbb{Z}_n$ is not a module but a complex in degrees ≥ 0. The fact that $H^0[L\mathbb{D}(\mathbb{Z}_n)] = 0$ corresponds to the fact that the naive definition of the dual gives $\mathbb{D}(\mathbb{Z}_n) = 0$. So, the naive definition does not see the hidden part of the dual which is $H^1[L\mathbb{D}(\mathbb{Z}_n)] = \mathbb{Z}_n$.

Observe that since the computation of the derived dual is in the setting of complexes of free finitely generated modules, the biduality works (by 2.3.5), so the canonical map $\mathbb{Z}_n \xrightarrow{L\iota_{\mathbb{Z}_n}} (L\mathbb{D})(L\mathbb{D})(\mathbb{Z}_n)$ is an isomorphism.

In 2.4 and 2.5 we will explain the ideas in the calculation of $L\mathbb{D}(\mathbb{Z}_n)$ so that we can see that they apply in many situations.

2.4. **Resolutions.** Here we repeat for any module M what we have been able to do for $\mathbb{Z}_n$. The precise, formal, solution of our wish to describe a module M in terms of maps between some (hopefully nicer) modules P^n. is the notion of a resolution of M.

2.4.1. *Exact complexes and short exact sequence.* A complex of $\Bbbk$-modules $C^\bullet = (\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots)$ is said to be exact if all of its cohomologies vanish, i.e., inclusion $B^n \subseteq Z^n$ is equality. (Exact complexes are also called exact sequences.)

2.4.2. *Examples.* Suppose that in the complex $C^\bullet$ all terms C^n, $n \neq 0$ are zero. Then $C^\bullet$ is exact iff $C^0 = 0$.

If $C^n = 0$ for $n \neq -1, 0$, then $C^\bullet$ is exact iff $d^{-1} : C^{-1} \to C^0$ is an isomorphism.

If $C^n = 0$ for $n \neq -1, 0, 1$, then $C^\bullet$ is exact iff $d^{-1} : C^{-1} \to C^0$ is injective, $d^0 : C^0 \to C^1$ is surjective, and in C^0 one has $\mathrm{Ker}(d^0) = Im(d^{-1})$. So all exact complexes with three terms (and such complexes are also called *short* exact sequences), are of the following form: module C^0 has a submodule C^{-1} and the quotient $C^1 = C^0/C^{-1}$.

2.4.3. *Resolutions.* A left resolution of a module M is an exact complex

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \xrightarrow{q} M \to 0 \to \cdots.$$

For example $\cdots \to 0 \to \underset{-1}{\mathbb{Z}} \xrightarrow{n} \underset{0}{\mathbb{Z}} \xrightarrow{q} \underset{1}{\mathbb{Z}_n} \to 0 \to \cdots$ (the numbers beneath are the positions in the complex) is a resolution of $\mathbb{Z}_n$.

We will also say that the complex $\cdots \to P^{-2} \to P^{-1} \to P^0 \to 0 \to \cdots$ together with a map q is a resolution of M. Another way to think of a resolution is as a morphism of complexes where we extend q to a map complexes

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & P^{-2} \arrow[r] \arrow[d] & P^{-1} \arrow[r] \arrow[d] & P^0 \arrow[r] \arrow[d, "q"'] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Here,

2.4.4. *Morphisms of complexes.* A map of complexes $f : A^\bullet \to B^\bullet$ is a system of maps f^n of the corresponding terms in complexes, which "preserves" the differential in the sense that in the diagram

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & A^{-2} \arrow[r] \arrow[d, "f^{-2}"'] & A^{-1} \arrow[r] \arrow[d, "f^{-1}"'] & A^0 \arrow[r] \arrow[d, "f^0"'] & A^1 \arrow[r] \arrow[d, "f^1"'] & \cdots \arrow[d] \\ \cdots \arrow[r] & B^{-2} \arrow[r] & B^{-1} \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & \cdots \end{tikzcd}$$

all squares commute in the sense that two possible ways of following arrows give the same result: $f^n \circ d_A^{n-1} = d_B^{n-1} \circ f^{n-1}$, for all n; i.e., $f \circ d = d \circ f$.

Now we have a category of complexes of $\Bbbk$-modules $\mathcal{C}^\bullet[\mathfrak{m}(\Bbbk)]$: objects are complexes and morphisms are maps of complexes.

2.4.5. *Free resolutions.* We consider subcategories of $\Bbbk$-modules $\mathcal{F}ree^l(\Bbbk) \subseteq \mathfrak{m}^l(\Bbbk) \subseteq \mathfrak{m}^l{}_{fg}(\Bbbk)$ consisting of free modules and of finitely generated modules. The intersection is $\mathcal{F}ree_{fg}{}^l(\Bbbk)$. Now we can say that a free resolution of M is a resolution $P^\bullet$ such that all P^i are free $\Bbbk$-modules, etc.

2.4.6. *Lemma.* Any module M has a free resolution.

Proof. (1) *There is a free module F and a surjective map $F \twoheadrightarrow M$* ("a free cover of M"). For this we choose any set $\mathcal{G} \subseteq M$ of generators of M (for instance $\mathcal{G} = M$), and let F be the free $\Bbbk$-module with the basis $\mathcal{G}$.

(2) Let $P^0 \xrightarrow{q} M$ be the map $F \to M$ from (1). If q has no kernel, we are done – we choose $P^k = 0,\ k < 0$. Otherwise we use again (1) to choose a free cover $P^{-1} \to \mathrm{Ker}(q)$, then ∂^{-1} is the composition $P^{-1} \to \mathrm{Ker}(q) \subseteq P^0$. Etc.

2.4.7. *Lemma.* If the ring $\Bbbk$ is noetherian any finitely generated module M has a resolution by free finitely generated modules.

Proof. (1) If $\Bbbk$-module M is finitely generated a free finitely generated module F and a surjective map $F \twoheadrightarrow M$. This is as before, except that we can now choose a set $\mathcal{G} \subseteq M$ of generators of M, to be a finite set.

To repeat the step (2), we need the new modules to be covered, such as $\mathrm{Ker}(q) \subseteq P^0$, $\mathrm{Ker}(\partial^{-1}) \subseteq P^{-1}, \ldots$ are finitely generated. That's what the noetherian assumption means: we say that a ring $\Bbbk$ is noetherian if any submodule of a finitely generated module is finitely generated.

2.4.8. *Projective modules.* We say that a $\Bbbk$-module P is projective if it is a summand of a free $\Bbbk$-module. So, free modules are projective and we get a larger class $\mathcal{P}roj(\Bbbk) \supseteq \mathcal{F}ree(\Bbbk)$. However it has the same good properties of free modules (so projective resolutions are as good as free resolutions!).

For instance: if P is a summand of a finitely generated free $\Bbbk$-module then the biduality map ι_P is again an isomorphism by sublemma 2.2.6.

2.4.9. *Finite resolutions.* The homological dimension of $\Bbbk$ is the infimum of all n such that any $\Bbbk$-module has a projective resolution of length $\leq n$ (i.e., $P^i = 0,\ i < -n$).

A field has dimension 0, $\mathbb{Z}$ has dimension one and $\mathbb{C}[x_1, ..., x_n]$ has dimension n. However the ring $\mathcal{O}(Y) = \mathbb{C}\oplus xy\mathbb{C}[x,y]\subseteq \mathbb{C}[x,y]$ of functions on the crossing $Y = \{xy = 0\}\subseteq\mathbb{A}^2$ has infinite dimension (because Y has a singularity).

2.5. **Derived category of $\Bbbk$-modules.** By now, we can compute the dual of any $\Bbbk$-module M – we can find a free resolution $P^\bullet$ following the proof of lemma 2.4.6, and then $L\mathbb{D}(M) \overset{\text{def}}{=} \mathbb{D}(P^\bullet)$. The basic idea is therefore to replace M by its resolution. In what sense are these the same things?

2.5.1. *Identification of module with its resolution.* I will explain it in two steps:
(i) already, the use of resolutions suggests that we have to pass from modules to the larger world of complexes of modules. The most natural way is to associate to a module M the complex $M^\#$ which is M in degree zero and 0 in other degrees. This gives a functor

$$\mathfrak{m}(\Bbbk) \ \to \mathcal{C}^*(\mathfrak{m}(\Bbbk)), \quad M\mapsto M^\#.$$

(However, we will usually denote $M^\#$ just by M again.)
(ii) Therefore the replacement of M by $P^\bullet$ is seen in the realm of complexes as the replacement of a complex $M^\#$ by $P^\bullet$. The relation between them is given by the map of complexes $M^\# \ \to P^\bullet$ from 2.4.3. This map can not be an isomorphisms since the complexes are really different, however it does become an isomorphism once we pass to cohomology: the only non-zero cohomology group of $P^\bullet$ is $H^0(P^\bullet) \cong M$, the same is true for $M^\#$ and moreover the morphism of complexes $M^\# \ \to P^\bullet$ induces id_M on $H^0(M^\#) \to H^0(P^\bullet)$.

2.5.2. *Derived category via inverting quasi-isomorphisms.* We say that a map of complexes $f : A^\bullet \ \to B^\bullet$ is a quasi-isomorphism ("qis") if the induced maps of cohomology groups $H^n(f) :\ H^n(A^\bullet) \to H^n(B^\bullet),\ n \in \mathbb{Z}$, are all isomorphisms.

So, $M^\#$ and $P^\bullet$ are "somewhat the same" in the sense that they are quasi-isomorphic. However, we would like them to be truly the same thing, i.e., we would like to proclaim all quasi-isomorphisms in $\mathcal{C}^\bullet(\mathfrak{m}(\Bbbk))$ to be isomorphisms! This is achieved by passing from $\mathcal{C}^\bullet(\mathfrak{m}(\Bbbk))$ to the so called derived category of $\Bbbk$-modules $D(\mathfrak{m}(\Bbbk))$.

2.5.3. *Localization of categories.* Localization of a ring A with respect to a subset $S\subseteq A$ is the ring A_S obtained by inverting all elements of S. More precisely, localization of a ring A with respect to $S\subseteq A$ is a pair of a ring A_S and a map of rings $A \xrightarrow{i} A_S$ such that $i(S)\subseteq (A_S)^* \overset{\text{def}}{=}$ the set of invertible elements of A_S. There may be many such pairs, and so we have to be still more precise, it is the universal such pair (i.e., the best such pair), in the sense that for each pair $(B, A \xrightarrow{k} B)$ such that $k(S) \subseteq B^*$, there is a unique map of rings $A_S \xrightarrow{\iota} B$ such that $k = \iota \circ i$.

Theorem. At least if A is commutative the localization of $S \subseteq A$ exists (and can be described).

The localization of a category $\mathcal{A}$ with respect to a class of morphisms $\mathcal{S} \subseteq Mor(\mathcal{A})$ is the (universal!) functor, i.e., morphism of categories, $\mathcal{A} \xrightarrow{i} \mathcal{A}_{\mathcal{S}}$ such that the images of all morphisms in $\mathcal{S}$ are isomorphisms in $\mathcal{A}_{\mathcal{S}}$ (i.e., have inverses in $\mathcal{A}_{\mathcal{S}}$). Again, localization exists and can be described under some conditions.

2.5.4. *Maps in the localized category.* To make this less abstract I will sketch how one goes about constructing $\mathcal{A}_{\mathcal{S}}$. However we will return to this more precisely.

Observe that a map in $\mathcal{A}$, say, $\alpha \in \mathrm{Hom}_{\mathcal{A}}(a, b)$ gives a map in $\mathcal{A}_{\mathcal{S}}$, the map is $i(\alpha) \in \mathrm{Hom}_{\mathcal{A}_{\mathcal{S}}}[i(a), i(b)]$. Moreover a wrong direction map $\sigma \in \mathrm{Hom}_{\mathcal{A}}(b, a)$ again gives a map from $i(a)$ to $i(b)$ in $\mathcal{A}_{\mathcal{S}}$, the map is $i(\sigma)^{-1} \in \mathrm{Hom}_{\mathcal{A}_{\mathcal{S}}}[i(a), i(b)]$. Since these are the only two kinds of maps that are asking to be in $\mathcal{A}_{\mathcal{S}}$, it is natural that all maps in $\mathcal{A}_{\mathcal{S}}$ should be generated from these two kinds of maps by using composition of maps.

This leads to the following idea: we will have $\mathrm{Ob}(\mathcal{A}_{\mathcal{S}}) = Ob(\mathcal{A})$ and the morphisms from a to b in $\mathcal{A}_{\mathcal{S}}$ will come from diagrams in $\mathcal{A}$

$$a \to x \leftarrow p \to \cdots q \to y \leftarrow b,$$

where all backwards maps are in $\mathcal{S}$. The last step is to decide when two such diagrams give the same map in $\mathcal{A}_{\mathcal{S}}$!

2.5.5. *Derived category of modules and complexes of free modules.* According to its definition $D(\mathfrak{m}(\Bbbk))$ looks like a somewhat abstract construction. Fortunately it turns out that there is a simple description of $D(\mathfrak{m}(\Bbbk))$ in terms of the category of complexes over free modules $\mathcal{C}^\bullet(\mathcal{F}ree(\Bbbk))$.

2.5.6. *Do we really want the derived category?* The historical origin of the idea is as we have introduced it: it is a good setting for doing calculations with complexes. However, the derived category $D(\mathcal{A})$ of a category $\mathcal{A}$ (say $\mathcal{A} = \mathfrak{m}(\Bbbk)$ as above), may be more "real" than the simple category $\mathcal{A}$ we started with. One indication is that there are pairs of very different categories $\mathcal{A}$ and $\mathcal{B}$ such that their derived categories $D(\mathcal{A})$ and $D(\mathcal{B})$ are canonically equivalent. For instance $\mathcal{A}$ and $\mathcal{B}$ could be the categories of graded modules for the symmetric algebra $S(V)$ and the exterior algebra $\overset{\bullet}{\wedge} V^*$ for dual vector spaces V and V^*. This turns out to be important, but there are more exciting examples: the relation between linear differential equations and their solutions, mirror symmetry.

2.5.7. *Bounded categories of complexes.* We say that a complex $C^\bullet$ is bounded from above if $C^n = 0,\ n >> 0$. The categories of such complexes is denoted $\mathcal{C}^-(\mathcal{A})$ and $D^-(\mathcal{A})$ (meaning that the complexes are allowed to stretch in the negative direction) Similarly one has $\mathcal{C}^+(\mathcal{A})$ and $D^+(\mathcal{A})$. We say that a complex $C^\bullet$ is bounded (or finite) if $C^n = 0$ for all but a finitely many $n \in \mathbb{Z}$, this gives $\mathcal{C}^b(\mathcal{A})$ and $D^b(\mathcal{A})$.

2.6. **Derived versions of constructions.**

2.6.1. *Improving an object M.* Let $\mathcal{A} = \mathfrak{m}(\Bbbk)$ and $M \in \mathcal{A}$. We improve M by replacing it with a complex $P^\bullet$ of free (or say, projective) modules. This can be described as

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \substack{\text{Projective}\\ \text{resolutions}} \arrow[d, "\delta"'] \\ \mathcal{C}^\bullet(\mathcal{A}) \arrow[r, "\gamma"] & \mathcal{C}^\bullet(\mathcal{P}roj(\mathcal{A})) \end{tikzcd}, \qquad \begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\beta"'] & (\cdots\to P^{-1}\to P^0 \xrightarrow{q} M\to 0\to\cdots) \arrow[d, "\delta"'] \\ (\cdots\to 0\to \underset{0}{M}\to 0\to\cdots) \arrow[r, "\gamma"] & (\cdots\to P^{-1}\to P^0\to 0\to\cdots). \end{tikzcd}$$

Combination of α and δ is intuitively clear: this is the description of M in terms of maps between projective modules. The other route is a more formal formulation of the same idea. Step β is the passage from modules to a larger world of complexes, and step γ is the replacement of a complex by a quasi-isomorphic complex.

2.6.2. *Any (additive) functor $\mathcal{D} : \mathcal{A} \to \mathcal{B}$ extends to complexes.* Let $\mathcal{A} = \mathfrak{m}(\Bbbk)$ and $\mathcal{B} = \mathfrak{m}(\Bbbk')$ be categories of modules over two rings, and let $\mathcal{D}$ be a way to construct from a module for $\Bbbk$ a module for $\Bbbk'$, i.e., a functor $\mathcal{D} : \mathcal{A} \to \mathcal{B}$. It extends to a functor from $\mathcal{A}$-complexes to $\mathcal{B}$-complexes $\mathcal{D}^\bullet : \mathcal{C}^\bullet(\mathcal{A}) \to \mathcal{C}^\bullet(\mathcal{B})$, that assigns to each $\mathcal{A}$-complex $A^\bullet = (\cdots \to A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots)$ a $\mathcal{B}$-complex

$$\mathcal{D}^\bullet(A^\bullet) = (\cdots \to \mathcal{D}(A^{-1}) \xrightarrow{\mathcal{D}(d^{-1})} \mathcal{D}(A^0) \xrightarrow{\mathcal{D}(d^0)} \mathcal{D}(A^1) \xrightarrow{\mathcal{D}(d^1)} \cdots).$$

(As we know, if $\mathcal{D}$ is contravariant – for instance if $\mathcal{D}$ is the duality $\mathcal{D}_\Bbbk$ – the formula for $\mathcal{D}^\bullet A^\bullet$ would involve switching n and $-n$).

The main point is that this really is a complex: since $\mathcal{D}$ is a functor it preserves compositions of morphisms, hence $\mathcal{D}(d^n)\circ\mathcal{D}(d^{n-1}) = \mathcal{D}(d^n\circ d^{n-1}) = \mathcal{D}(0) = 0$. Asking that $\mathcal{D}$ is additive i.e., $\mathcal{D}(A'\oplus A'') = \mathcal{D}(A')\oplus\mathcal{D}(A''),\ A', A'' \in \mathcal{A}$, is needed for the last step $\mathcal{D}(0) = 0$.

2.6.3. *Left derived version $L\mathcal{D}$ of $\mathcal{D}$.* It really means that we do not apply $\mathcal{D}$ directly to M but to its improved version $P^\bullet$:

$$\begin{tikzcd} M \arrow[r, "\in"] & \mathcal{A} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \substack{\text{Projective}\\ \text{resolutions}} \arrow[d, "\delta"'] & & \\ & \mathcal{C}^\bullet(\mathcal{A}) \arrow[r, "\gamma"] & \mathcal{C}^\bullet(\mathcal{P}roj(\mathcal{A})) \arrow[r, "\mathcal{D}^\bullet"] & \mathcal{C}^\bullet(\mathcal{B}) \arrow[r, "\ni"] & \mathcal{D}^\bullet(P^\bullet) \overset{\text{def}}{=} L\mathcal{D}(M). \end{tikzcd}$$

2.6.4. *Left and right derived functors.* In order to say that $L\mathcal{D}$ is really an improvement of $\mathcal{D}$, we need to know that $H^0[L\mathcal{D}(M)] = \mathcal{D}(M)$, then $L\mathcal{D}(M)$ contains the information on $\mathcal{D}(M)$ and also a "hidden part" given by higher cohomologies $H^i[L\mathcal{D}(M)],\ i > 0$.

This is going to be true precisely if $\mathcal{D}$ has a property called right exactness (duality $\mathcal{D}$ is right exact!). There are important functors which are not right exact but have a "dual" property of left exactness, they will require a "dual" strategy: a right resolution of M:

$$\cdots \to M \to I^0 \to I^1 \to \cdots$$

by injective modules. We'll be back to that.

2.7. **Duality for** $\Bbbk = \mathbb{C}[x_1, ..., x_n]$**.** The commutative ring $\Bbbk = \mathbb{C}[x_1, ..., x_n]$ is the algebra of functions on the n-dimensional affine space $\mathbb{A}^n \overset{\text{def}}{=} \mathbb{C}^n$. Natural examples of $\Bbbk$-modules have geometric meaning.

2.7.1. *Affine algebraic varieties.* We say that an affine algebraic variety is a subset Y of some $\mathbb{A}^n$ which is given by polynomial conditions: $Y = \{z = (z_1, ..., z_n) \in \mathbb{C}^n;\ f_1(z) = \cdots = f_c(z) = 0\}$. The set $\mathcal{I}_Y$ of functions that vanish on Y is an ideal in $\Bbbk$ (i.e., a $\Bbbk$-submodule of the $\Bbbk$-module $\Bbbk$). We define the ring $\mathcal{O}(Y)$ of polynomial functions on Y as the all restrictions $f|Y$ of polynomials $f \in \Bbbk$ to Y. So $\mathcal{O}(Y) = \Bbbk/\mathcal{I}_Y$ is also a module for $\Bbbk = \mathcal{O}(\mathbb{A}^n)$.

2.7.2. *Duality.* We will consider the $\Bbbk$-module $\mathcal{O}(Y)$ where Y is the origin in $\mathbb{A}^n$. Then $\mathcal{I}_Y = \sum\ x_i{\cdot}k$ and therefore $\mathcal{O}(Y) = \Bbbk/\sum\ x_i{\cdot}k$ is isomorphic to $\mathbb{C}$ as a ring ($\mathbb{C}$-valued functions on a point!). However it is more interesting as a $\Bbbk$-module.

2.7.3. $n = 1$. Here $\mathbb{C}[x]$ and $\mathcal{I}_Y = x\mathbb{C}[x]$, so we have a resolution $\cdots \to 0 \to \mathbb{C}[x] \xrightarrow{x} \mathbb{C}[x] \xrightarrow{q} \mathcal{O}(Y) \to 0 \to \cdots$ and the computation of the dual of $\mathcal{O}(Y)$ is the same as in the case of $\mathbb{Z}_n$. One finds that $\mathbb{D}[\mathcal{O}(Y)] \cong\ \mathcal{O}(Y)[-1]$.

2.7.4. $n = 2$. Then $\mathcal{O}(\mathbb{A}^2) = \mathbb{C}[x, y]$ and $\mathcal{O}(Y) = \mathbb{C}[x, y]/\langle x, y\rangle = \mathbb{C}[x, y]/(x\mathbb{C}[x, y] + y\mathbb{C}[x, y]) = \Bbbk/(x\Bbbk + y\Bbbk)$. The kernel of the covering $P^0 = \Bbbk \xrightarrow{q} \mathcal{O}(Y)$ is $x\Bbbk + y\Bbbk$. We can cover it turn with $P^0 = \Bbbk{\oplus}\Bbbk \xrightarrow{\alpha} x\Bbbk + y\Bbbk, \alpha(f, g) = x\alpha + y\beta$. This covering still contains surplus: $\mathrm{Ker}(\alpha) =\ \{(-yh, xh);\ h \in \Bbbk\}$. However, this is a free module so the next covering $P^{-2} = \Bbbk \xrightarrow{\beta} \mathrm{Ker}(\alpha){\subseteq}P^{-1},\ \beta(h)(-yh, xh)$. This gives a resolution

$$\cdots \to 0 \to \mathbb{C}[x, y] \xrightarrow{\beta} \mathbb{C}[x, y]{\oplus}\mathbb{C}[x, y] \xrightarrow{\alpha} \mathbb{C}[x, y] \xrightarrow{q} \mathcal{O}(Y) \to 0 \to \cdots.$$

As a complex this resolution is

$$P^\bullet = [\cdots \to 0 \to \underset{-2}{\mathbb{C}[x, y]} \xrightarrow{\beta=(-y,x)} \underset{-1}{\mathbb{C}[x, y]{\oplus}\mathbb{C}[x, y]} \xrightarrow{\alpha=(x,y)} \underset{0}{\mathbb{C}[x, y]} \to 0 \to \cdots].$$

Therefore, computing the adjoints by lemma 2.3.3 gives

$$L\mathbb{D}[\mathcal{O}(Y)] = \mathbb{D}P^\bullet = [\cdots \to 0 \to \underset{0}{\mathbb{C}[x,y]} \xrightarrow{\alpha^*=(x,y)} \underset{1}{\mathbb{C}[x,y]\oplus\mathbb{C}[x,y]} \xrightarrow{\beta^*=(-y,x)} \underset{2}{\mathbb{C}[x,y]} \to 0 \to \cdots].$$

The cohomology of this complex is easy to compute (nothing new!), it gives

Lemma. $L\mathbb{D}[\mathcal{O}(Y)] \cong \mathcal{O}(Y)[-2]$.

2.7.5. *The general n.* The resolutions above for $n = 1, 2$ are example of the Koszul complex which we will meet later.

2.7.6. *Geometric nature of integers.* For the ring $\mathbb{C}[x_1, ..., x_n] = \mathcal{O}(\mathbb{A}^n)$ of functions on the n-dimensional affine space, the derived dual of the $\mathbb{C}[x_1, ..., x_n]$-module $\mathcal{O}(origin)$ is the same module $\mathcal{O}(origin)$ shifted to the degree n. So the shift is clearly the codimension of the origin in $\mathbb{A}^n$, and it is equal to n because $\dim(\mathbb{A}^n) = n$.

In view of the similarity of computations for $\mathcal{O}(\mathbb{A}^1) = \mathbb{C}[x]$ and for $\mathbb{Z}$, we may expect that $\mathbb{Z}$ is also a ring of functions on some geometric object, and that its dimension is one. So $\mathbb{Z}$ should correspond to some geometric object which we will denote $\mathrm{Spec}(\mathbb{Z})$, and $\mathrm{Spec}(\mathbb{Z})$ is some sort of a curve.

2.8. **Homology and derived constructions.**

2.8.1. *Resolutions and triangulations.* The idea is the same – explain complicated objects in terms of combining simple ones. This can be done in several ways so in the end one has to check that whatever we produced is independent of choices. (This we leave for later.)

3. **Sheaves**

Sheaves are a machinery which addresses an essential problem – the relation between local and global information – so they appear throughout mathematics.

3.1. **Definition.**

3.1.1. *Smooth functions on* $\mathbb{R}$. Let X be $\mathbb{R}$ or any smooth manifold. The notion of smooth functions on X gives the following data:

- for each open $U \subseteq X$ an algebra $C^\infty(U)$ (the smooth functions on U),
- for each inclusion of open subsets $V \subseteq U \subseteq X$ a map of algebras $C^\infty(U) \xrightarrow{\rho^U_V} C^\infty(V)$ (the restriction map);

and these data have the following properties

(1) (transitivity of restriction) $\rho^U_V \circ \rho^U_V = \rho^U_W$ for $W \subseteq V \subseteq U$,
(2) (gluing) if the functions $f_i \in C^\infty(U_i)$ on open subsets $U_i \subseteq X,\ i \in I$, are compatible in the sense that $f_i = f_j$ on the intersections $U_{ij} \stackrel{\text{def}}{=} U_i \cap U_j$, then they glue into a unique smooth function f on $U = \cup_{i \in I} U_i$.

The context of dealing with objects which can be restricted and glued compatible pieces is formalized in the notion of

3.1.2. *Sheaves on a topological space.* A sheaf of sets $\mathcal{S}$ on a topological space $(X, \mathcal{T})$ consists of the following data:

- for each open $U \subseteq X$ a set $\mathcal{S}(U)$,
- for each inclusion of open subsets $V \subseteq U \subseteq X$ a map $\mathcal{S}(U) \xrightarrow{\rho^U_V} \mathcal{S}(V)$ (called the restriction map);

and these data are required to satisfy

(1) (transitivity of restriction) $\rho^U_V \circ \rho^U_V = \rho^U_W$ for $W \subseteq V \subseteq U$,
(2) (gluing) Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of an open $U \subseteq X$. For a family of elements $f_i \in \mathcal{S}(U_i),\ i \in I$, compatible in the sense that $\rho^{U_i}_{U_{ij}} f_i = \rho^{U_i}_{U_{ij}} f_j$ in $\mathcal{S}(U_{ij})$ for $i, j \in I$; there is a unique $f \in \mathcal{S}(U)$ such that on the intersections $\rho^{U_i}_{U_{ij}} f = f_i$ in $\mathcal{S}(U_i),\ i \in I$.
(3) $\mathcal{S}(U) = \emptyset$.

We can equally define sheaves of abelian groups, rings, modules, etc – only the last, and least interesting requirement has to be modified, say in abelian groups we would ask that $\mathcal{S}(U)$ is the trivial group $\{0\}$.

3.1.3. *Examples.* (1) *Structure sheaves.* On a topological space X one has a sheaf of continuous functions C_X. If X is a smooth manifold there is a sheaf C_X^∞ of smooth functions, etc., holomorphic functions $\mathcal{H}_X$ on a complex manifold, "polynomial" functions $\mathcal{O}_X$ on an algebraic variety. In each case the topology on X and the sheaf contain all information on the structure of X.

(2) The constant sheaf S_X on X associated to a set S: $S_X(U)$ is the set of *locally* constant functions from U to X.

(3) Constant functions do not form a sheaf, neither do the functions with compact support. A given class $\mathcal{C}$ of objects forms a sheaf if it is defined by local conditions. For instance, being a (i) function with values in S, (ii) non-vanishing (i.e., invertible) function, (iii) solution of a given system $(*)$ of differential equations; are all local conditions: they can be checked in a neighborhood of each point.

3.2. **Global sections functor $\Gamma : \mathcal{S}heaves(X) \to \mathcal{S}ets$.** Elements of $\mathcal{S}(U)$ are called the sections of a sheaf $\mathcal{S}$ on $U \subseteq X$ (this terminology is from classical geometry). By $\Gamma(X, \mathcal{S})$ we denote the set $\mathcal{S}(X)$ of global sections.

The construction $\mathcal{S} \mapsto \Gamma(\mathcal{S})$ means that we are looking at global objects of a given class of objects defined by local conditions. We will see that it has a hidden part, the cohomology $\mathcal{S} \mapsto H^\bullet(X, \mathcal{S})$ of sheaves on X.

Γ recovers connected components i.e., H_0. On a smooth manifold X, $\Gamma(C^\infty) = C^\infty(X)$ is huge and that will be all: $H^{>0}$ will turn out to be zero (nothing hidden). The holomorphic setting is more subtle in this sense.

3.2.1. *Solutions of differential equations.* Solutions of a system $(*)$ of differential equation on X form a sheaf $\mathcal{S}ol_{(*)}$. If X is an interval in $\mathcal{R}$ and $(*)$ is one equation of the form $y^{(n)} + a_1(t) y^{(n-1)} + \cdots + a_0(t) = 0$ with $a_i \in C^\infty(X)$ then any point $c \in X$ gives evaluation isomorphism of vector spaces $\mathcal{S}ol_{(*)}(X) \xrightarrow{E_c} \mathbb{C}^n$ by $E_c(y) = (y(c), ..., y^{(n-1)}(c))$ (solutions correspond to initial conditions!).

Lemma. $\mathcal{S}ol_{(*)}$ is a constant sheaf on X.

On the other hand let $(*)$ be the equation $zy' = \lambda y$ considered as equation in holomorphic functions on $X = \mathbb{C}^*$. On any disc $c \in D \subseteq X$, evaluation still gives $\mathcal{S}ol_{(*)}(D) \xrightarrow{\cong} \mathbb{C}$, the solutions are multiples of functions z^λ defined using a branch of logarithm on D). However, $\Gamma(X, \mathcal{S}ol_{(*)}) = 0$ if $\lambda \notin \mathbb{Z}$ (any global solution would change by a factor of $e^{2\pi i \lambda}$ as we move once around the origin). So locally there is the expected amount but nothing globally.

3.3. **Projective line $\mathbb{P}^1$ over $\mathbb{C}$.** $\mathbb{P}^1 = \mathbb{C} \cup \infty$ can be covered by $U_1 = U = \mathbb{C}$ and $U_2 = V = \mathbb{P}^1 - \{0\}$. We think of $X = \mathbb{P}^1$ as a complex manifold by identifying U and V with $\mathbb{C}$ using coordinates u, v such that on $U \cap V$ one has $uv = 1$.

3.3.1. *Lemma.* $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$.

Proof. (1) *Proof using a cover.* A holomorphic function f on X restricts to $f|U = \sum_{n\geq 0} \alpha_n u^n$ and to $f|V = \sum_{n\geq 0} \beta_n v^n$. On $U \cap V = \mathbb{C}^*$, $\sum_{n\geq 0} \alpha_n u^n = \sum_{n\geq 0} \beta_n u^{-n}$, and therefore $\alpha_n = \beta_n = 0$ for $n \neq 0$.

(2) *Proof using maximum modulus principle.* The restriction of a holomorphic function f on X to $U = \mathbb{C}$ is a bounded holomorphic function (since X is compact), hence a constant.

3.4. **Čech cohomology of a sheaf $\mathcal{A}$ with respect to a cover $\mathcal{U}$.** Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X. We will use finite intersections $U_{i_0,\dots,i_p} \stackrel{\text{def}}{=} U_{i_0} \cap \cdots \cap U_{i_p}$.

3.4.1. *Calculations of global sections using a cover.* Motivated by the calculation of global sections in 3.3.1, to a sheaf $\mathcal{A}$ on X we associate

- Set $C^0(\mathcal{U}, \mathcal{A}) \stackrel{\text{def}}{=} \prod_{i\in I} \mathcal{A}(U_i)$ whose elements are systems $f = (f_i)_{i\in I}$ with one section $f_i \in \mathcal{A}(U_i)$ for each open set U_i,
- $C^1(\mathcal{U}, \mathcal{A}) \stackrel{\text{def}}{=} \prod_{(i,j)\in I^2} \mathcal{A}(U_{ij})$ whose elements are systems $g = (g_{ij})_{I^2}$ of sections g_{ij} on all intersections U_{ij}.

Now, if $\mathcal{A}$ is a sheaf of abelian groups we can reformulate the calculations of global sections of $\mathcal{A}$ in terms of "linear algebra data":

$$\Gamma(\mathcal{A}) \xrightarrow{\cong} \operatorname{Ker}[C^0(\mathcal{U}, \mathcal{A}) \xrightarrow{d} C^1(\mathcal{U}, \mathcal{A})]$$

for the map d which sends $f = (f_i)_I \in C^0$ to $df \in C^1$ with

$$(df)_{ij} = \rho^{U_j}_{U_{ij}} f_j - \rho^{U_i}_{U_{ij}} f_i.$$

More informally, $(df)_{ij} = f_j|U_{ij} - f_i|U_{ij}$.

3.4.2. *Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{A})$.* Emboldened, we try more of the same and define the abelian groups

$$\check{C}^n(\mathcal{U}, \mathcal{A}) \stackrel{\text{def}}{=} \prod_{(i_0,\dots,i_n)\in I^n} \mathcal{A}(U_{i_0,\dots,i_n})$$

of systems of sections on multiple intersections, and relate them by the maps $\check{C}^n(\mathcal{U}, \mathcal{A}) \xrightarrow{d^n} \check{C}^{n+1}(\mathcal{U}, \mathcal{A})$ which sends $f = (f_{i_0,\dots,i_n})_{I^n} \in C^n$ to $d^n\ f \in \check{C}^{n+1}$ with

$$(d^n\ f)_{i_0,\dots,i_{n+1}} = \sum_{s=0}^{n+1} (-1)^s f_{i_0,\dots,i_{s-1},i_{s+1},\dots,i_{n+1}}.$$

3.4.3. *Lemma.* $(\check{C}^\bullet(\mathcal{U}, \mathcal{A}), d^\bullet)$ is a complex, i.e., $d.\circ d^{n-1} = 0$.

3.4.4. *Čech cohomology* $\check{H}^\bullet(X,\mathcal{U};\mathcal{A})$. It is defined as the cohomology of the Čech complex $\check{C}^\bullet(\mathcal{U},\mathcal{A})$. We have already observed that

3.4.5. *Lemma.* $\check{H}^0(X,\mathcal{U};\mathcal{A}) = \Gamma(\mathcal{A})$.

3.4.6. *The "small Čech complex".* If the set I has a complete ordering, we can choose in $\check{C}^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \prod_{(i_0,\dots,i_n)\in I^n} \mathcal{A}(U_{i_0,\dots,i_n})$ a subgroup $C^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \prod_{i_0<\dots<i_n} \mathcal{A}(U_{i_0,\dots,i_n})$. This is what we will usually use in computations since it is smaller but it also computes the Čech cohomology:

3.4.7. *Lemma.* (a) $C^\bullet(\mathcal{U},\mathcal{A})\subseteq\check{C}^\bullet(\mathcal{U},\mathcal{A})$ is a subcomplex (i.e., it is invariant under the differential). (b) Map of complexes $C^\bullet(\mathcal{U},\mathcal{A})\subseteq\check{C}^\bullet(\mathcal{U},\mathcal{A})$ is a quasi-isomorphism.

Proof. Intuitively plausible since the data in $\check{C}$ which is not in $C^\bullet$ is a duplication of data in $C^\bullet$.

Lemma. $\check{H}^\bullet(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}) = \check{H}^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$.

Proof. Since the cover we use $\mathcal{U} = \{U_1,U_2\}$ has two elements, $C^n = 0$ for $n > 1$. We know $\check{H}^0 = \Gamma(\mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$, so it remains to understand $\check{H}^1 = C^1/dC^0 = \mathcal{O}(U\cap V)/[\mathcal{O}(U)+\mathcal{O}(V)$, i.e., all Laurent series $\phi = \sum_{-\infty}^{+\infty} \gamma_n u^n$ that converge on $\mathbb{C}^*$, modulo the series $\sum_0^{+\infty} \lambda_n u^n$ and $\sum_0^{+\infty} \beta_n u^{-n}$, that converge on $\mathbb{C}$ and on $\mathbb{P}^1 - 0$. However, if a Laurent series $\phi = \sum_{-\infty}^{+\infty} \gamma_n u^n$ converges on $\mathbb{C}^*$,then Laurent series $\phi^+ = \sum_0^{+\infty} \gamma_n u^n$ converges on $\mathbb{C}$, and $\phi^- = \sum_{-\infty}^{-1} \gamma_n u^n$ converge on $\mathbb{C}^*\cup\infty$.

3.5. **Cohomology of vector bundles.**

3.5.1. *Moebius strip.* It projects to the central curve S^1 and the fibers are real lines.

3.5.2. *Vector bundle over space X.* One can extend various notions to relative setting over some "base" X. A reasonable notion of a "vector space over a set X" is a collection $V = (V_x)_{x\in X}$ of vector spaces, one for each point of X. Then the total space $V = \sqcup_{x\in X} V_x$ maps to X and the fibers are vector spaces. If X is a topological space, we want the family of V_x to be continuous in x. This leads to the notion of a vector bundle over a topological space, and similarly over a manifold.

3.5.3. *Examples.* On each manifold X there are the tangent and cotangent vector bundles TX, T^*X. In terms of local coordinates x_i at a, the fibers are $T_aX = \oplus\ \mathbb{R}\frac{\partial}{\partial x_i}$ and $T_aX = \oplus\ \mathbb{R}dx_i$.

Moebius strip is a line bundle over S^1. One can describe it in terms of gluing of trivial vector bundles on an open cover.

This gives a general construction of vector bundles V by gluing trivial vector bundles $V_i = U_i{\times}\Bbbk^n$ on an open cover $\mathcal{U} = (U_i)_{i\in I}$, and the transition functions $\phi_{ij} : U_{ij} \rightarrow GL_n(\Bbbk)$. (Here $\Bbbk = \mathbb{R}$ or $\mathbb{C}$.)

$V = [\sqcup_{i\in I} U_i{\times}\Bbbk^n]/\sim$. where $V_i \ni (u_i, z) \sim (u_j, w) \in V_j$ if $u_i = u_j$ and $z = \phi_{ij}(u_j){\cdot}w$.

3.5.4. *Sheaf $\mathcal{V}$ associate to a vector bundle V.* Over an open $U{\subseteq}X$, $\mathcal{V}(U) \overset{\text{def}}{=}$ "sections of the vector bundle V over U". If V is obtained by gluing trivial vector bundles $V_i = U_i{\times}\mathbb{C}^n$ by transition functions ϕ_{ij}, then $\mathcal{V}(U)$ consists of all systems of $f_i \in \mathcal{H}(U_i \cap U, \mathbb{C}^n)$ such that on all intersections $U_{ij} \cap U$ one has $f_i = \phi_{ij} f_j$.

3.5.5. *Line bundles L_n on $\mathbb{P}^1$.* On $\mathbb{P}^1$ let L_n be the vector bundle obtained by gluing trivial vector bundles $U{\times}\mathbb{C}$, $V{\times}\mathbb{C}$ over $U \cap V$ by identifying $(u, \xi) \in U{\times}\mathbb{C}$ and $(v, \zeta) \in V{\times}\mathbb{C}$ if $uv = 1$ and $\zeta = u^n{\cdot}\xi$. So for $U_1 = U$ and $U_s = V$ one has $\phi_{12}(u) = u^n$, $U \in U \cap V{\subseteq}U$. Let $\mathcal{L}_n$ be the sheaf of holomorphic sections of L_n.

3.5.6. *Lemma.* (a) $\Gamma(\mathbb{P}^1, \mathcal{L}_n) \cong \mathbb{C}_n[x, y] \overset{\text{def}}{=}$ homogeneous polynomials of degree n. So, it is zero if $n < 0$ and for $n \geq 0$ the dimension is $n + 1$.

(b) $H^1(\mathbb{P}^1,\mathcal{U}; \mathcal{L}_n) \cong$?.

3.5.7. *Remark.* For distinct points $P_1, ..., P_n$ on $\mathbb{P}^1$, and integers D_i, define the sheaf $\mathcal{L} = \mathcal{O}(\sum\ D_iP_i)$ by $\mathcal{L}(U) \overset{\text{def}}{=}$ "all holomorphic functions f on $U - \{P_1, ..., P_n\}$, such that $\text{ord}_{P_i} f \geq -D_i$. Then $\mathcal{O}(\sum\ D_iP_i) \cong \mathcal{L}_{\sum\ D_i}$.

3.6. **Geometric representation theory.** SL_2 acts on $\mathbb{C}_2$ and therefore on (i) $\mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x, y]$, and (ii) each $\mathbb{C}_n[x, y]$; on (iii) $\mathbb{P}^1 =$ lines in $\mathbb{C}^2$, and less obviously on (iv) each $\mathcal{L}_n$, hence also on (v) each $H^\bullet(\mathbb{P}^1, \mathcal{L}_n)$. In fact,

3.6.1. *Lemma.* $\mathbb{C}_n[x, y] = \Gamma(\mathbb{P}^1, \mathcal{L}_n), n = 0, 1, 2, ..$ is the list of

(1) all irreducible finite dimensional holomorphic representations of SL_2.
(2) all irreducible finite dimensional representations of $SU(2)$ on complex vector spaces.

3.6.2. *Borel-Weil-Bott theorem.* For each semisimple (reductive) complex group G there is a space $\mathcal{B}$ (the flag variety of G) such that all irreducible finite dimensional holomorphic representations of G are obtained as global sections of all line bundles on $\mathcal{B}$.

3.7. **Cohomology of the constant sheaf is dual to homology.**

Lemma. For $X = S^1$, $\check{H}^*(X,\mathcal{U}; \Bbbk_X)$ is dual to $H_*(X, \Bbbk)$.

4. **D-modules**

A "D-module" on a space X means a module for the ring D_X of differential operators (or a sheaf of rings $\mathcal{D}_X$).

4.1. **Intro.**

4.1.1. (i) The origin of the theory is the idea of Sato to study a function f on X, by algebraic study of a D_X-module M_f associated to f. (ii) Similarly, a system $(*)$ of *linear* differential equations on X, has an algebraic reformulation – a module M_* for the ring of differential operators.

These two ideas may be viewed as parallel to the classical ideas: (i) algebraic number theory studies numbers via the algebraic equations they satisfy, (ii) once we observe that the choice of coordinates can be misleading, we pass from the study of systems of linear equations to linear algebra on vector spaces.

4.1.2. We will use D-modules as an illustration of usefulness of homological algebra:

(1) **Riemann-Hilbert correspondence.** The notion of a solution of a system $(*)$ of homogeneous linear differential equations, can be formulated in terms of an associated D-module M_*. This algebraic setting offers an extended meaning of solutions: the solutions of differential equations with singularities form a complex (rather then a single vector space). The higher cohomologies of this complex can be called "higher solutions" of $(*)$.

The strength of this is illustrated by the observation that on the level of complexes (i.e., when one uses the language of derived categories), the process of passing from differential equations to solutions gives an identification (the Riemann-Hilbert correspondence), of two realms:
   (a) differential equations with *regular* singularities,
   (b) constructible sheaves (a topological notion).

(2) **Functoriality.** As a tool for differential equations, we want the ability
   (a) to integrate a given system $(*)$ in certain directions,
   (b) to restrict $(*)$ to submanifolds,
   (c) to glue the information from various restrictions into information on the the original system $(*)$.

This amounts, abstractly speaking, to the following "functoriality" of differential equations (or D-modules):
   - when two spaces are related by a map $\pi : X \to Y$ we want to be able move differential equations from one to another.

This is well understood in the algebro-geometric setting of D-modules, and again, it is most efficient when performed on the level of derived categories.

In the exposition bellow we will intertwine (1) and (2), and use the RH-correspondence as an additional motivation for functoriality. Moreover, we will need to notice that sheaves have the same kind of functoriality as D-modules.

4.2. D-modules and differential equations.

4.2.1. *The algebra $D_{\mathbb{A}^n}$.* Let X be the affine space $\mathbb{A}^n$ over $\mathbb{C}$. Let D_X the ring of linear differential operators with polynomial coefficients, hence $D_X = \oplus_{I,J}\ \mathbb{C}x^I\partial^J$ and

4.2.2. *Lemma.* D_X is the $\mathbb{C}$-algebra generated by x_i's,∂_i's and the relations:

$$\text{(i) } x_i\text{'s commute, (ii) } \partial_i\text{'s commute, (iii) } [\partial_i, x_j] = \delta_{ij}.$$

4.2.3. *Natural modules for D_X.* These are various function spaces:

- (i) for U open in $\mathbb{R}^n$: $C^\infty(U)$, $\mathcal{D}(U) \overset{\text{def}}{=}$ the distributions on U, hyper-functions on U, etc;
- (ii) for V open in $\mathbb{C}^n$: $\mathcal{H}(V) =$ holomorphic functions on V,
- (iii) for W Zariski-open in $\mathbb{C}^n$ (i.e., the complement of some affine algebraic subvariety): $\mathcal{O}(W) \overset{\text{def}}{=}$ polynomial functions on W.

Observe that these examples of spaces of functions actually form sheaves of D_X-modules: restriction of functions is a map of D_X-modules.

4.2.4. *A system of linear differential equations with polynomial coefficients.* Such system

$$(*) \qquad \sum_{i=1}^{q} d_{ij}y_i = 0,\ 1 \le j \le p;$$

of p equations in q unknowns, is given by a matrix $d = (d_{ij}) \in M_{pq}(D_X)$. One can consider solutions in any D-module F:

$$Sol_*(\mathcal{F}) = \{f \in \mathcal{F}^q,\ \sum_{i=1}^{q} d_{ij}f_i = 0,\ 1 \le j \le p\}.$$

Moreover, if $\mathcal{F}$ is a sheaf, then we can define a sheaf of solutions $Sol_*(\mathcal{F})$ – its sections on $U\subseteq X$ are the solutions on U:

$$(Sol_*(\mathcal{F}))(U) \overset{\text{def}}{=} Sol_*(\mathcal{F}(U)).$$

4.2.5. *The D-module associated to a system* $(*)$. This passage $(*)\mapsto M_*$, from linear differential equations to D-modules, is the same as the passage from systems of linear equations to linear algebra (you loose the coordinates). Having q unknowns y_i, translates into a free D_X-module $F = (D_X)^q = \oplus_1^q \ D_X{\cdot}E_i$, where we denote its canonical D_X-basis by E_i. Imposing the equations $(*)$ translates into passing to the quotient $M_* \stackrel{\text{def}}{=} F/R$ by the relations submodule $R \stackrel{\text{def}}{=} \sum_1^p \ D_X{\cdot}(\sum_{i=1}^q \ d_{ij}E_i) \subseteq F$.

Denote by F_i be image of E_i in the quotient M_*. These elements generate $M_* = \sum \ D_X{\cdot}F_i$. Observe that F_i's form a "formal" solution of $(*)$, i.e., one has $\sum_{i=1}^q \ d_{ij}F_i = 0$ in M_*. This is actually the "universal" solution:

4.2.6. *Lemma.* For any D-module F:

$$Sol_*(\mathcal{F}) \cong \ \mathrm{Hom}_{D_X}(M, \mathcal{F}).$$

4.2.7. *Example.* For $(*)$ given by $y' = 0$, i.e., $\partial y = 0$, one has $F = D_X$ and $R = D_X{\cdot}\partial\subseteq F$, hence $M = D_X/D_X{\cdot}\partial \xleftarrow{\cong} \mathbb{C}[x]$. Now,

$$Sol_*(C^\infty(\mathbb{R})) = \ constant\ functions\ on\ \mathbb{R} = \ \mathrm{Hom}_{D_X}(M, C^\infty(\mathbb{R})).$$

4.3. **Higher solutions.** Singularities of differential equations will affect solutions in two ways:

- (i) there may be local solutions but no global solutions (due to "monodromy", i.e., a non-consistent behavior of solutions around a singularity),
- (ii) there may be no ordinary solutions but there may be higher ("hidden") solutions.

For (i), we saw that the equation $zy' = \lambda y$ has a singularity ($\stackrel{\text{def}}{=}$ bad behavior), at 0 (for $\lambda \notin \mathbb{Z}$): on a disc D that does not contain 0, solutions form a 1-dimensional vector space, but if $D \ni 0$ then there are no solutions. The simplest example of (ii) is

4.3.1. *δ-function.* We consider the simplest singular differential equations $zy = 0$. Its solutions in distributions are the multiples of the δ-function. If we view the problem in the holomorphic functions $\mathcal{H}(\mathbb{A}^1)$, then there are no solutions though morally there should be (the δ-function). To find these "hidden" solution we introduce the

4.3.2. *The derived notion of solutions.* We will derive the notion of solution using the algebraic formulation $Sol_*(\mathcal{F}) \cong \ \mathrm{Hom}_{D_X}(M_*, \mathcal{F})$. Our derived notion of solutions is

$$LSol_*(\mathcal{F}) \stackrel{\text{def}}{=} \ \mathrm{Hom}_{D_X}(P^\bullet, \mathcal{F})$$

for a free resolution $P^\bullet$ of M_*.

4.3.3. *δ-function.* Here $M_* = D_X/z{\cdot}D_X \xleftarrow{\cong} \mathbb{C}[\partial]$ has a resolution $\cdots \to 0 \to D_X \xrightarrow{R_z} D_X \xrightarrow{q} M_* \to 0 \to \cdots$, so $P^\bullet = (\cdots \to 0 \to \underset{-1}{D_X} \xrightarrow{R_z} \underset{0}{D_X} \to 0 \to \cdots)$ is a free resolution and therefore,

$$LSol_*(\mathcal{F}) \overset{\text{def}}{=} \text{Hom}(P^\bullet, \mathcal{F}) = (\cdots \to 0 \to \underset{0}{\text{Hom}_{D_X}(D_X, \mathcal{F})} \xrightarrow{z} \underset{1}{\text{Hom}_{D_X}(D_X, \mathcal{F})} \to 0 \to \cdots)$$
$$\cong (\cdots \to 0 \to \underset{0}{\mathcal{F}} \xrightarrow{z} \underset{1}{\mathcal{F}} \to 0 \to \cdots).$$

Therefore, $H^0[LSol_*(\mathcal{F})] = \text{Ker}(\mathcal{F} \xrightarrow{z} \mathcal{F})$ and $H^1[LSol_*(\mathcal{F})] = \mathcal{F}/z\mathcal{F}$.

So, if $\mathcal{F} = \mathcal{H}(\mathbb{A}^1)$, then $H^0 = \text{Ker}(z)$ is zero, and the the evaluation at zero map $H^1 = \mathcal{H}(\mathbb{A}^1)/z{\cdot}\mathcal{H}(\mathbb{A}^1) \xrightarrow{ev_0} \mathbb{C}$ is an isomorphism. So, there are no solutions, but there is a higher solution. The whole complex $LSol_*(\mathcal{H}(\mathbb{A}^1))$ is quasi-isomorphic to $\mathcal{H}(\mathbb{A}^1)/z\mathcal{H}(\mathbb{A}^1)[-1] \cong \mathbb{C}[-1]$, via the quasi-isomorphism

$$\begin{array}{ccccccccccc}
(\cdots & \longrightarrow & 0 & \longrightarrow & \underset{0}{\mathcal{H}(\mathbb{A}^1)} & \xrightarrow{L_z} & \underset{1}{\mathcal{H}(\mathbb{A}^1)} & \longrightarrow & 0 & \longrightarrow & \cdots) \\
\downarrow & & \downarrow & & \downarrow & & {\scriptstyle ev_0}\downarrow & & \downarrow & & \downarrow \\
(\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{C} & \longrightarrow & 0 & \longrightarrow & \cdots)
\end{array}.$$

We can say that the δ-function appears in the holomorphic picture in degree 1 (the codimension of the point!). In contrast, in the distribution picture it appears in degree zero: for $\mathcal{F} = \mathcal{D}(\mathbb{R})$ one has $H^0[LSol_*(\mathcal{F})] = \text{Ker}(\mathcal{F} \xrightarrow{z} \mathcal{F}) = \mathbb{C}\delta_0$ and $H^1[LSol_*(\mathcal{F})] = \mathcal{F}/z\mathcal{F} = 0$.

4.4. **Riemann-Hilbert correspondence: differential equations are the same as solutions.** Let X be a smooth complex algebraic variety. On the category of D_X-modules we will consider the functor $\mathcal{S}ol(M) \overset{\text{def}}{=} \text{Hom}_{\mathcal{D}_X}(M, \mathcal{H}_X)$ of solutions in the sheaf of holomorphic functions $\mathcal{H}_X$, and the corresponding derived construction of complexes of solution sheaves

$$L\mathcal{S}ol(M) \overset{\text{def}}{=} \mathcal{S}ol(P^\bullet) = \text{Hom}_{\mathcal{D}_X}(P^\bullet, \mathcal{H}_X),$$

which is defined by replacing M with a projective resolution $P^\bullet$.

4.4.1. *Theorem.* The derived construction of solutions in the sheaf of holomorphic functions $L\mathcal{S}ol$ gives a (contravariant) equivalence of derived categories

$$D^b[\mathfrak{m}_{RS}(\mathcal{D}_X)] \xrightarrow{L\mathcal{S}ol} D^b[Sheaves_c(X)].$$

This requires some explanation, or maybe a lot.

4.5. **Differential equations (or D-modules) with Regular Singularities.** Notation "RS" will mean "with regular singularities", and we now explain which D-modules are said to have regular singularities.

4.5.1. *Regular singular differential equations on curves.* Let X be open in $\mathbb{C}$ and consider a differential equation $(*)$ on X

$$Y^{(n)} + A_1(z)Y^{(n-1)} + \cdots + A_{n-1}(z)Y' + A_0(z)Y = 0;$$

with $Y \in \mathcal{H}(X, \mathbb{C}^n) = \mathcal{H}(X)^n$ (a column vector) and the (matrix valued) coefficient functions $A_i \in \mathcal{H}(X, M_n) = M_n(\mathcal{H}(X))$.

Consider a point $a \in \mathbb{C} - X$ such that X contains some punctured disc $D^* = D - \{a\}$ around a. Choose a local coordinate z on D centered at a. The coefficients of the equation need not be holomorphic at a, i.e., "$(*)$ may have a singularity at a". We say that $(*)$ has a regular singularity at a if the equation can be rewritten in terms of the coordinate z and $\partial = \partial/\partial z$ near a, so that it is of the type $dy = 0$ where d can be written in the form

$$d = \ (z\partial)^n + B_1(z)(z\partial)^{n-1} + \cdots + B_{n-1}(z)(z\partial) + B_n(z)$$

with *rational* (matrix valued) functions B_i.

Here, the appearance of z in front of ∂ causes a singularity (remember the example $d = z\partial - \lambda$). *Regular* means that we do not allow z^2 in front of ∂!

4.5.2. *Example.* Solve $(z^n\partial - 1)y = 0$ for $n = 0, 1, 2, 3$; and compare how complicated are the solutions.

4.5.3. *Frobenius method.* There is a satisfactory method for solving differential equations with RS. It usually appears as the first example of usefulness of linear algebra (the only simpler example I can think of is the Frobenius method for difference equations).

4.5.4. *D-modules on curves.* Via the relation between D-modules and differential equations, we will say that a D-module on X has RS if it comes from a differential equation with RS.

4.5.5. *Algebraic varieties.* For a general X, a D_X-module is said to have RS if its restriction to any algebraic curve in X has RS.

4.5.6. *Remarks.* (1) Differential equations with regular singularities are omnipresent in mathematics (Gauss-Manin connections and the Picard-Fuchs equations in algebraic geometry, differential equations that define hypergeometric functions and all classes of special functions,...). However, the equations of mathematical physics are usually irregular, and the irregular ones are less understood already in dimension 2.

(2) The general definition of RS uses the idea of restricting a D-module. This is a part of the functoriality property of D-modules:

4.6. **Functoriality of D-modules.** This means the correspondences of (complexes of) D-modules on spaces X and Y, that come from a map $X \xrightarrow{\pi} Y$. Here we describe the functoriality explicitly when X and Y are affine spaces. Next, in 4.7 we list the additional ingredients needed when X and Y are arbitrary smooth algebraic varieties.

4.6.1. *Affine spaces.* Let $X = \mathbb{A}^n \xrightarrow{\pi} Y = \mathbb{A}^m$. Then there are two direct image (push-forward) constructions $\mathfrak{m}(D_X) \xrightarrow{\pi_!,\pi_*} \mathfrak{m}(D_Y)$, and two inverse image (pull-back) constructions $\mathfrak{m}(D_X) \xleftarrow{\pi^!,\pi^*} \mathfrak{m}(D_Y)$. Among the properties are

(1) $(\sigma\circ\tau)_? = \sigma_?\circ\tau_?$ and $(\sigma\circ\tau)^? = \tau^?\circ\sigma^?$ for $? \in \{!, *\}$.
(2) These functors are best used in the larger world of complexes.

The graph of π gives a factorization of π into a composition of an embedding and a projection

$$X \xrightarrow{i:x\mapsto(x,\pi(x))} (X\times Y) \xrightarrow{q=pr_2} Y,$$

so by (1) it suffices to explain $\pi_?$ and $\pi^?$ when π is an embedding or a projection.

4.6.2. *Embeddings.* Let $X = \mathbb{A}^n$ and $Z = \mathbb{A}^p$ have coordinates x_i and z_j, so that on $Y = X\times Z = \mathbb{A}^{n+p}$ one has x_i, z_j's. Let $i : X\hookrightarrow Y,\ a\mapsto(a,0)$. (If $i(a) = (a,\pi(a))$ we can change the coordinates to x_i's and $z_j = y_j - \pi_j(x_1, ..., x_n)$, to be in the above situation.) Now

Lemma. (a) Define both $i_*(M)$ and $i_!(M)$ as $\oplus_J\ (\frac{\partial}{\partial z})^J M \cong\ \mathbb{C}[\frac{\partial}{\partial z}]\otimes_{\mathbb{C}} M$. These are functors $\mathfrak{m}(D_X) \xrightarrow{i_!=i_*} \mathfrak{m}(D_Y)$.

(b) $i^o(N) \stackrel{\text{def}}{=} N/\sum\ y_j N$ defines a functor $\mathfrak{m}(D_X) \xleftarrow{i^o} \mathfrak{m}(D_Y)$.

4.6.3. *Projections.* Let $X = \mathbb{A}^n$ and $Z = \mathbb{A}^m$ have coordinates x_i and z_j, so that on $Y = Z\times X$ one has x_i, z_j's. Let $q : Y\twoheadrightarrow X,\ (a, b)\mapsto b$ be the second projection.

Lemma. (a) $q_*(N) = \stackrel{\text{def}}{=} N/\sum\ \partial_j N$ defines a functor $\mathfrak{m}(D_X) \xleftarrow{q_*} \mathfrak{m}(D_Y)$.

(b) $q^o M \stackrel{\text{def}}{=} \oplus_J\ z^J M \cong\ \mathbb{C}[z_1, ..., z_p]\otimes_{\mathbb{C}} M$, defines a functor $\mathfrak{m}(D_X) \xrightarrow{q^o} \mathfrak{m}(D_Y)$.

4.6.4. *What does the functoriality mean for functions or differential equations?* Let us look for justification of the above definitions on grounds of algebra or of our experience with functions/differential equations.

- Growth from a smaller to a larger space:
  - $\boldsymbol{i_*}$. The formula comes from the idea that we can already act on M by x_i, ∂_{x_i}, and also by any function f on Y (we just need to first restrict it to X), so we have to add the action by ∂_{z_j}'s, and this we do by "formally" adding the derivatives normal to X in Y.

- $\boldsymbol{q^o}$. Similarly, we can already act on M by x_i, ∂_{x_i}, and also it is natural that ∂_{z_j}'s should act by 0 on M since the q-pull-back means "extending M to Y in a way constant along the fibers". So we only need to "formally" add the functions z_j on the fibers of q.

• Reduction from a larger space to a smaller one:

- $\boldsymbol{q_*}$. The natural ways of producing from a function ϕ on Y a function $q_!\phi$ on X boil down to taking average over the fibers: $(q_!\phi)(x_1, ..., x_n) = \int\ \phi(x_1, ..., x_n, z_1, ..., z_p)\ dz_1 \cdot\cdot\cdot dz_p$. Let us look only at $q : \mathbb{R} \to \mathrm{pt}$, so that $q_!\phi = \int_{\mathbb{R}}\ \phi(z)\ dz$, in order that it makes sense we need to be in the world of (something like) the compactly supported functions $C_c^\infty(\mathbb{R})$. Then the integral $q_!$ kills derivatives: $q_!(\partial\phi) = \phi(b) - \phi(a) = 0$ for a sufficiently large interval (a, b). In honor of this fact we have defined q_*N as $N/\sum\ \partial_{z_j}N$, i..e., the quotient obtained by killing all fiberwise derivatives in N.
- $\boldsymbol{i^o}$. The restriction i^oN of N to X should kill all z_j's since the restriction of these functions to X is 0, this motivates $N/\sum\ \partial_{z_j}N$.

4.6.5. *Derived functoriality.* It turns out that the "growth" functors above are exact (see 7.1), so we are only interested in deriving the "reduction" functors. We use left derived functors, i.e., we replace module N by its projective (usually free) resolution $P^\bullet$. So,

$$Lq_*(N) \overset{\text{def}}{=} q_*P^\bullet \qquad \text{and} \qquad Li^o(N) \overset{\text{def}}{=} i^oP^\bullet.$$

4.7. **D-modules on a smooth algebraic variety** $\boldsymbol{X}$**.** When working with arbitrary smooth algebraic varieties one needs (i) to replace the ring of differential operators with the sheaf of differential operators, (ii) use derived categories of D-modules.

On an algebraic variety X there may not be enough global differential operators (for instance a curve of genus > 1), so one is forced to look at local ones, i.e., at the sheaf $\mathcal{D}_X$ of differential operators on X. (Remember how few global holomorphic functions there are on $X = \mathbb{P}^1$!) A D-module on X will now mean a $\mathcal{D}_X$-module.

4.7.1. *Sheafifying algebra.* We define the notions of (i) a sheaf of rings $\mathcal{A}$ on X, (ii) a module $\mathcal{M}$ for a sheaf of rings $\mathcal{A}$, (iii) the category $\mathfrak{m}(\mathcal{A})$ of $\mathcal{A}$-modules.

4.7.2. *Sheaf of rings $\mathcal{D}_X$.* Let X be an algebraic variety, i.e., locally X can be identified with an affine algebraic variety (i.e., locally X is the set of solutions of several polynomial equations in some $\mathbb{C}^n$), and assume that X is smooth (i.e., locally looks like an open part of some $\mathbb{C}^m$). Then we know what "differential operators with polynomial coefficients" means in local coordinates. So, for any U Zariski-open in X we can define $\mathcal{D}_X(U) \overset{\text{def}}{=} D_U$.

4.7.3. *Functoriality of D-modules.* Let $X \xrightarrow{\pi} Y$ be a map of smooth algebraic varieties. When working with general algebraic varieties one is forced to use derived categories! So now the statement is:

Theorem. There are two direct image (push-forward) constructions

$$D^b[\mathfrak{m}(\mathcal{D}_X)] \xrightarrow{\pi_!,\pi_*} D^b[\mathfrak{m}(\mathcal{D}_Y)],$$

and two inverse image (pull-back) constructions

$$D^b[\mathfrak{m}(\mathcal{D}_X)] \xleftarrow{\pi^!,\pi^*} D^b[\mathfrak{m}(\mathcal{D}_Y)].$$

4.8. **Local systems.** Now we turn to the structure of solutions of differential equations. The solutions of ("nice") differential equations are complexes of constructible sheaves. A sheaf is said to be constructible if it can be obtained by gluing together "local systems". We will see that local systems have an algebraic description as representations of groups.

4.8.1. *Definition.* A sheaf $\mathcal{L}$ on a topological space X is said to be a local system if it is locally constant, i.e., X can be covered by open sets U such that the restriction $\mathcal{L}|U$ is isomorphic to some constant sheaf on U.

4.8.2. *Example.* The basic examples of local systems are the sheaves $\mathcal{S}_\lambda$ of solutions of $(z\partial - \lambda)y = 0$ on $X = \mathbb{C}^*$. Locally (say on each disk in X), they are constant sheaves – but not globally. For that observe that following a section around the origin gives the μ-multiple of the section, where the number $\mu = e^{2\pi i\lambda}$ is called the monodromy of the local system $\mathcal{S}_\lambda$ (on the positively oriented loop around the origin).

We will now restate this formally in the general case. In general monodromy will not be just a number $\mu \in \mathbb{C}^*$, but a representation of the so called fundamental group of X.

4.8.3. *Monodromy representation attached to a local system.* Let $\mathcal{L}$ be a local system (of vector spaces) on X and fix a point $b \in X$. Since $\mathcal{L}$ is locally isomorphic to a constant sheaf, there is a neighborhood D of a such that there is an isomorphism of $\mathcal{L}|D$ with a constant sheaf V_D on D for some vector space V. We assume that X is locally connected, so that one can choose D connected.

Let γ be an oriented loop in X starting and ending at b. We cover γ with some connected open sets $D_i,\ 0 \le i \le n$, so that

(1) $D_0 = D = D_n$
(2) $D_i \cap D_{i-1}$ is non-empty and connected,
(3) sequence $D_0, ..., D_n$ follows the direction of γ.

This gives an operator (monodromy operator along the sequence of D_i's) $\mu_{\gamma,D_0,\dots,D_n}$ on $\mathcal{L}(D)$ defined by: Any $s \in \mathcal{L}(D)$ extends in a unique way to a sequence $s_i \in \mathcal{L}(D_i)$ so that $s_0 = s$ and $s_i = s_{i-1}$ on $D_i \cap D_{i-1}$. Now, the action of the monodromy is the evolution of the section along the loop γ (or along the sequence $D_0, ..., D_n$):

$$\mu_{\gamma,D_0,\dots,D_n} s \stackrel{\text{def}}{=} s_n.$$

Now one observes that

(1) $\mu_{\gamma,D_0,\dots,D_n} = \mu_\gamma$, i.e., it depends only on μ and not on the choice of D_i's.
(2) Actually, if γ_1 and γ_2 are close enough (so that both lie in the same D_i's), then $\mu_{\gamma_1} = \mu_{\gamma_2}$.
(3) As a consequence we note that if γ_1 can be continuously deformed to γ_2 then $\mu_{\gamma_1} = \mu_{\gamma_2}$ (we can pass from one to another through a sequence of close enough curves). Therefore,
(4) μ_γ only depends on the homotopy class $[\gamma]$ of the loop γ (we say that γ_1 and γ_2 are homotopic if γ_1 can be continuously deformed to γ_2, this is an equivalence relation on loops through b).
(5) The set $\pi_1(X,b)$ of all homotopy classes of loops through b is a group: one can compose and reverse loops up to a deformation.
(6) $\mu : \pi_1(X,b) \to GL(\mathcal{L}(D))$ is a representation of the fundamental group on the vector space $\mathcal{L}(D))$, i.e.,
   - $\mu_{[\gamma]}$ is a linear operator,
   - $\mu_b = id_{\mathcal{L}(D)}$ for the constant loop b, and
   - $\mu_{[\gamma_2]\cdot[\gamma_1]} = \mu_{[\gamma_2]} \circ \mu_{[\gamma_1]}$.

One can check that $\mathcal{L}$ can be reconstructed from the representation μ, actually

4.8.4. *Lemma.* Monodromy gives a bijection of isomorphism classes of local systems on X and isomorphism classes of representations of the fundamental group.

A more fancy version is (the additional information is that monodromy preserves all relations between objects):

4.8.5. *Lemma.* Monodromy gives an equivalence of categories of $\mathcal{L}oc\mathcal{S}yst(X)$ of local systems on X and $Rep(\Pi_1(X,b))$ of representations of the fundamental group.

4.9. **Constructible sheaves.** "$Sheaves_c(X)$" means "constructible sheaves" on X. A sheaf $\mathcal{A}$ on X is constructible if there is a stratification of X, i.e., a decomposition $X = \sqcup_{i\in I} X_i$ into smooth algebraic subvarieties, such that the restriction of $\mathcal{A}$ to each stratum is locally isomorphic to a constant sheaf (i.e., it is a "local system").

A typical example is the sheaf of solutions of $zy' = \lambda y$ on $X = \mathbb{C}$.

4.9.1. *Remarks.* (1) For $1 = b \in X\mathbb{C}^*$ one has $\pi_1(*X,b) \cong \mathbb{Z}$ since any loop can be deformed to a loop which runs on the unit circle n times around the origin for some $n \in \mathbb{Z}$. A representation of $\pi_1(*X,b) \cong \mathbb{Z}$ on a vector space V is the same as an invertible linear operator on V and if $\dim(V) = 1$ it is the same as a non-zero number.

(2) The definition of constructible sheaves uses the notion of a restriction of a sheaf to a subvariety. This is a particular case of

4.10. **Functoriality of sheaves.** Let $X \xrightarrow{\pi} Y$ be a map of topological spaces. Then there are two direct image (push-forward) constructions $\mathcal{S}heaves(X) \xrightarrow{\pi_!,\pi_*} \mathcal{S}heaves(Y)$, and two inverse image (pull-back) constructions $\mathcal{S}heaves(X) \xleftarrow{\pi^!,\pi^*} \mathcal{S}heaves(Y)$. Among the properties are

(1) $(\sigma\circ\tau)_? = \sigma_?\circ\tau_?$ and $(\sigma\circ\tau)^? = \tau^?\circ\sigma^?$ for $? \in \{!, *\}$.
(2) These functors are best used in the larger world of complexes.

As usual one becomes familiar with these by studying the particular cases of an embedding or a projection.

Lemma. (a) $\pi_*(\mathcal{M})\ (V) \overset{\text{def}}{=} \mathcal{M}(\pi^{-1}V)$ defines a functor $\mathcal{S}heaves(X) \xrightarrow{\pi_*} \mathcal{S}heaves(Y)$.

(b) $\underline{\pi^{-1}}(\mathcal{N})\ (U) \overset{\text{def}}{=} \lim\limits_{\substack{\rightarrow \\ V\supseteq\pi(U)}} \mathcal{N}(V)$ defines a functor $pre\mathcal{S}heaves(X) \xrightarrow{\pi^{-1}} pre\mathcal{S}heaves(Y)$.

4.10.1. *Remarks.* (1) Actually, the second construction can be corrected (improved) to give a functor $\mathcal{S}heaves(X) \xrightarrow{\pi^{-1}} \mathcal{S}heaves(Y)$. More about it later. It will turn out that the operation π^{-1} has a more complicated definition but a much simpler behavior then π_* (π^{-1} is exact while π_* is only left exact).

(2) For $S\overset{i}{\subseteq}X$ by the restriction of a sheaf $\mathcal{A}$ on X to S we mean the sheaf $\mathcal{A}|S \overset{\text{def}}{=}\ i^{-1}\mathcal{A}$.

Part 2. Categories

Our first goal is to restrict a sheaf to a point, this involves the idea of a limit in a category. Section 5 covers basic categorical tricks, one of which is the appropriate notion of limits.

Next section 6 deals with our main goal. We want to extend the ideas that work in categories of modules over a ring to categories of sheaves of abelian groups (or sheaves of modules), and maybe some other categories. This is achieved by a more abstract setting of abelian categories which gives a good (but not at all most general) framework for homological algebra.

Finally, in section 8 we check that the category of sheaves of abelian groups on a given topological space is an abelian category and that homological algebra can be used in this category (it has enough injectives).

4.10.2. *Why categories?* The notion of a category is misleadingly elementary. It formalizes the idea that we study certain kind of objects (i.e., endowed with some specified structures) and that it makes sense to go from one such object to another via something (a "morphism") that preserves the relevant structures. Since this is indeed what we usually do, the *language* of categories is convenient.

However, soon one finds that familiar notions and constructions (such as (i) empty set, (ii) union of sets, (iii) product of sets, (iv) abelian group, ...) **categorify**, i.e., have analogues (and often more then one) in general categories (respectively: (i) initial object, final object, zero object; (ii) sum of objects or more generally a direct (inductive) limit of objects; (iii) product of objects or more generally the inverse (projective) limit of objects; (iv) additive category, abelian category; ...). This enriched language of categories was recognized as fundamental for describing various complicated phenomena, and the study of special kinds of categories mushroomed to the level of the study of functions with various properties in analysis.

5. Categories

5.1. **Categories.** A category $\mathcal{C}$ consists of

(1) a class $Ob(\mathcal{C})$ whose elements are called objects of $\mathcal{C}$,
(2) for any $a, b \in Ob(\mathcal{C})$ a set $\mathrm{Hom}_{\mathcal{C}}(a, b)$ whose elements are called morphisms ("maps") from a to b,
(3) for any $a, b, c \in Ob(\mathcal{C})$ a function $\mathrm{Hom}_{\mathcal{C}}(b, c) \times \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}}(a, c)$, called composition,
(4) for any $a \in Ob(\mathcal{C})$ an element $1_a \in \mathrm{Hom}_{\mathcal{C}}(a, a)$,

such that the composition is associative and 1_a is a neutral element for composition.

5.1.1. *Examples.* (1) Categories of sets with additional structures: $\mathcal{S}ets$, $\mathcal{A}b$, $\mathfrak{m}(\Bbbk)$ for a ring $\Bbbk$ (denoted also $\mathcal{V}ect(\Bbbk)$ if $\Bbbk$ is a field), $\mathcal{G}roups$, $\mathcal{R}ings$, $\mathcal{T}op$, $\mathcal{O}rd\mathcal{S}ets \stackrel{\text{def}}{=}$ category of ordered sets, ... (2) Another kind of examples: any partially ordered set $(I, \leq)$ defines a category with $Ob = I$ and $\mathrm{Hom}(a, b) =$ point (call this point (a, b)) if $a \leq b$ and $\emptyset$ otherwise. (3) Sheaves of sets on X, Sheaves of abelian groups on X, Constructible sheaves on X, ...

A category $\mathcal{C}$ is said to be small if $Ob(\mathcal{C})$ is a set. Instead of $a \in \mathrm{Ob}(\mathcal{C})$ we will just say $a \in \mathcal{C}$.

5.2. **Objects.**

5.2.1. *Some special objects and maps.* We say that $i \in \mathcal{C}$ is an initial object if for any $a \in \mathcal{C}$ set $\mathrm{Hom}_{\mathcal{C}}(i, a)$ has one element. $t \in \mathcal{C}$ is a terminal object if for any $a \in \mathcal{C}$ set $\mathrm{Hom}_{\mathcal{C}}(a, t)$ has one element. We say that $z \in \mathcal{C}$ is a zero object if it is both initial and terminal.

A map $\phi \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ is said to be an isomorphism if it is invertible, i.e., if there is a $\psi \in \mathrm{Hom}_{\mathcal{C}}(b, a)$ such that ...

Lemma. Initial object in $\mathcal{C}$ (if it exists) is unique up to a canonical isomorphism, i.e., for any two initial objects i, j in $\mathcal{C}$ there is a canonical isomorphism. (The same for terminal and zero objects.)

5.2.2. *The useful notion of "equality" of objects.* This introduces an essential subtlety: while in a set two elements may be equal or different, in a category the corresponding notion for two objects has versions: (i) the same, (ii) isomorphic, (iii) isomorphic by a canonical (given) isomorphism. It turns out that (i) is too restrictive, (ii) is too lax and (iii) is the most useful. So we will often say that $a = b$ and mean that we have in mind a specific isomorphism $\phi : a \rightarrow b$.

5.2.3. *Sums and products.*

5.2.4. *Products.* A product of objects a and b in $\mathcal{C}$ is a triple (Π, p, q) where $\Pi \in \mathcal{C}$ is an object while $p \in \mathrm{Hom}_{\mathcal{C}}(\Pi, a)$, $q \in \mathrm{Hom}_{\mathcal{C}}(\Pi, b)$ are maps such that for any $x \in \mathcal{C}$ the function

$$\mathrm{Hom}_{\mathcal{C}}(x, \Pi) \ni \phi \mapsto (p \circ \phi, q \circ \phi) \in \mathrm{Hom}_{\mathcal{C}}(x, a) \times \mathrm{Hom}_{\mathcal{C}}(x, b)$$

is a bijection.

Remarks. (0) If $\mathcal{C} = \mathcal{S}ets$ then the product of sets $\Pi = a \times b$ together with projections p, q satisfies this property.

(1) While we expect that a product of a and b should be a specific object built from a and b, this is not what the categorical definition above says. For given a, b there may be many triples (Π, p, q) satisfying the product property – however it is easy to see that any two such $(\Pi_i, p_i, q_i), i = 1, 2$; are related by a *canonical isomorphism* $\phi : \Pi_1 \longrightarrow \Phi_2$ provided by the the defining property of the product. This is another example of 5.2.2.

(2) We often abuse the language and say that "Π is product of a and b", what we mean is that we remember the additional data p, q but are too lazy to mention these. Moreover, we may even denote Π by $a \times b$ suggesting (in general incorrectly) that the product can be constructed naturally from a and b – what we mean by notation $a \times b$ is *some* object Π supplied with maps p, q with properties as above (again an example of the idea 5.2.2).

(3) The product of a and b is an example of a standard construction of "an object defined by a universal property" (in this case the property that a map into a product is the same as a pair of maps into a and b), or "an object (co)representing a functor" (in this case Π corepresents a contravariant functor $\mathcal{C} \ni x \mapsto F(x) \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{C}}(x, a) \times \text{Hom}_{\mathcal{C}}(x, b) \in \mathcal{S}ets$, in the sense that the functor F is identified with the functor $\text{Hom}_{\mathcal{C}}(-, \Pi)$ that one gets from Π). As we see, "an object defined by a universal property": (i) is not really one object but a system of various objects related by (compatible) isomorphisms, (ii) is not really just one object but an object supplied with some additional data consisting of some morphisms (such as p, q above).

(4) A product of a and b in $\mathcal{C}$ need not exist!

5.2.5. *Sums.* A sum of objects a and b in $\mathcal{C}$ is a triple (Σ, i, j) where $\Sigma \in \mathcal{C}$ is an object while $i \in \text{Hom}_{\mathcal{C}}(a, \Sigma)$, $j \in \text{Hom}_{\mathcal{C}}(b, \Sigma)$ are maps such that for any $x \in \mathcal{C}$ the function

$$\text{Hom}_{\mathcal{C}}(\Sigma, x) \ni \phi \mapsto (\phi \circ i, \phi \circ j) \in \text{Hom}_{\mathcal{C}}(a, x) \times \text{Hom}_{\mathcal{C}}(b, x)$$

is a bijection.

Example. In $\mathcal{S}ets$ the sums exist and the sum of a and b is the disjoint union $a \sqcup b$.

5.2.6. *Sums and products of families of objects.* This is the same as for two objects. A product of $a_i \in \mathcal{C}$, $i \in I$ is a pair $(P, (p_i)_{i \in I}))$ where $p_i : P \longrightarrow a_i$ gives a bijection

$$\text{Hom}_{\mathcal{C}}(x, P) \ni \phi \mapsto (p_i \circ \phi)_{i \in I} \in \Pi_{i \in I} \text{Hom}_{\mathcal{C}}(x, a_i).$$

We use the notation $\prod_{i \in I} a_i$. A sum of $a_i \in \mathcal{C}$, $i \in I$ is a pair $(S, (j_i)_{i \in I}))$ where $j_i : a_i \longrightarrow S$ gives a bijection

$$\text{Hom}_{\mathcal{C}}(S, x) \ni \phi \mapsto (\phi \circ j_i)_{i \in I} \in \Pi_{i \in I} \text{Hom}_{\mathcal{C}}(a_i, x).$$

The notation is $\sqcup_{i \in I} a_i$ or $\oplus_{i \in I} a_i$.

Lemma. For a ring $\Bbbk$ category $\mathfrak{m}(\Bbbk)$ has sums and products. The product $\prod_{i\in I} M_i$ is (as a set) just the product of sets, so it consists of all families $m = (m_i)_{i\in I}$ with $m_i \in M_i,\ i \in I$ (such family is often written as a possibly infinite sum $\sum_{i\in I} m_i \overset{\text{def}}{=} (m_i)_{i\in I}$). The sum $\oplus_{i\in I} a_i$ happens to be the submodule of $\prod_{i\in I} M_i$ consisting of all finite families $m = (m_i)_{i\in I}$, i.e.families such that $m_i = 0$ for all but finitely many $i \in I$.

In this way we get familiar with categorical constructions: by checking what they mean in familiar categories.

5.3. **Limits.** Consider a system of increasing subsets $A_0 \subseteq A_1 \subseteq \cdots$ of a set A, we will say that its limit $\lim\limits_{\rightarrow} A_i$ is the subset $\cup_{i\geq 0} A_i$ of A. Similarly, the limit of a decreasing sequence of subsets $B_0 \supseteq B_1 \subseteq \cdots$ of A, will be the subset $\lim\limits_{\leftarrow} B_i \overset{\text{def}}{=} \cap_{i\geq 0} B_i$ of A. Now we give precise meaning to this:

5.3.1. *Inductive limits.* An inductive system of objects of $\mathcal{C}$ over a partially ordered set $(I, \leq)$, consists of objects $a_i \in \mathcal{C},\ i \in I$; and maps $\phi_{ji} : a_i \to a_j$ for all $i \leq j$ in I; such that $\phi_{ii} = 1_{a_i},\ i \in I$ and $\phi_{kj} \circ \phi_{ji} = \phi_{ki}$ when $i \leq j \leq k$.

Its limit is a pair $(a, (\rho_i)_{i\in I})$ of $a \in \mathcal{C}$ and maps $\rho_i : a_i \to a$ such that

- $\rho_j \circ \phi_{ji} = \rho_i$ for $i \leq j$, and moreover
- $(a, (\rho_i)_{i\in I})$ is universal with respect to this property in the sense that for any $(a', (\rho'_i)_{i\in I})$ that satisfies $\rho'_j \circ \phi_{ji} = \rho'_i$ for $i \leq j$, there is a unique map $\rho : a \to a'$ such that $\rho'_i = \rho \circ \rho_i,\ i \in I$.

Informally, we write $\lim\limits_{\rightarrow\ I,\leq} a_i = a$.

5.3.2. *Example.* A system of increasing subsets $A_0 \subseteq A_1 \subseteq \cdots$ of A really form an inductive system in the category $\mathcal{S}ets$ (and over $\mathbb{N}$ with the standard order) and $\lim\limits_{\rightarrow} A_i = \cup_{i\geq 0} A_i$.

5.3.3. *Projective limits.* A projective system of objects of $\mathcal{C}$ over a partially ordered set $(I, \leq)$, consists of objects $a_i \in \mathcal{C},\ i \in I$; and maps $\phi_{ij} : a_j \to a_i$ for all $i \leq j$ in I; such that $\phi_{ii} = 1_{a_i},\ i \in I$ and $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ when $i \leq j \leq k$.

Its limit is a pair $(a, (\sigma_i)_{i\in I})$ of $a \in \mathcal{C}$ and maps $\sigma_i : a \to a_i$ such that

- $\phi_{ij} \circ \sigma_j = \sigma_i$ for $i \leq j$, and
- $(a, (\sigma_i)_{i\in I})$ is universal in the sense that for any $(a', (\sigma'_i)_{i\in I})$ that satisfies $\phi_{ij} \circ \sigma'_j = \sigma'_i$ for $i \leq j$, there is a unique map $\sigma : a' \to a$ such that $\sigma'_i = \sigma_i \circ \sigma,\ i \in I$.

Informally, $\lim\limits_{\leftarrow\ I,\leq} a_i = a$.

Example. A decreasing sequence of subsets $B_0 \supseteq B_1 \subseteq \cdots$ of A forms a projective system and $\lim\limits_{\leftarrow} B_i = \cap_{i\geq 0} B_i$ of A.

In these basic examples we have: $\lim\limits_{\rightarrow}$ = growth) and $\lim\limits_{\leftarrow}$ = decline, but it can also go the opposite way:

5.3.4. *Lemma.* (a) Let $(I, \leq)$ be $\{1, 2, 3, ...\}$ with the order $i \leq j$ if i divides j. In $\mathcal{A}b$ let $A_i = \mathbb{Q}/\mathbb{Z}$ for all $i \in I$, and let ϕ_{ji} be the multiplication by j/i when i divides j. This is an inductive system and $\lim\limits_{\rightarrow} A_i =?$.

(b) Let $(I, \leq)$ be $\mathbb{N} = \{0, 1, ...\}$ with the standard order. In $\mathcal{R}ings$ let $\mathbb{k}_n = \mathbb{C}[x]/x^{n+1}$ and for $i \leq j$ let ϕ_{ij} be the obvious quotient map. This is a projective system and $\lim\limits_{\leftarrow} \mathbb{k}_i =?$.

5.3.5. *Limits are functorial.* Let $\mathcal{IS}_{(I,\leq)}(\mathcal{C})$ be the category of inductive systems in the category $\mathcal{C}$ and over a partially ordered set $(I, \leq)$. Objects are the inductive systems $(A_i)_{i\in I}, (\phi_{ji})_{i\leq j}$ and a map μ from $(A'_i)_{i\in I}, (\phi'_{ji})_{i\leq j}$ to $(A''_i)_{i\in I}, (\phi''_{ji})_{i\leq j}$ is a system of maps $\mu_i : A'_i \rightarrow A''_i$, $i \in I$, which intertwine the structure maps of the two inductive systems i.e., for $i \leq j$ the diagram

$$\begin{array}{ccc} A'_i & \xrightarrow{\mu_i} & A''_i \\ {\scriptstyle \phi'_{ji}}\downarrow & & \downarrow{\scriptstyle \phi''_{ji}} \\ A'_j & \xrightarrow{\mu_i} & A''_j \end{array}$$

commutes.

Lemma. If limits of both systems exist then a map μ of systems defines a map $\lim\limits_{\rightarrow} A'_i \xrightarrow{\lim\limits_{\rightarrow} \mu_i} \lim\limits_{\rightarrow} A''_i$.

Proof. By definition of $\lim\limits_{\rightarrow}$.

5.3.6. *Limits in sets, abelian groups, modules and such.* In the category $\mathcal{S}ets$ one has inductive and projective limits (i.e., each inductive or projective system has a limit):

5.3.7. *Lemma.* Let $(I, \leq)$ be a partially ordered set such that for any $i, j \in I$ there is some $k \in I$ such that $i \leq k \geq j$.

(a) (Construction of projective limits of sets.) Let $(A_i)_{i\in I}$ and maps $(\phi_{ij} : A_j \rightarrow A_i)_{i\leq j}$ be a projective system of sets. Then

$$\lim_{\leftarrow} A_i \text{ is the subset of } \prod_{i\in I} A_i \text{ consisting of all families } a = (a_i)_{i\in I} \text{ such that } \phi_{ij} a_j = a_i.]$$

b) (Construction of inductive limits of sets.) Let the family of sets $(A_i)_{i\in I}$ and maps $(\phi_{ji} : A_i \rightarrow A_j)_{i\leq j}$ be an inductive system of sets.

(1) Show that the relation $\sim$ defined on the disjoint union $\sqcup_{i\in I}\ A_i \stackrel{\text{def}}{=} \cup_{i\in I}\ A_i\times\{i\}$ by
  - $(a,i)\sim(b,j)$ (for $a\in A_i,\ b\in A_j$), if there is some $k\ge i,j$ such that "$a=b$ in A_k", i.e., if $\phi_{ki}a=\phi_{kj}b$,

  is an equivalence relation.

(2) Show that $\lim\limits_{\rightarrow} A_i$ is the quotient $[\sqcup_{i\in I}\ A_i]/\sim$ of the disjoint union by the above equivalence relation.

5.3.8. *Remarks.* (1) The above lemma gives simple descriptions of limits, $\lim\limits_{\leftarrow} A_i$ is a subset of the product $\prod A_i$, and $\lim\limits_{\rightarrow} A_i$ is a quotient of a sum (=disjoint union). Even better, $\lim\limits_{\rightarrow} A_i$ can be described by

- for $i\in I$, any $a\in A_i$ defines an element $\overline{a}$ of $\lim\limits_{\rightarrow} A_i$,
- all elements of $\lim\limits_{\rightarrow} A_i$ arise in this way, and
- for $a\in A_i$ and $b\in A_j$ one has $\overline{a}=\overline{b}$ iff for some $k\in I$ with $i\le k\ge j$ one has $a=b$ in A_k.

(This is just a simple retelling of the lemma, here $\overline{a}$ is the image of (a,i) in the quotient $[\sqcup_{i\in I}\ A_i]/\sim$.)

(2) The same existence and description of limits works in many categories such as abelian groups, $\Bbbk$-modules, $\mathcal{G}roups$, etc (these are all categories where objects are sets with some additional structure). For instance for an inductive system of abelian groups A_i over $(I,\le)$, the inductive limit $\lim\limits_{\rightarrow} A_i$ always exists, it can be described as in (1), but one has to also explain what is the group structure (addition) on the set $[\sqcup_{i\in I}\ A_i]/\sim$. This is clear, if $a\in A_i$ and $b\in A_j$ then $\overline{a}+\overline{b}=\overline{\phi_{ki}a+\phi_{kj}b}$ for any k with $i\le k\ge j$.

(c) Actually the lemma generalizes to limits in an arbitrary category (see lemma 5.3.12 bellow), but then we have to explicitly ask for the existence of some standard constructions. For instance the category of finite dimensional vector spaces (or even simpler, the category of finite sets) clearly does not have infinite sums or products, hence can not have limits.

5.3.9. *Stalks of a sheaf.* We want to restrict a sheaf $\mathcal{F}$ on a topological space X to a point $a\in X$. The restriction $\mathcal{F}|a$ is a sheaf on a point, so it just one set $\mathcal{F}_a \stackrel{\text{def}}{=} (\mathcal{F}|a)(\{a\})$ called the stalk of $\mathcal{F}$ at a. What should $\mathcal{F}_a$ be? It has to be related to all $\mathcal{F}(U)$ where $U\subseteq X$ is is open and contains a, and $\mathcal{F}(U)$ should be closer to $\mathcal{F}_a$ when U is a smaller neighborhood. A formal way to say this is that

- (i) the set $\mathcal{N}_a$ of neighborhoods of a in X is partially ordered by $U\le V$ if $V\subseteq U$, (ii) the values of $\mathcal{F}$ on neighborhoods $(\mathcal{F}(U))_{U\in\mathcal{N}_a}$ form an inductive system, (iii) we define the stalk by $\mathcal{F}_a \stackrel{\text{def}}{=} \lim\limits_{\substack{\rightarrow\\ U\in\mathcal{N}_a}} \mathcal{F}(U)$.

Lemma. (a) The stalk of a constant sheaf of sets $S_{\mathbb{R}^n}$ at any point is canonically identified with the set S.

(b) The stalk of a the sheaf $\mathcal{H}_{\mathbb{C}}$ of holomorphic functions at the origin is canonically identified with the ring of convergent power series. ("Convergent" means that the series converges on *some* disc around the origin.)

5.3.10. *Fibered products as limits.* First, notice that sums and products are special cases of limits:

Lemma. Let $a_i \in \mathcal{C},\ i \in I$. If we supply I with the discrete partial order (i.e., $i \leq j$ iff $i = j$), then $\underset{\rightarrow}{\lim}\ a_i$ is the same as $\oplus_{i\in I} a_i$, and $\underset{\leftarrow}{\lim}\ a_i$ is the same as $\prod_{i\in I} a_i$.

Next, given $a_1 \xrightarrow{p} b \xleftarrow{q} a_2$ in $\mathcal{C}$, we say that the fibered product of a_1 and a_2 over the base b (denoted $a_1 \times_b a_2$) is the projective limit of the projective system $a_i,\ i \in I$ where

- (i) $I = \{1, 2, k\}$ and $a_k = b$,
- (ii) the only nontrivial inequalities in I are $1 \geq k \leq 2$.

Lemma. (a) In $\mathcal{S}ets$ fibered products exist and the fibered product of $A \xrightarrow{p} S \xleftarrow{q} B$ is the set $A \underset{S}{\times} B = \{...\}$.

(b) If $A \xrightarrow{\subseteq} S \xleftarrow{\supseteq} B$ are inclusions of subsets of S the fibered product $A \underset{S}{\times} B =\ A \cap B$ is just the intersection $A \cap B$.

5.3.11. *Existence and construction of limits.* The following lemma gives a criterion for existence of limits in a category $\mathcal{C}$, and a way to describe them in terms of simpler constructions. The particular case when $\mathcal{C}$ is the category of sets is the lemma 5.3.6 – it is less abstract since we assume some familiar properties of sets. The general proof is "the same".

5.3.12. *Lemma.* (a) If a category $\mathcal{C}$ has

(1) products of families of objects and
(2) any pair of maps $\mu, \nu \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ has an equalizer (i.e., a map $e \xrightarrow{\sigma} a$ universal among all maps ϕ into a such that $\mu \circ \phi = \nu \circ \phi$),

then $\mathcal{C}$ has projective limits (and they can be described in terms of products and equalizers).

(b) Dually, if a category $\mathcal{C}$ has

(1) sums of families of objects and
(2) any pair of maps $\mu, \nu \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ has a coequalizer (i.e., a map $b \xrightarrow{\sigma} c$ universal among all maps ϕ from b such that $\phi \circ \mu = \phi \circ \nu$),

then $\mathcal{C}$ has inductive limits (and they can be described in terms of sums and coequalizers).

5.4. **Categories and sets.** Some of the relations:

- All sets form a category $\mathcal{S}ets$ which can be viewed as the basic example of a category.
- Each set S defines a (small) category $\underline{S}$ with $Ob(\underline{S}) = S$ and for $a, b \in S$ $\mathrm{Hom}_{\underline{S}}(a,b)$ is $\{1_a\}$ if $a = b$ and it is empty otherwise. In the opposite (and more stupid) direction, each small category $\mathcal{C}$ gives a set $Ob(\mathcal{C})$ (we just forget the morphisms).
- The structure of a category can be viewed as a more advanced version of the structure of a set.

5.4.1. *Question.* If all sets form a structure more complicated then a set – a category $\mathcal{S}ets$, what do all categories form? (All categories form a more complicated structure, a 2-category. Moreover all n-categories form an $(n+1)$-category ...)

5.4.2. *Operations on categories.* The last remark suggests that what we can do with sets, we should be able to do with categories (though it may get more complicated).

For instance the product of sets lifts to a notion of a product of categories $\mathcal{A}$ and $\mathcal{B}$, The category $\mathcal{A}\times\mathcal{B}$ has $Ob(\mathcal{A}\times\mathcal{B}) = Ob(\mathcal{A})\times Ob(\mathcal{B})$ and $\mathrm{Hom}_{\mathcal{A}\times\mathcal{B}}[(a',b'),(a'',b'')] \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{A}}[a',a'']\times\mathrm{Hom}_{\mathcal{B}}[b',b'']$.

However, since we are dealing with a finer structure there are operations on categories that do not have analogues in sets. Say the dual (opposite) category of $\mathcal{A}$ is the category $\mathcal{A}^o$ with $Ob(\mathcal{A}^o) = Ob(\mathcal{A})$, but

$$\mathrm{Hom}_{\mathcal{A}^o}(a,b) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{A}}(b,a).$$

This is the formal meaning of the observation that reversing the arrows gives a “duality operation” for constructions in category theory. For instance, a projective system in $\mathcal{A}$ is the same as an inductive system in $\mathcal{A}^o$, a sum in $\mathcal{A}$ is the same as a product in $\mathcal{A}^o$, etc. This is useful: for any statement we prove for projective systems there is a “dual” statement for inductive systems which is automatically true.

5.5. **Functors.** The analogue on the level of categories of a function between two sets is a functor between two categories.

A functor F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of

- for each object $a \in \mathcal{A}$ an object $F(a) \in \mathcal{B}$,
- for each map $\alpha \in \mathrm{Hom}_{\mathcal{A}}(a',a'')$ in $\mathcal{A}$ a map $F(\alpha) \in \mathrm{Hom}_{\mathcal{B}}(Fa',Fa'')$

such that F preserves compositions and units, i.e., $F(\beta\circ\alpha) = F(\beta)\circ F(\alpha)$ and $F(1_a) = 1_{Fa}$.

5.5.1. *Examples.* (1) A functor means some construction, say a map of rings $\Bbbk \xrightarrow{\phi} \langle$ gives

- a pull-back functor $\phi^* : \mathfrak{m}(\langle) \to \mathfrak{m}(\Bbbk)$ where $\phi^* N = N$ as an abelian group, but now it is considered as module for $\Bbbk$ via ϕ.
- a push-forward functor $\phi_* : \mathfrak{m}(\Bbbk) \to \mathfrak{m}(\langle)$ where $\phi_* M \overset{\text{def}}{=} \langle \otimes_{\Bbbk} M$. This is called "change of coefficients".

To see that these are functors, we need to define them also on maps. So, a map $\beta : N' \to N''$ in $\mathfrak{m}(\langle)$ gives a map $\phi^*(\beta) : \phi^*(N') \to \phi^*(N'')$ in $\mathfrak{m}(\Bbbk)$ which as a function between sets is really just $\beta : N' \to N''$. On the other hand, $\alpha : M' \to M''$ in $\mathfrak{m}(\Bbbk)$ gives $\phi_*(\alpha) : \phi_*(M') \to \phi_*(M'')$ in $\mathfrak{m}(\langle)$, this is just the map $1_{\langle} \otimes \alpha : \ \langle \otimes_{\Bbbk} M' \to \langle \otimes_{\Bbbk} M'',\ c \otimes x \mapsto c \otimes \alpha(x)$.

Here we see a general feature: functors often come in pairs ("adjoint pairs of functors") and usually one of them is stupid and the other one an interesting construction.

(2) For any category $\mathcal{A}$ there is the identity functor $1_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. Two functors $\mathcal{A} \xrightarrow{F} \mathcal{B}$ and $\mathcal{B} \xrightarrow{G} \mathcal{C}$ can be composed to a functor $\mathcal{A} \xrightarrow{G \circ F} \mathcal{C}$.

(3) An object $a \in \mathcal{A}$ defines two functors, $\mathrm{Hom}_{\mathcal{A}}(a, -) : \ \mathcal{A} \to \mathcal{S}ets$, and $\mathrm{Hom}_{\mathcal{A}}(-, a) : \mathcal{A}^o \to \mathcal{S}ets$. Moreover, $\mathrm{Hom}_{\mathcal{A}}(-, -)$ is a functor from $\mathcal{A}^o \times \mathcal{A}$ to sets!

(4) For a ring $\Bbbk$, tensoring is a functor $- \otimes_{\Bbbk} - : \ \mathfrak{m}^r(\Bbbk) \times \mathfrak{m}^l(\Bbbk) \to \mathcal{A}b$.

5.5.2. *Contravariant functors.* We say that a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is given by assigning to any $a \in \mathcal{A}$ some $F(a) \in \mathcal{B}$, and for each map $\alpha \in \mathrm{Hom}_{\mathcal{A}}(a', a'')$ in $\mathcal{A}$ a map $F(\alpha) \in \mathrm{Hom}_{\mathcal{B}}(Fa'', Fa')$ – notice that we have changed the direction of the map so now we have to require $F(\beta \circ \alpha) = F(\alpha) \circ F(\beta)$ (and $F(1_a) = 1_{Fa}$).

This is just a way of talking, not a new notion since a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is the same as a functor F from $\mathcal{A}$ to $\mathcal{B}^o$ (or a functor F from $\mathcal{A}^o$ to $\mathcal{B}$).

5.5.3. *Categorification of "subset".* The categorical analogue of the notion of a subset of a set is the notion of a (full) subcategory of a category.

A subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is given by a subclass $Ob(\mathcal{C}') \subseteq Ob(\mathcal{C})$ and for any $a, b \in Ob(\mathcal{C}')$ a subset $\mathrm{Hom}_{\mathcal{C}'}(a, b) \subseteq \mathrm{Hom}_{\mathcal{C}}(a, b)$ such that $\mathrm{Hom}_{\mathcal{C}'}(a, a) \ni 1_a,\ a \in \mathcal{C}'$, and the sets $\mathrm{Hom}_{\mathcal{C}'}(a, b),\ a, b \in \mathcal{C}'$ are closed under the composition in $\mathcal{C}$.

A full subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is a subcategory $\mathcal{C}'$ such that $\mathrm{Hom}_{\mathcal{C}'}(a, b) \subseteq \mathrm{Hom}_{\mathcal{C}}(a, b)$ for $a, b \in \mathcal{C}'$.

5.5.4. *Examples.* (1) $\mathcal{F}ree(\Bbbk) \subseteq \mathfrak{m}(\Bbbk)$. (2) Category $\mathcal{C}$ defines subcategory $\mathcal{C}^*$ where objects are the same and morphisms are the isomorphisms from $\mathcal{C}$.

5.5.5. *Some categorifications of "injection", "surjection" and "bijection".* The following properties of a functor $F : \mathcal{A} \hookrightarrow \mathcal{B}$ are categorical analogues of the injectivity property of a function:

- F is faithful (or "an embedding of categories") if all maps $\mathrm{Hom}_{\mathcal{A}}(a', a'') \to \mathrm{Hom}_{\mathcal{B}}(Fa', Fa'')$, $a', a'' \in \mathcal{A}$ are injective.
- F is called fully faithful (or a full embedding) if all maps $\mathrm{Hom}_{\mathcal{A}}(a', a'') \to \mathrm{Hom}_{\mathcal{B}}(Fa', Fa'')$ are bijections.

Analogue of surjectivity:

- F is said to be essentially surjective, if it is surjective on isomorphism classes of objects, i.e., any $b \in \mathcal{B}$ is isomorphic to Fa for some $a \in \mathcal{A}$.

Analogue of a bijection:

- F is said to be an equivalence of categories if it is essentially surjective and fully faithful.

Here we took a point of view that a bijection is a function which is both injective and surjective.

5.5.6. *Examples.* For $\phi : \Bbbk \to \langle$ consider $\phi^* : \mathfrak{m}(\langle) \to \mathfrak{m}(\Bbbk)$. For instance if ϕ is $\mathbb{Z} \hookrightarrow \Bbbk$ or $\mathbb{R} \hookrightarrow$
C then ϕ^* is the forgetful functor $\mathfrak{m}(\Bbbk) \to \mathcal{A}b$ or $\mathcal{V}ect_{\mathbb{C}} \to \mathcal{V}ect_{\mathbb{R}}$. is faithful but not full. ϕ^* is always faithful but in our examples it is neither full nor essentially surjective.

Example. Let $\Bbbk$ be a field and $\mathcal{V}_{\Bbbk}$ the category such that $Ob(\mathcal{V}_k) = \mathbb{N} = \{0, 1, ...\}$ and $\mathrm{Hom}(n, m) = M_{mn}$ (matrices with m rows and n columns. It is equivalent to the category $\mathcal{V}ect_{\Bbbk}^{fg}$ of finite dimensional vector spaces by the functor $\mathcal{V}_{\Bbbk} \xrightarrow{\iota} \mathcal{V}ect_{\Bbbk}^{fg}$, here $\iota(n) = \Bbbk^n$ and for a matrix $\alpha \in M_{mn}$, $\iota_\alpha : \Bbbk^m \to \Bbbk^n$ is the multiplication by α.

Notice that $\mathcal{V}_{\Bbbk}$ and $\mathcal{V}ect_{\Bbbk}^{fg}$ seem very different (say only the first one is small), their content is the same (the linear algebra), but one is more convenient for computation and the other for thinking. Historically, equivalence ι is roughly the observation that one can do linear algebra without always choosing coordinates.

5.6. **Natural transformations of functors ("morphisms of functors").** A natural transformation η of a functor $F : \mathcal{A} \to \mathcal{B}$ into a functor $G : \mathcal{A} \to \mathcal{B}$ consists of maps $\eta_a \in \mathrm{Hom}_{\mathcal{B}}(Fa, Ga)$, $a \in \mathcal{A}$ such that for any map $\alpha : a' \to a''$ in $\mathcal{A}$ the following diagram commutes

$$\begin{array}{ccc} F(a') & \xrightarrow{F(\alpha)} & F(a'') \\ {\scriptstyle \eta_{a'}}\downarrow & & {\scriptstyle \eta_{a''}}\downarrow \\ G(a') & \xrightarrow{G(\alpha)} & G(a'') \end{array} \quad , \quad i.e., \quad \eta_{a''} \circ F(\alpha) = G(\alpha) \circ \eta_{a'}.$$

So, η relates values of functors on objects in a way compatible with the values of functors on maps. In practice, any "natural" choice of maps η_a will have the compatibility property.

5.6.1. *Example.* For the functors $\phi_* M = \langle \otimes_{\Bbbk} M$ and $\phi^* N = N$ from 5.5.1(1), there are canonical morphisms of functors

$$\alpha:\ \phi_*\circ\phi^* \to 1_{\mathfrak{m}(\langle)}, \qquad \phi_*\circ\phi^*(N) = \langle\otimes_{\Bbbk} N \xrightarrow{\alpha_N} N = 1_{\mathfrak{m}(\langle)}(N)$$

is the action of $\langle$ on N and

$$\beta:\ 1_{\mathfrak{m}(\Bbbk)} \to \phi^*\circ\phi_*, \qquad \phi^*\circ\phi_*(M) = \langle\otimes_{\Bbbk} M \xleftarrow{\beta_M} M = 1_{\mathfrak{m}(\mathbb{M})}(M)$$

is the map $m \mapsto 1_{\langle}\otimes m$.

For any functor $F : \mathcal{A} \to \mathcal{B}$ one has $1_F : F \to F$ with $(1_F)_a = 1_{Fa} : Fa \to Fa$. For three functors F, G, H from $\mathcal{A}$ to $\mathcal{B}$ one can compose morphisms $\mu : F \to G$ and $\nu : G \to H$ to $\nu\circ\mu : F \to H$

5.7. **Adjoint functors.** This is often the most useful categorical idea.

5.7.1. *Useful definition.* An adjoint pair of functors is a pair of functors $(\mathcal{A} \xrightarrow{F} \mathcal{B}, \mathcal{B} \xrightarrow{G} \mathcal{A})$ together with "natural identifications"

$$\zeta_{a,b}:\ \mathrm{Hom}_{\mathcal{B}}(Fa, b) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(a, Gb),\ a \in \mathcal{A},\ b \in \mathcal{B}.$$

Here,

(1) "natural" means behaving naturally in a and b, and by this we mean that ζ is a natural transform of functors $\zeta:\ \mathrm{Hom}_{\mathcal{B}}(F-, -) \to \mathrm{Hom}_{\mathcal{A}}(-, G-)$ from $\mathcal{A}^o \times \mathcal{B}$ to sets.
(2) "identification" means that each function $\zeta_{a,b}$ is a bijection.

We say that F is the left adjoint of G and that that G is the left adjoint of F (in the identity of homomorphisms F appears on the left in Hom and G on the right).

5.7.2. *Lemma.* Functors (ϕ_*, ϕ^*) from 5.5.1(1) form an adjoint pair, i.e., there is a canonical identification

$$\mathrm{Hom}_{\mathfrak{m}(\langle)}(\phi_* M, N) \xrightarrow{\eta_{M,N}} \mathrm{Hom}_{\mathfrak{m}(\Bbbk)}(M, \phi^* N), \qquad M \in \mathfrak{m}(\Bbbk),\ N \in \mathfrak{m}(\langle).$$

If $\langle\otimes_{\Bbbk} M \xrightarrow{\sigma} N,\ M \xrightarrow{\tau} N$, then $\mathrm{Hom}_{\mathfrak{m}(\langle)}(\langle\otimes_{\Bbbk} M, N) \xrightarrow{\eta_{M,N}} \mathrm{Hom}_{\mathfrak{m}(\Bbbk)}(M, N)$ by

$$\eta(\sigma)(m) =\ \sigma(1\otimes m) \quad \text{and} \quad \eta^{-1}(\tau)(c\otimes m) = c\tau(m), \quad m \in M,\ c \in \langle.$$

5.7.3. *Remark.* As in this example, often an adjoint pair appears in the following way: there is an obvious functor A (so obvious that we usually do not pay it any attention), but it has an adjoint B which is an interesting construction. The point is that this "interesting construction" B is intimately tied to the original "stupid" functor A, hence the properties of B can be deduced from the properties of the original simpler construction A. In fact B is produced from A in an explicit way as the following lemma shows.

5.7.4. What is the relation between morphisms of functors $\phi_*\circ\phi^* \xrightarrow{\alpha} 1_{\mathfrak{m}(\langle)}$ and $\phi^*\circ\phi_* \xleftarrow{\beta} 1_{\mathfrak{m}(\Bbbk)}$, from 5.6.1, and the isomorphism of functors $\mathrm{Hom}_{\mathfrak{m}(\langle)}(\phi_*-,-) \xrightarrow{\eta} \mathrm{Hom}_{\mathfrak{m}(\Bbbk)}(-,\phi^*-)$ from 5.7.2 ? They are really the same thing, i.e., two equivalent ways to describe adjointness.

5.7.5. *Lemma.* (Existence of the right adjoint.)
(a) If F has a right adjoint F then for each $b \in \mathcal{B}$ the functor

$$\mathrm{Hom}_{\mathcal{B}}(F-,b) : \mathcal{A} \to \mathcal{S}ets,\ a \mapsto \mathrm{Hom}_{\mathcal{B}}(Fa,b)$$

is representable (see 5.9).

(b) Suppose that for each $b \in \mathcal{B}$ the functor $\mathrm{Hom}_{\mathcal{B}}(F-,b) : \mathcal{A} \to \mathcal{S}ets$ is representable. For each $b \in B$ choose a representing object $Gb \in \mathcal{A}$, then G is a functor from $\mathcal{B}$ to $\mathcal{A}$ and it s the right adjoint of F.

(c) The right adjoint of F, if it exists, is unique up to a canonical isomorphism.

Of course the symmetric claims hold for left adjoints.

5.7.6. *Left adjoints of forgetful functors.* We say that a functor $\mathcal{F}$ is forgetful if it consists in dropping part of the structure of an object. Bellow we will denote its left adjoint by $\mathcal{G}$. Standard construction (that add to the structure of an object), are often adjoints of forgetful functors

(1) If $\mathcal{F} : \mathfrak{m}(\Bbbk) \to \mathcal{S}ets$ then $\mathcal{G}$ sends set S to the the free $\Bbbk$-module $\Bbbk[S] = \oplus_{s\in S}\ \Bbbk{\cdot}s$ with a basis S.
(2) Let $\Bbbk$ be a commutative ring. For $\mathcal{F} : \Bbbk - \mathcal{C}om\mathcal{A}lg \to \mathcal{S}et$ from commutative $\Bbbk$-algebras to sets, $\mathcal{G}$ sends a set S to the polynomial ring $\Bbbk[x_s,\ s \in S]$ where variables are given by all elements of S.
(3) If $\mathcal{F} : \Bbbk - \mathcal{C}om\mathcal{A}lg \to \mathfrak{m}(\Bbbk)$, then for a $\Bbbk$-module M, $\mathcal{G}(M)$ is the symmetric algebra $S(M)$. (To get exterior algebras in the same way one needs the notion of super algebras.)
(4) For the functor $\mathcal{F} : \Bbbk - \mathcal{A}lg \to \mathfrak{m}(\Bbbk)$ from $\Bbbk$-algebras to $\Bbbk$-modules, $\mathcal{G}(M)$ is the tensor algebra $S(M)$.
(5) Forgetful functor $\mathcal{F} : \mathcal{T}op\mathcal{S}ets$ has a left adjoint $\mathcal{D}$ that sends a set S to S with the discrete topology, and also the right adjoint C such that $C(S)$ is S with the topology such that only S and ϕ are open.

5.7.7. *Functors between categories of modules.* (1) For $\phi : \Bbbk \to \langle$, one has adjoint triple $(\phi_*, \phi^*, \phi_\star)$ with $\phi_\star(M) \stackrel{\text{def}}{=} \mathrm{Hom}_{\Bbbk}(\langle, M)$, i.e., (ϕ_*, ϕ^*) and $(\phi^*, \phi_\star)$ are adjoint pairs. So ϕ^* has both a left and a right adjoint and they are very different.

(2) $(\Bbbk, \langle)$-bimodule X gives $X_* : \mathfrak{m}(\langle) \to \mathfrak{m}(\Bbbk)$, with $X_*(N) \stackrel{\text{def}}{=} X\otimes_{\langle} N$, what is its right adjoint?

5.7.8. *More categorifications of "injection", "surjection", "bijection".*

5.8. **Higher categories.** Notice that

(1) The class $\mathcal{C}at$ of all categories has a category like structure with $Ob(\mathcal{C}at) \stackrel{\text{def}}{=}$ categories, and (for any two categories $\mathcal{A}$ and $\mathcal{B}$) $\text{Hom}_{\mathcal{C}at}(\mathcal{A},\mathcal{B}) \stackrel{\text{def}}{=} \mathit{Funct}(\mathcal{A},\mathcal{B}) \stackrel{\text{def}}{=}$ functors from $\mathcal{A}$ to $\mathcal{B}$. However, the class $\text{Hom}_{\mathcal{C}at}(\mathcal{A},\mathcal{B})$ need not be a set unless the categories $\mathcal{A}$ and $\mathcal{B}$ are small.
(2) Moreover, for any two categories $\mathcal{A}$ and $\mathcal{B}$, $\text{Hom}_{\mathcal{C}at}(\mathcal{A},\mathcal{B}) = \mathit{Funct}(\mathcal{A},\mathcal{B})$ is actually a category with $Ob(\mathit{Funct}(\mathcal{A},\mathcal{B}))$ = functors from $\mathcal{A}$ to $\mathcal{B}$, and for $F,G \in \mathit{Funct}(\mathcal{A},\mathcal{B})$, the morphisms $\text{Hom}_{\mathit{Funct}(\mathcal{A},\mathcal{B})}(F,G) \stackrel{\text{def}}{=}$ natural transforms from F to G.

The two structures (1) and (2) (together with natural compatibility conditions between them) make $\mathcal{C}at$ into what is called a 2-category. We leave this notion vague as it is not central to what we do now. (We will see that complexes and topological spaces are also 2-categories.)

Geometrically, a category $\mathcal{A}$ defines a 1-dimensional simplicial complex $|\mathcal{A}|$ (the nerve of $\mathcal{A}$) where vertices=$Ob(\mathcal{A})$ and (directed) edges between vertices a and b are given by $\text{Hom}_{\mathcal{C}}(a,b)$. A 2-category $\mathcal{B}$ defines a 2-dimensional topological object $|\mathcal{B}|$ (the nerve of $\mathcal{B}$). If say, $\mathcal{B} = \mathcal{C}at$ then vertices $Ob(\mathcal{C}at)$ = categories, edges between vertices $\mathcal{A}$ and $\mathcal{B}$ corresponds to all functors from $\mathcal{A}$ to $\mathcal{B}$, and for two functors $\mathcal{A} \xrightarrow{F,G} \mathcal{B}$ (i.e., two edges from the point $\mathcal{A}$ to the point $\mathcal{B}$), morphisms $F \xrightarrow{\eta} G$ correspond to (directed) 2-cells in $|\mathcal{C}at|$ whose boundary is the union of edges corresponding to F and G. So one dimensional topology controls the level of our thinking when we use categories, 2-dimensional when we use 2-categories and in this way one can continue to define more complicated frameworks for thinking of mathematics, modeled on more complicated topology: the nerve of an n-category is an n-dimensional topological object for $n = 0,1,2,...,\infty$ (here "1-category" means just "category" and a 0-category is a set).

5.9. **Construction (description) of objects via representable functors.** Yoneda lemma bellow says that passing from an object $a \in \mathcal{A}$ to the corresponding functor $\text{Hom}_{\mathcal{A}}(-,a)$ does not loose any information – a can be recovered from the functor $\text{Hom}_{\mathcal{A}}(-,a)$. This has the following applications:

(1) One can describe an object a by describing the corresponding functor $\text{Hom}_{\mathcal{A}}(-,a)$. This turns out to be **the** *most natural* description of a.
(2) One can start with a functor $F : \mathcal{A}^o \to \mathcal{S}ets$ and ask whether it comes from some objects of a. (Then we say that a represents F and that F is representable).
(3) Functors $F : \mathcal{A}^o \to \mathcal{S}ets$ behave somewhat alike the objects of $\mathcal{A}$, and we can think of their totality as a natural enlargement of $\mathcal{A}$ (like one completes $\mathbb{Q}$ to $\mathbb{R}$).

5.9.1. *Category* $\widehat{\mathcal{A}}$. To a category $\mathcal{A}$ one can associate a category $\widehat{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{F}unct(\mathcal{A}^o, \mathcal{S}ets)$ of contravariant functors from $\mathcal{A}$ to sets. Recall that each object $a \in \mathcal{A}$ defines a functor $\iota_a = \operatorname{Hom}_{\mathcal{A}}(-,a) \in \widehat{\mathcal{A}}$. The following statement essentially says that one can recover a form the functor $\operatorname{Hom}_{\mathcal{A}}(-,a)$, i.e., that this functor contains all information about a.

5.9.2. *Theorem.* (Yoneda lemma)
(a) Construction ι is a functor and it is a full embedding $\iota : \mathcal{A} \to \widehat{\mathcal{A}}$.

(b) Functor $\operatorname{Hom}_{\mathcal{A}}(-,a)$ determines a up to a unique isomorphism.

5.9.3. *Representable functors.* We will say that a functor $F : \mathcal{A}^o \to \mathcal{S}ets$ is representable if there is some $a \in \mathcal{A}$ and an isomorphism of functors $\eta : \operatorname{Hom}_{\mathcal{A}}(-,a) \to F$. Then we say that a represents F. This is the basic categorical trick used to describe an object a up to a canonical isomorphism: instead of describing a we describe a functor F isomorphic to $\operatorname{Hom}_{\mathcal{A}}(-,a)$.

For instance a product of a and b is an object that represents the functor

$$\mathcal{A} \ni x \mapsto \operatorname{Hom}(x,a) \times \operatorname{Hom}_{\mathcal{A}}(x,b) \in \mathcal{S}ets.$$

More generally, a projective system $a_i,\ i \in I$, in $\mathcal{A}$ defines a functor $\mathbf{\underleftarrow{lim}}\ a_i \in \widehat{\mathcal{A}}$ by

$$(\mathbf{\underleftarrow{lim}}\ a_i)(x) \stackrel{\text{def}}{=} \underleftarrow{\lim}\ \operatorname{Hom}_{\mathcal{A}}(x,a_i).$$

Here, the limit on the right is in sets and we know that the category $\mathcal{S}ets$ has limits! (there is no limit on the LHS – this is just the notation for a new functor). From definition one finds that $\mathbf{\underleftarrow{lim}}\ a_i$ is representable iff $\underleftarrow{\lim}\ a_i$ exists (and then the limit represents the functor).

Dually, a functor $G : \mathcal{A} \to \mathcal{S}ets$ is said to be corepresentable if there is some $a \in \mathcal{A}$ and an isomorphism of functors $\eta : \operatorname{Hom}_{\mathcal{A}}(a,-) \to G$. Then we say that a corepresents F. A sum of a and b is an object that corepresents the functor

$$\mathcal{A} \ni x \mapsto \operatorname{Hom}(a,x) \times \operatorname{Hom}_{\mathcal{A}}(b,x) \in \mathcal{S}ets.$$

An inductive system $a_i,\ i \in I$, in $\mathcal{A}$ defines a functor $\mathbf{\underrightarrow{lim}}\ a_i : \mathcal{A} \to \mathcal{S}ets$ by

$$(\mathbf{\underrightarrow{lim}}\ a_i)(x) \stackrel{\text{def}}{=} \underleftarrow{\lim}\ \operatorname{Hom}_{\mathcal{A}}(a_i,x)$$

which is corepresentable by $\underrightarrow{\lim}\ a_i$ (iff it exists).

So, we say that we represent contravariant functors from $\mathcal{A}$ to $\mathcal{S}ets$, and that we we corepresent covariant functors from $\mathcal{A}$ to $\mathcal{S}ets$. One can also omit "co" and talk about representing both times (there is no confusion since the nature of functors distinguishes the two situations).

5.10. **Completion of a category $\mathcal{A}$ to $\widehat{\mathcal{A}}$.** This is a categorical analogue of one of the basic tricks in analysis: since among functions one can not find beauties like the δ-functions, we extend the notion of of functions by adding distributions. Remember that the distributions on an open $U\subseteq\mathbb{R}^n$ are the (nice) linear functionals on the vector space of of (nice) functions: $\mathcal{D}(U,\mathbb{C})\subseteq C_c^\infty(U,\mathbb{C})^* = \mathrm{Hom}_{\mathbb{C}}[C_c^\infty(U),\mathbb{C}]$.

The translation of this procedure in the categorical setting is the observation that the category $\widehat{\mathcal{A}}$ may contain many beauties that should morally be in $\mathcal{A}$. One example will be a way of treating inductive systems in $\widehat{\mathcal{A}}$. In particular we will see inductive systems of infinitesimal geometric objects that underlie the differential calculus.

5.10.1. *Limits.* For instance, a limit of an inductive system in $\mathcal{A}$ need not exist in $\mathcal{A}$ but it always exists in a larger category $\widehat{\mathcal{A}}$. An inductive system $a_i,\ i\in I$, in $\mathcal{A}$ defines an object in $\mathcal{A}$ if the limit $\lim\limits_{\rightarrow} a_i$ exists, however it always defines a functor $\mathop{\mathbf{lim}}\limits_{\rightarrow} a_i \in \widehat{\mathcal{A}}$ as above. This allows us to <u>*think*</u> of the inductive system $a_i,\ i\in I$, in $\mathcal{A}$, *as if* it were an object $\mathop{\mathbf{lim}}\limits_{\rightarrow} a_i$ in $\mathcal{A}$ – this object would be characterized by the property that the maps into it are systems of compatible maps into a_i's:

$$\mathrm{Hom}_{\mathcal{A}}(x,\ \mathop{\mathbf{lim}}\limits_{\rightarrow} a_i) = \lim\limits_{\rightarrow} \mathrm{Hom}_{\mathcal{A}}(x,a_i),\quad x\in\mathcal{A},$$

and by Yoneda lemma it would really characterize the object. For this reason an inductive system in $\mathcal{A}$ is called an ind-object of $\mathcal{A}$ (while it really gives an object of $\widehat{\mathcal{A}}$).

Similarly one calls projective systems pro-objects of $\mathcal{A}$. However, the completion $\widehat{\mathcal{A}}$ is a good setting only for inductive systems – projective ones are related to covariant functors from $\mathcal{A}$ to sets as we will see in the following example.

5.10.2. *Affine schemes.* The geometry we use here is the algebraic geometry. Its geometric objects are called schemes and they are obtained by gluing today schemes of a somewhat special type, which are called affine schemes (like manifolds are all obtained by gluing open pieces of $\mathbb{R}^n$'s). An affine schemes S over $\mathbb{C}$ is determined by its algebra of functions $\mathcal{O}(S)$ which is a $\mathbb{C}$-algebra. Moreover, any commutative $\mathbb{C}$-algebra A is the algebra of functions on some scheme – the scheme is called the spectrum of A and denoted $\mathrm{Spec}(A)$. So, affine algebras are really the same as commutative $\mathbb{C}$-algebras, except that a map of affine schemes $X\xrightarrow{\phi}Y$ defines a map of functions $\mathcal{O}(Y)\xrightarrow{\phi^*}\mathcal{O}(X)$ in the opposite direction (the pull-back $\phi^*(f) = f\circ\phi$). The statement "information contained in two kinds of objects is the same but the directions reverse when one passes from geometry to algebra" is categorically stated as categories $\mathcal{S}chemes_{\mathbb{C}}$ and $(\mathcal{C}om\mathcal{A}lg_{\mathbb{C}})^o$ are equivalent.

We will simplify this kind of thinking and *define* the category of affine schemes over $\mathbb{C}$ as the the opposite of the category of commutative $\mathbb{C}$-algebras. The part that I skipped is how one develops a geometric point of view on affine schemes defined in this way. Basically one thinks geometrically and translates geometric ideas in algebra, and proves geometric theorems in algebra. Once sufficiently many geometric statements are verified in this way one can build up on these and do everything in geometry.

5.10.3. *Schemes over $\mathbb{C}$ (or over any commutative ring $\Bbbk$).* Let us repeat the above announced steps. The category $\mathcal{A}ff\mathcal{S}chemes_{\mathbb{C}}$ of affine $\mathbb{C}$-schemes is defined as $\mathcal{C}om\mathcal{A}lg_{\mathbb{C}}^{o}$. We denote the canonical contravariant functor $\mathcal{C}om\mathcal{A}lg_{\mathbb{C}}^{o} \to \mathcal{A}ff\mathcal{S}chemes_{\mathbb{C}}$, by Spec, and in the opposite direction one has the operation of taking functions on a scheme $\mathcal{A}ff\mathcal{S}chemes_{\mathbb{C}} \xrightarrow{\mathcal{O}} \mathcal{C}om\mathcal{A}lg_{\mathbb{C}}^{o}$. For an algebra A in we think of $\mathrm{Spec}(A) \in \mathcal{S}chemes_{\mathbb{C}}$ as a geometric object such that A is the algebra of functions on it: $\mathcal{O}(\mathrm{Spec}(A)) = A$.

Now we consider some projective systems of algebras and the corresponding inductive systems of affine schemes.

5.10.4. *Some projective systems of algebras.* Formal power series $\mathbb{C}[[x]]$ is $\lim\limits_{\leftarrow} \mathbb{C}[x]/x^n$ in $\mathcal{C}om\mathcal{A}lg_{\mathbb{C}}$, however the projective system of commutative $\mathbb{C}$-algebras $\mathbb{C}[x]/x^n,\ n = 1, 2, ...$ is itself something like a commutative $\mathbb{C}$-algebra. It is a pro-object in $\mathcal{C}om\mathcal{A}lg_{\mathbb{C}}$ and it defines a functor $\mathop{\mathbf{lim}}\limits_{\leftarrow} \mathbb{C}[x]/x^n : \mathcal{C}om\mathcal{A}lg_{\mathbb{C}} \to \mathcal{S}ets$ by

$$(A) \stackrel{\text{def}}{=} \mathop{\mathbf{lim}}\limits_{\leftarrow} \mathrm{Hom}(\mathbb{C}[x]/x^n, A) = \text{ nilpotent elements of } A.$$

Equivalently, one can think of the pro-object $\mathop{\mathbf{lim}}\limits_{\leftarrow} \mathbb{C}[x]/x^n$ as a topological $\mathbb{C}$-algebra $\underline{(\mathbb{C}[[[x]], \mathcal{T})}$ with the topology $\mathcal{T}$ which one can describe by: a basis of neighborhoods of 0 is given by all ideals $x^n\mathbb{C}[[x]],\ n > 0$. The reason is that any topological algebra $\mathcal{A}$ also gives an a functor $\underline{\mathcal{A}} :\ \mathcal{C}om\mathcal{A}lg_{\mathbb{C}} \to \mathcal{S}ets$ via

$$\mathcal{C}om\mathcal{A}lg_{\mathbb{C}} \ni A \mapsto \underline{\mathcal{A}}(A) \stackrel{\text{def}}{=} \mathrm{Hom}_{Top.Algebras}(\mathcal{A}, A)$$

where A is considered as a topological algebra with discrete topology. Now we have a way of comparing a pro-system and a topological algebra by comparing the corresponding functors, and we find that the functors are the same

$$\mathop{\mathbf{lim}}\limits_{\leftarrow} \mathbb{C}[x]/x^n = \{\text{nilpotents in } -\} = \underline{(\mathbb{C}[[[x]], \mathcal{T})}.$$

5.10.5. *Some injective systems of geometric objects.* The n^{th} infinitesimal neighborhood $IN^n_{\mathbb{A}^1}(0)$ of the point 0 in the line $\mathbb{A}^1$ is the scheme defined by the algebra $\mathbb{C}[x]/x^{n+1}$

$$IN^n_{\mathbb{A}^1}(0) \stackrel{\text{def}}{=} \mathrm{Spec}(\mathbb{C}[x]/x^{n+1}) \quad \textit{hence} \quad \mathcal{O}(IN^n_{\mathbb{A}^1}(0)) = \mathbb{C}[x]/x^{n+1}.$$

Now, pro-algebra $\mathop{\mathbf{lim}}\limits_{\leftarrow} \mathbb{C}[x]/x^n$ in $\mathcal{C}om\mathcal{A}lg_{\mathbb{C}}$, is thought of as the algebra of functions on the ind-object in schemes $\mathop{\mathbf{lim}}\limits_{\rightarrow} IN^n_{\mathbb{A}^1}(0)$ which we call the formal neighborhood $FN_{\mathbb{A}^1}(0)$ of 0 in $\mathbb{A}^1$:

$$FN_{\mathbb{A}^1}(0) \stackrel{\text{def}}{=} \mathop{\mathbf{lim}}\limits_{\rightarrow} IN^n_{\mathbb{A}^1}(0) \quad \text{and} \quad \mathcal{O}(FN_{\mathbb{A}^1}(0)) \stackrel{\text{def}}{=} \mathop{\mathbf{lim}}\limits_{\leftarrow} \mathcal{O}(IN^n_{\mathbb{A}^1}(0)) = \mathop{\mathbf{lim}}\limits_{\leftarrow} \mathbb{C}[x]/x^{n+1}.$$

It turns out that the finite infinitesimal neighborhoods and formal neighborhoods are precisely what the classics were thinking about when they based calculus on infinitesimally small quantities – the only problem that obscured the rigorous basis of the differential calculus on infinitesimals was the lack of the language of schemes.

6. **Abelian categories**

An abelian category is a category $\mathcal{A}$ which has the formal properties of the category $\mathcal{A}b$, i.e., we can do in $\mathcal{A}$ all computations that one can do in $\mathcal{A}b$.

6.1. **Additive categories.** Category $\mathcal{A}$ is additive if

- (A0) For any $a, b \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(a, b)$ has a structure of abelian group such that then compositions are bilinear.
- (A1) $\mathcal{A}$ has a zero object,
- (A2) $\mathcal{A}$ has sums of two objects,
- (A3) $\mathcal{A}$ has products of two objects,

6.1.1. *Lemma.* (a) Under the conditions (A0),(A1) one has (A2) $\Leftrightarrow$ (A3).

(b) In an additive category $a \oplus b$ is canonically the same as $a \times b$,

For additive categories $\mathcal{A}, \mathcal{B}$ a functor $F : \mathcal{A} \to \mathcal{B}$ is additive if the maps $\mathrm{Hom}_{\mathcal{A}}(a', a'') \to \mathrm{Hom}_{\mathcal{B}}(Fa', Fa'')$ are always morphisms of abelian groups.

6.1.2. *Examples.* (1) $\mathfrak{m}(\Bbbk)$, (2) $\mathcal{F}ree(\Bbbk)$, (3) $\mathcal{F}ilt\mathcal{V}ect_{\Bbbk} \overset{\text{def}}{=}$ filtered vector spaces over $\Bbbk$.

6.2. **(Co)kernels and (co)images.** In module categories a map has kernel, cokernel and image. To incorporate these notions into our project of defining abelian categories we will find their abstract formulations.

6.2.1. *Kernels: Intuition.* Our intuition is based on the category of type $\mathfrak{m}(\Bbbk)$. For a map of $\Bbbk$-modules $M \xrightarrow{\alpha} N$

- the kernel $\mathrm{Ker}(\alpha)$ is a subobject of M,
- the restriction of α to it is zero,
- and this is the largest subobject with this property

6.2.2. *Categorical formulation.* Based on this, our general definition (in an additive category $\mathcal{A}$), of "k is a kernel of the map $a \xrightarrow{\alpha} b$", is

- we have a map $k \xrightarrow{\sigma} M$ from k to M,
- if we follow this map by α the composition is zero,

- map $k \xrightarrow{\sigma} M$ is universal among all such maps, in the sense that
  - $\cdot$ all maps into a, $x \xrightarrow{\tau} a$, which are killed by α,
  - $\cdot$ factor uniquely through k (i.e., through $k \xrightarrow{\sigma} a$).

  So, all maps from x to a which are killed by α are obtained from σ (by composing it with some map $x \rightarrow k$). This is the "universality" property of the kernel.

6.2.3. *Reformulation in terms of representability of a functor.* A compact way to restate the above definition is:

- The kernel of $a \xrightarrow{\alpha} b$ is any object that represents the functor

$$\mathcal{A} \ni x \mapsto {}_{\alpha}\mathrm{Hom}_{\mathcal{A}}(x, a) \stackrel{\text{def}}{=} \{\gamma \in \mathrm{Hom}_{\mathcal{A}}(x, a);\ \alpha\circ\gamma = 0\ \}.$$

One should check that this is the same as the original definition.

We denote the kernel by $\mathrm{Ker}(\alpha)$, but as usual, remember that

- this is not one specific object – it is only determined up to a canonical isomorphism,
- it is not only an object but a pair of an object and a map into a

.

6.2.4. *Cokernels.* In $\mathfrak{m}(\Bbbk)$ the cokernel of $M \xrightarrow{\alpha} N$ is $N/\alpha(M)$. So N maps into it, composition with α kills it, and the cokernel is universal among all such objects. When stated in categorical terms we see that we are interested in the functor

$$x \mapsto \mathrm{Hom}_{\mathcal{A}}(b, x)_{\alpha} \stackrel{\text{def}}{=} \{\tau \in \mathrm{Hom}_{\mathcal{A}}(b, x);\ \tau\circ\alpha = 0\ \},$$

and the formal definition is symmetric to the definition of a kernel:

- The cokernel of f is any object that represents the functor $\mathcal{A} \ni x \mapsto \mathrm{Hom}_{\mathcal{A}}(b, x)_{\alpha}$.

So this object $\mathrm{Coker}(\alpha)$ is supplied with a map $b \rightarrow \mathrm{Coker}(\alpha)$ which is universal among maps from b that kill α.

6.2.5. *Images and coimages.* In order to define the image of α we need to use kernels and cokernels. In $\mathfrak{m}(\Bbbk)$, $Im(\alpha)$ is a subobject of N which is the kernel of $N \rightarrow \alpha(M)$. We will see that the categorical translation obviously has a symmetrical version which we call coimage. Back in $\mathfrak{m}(\Bbbk)$ the coimage is $M/\mathrm{Ker}(\alpha)$, hence there is a canonical map $Coim(\alpha) = M/\mathrm{Ker}(\alpha) \rightarrow Im(\alpha)$, and it is an isomorphism. This observation will be the final ingredient in the definition of abelian categories. Now we define

- Assume that α has cokernel $b \rightarrow \mathrm{Coker}(\alpha)$, the image of α is $Im(\alpha) \stackrel{\text{def}}{=} \mathrm{Ker}[b \rightarrow \mathrm{Coker}(\alpha)]$ (if it exists).
- Assume that α has kernel $\mathrm{Ker}(\alpha) \rightarrow a$, the coimage of α is $Coim(\alpha) \stackrel{\text{def}}{=} \mathrm{Coker}[\mathrm{Ker}(\alpha) \rightarrow a]$. (if it exists).

6.2.6. *Lemma.* If α has image and coimage, there is a canonical map $Coim(\alpha) \to Im(\alpha)$, and it appears in a canonical factorization of α into a composition

$$a \to Coim(\alpha) \to Im(\alpha) \to b.$$

6.2.7. *Examples.* (1) In $\mathfrak{m}(\Bbbk)$ the categorical notions of a (co)kernel and image have the usual meaning, and coimages coincide with images.

(2) In $\mathcal{F}ree(\Bbbk)$ kernels and cokernels need not exist.

(3) In $\mathcal{FV} \stackrel{\text{def}}{=} \mathcal{F}ilt\mathcal{V}ect_{\Bbbk}$ for $\phi \in \mathrm{Hom}_{\mathcal{FV}}(M_*, N_*)$ (i.e., $\phi : M \to N$ such that $\phi(M_k) \subseteq N_k,\ k \in \mathbb{Z}$), one has

- $\mathrm{Ker}_{\mathcal{FV}}(\phi) = \mathrm{Ker}_{\mathcal{V}ect}(\phi)$ with the induced filtration $\mathrm{Ker}_{\mathcal{FV}}(\phi)_n = \mathrm{Ker}_{\mathcal{V}ect}(\phi) \cap M_n$,
- $\mathrm{Coker}_{\mathcal{FV}}(\phi) = N/\phi(M)$ with the induced filtration $\mathrm{Coker}_{\mathcal{FV}}(\phi)_n =$ image of N_n in $N/\phi(M) = [N_n + \phi(M)]/\phi(M) \cong N_n/\phi(M) \cap N_n$.
- $Coim_{\mathcal{FV}}(\phi) = M/\mathrm{Ker}(\phi)$ with the induced filtration $Coim_{\mathcal{FV}}(\phi)_n =$ image of M_n in $M/\mathrm{Ker}(\phi) = M_n + \mathrm{Ker}(\phi)/\mathrm{Ker}(\phi) \cong = M_n/M_n \cap \mathrm{Ker}(\phi)$,
- $Im_{\mathcal{FV}}(\phi) = Im_{\mathcal{V}ect}(\phi) \subseteq N$, with the induced filtration $Im_{\mathcal{FV}}(\phi)_n = Im_{\mathcal{V}ect}(\phi) \cap N_n$.

Observe that the canonical map $Coim_{\mathcal{FV}}(\phi) \to Im_{\mathcal{FV}}(\phi)$ is an isomorphism of vector spaces $M/\mathrm{Ker}(\phi) \to Im_{\mathcal{V}ect}(\phi)$, however the two spaces have filtrations induced from filtrations on M and N respectively, and these need not coincide.

For instance one may have M and N be two filtrations on the same space V, if $M_k \subseteq N_k$ then $\phi = 1_V$ is a map of filtered spaces $M \to N$ and Ker $= 0$Coker so that $Coim_{\mathcal{FV}}(\phi) = M$ and $Im_{\mathcal{FV}}(\phi) = N$ and the map $Coim_{\mathcal{FV}}(\phi) \to Im_{\mathcal{FV}}(\phi)$ is the same as ϕ, but ϕ is an isomorphism iff the filtrations coincide: $M_k = N_k$.

6.3. **Abelian categories.** Category $\mathcal{A}$ is abelian if

- (A0-3) It is additive,
- It has kernels and cokernels (hence in particular it has images and coimages!),
- The canonical maps $Coim(\phi) \to Im(\phi)$ are isomorphisms

6.3.1. *Examples.* Some of the following are abelian categories: (1) $\mathfrak{m}(\Bbbk)$ including $\mathcal{A}b = \mathfrak{m}(\mathbb{Z})$. (2) $\mathfrak{m}_{fg}(\Bbbk)$ if $\Bbbk$ is noetherian. (3) $\mathcal{F}ree(\Bbbk) \subseteq \mathcal{P}roj(\Bbbk) \subseteq \mathfrak{m}(\Bbbk)$. (4) $\mathcal{C}^\bullet(\mathcal{A})$. (5) Filtered vector spaces.

6.4. **Abelian categories and categories of modules.**

6.4.1. *Exact sequences in abelian categories.* Once we have the notion of kernel and cokernel (hence also of image), we can carry over from module categories $\mathfrak{m}(\Bbbk)$ to general abelian categories our homological train of thought. For instance we say that

- a map $i : a \to b$ makes a into a subobject of b if $\mathrm{Ker}(i) = 0$ (we denote it $a \hookrightarrow b$ or even informally by $a \subseteq b$, one also says that i is a monomorphism or informally that it is an inclusion),
- a map $q : b \to c$ makes c into a quotient of b if $\mathrm{Coker}(q) = 0$ (we denote it $b \twoheadrightarrow c$ and say that q is an epimorphism or informally that q is surjective),
- the quotient of b by a subobject $a \xrightarrow{i} b$ is $b/a \stackrel{\text{def}}{=} \mathrm{Coker}(i)$,
- a complex in $\mathcal{A}$ is a sequence of maps $\cdots A^n \xrightarrow{d^n} A^{n+1} \to \cdots$ such that $d^{n+1} \circ d^n = 0$, its cocycles, coboundaries and cohomologies are defined by $B^n = Im(d^n)$ is a subobject of $Z^n = \mathrm{Ker}(d^n)$ and $H^n = Z^n/B^n$;
- sequence of maps $a \xrightarrow{\mu} b \xrightarrow{\nu} c$ is exact (at b) if $\nu \circ \mu = 0$ and the canonical map $Im(\mu) \to \mathrm{Ker}(\nu)$ is an isomorphism.

Now with all these definitions we are in a familiar world, i.e., they work as we expect. For instance, sequence $0 \to a' \xrightarrow{\alpha} a \xrightarrow{\beta} a'' \to 0$ is exact iff a' is a subobject of a and a'' is the quotient of a by a', and if this is true then

$$\mathrm{Ker}(\alpha) = 0, \mathrm{Ker}(\beta) = a', \mathrm{Coker}(\alpha) = a'', \mathrm{Coker}(\beta) = 0, \mathrm{Im}(\alpha) = a', \mathrm{Im}(\beta) = a''.$$

The difference between general abelian categories and module categories is that while in a module category $\mathfrak{m}(\Bbbk)$ our arguments often use the fact that $\Bbbk$-modules are after all abelian groups and sets (so we can think in terms of their elements), the reasoning valid in any abelian category has to be done more formally (via composing maps and factoring maps through intermediate objects). However, this is mostly appearances – if we try to use set theoretic arguments we will not go wrong:

6.4.2. *Theorem.* [Mitchell] Any abelian category is equivalent to a full subcategory of some category of modules $\mathfrak{m}(\Bbbk)$.

7. **Exactness of functors and the derived functors**

7.1. **Exactness of functors.** As we have observed, any functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories extends to a functor between the corresponding categories of complexes $\mathcal{C}^\bullet(F) : \mathcal{C}^\bullet(\mathcal{A}) \to \mathcal{C}^\bullet(\mathcal{B})$. We would like next to extend it to a functor between derived categories $D(F) : D(\mathcal{A}) \to D(\mathcal{B})$ – this is what one calls the derived version of F. We may denote it again by F, or by $D(F)$, or use some other notation which reflects on the way we produce the extension.

This extension is often obtained using exactness properties of F, i.e., it will depend on how much does F preserves exact sequences. Say, if F is exact then $D(F)$ is obvious: it is just the functor F applied to complexes. If F is right-exact $D(F)$ is the "left derived functor LF" obtained by replacing objects with projective resolutions. If F is left-exact, $D(F)$ is the "right derived functor RF" obtained by replacing objects with injective resolutions (see 7.5).

7.1.1. *Exactness of a sequence of maps.* A sequence of maps in an abelian category $M_a \xrightarrow{\alpha_a} M_{a+1} \to \cdots \to M_{b-1} \xrightarrow{\alpha_{b-1}} M_b$ is said to be exact at M_i (for some i with $a < i < b$) if $Im(\alpha_{i-1}) = \mathrm{Ker}(\alpha_i)$. The sequence is said to be exact if it is exact at all $M_i,\ a < i < b$. (The sequence may possibly be infinite in one or both directions.)

7.1.2. *Exact functors.* Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. (One can think of the case where $\mathcal{A} = \mathfrak{m}(\Bbbk)$ and $\mathcal{B} = \mathfrak{m}(\langle)$ since the general case works the same.)

We will say that F is exact if it preserves short exact sequences, i.e., for any SES $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ in $\mathcal{A}$, its F-image in $\mathcal{B}$ is exact, i.e., the sequence $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ is a SES in $\mathcal{B}$.

Example. The pull-back functors ϕ^* from 5.5.1 are exact since they do not change the structure of abelian groups and the exactness for modules only involves the level of abelian groups.

In practice few interesting functors are exact so we have to relax the notion of exactness:

7.2. **Left exact functors.** We say, that F is left exact if for any SES its F-image $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ is exact except possibly in the A''-term, i.e., $F(\beta)$ need not be surjective.

7.2.1. *Lemma.* The property of left exactness is the same as asking that F preserves exactness of sequences of the form $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C''$, i.e., if $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A''$ is exact in $\mathcal{A}$, its F-image $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'')$ is exact in $\mathcal{B}$.

Proof. The property in the lemma seems a priori stronger because it produces the same conclusion in a larger number of cases. However it follows from the definition of left exactness by a diagram chase which uses left exactness in two places.

7.2.2. *Example: Invariants are left exact.* For a group G let $Rep_{\Bbbk}(G)$ be the category of all representations of G over a field $\Bbbk$. A representations of G is a pair (V, π) of a vector space V and a map of groups $G \xrightarrow{\pi} GL(V)$. We often denote $\pi(g)v$ by gv, and we omit V or π from the notation for representations. Representations of G are the same as modules for the algebra $\Bbbk[G] \overset{\text{def}}{=} \oplus_{g\in G} \Bbbk g$ (multiplication is obvious). So short exact sequences, etc., make sense in $Rep_{\Bbbk}(G)$.

Lemma. The functor of invariants, $I : Rep_{\Bbbk}(G) \to \mathcal{V}ect_{\Bbbk}$, by $I(V, \pi) \overset{\text{def}}{=} V^G \overset{\text{def}}{=} \{v \in V,\ gv = v,\ g \in G\}$ is left exact.

Proof is easy. It is more interesting to see how exactness fails at the right end.

Counterexample. For $G = \mathbb{Z}$, a representation is the same as a vector space V with an invertible linear operator A $(= \pi(1))$. Therefore, $I(V, A)$ is the 1-eigenspace V_1 of A. Short exact sequences in $Rep_{\Bbbk}(\mathbb{Z})$ are all isomorphic to the ones of the form $0 \rightarrow (V', A|V') \rightarrow (V, A) \rightarrow (V'', \overline{A}) \rightarrow 0$, i.e., one has a vector space V with an invertible linear operator A, an A-invariant subspace V' (we restrict A to it), and the quotient space $V'' = V/V'$ (we factor A to it). So the exactness on the right means that any $w \in V/V'$ such that $\overline{A}w = w$ comes from some v in V such that $Av = v$.

If $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ with $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $V' = \mathbb{C}e_1$ then $0 \rightarrow I(V', A|V') \rightarrow I(V, A) \rightarrow I(V'', \overline{A}) \rightarrow 0$, is just $0 \rightarrow \mathbb{C}e_1 \xrightarrow{id} \mathbb{C}e_1 \xrightarrow{0} V'' \rightarrow 0$, and the exactness fails on the right.

The principle "Invariants are left exact". It applies to many other situations. Also, since

$$I(V) = \operatorname{Hom}_G(\Bbbk, V),$$

it is a special case of the next lemma. The meaning of the last equality is:

- $\Bbbk$ denotes the trivial one dimensional representation of G on the vector space $\Bbbk$.
- Moreover, the equality notation $I(V) = \operatorname{Hom}_G(\Bbbk, V)$ is only a remainder of a more precise statement: there is a canonical isomorphism of functors $I \xrightarrow{\eta} \operatorname{Hom}_{Rep_{\Bbbk}(G)}(\Bbbk, -)$ from $Rep_{\Bbbk}(G)$ to $\mathcal{V}ect_{\Bbbk}$.
- The map η_V sends a G-fixed vector $w \in I(V)$ to a linear map $\eta_V(w) : \Bbbk \rightarrow V$, given by multiplying w with scalars: $\Bbbk \ni c \mapsto c \cdot v \in V$. One easily checks that η_V is an isomorphism of vector spaces.

7.2.3. *Lemma.* Let $\mathcal{A}$ be an abelian category, for any $a \in \mathcal{A}$, the functor

$$\operatorname{Hom}_{\mathcal{A}}(a, -) : \mathcal{A} \rightarrow \mathcal{A}b,$$

is left exact!

Proof. For an exact sequence $0 \rightarrow b' \xrightarrow{\alpha} b \xrightarrow{\beta} b'' \rightarrow 0$ we consider the corresponding sequence $\operatorname{Hom}_{\mathcal{A}}(a, b') \xrightarrow{\alpha_*} \operatorname{Hom}_{\mathcal{A}}(a, b) \xrightarrow{\beta_*} \operatorname{Hom}_{\mathcal{A}}(a, b'')$.

(1) α_* is injective. if $a \xrightarrow{\mu} b'$ and $0 = \alpha_*(\mu) \stackrel{\text{def}}{=} \alpha \circ \mu$, then μ factors through the kernel $\operatorname{Ker}(\alpha)$ (by the definition of the kernel). However, $\operatorname{Ker}(\alpha) = 0$ (by definition of a short exact sequence), hence $\mu = 0$.

(2) $\operatorname{Ker}(\beta^*) = Im(\alpha_*)$. First, $\beta_* \circ \alpha_* = (\beta \circ \alpha)_* = 0_* = 0$, hence $Im(\alpha_*) \subseteq \operatorname{Ker}(\beta^*)$. If $a \xrightarrow{\nu} b$ and $0 = \beta_*(\nu)$, i.e., $0 = \beta \circ \nu$, then ν factors through the kernel $\operatorname{Ker}(\beta)$. But $\operatorname{Ker}(\beta) = a'$ and the factorization now means that ν is in $Im(\alpha_*)$.

7.2.4. *Counterexample.* Let $\mathcal{A} = \mathcal{A}b$ and apply $\operatorname{Hom}(a, -)$ for $a = \mathbb{Z}/2\mathbb{Z}$ to $0 \rightarrow 2\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \rightarrow 0$. Then $id_{\mathbb{Z}/2\mathbb{Z}}$ does not lift to a map from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$. So β_* need not be surjective.

7.3. **Right exact functors.** F is right exact if it satisfies one of two equivalent properties

(1) F-image $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ of a SES is exact except possibly in the A'-term, i.e., $F(\alpha)$ may fail to be injective.
(2) F preserves exactness of sequences of the form $C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$.

Version (1) is easier to check and (2) is easier to apply.

7.3.1. *Lemma.* Tensoring is right exact in each argument, i.e., for any left $\Bbbk$-module M the functor $M \otimes_{\Bbbk} - : \mathfrak{m}^r(\Bbbk) \to \mathcal{A}b$ is right exact, and so is $-\otimes_{\Bbbk} N : \mathfrak{m}(\Bbbk) \to \mathcal{A}b$ for any right $\Bbbk$-module N.

7.3.2. *Contravariant case.* Let us state the definition of right exactness also in the case that F is a contravariant from $\mathcal{A}$ to $\mathcal{B}$. The choice of terminology is such that one requires that the functor $F : \mathcal{A} \to \mathcal{B}^o$ is exact, this boils down to asking (again) that exactness is preserved except possibly at A' (the left end of the original sequence. So we need one of the following equivalent properties

(1) For a SES $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ in $\mathcal{A}$, its F-image $F(0) \to F(A'') \xrightarrow{F(\beta)} F(A) \xrightarrow{F(\alpha)} F(A') \to F(0)$ is exact except possibly at $F(A')$, i.e., $F(\alpha)$ may fail to be surjective.
(2) F preserves exactness of sequences of the form $C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$.

The most important example is

7.3.3. *Lemma.* For any $M \in \mathcal{A}$, the contravariant functor $G_M = \mathrm{Hom}_{\mathcal{A}}(-, M)$ is right exact.

7.4. **Projectives and the existence of projective resolutions.** Let $\mathcal{A}$ be an abelian category.

7.4.1. *Projectives.* We say that $p \in \mathcal{A}$ is a projective object if the functor $\mathrm{Hom}_{\mathcal{A}}(p, -) : \mathcal{A} \to \mathcal{A}b$ is exact. Since $\mathrm{Hom}_{\mathcal{A}}(p, -)$ is known to be always left exact, what we need is that for any short exact sequence $0 \to a \xrightarrow{\alpha} b \xrightarrow{\beta} c \to 0$ map $\mathrm{Hom}(p, b) \to \mathrm{Hom}(p, c)$ is surjective. In other words, if c is a quotient of b then any map from p to the quotient $p \xrightarrow{\gamma} c$ lifts to a map to b, i.e., there is a map $p \underset{\tilde{\gamma}}{\to} b$ such that $\gamma = \beta \circ \tilde{\gamma}$ for the quotient map $b \xrightarrow{\beta} c$.

7.4.2. *Lemma.* $\oplus_{i \in I}\, p_i$ is projective iff all summands p_i are projective.

This definition of projectivity generalizes our earlier definition in module categories since

7.4.3. *Lemma.* For a $\Bbbk$-module P, functor $\mathrm{Hom}_{\mathfrak{m}(\Bbbk)}(P,-):\ \mathfrak{m}(\Bbbk)\to\mathcal{A}b$ is exact iff P is a summand of a free module.

We say that abelian category $\mathcal{A}$ has enough projectives if any object is a quotient of a projective object.

corr Module categories have enough projectives.

The importance of "enough projectives" comes from

7.4.4. *Lemma.* For an abelian category $\mathcal{A}$ the following is equivalent

(1) Any object of $\mathcal{A}$ has a projective resolution (i.e., a left resolution consisting of projective objects).
(2) $\mathcal{A}$ has enough projectives.

7.5. **Injectives and the existence of injective resolutions.** Dually, we say that $i\in\mathcal{A}$ is an injective object if the functor $\mathrm{Hom}_{\mathcal{A}}(-,i):\ \mathcal{A}\to\mathcal{A}b^o$ is exact.

Again, since $\mathrm{Hom}_{\mathcal{A}}(-,i)$ is always right exact, we need for any short exact sequence $0\to a\xrightarrow{\alpha} b\xrightarrow{\beta} c\to 0$ that the map $\mathrm{Hom}(b,p)\xrightarrow{\alpha^*}\mathrm{Hom}(a,p),\ \alpha^*(\phi)=\phi\circ\alpha$; be surjective. This means that if a is a subobject of b then any map $a\xrightarrow{\gamma} i$ from a subobject a to i extends to a map from b to i, i.e., there is a map $b\xrightarrow{\tilde{\gamma}} i$ such that $\gamma=\ \tilde{\gamma}\circ\alpha$. So, an object i is injective if each map from a subobject $a'\hookrightarrow a$ to i, extends to the whole object a.

7.5.1. *Example.* $\mathbb{Z}$ is projective in $\mathcal{A}b$ but it is not injective in $\mathcal{A}b$: $\mathbb{Z}\subseteq\frac{1}{n}\mathbb{Z}$ and the map $1_{\mathbb{Z}}:\mathbb{Z}\to\mathbb{Z}$ does not extend to $\frac{1}{n}\mathbb{Z}\to\mathbb{Z}$.

7.5.2. *Lemma.* Product $\prod_{i\in I}\ J_i$ is injective iff all factors J_i are injective.

7.5.3. *Lemma.* A $\mathbb{Z}$-module I is injective iff I is divisible, i.e., for any $a\in I$ and $n\in\{1,2,3,...\}$ there is some $\tilde{a}\in I$ such that $a=n\cdot\tilde{a}$. (i.e., multiplication $n:I\to I$ with $n\in\{1,2,3,...\}$ is surjective.)

The proof will use the Zorn lemma which is an essential part of any strict definition of set theory:

- Let $(I,\leq)$ be a (non-empty) partially ordered set such that any chain J in I (i.e., any totally ordered subset) is dominated by some element of I (i.e., there is some $i\in I$ such that $i\geq j,\ j\in J$). Then I has a maximal element.

Proof. For any $a\in I$ and $n>0$ we can consider $\frac{1}{n}\mathbb{Z}\supseteq\mathbb{Z}\xrightarrow{\alpha} I$ with $\alpha(1)=a$. If I is injective then α extends to $\widetilde{\alpha}:\frac{1}{n}\mathbb{Z}\to I$ and $a=n\widetilde{\alpha}(\frac{1}{n})$.

Conversely, assume that I is divisible and let $A\supseteq B\xrightarrow{\beta} I$. Consider the set $\mathcal{E}$ of all pairs (C,γ) with $B\subseteq C\subseteq A$ and $\gamma:C\to I$ an extension of β. It is partially ordered with $(C,\gamma) \leq (C',\gamma')$ if $C{\subseteq}C'$ and γ' extends γ. From Zorn lemma and the following observations it follows that $\mathcal{E}$ has an element (C,γ) with $C = A$:

(1) For any totally ordered subset $\mathcal{E}'{\subseteq}\mathcal{E}$ there is an element $(C,\gamma) \in \mathcal{E}$ which dominates all elements of $\mathcal{E}'$ (this is clear: take $C = \cup_{(C',\gamma')\in\mathcal{E}'} C'$ and γ is then obvious).

(2) If $(C,\gamma) \in \mathcal{E}$ and $C \neq A$ then (C,γ) is not maximal:
   - choose $a \in A$ which is not in C and let $\widetilde{C} = C + \mathbb{Z}{\cdot}a$ and $C \cap \mathbb{Z}{\cdot}a = \mathbb{Z}{\cdot}na$ with $n \geq 0$. If $n = 0$ then $\widetilde{C} = C{\oplus}\mathbb{Z}{\cdot}a$ and one can extend γ to C by zero on $\mathbb{Z}{\cdot}a$. If $n > 0$ then $\gamma(na) \in I$ is n-divisible, i.e., $\gamma(na) = nx$ for some $x \in I$. Then one can extend γ to $\widetilde{C}$ by $\widetilde{\gamma}(a) = x$ (first define a map on $C{\oplus}\mathbb{Z}{\cdot}a$, and then descend it to the quotient $\widetilde{C}$).

7.5.4. We say that abelian category $\mathcal{A}$ has enough injectives if any object is a subobject of an injective object.

7.5.5. *Lemma.* For an abelian category $\mathcal{A}$ the following is equivalent

(1) Any object of $\mathcal{A}$ has an injective resolution (i.e., a right resolution consisting of injective objects).
(2) $\mathcal{A}$ has enough injectives.

7.5.6. *Lemma.* For any abelian group M denote $\widehat{M} = \mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z})$. Then the canonical map $M \xrightarrow{\rho} \widehat{\widehat{M}}$ is injective.

(b) Category of abelian groups has enough injectives.

Proof. (a) For $m \in M,\ \chi \in \widehat{M}$, $\rho(m)\ (\chi) \stackrel{\text{def}}{=} \chi(m)$. So, $\rho(m) = 0$ means that m is killed by each $\chi\widehat{M}$ ("each character of M"). If $m \neq 0$ then $\mathbb{Z}{\cdot}m$ is isomorphic to $\mathbb{Z}$ or to one of $\mathbb{Z}/n\mathbb{Z}$, in each case we can find a $\mathbb{Z}{\cdot}m \xrightarrow{\chi_0} \mathbb{Q}/\mathbb{Z}$ which is $\neq 0$ on the generator m. Since $\mathbb{Q}/\mathbb{Z}$ is injective we can extend χ_0 to M.

(b) To M we associate a huge injective abelian group $I_M = \prod_{x\in\widehat{M}} \mathbb{Q}/\mathbb{Z}{\cdot}x = (\mathbb{Q}/\mathbb{Z})^{\widehat{M}}$, its elements are $\widehat{M}$-families $c = (c_\chi)_{\chi\in\widehat{M}}$ of elements of $\mathbb{Q}/\mathbb{Z}$ (we denote such family also as a (possibly infinite) formal sum $\sum_{\chi\in\widehat{M}} c_\chi{\cdot}\chi$). By part (a), canonical map io is injective

$$M \xrightarrow{\iota} I_M, \qquad \iota(m) = (\chi(m))_{\chi\in\widehat{M}} = \sum_{\chi\in\widehat{M}} \chi(m){\cdot}\chi, \quad m \in M.$$

7.5.7. *Lemma.* Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Its right adjoint preserves injectivity and its left adjoint preserves projectivity.

Proof. For the right adjoint G and an injective $b \in \mathcal{B}$, functor $\mathrm{Hom}_{\mathcal{A}}[-,Gb] \cong \mathrm{Hom}[F-,b]$ is exact.

corr For a map of rings $\Bbbk \xrightarrow{\phi} \langle$ functor $\phi_* : \mathfrak{m}(\Bbbk) \to \mathfrak{m}(\langle), \quad \phi_*(M) = \langle \otimes_\Bbbk M$ preserves projectivity and $\phi_\star : \mathfrak{m}(\Bbbk) \to \mathfrak{m}(\langle), \quad \phi_\star(M) = \mathrm{Hom}_\Bbbk(\langle, M)$ preserves injectivity.

Proof. These are the two adjoints of the forgetful functor ϕ^*.

7.5.8. *Theorem.* Module categories $\mathfrak{m}(\Bbbk)$ have enough injectives.

Proof. The problem will be reduced to the case $\Bbbk = \mathbb{Z}$ via the canonical map of rings $\mathbb{Z} \xrightarrow{\phi} \Bbbk$. Any $\Bbbk$-module M gives a $\mathbb{Z}$-module $\phi^* M$, and by lemma 7.5.6 there is an embedding $M \xrightarrow{\iota} I_M$ into an injective abelian group. Moreover, by corollary 7.5.7 $\phi_\star I_M$ is an injective $\Bbbk$-module. So it suffices to have an embedding $M \hookrightarrow \phi_\star I_M$.

The adjoint pair $(\phi^*, \phi_\star)$ gives a map of $\Bbbk$-modules $M \xrightarrow{\zeta} \phi_\star \phi^* M = \mathrm{Hom}_\mathbb{Z}(\Bbbk, M)$, by $\zeta(m)\ c = cm,\ m \in M,\ c \in \Bbbk$. It remains to check that both maps in the composition $M \xrightarrow{\zeta} \phi_\star(M) \xrightarrow{\phi_\star(\iota)} \phi_\star(I_M)$ are injective. For ζ it is obvious since $\zeta(m)\ 1_\Bbbk = m$, and for $\phi_\star$ we recall that $\mathrm{Hom}_\mathbb{Z}(\Bbbk, -)$ is left exact, hence takes injective maps to injective maps.

7.5.9. *Examples.* (1) An injective resolution of the $\mathbb{Z}$-module $\mathbb{Z}$: $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/Z \to 0$.

(2) Injective resolutions are often big, hence more difficult to use in specific calculations then say, the free resolutions. We will need them mostly for the functor $\Gamma(X, -)$ of global sections of sheaves, and the functors $\mathrm{Hom}_\mathcal{A}(a, -)$.

7.6. **Exactness and the derived functors.** This is a preliminary motivation for the precise construction of derived functors in the next chapters.

7.6.1. *Left derived functor RF of a right exact functor F.* We observe that if F is right exact then the correct way to extend it to a functor on the derived level is the construction $LF(M) \overset{\text{def}}{=} F(P^\bullet)$, i.e., replacement of the object by a projective resolution. "Correct" means here that LF is really more then F – it contains the information of F in its zero$^{\text{th}}$ cohomology, i.e., $L^0 F \cong F$ for $L^i F(M) \overset{\text{def}}{=} H^i[LF(M)]$. Letter L reminds us that we use a left resolution.

7.6.2. *Lemma.* If the functor $F : \mathcal{A} \to \mathcal{B}$ is right exact, there is a canonical isomorphism of functors $H^0(LF) \cong F$.

Proof. Let $\cdots \to P^{-2} \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{q} M \to 0$ be a projective resolution of M.

(A) *The case when F is covariant.* Then $LF(M) = F[\cdots \to P^{-2} \to P^{-1} \xrightarrow{d^{-1}} P^0 \to 0 \to \cdots]$ equals

$$[\cdots \to F(P^{-2}) \to F(P^{-1}) \xrightarrow{F(d^{-1})} F(P^0) \to 0 \to \cdots],$$

so $H^0[LF(M)] = F(P^0)/F(d^{-1})F(P^{-1})$.

If we apply F to the exact sequence $P^{-1} \xrightarrow{d^{-1}} P^0 \to M \xrightarrow{q} 0$, the right exactness gives an exact sequence $F(P^{-1}) \xrightarrow{F(d^{-1})} F(P^0) \xrightarrow{F(q)} F(M) \to 0$. Therefore, $F(q)$ factors to a canonical map $F(P^0)/F(d^{-1})F(P^{-1}) \to F(M)$ which is an isomorphism.

(B) *The case when F is contravariant.* This is similar, $LF(M) = F(\cdots \to P^{-2} \to P^{-1} \xrightarrow{d^{-1}} P^0 \to 0 \to \cdots)$ equals

$$\cdots \to 0 \to F(P^{-0}) \xrightarrow{d^{-1}} F(P^{-1}) \to F(P^{-2}) \to 0 \to \cdots,$$

and we get $H^0[LF(M)] = \operatorname{Ker}[F(P^{-0}) \xrightarrow{d^{-1}} F(P^{-1})]$. However, applying F to the exact sequence $P^{-1} \xrightarrow{d^{-1}} P^0 \to M \to 0$ gives an exact sequence $0 \to F(M) \to F(P^0) \xrightarrow{F(d^{-1})} F(P^{-1})$. So the canonical map $F(M) \to \operatorname{Ker}[F(P^{-0}) \xrightarrow{d^{-1}} F(P^{-1})]$ is an isomorphism.

7.6.3. *Remark.* As we see the argument is categorical and would not simplify if we only considered module categories.

7.6.4. *Example.* Recall the functor $i^o : \mathfrak{m}(D^1_{\mathbb{A}}) \to \mathfrak{m}(D_{\mathbb{A}^0})$, $i^oM = M/xM$. It is right exact by 7.3.1 since $M/xM \cong M \otimes_{\Bbbk[x]} \Bbbk[x]/x\Bbbk[x]$, and therefore $H^0[Li^o(M)] \cong i^o(M)$ by the preceding lemma.

7.6.5. *Right derived functor RF of a left exact functor F.* Obviously, we want to define for any left exact functor $F : \mathcal{A} \to \mathcal{B}$ a right derived functor RF by replacing an object by its injective resolution. Then, as above $H^0(RF) \cong F$.

8. **Abelian category of sheaves of abelian groups**

For a topological space X we will denote by $\mathcal{S}h(X) = \mathcal{S}heaves(X, \mathcal{A}b)$ the category of sheaves of abelian groups on X. Since a sheaf of abelian groups is something like an abelian group smeared over X we hope to $\mathcal{S}h(X)$ is again an abelian category. When attempting to construct cokernels, the first idea does not quite work – it produces something like a sheaf but without the gluing property. This forces us to

- (i) generalize the notion of sheaves to a weaker notion of a presheaf,
- (ii) find a canonical procedure that improves a presheaf to a sheaf.

(We will also see that a another example that requires the same strategy is the pull-back operation on sheaves.)

Now it is easy to check that we indeed have an abelian category. What allows us to compute in this abelian category is the lucky break that one can understand kernels, cokernels, images and exact sequences just by looking at the stalks of sheaves.

8.1. **Categories of sheaves.**

A presheaf of sets $\mathcal{S}$ on a topological space $(X, \mathcal{T})$ consists of the following data:

- for each open $U \subseteq X$ a set $\mathcal{S}(U)$,
- for each inclusion of open subsets $V \subseteq U \subseteq X$ a map $\mathcal{S}(U) \xrightarrow{\rho^U_V} \mathcal{S}(V)$ (called the restriction map);

and these data are required to satisfy

- (Sh0)(Transitivity of restriction) $\rho^U_V \circ \rho^U_V = \rho^U_W$ for $W \subseteq V \subseteq U$

A sheaf of sets on a topological space $(X, \mathcal{T})$ is a presheaf $\mathcal{S}$ which also satisfies

- (Sh1) (Gluing) Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of an open $U \subseteq X$ (We denote $U_{ij} = U_i \cap U_j$ etc.). We ask that any family of compatible sections $f_i \in \mathcal{S}(U_i)$, $i \in I$, glues uniquely. This means that if sections f_i agree on intersections in the sense that $\rho^{U_i}_{U_{ij}} f_i = \rho^{U_i}_{U_{ij}} f_j$ in $\mathcal{S}(U_{ij})$ for any $i, j \in I$; then there is a unique $f \in \mathcal{S}(U)$ such that $\rho^U_{U_i} f = f_i$ in $\mathcal{S}(U_i)$, $i \in I$.
- $\mathcal{S}(\emptyset)$ is a *point*.

8.1.1. *Remarks.* (1) Presheaves of sets on X form a category $pre\mathcal{S}heaves(X, \mathcal{S}ets)$ when $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ consists of all systems $\phi = (\phi_U)_{U \subseteq X \ open}$ of maps $\phi_U : \mathcal{A}(U) \to \mathcal{B}(U)$ which are compatible with restrictions, i.e., for $V\subseteq U$

$$\begin{array}{ccc} \mathcal{A}(U) & \xrightarrow{\phi_U} & \mathcal{B}(U) \\ \rho^U_V\Big\downarrow & & \rho^U_V\Big\downarrow \\ \mathcal{A}(V) & \xrightarrow{\phi_V} & \mathcal{B}(V) \end{array} .$$

(One reads the diagram above as : "the diagram ... commutes".) The sheaves form a full subcategory $pre\mathcal{S}heaves(X,\mathcal{S}ets)$ of $\mathcal{S}heaves(X,\mathcal{S}ets)$.

(2) We can equally define categories of sheaves of abelian groups, rings, modules, etc. For a sheaf of abelian groups we ask that all $\mathcal{A}(U)$ are abelian groups, all restriction morphisms are maps of abelian groups, and we modify the least interesting requirement (Sh2): $\mathcal{S}(\phi)$ is the trivial group $\{0\}$. In general, for a category $\mathcal{A}$ one can define categories $pre\mathcal{S}heaves(X,\mathcal{A})$ and $\mathcal{S}heaves(X,\mathcal{A})$ similarly (the value on $\emptyset$ should be the final object of $\mathcal{A}$).

8.2. **Sheafification of presheaves.** We will use the wish to pull-back sheaves as a motivation for a procedure that improves presheaves to sheaves.

8.2.1. *Functoriality of sheaves.* Recall that for any map of topological spaces $X \xrightarrow{\pi} Y$ one wants a pull-back functor $\mathcal{S}heaves(Y) \xrightarrow{\pi^{-1}} \mathcal{S}heaves(X)$. As we have seen in the definition of a stalk of a sheaf (pull ,back to a point), the natural formula is

$$\underline{\pi^{-1}}(\mathcal{N})\ (U) \overset{\text{def}}{=} \lim_{\substack{\to \\ V\supseteq\pi(U)}} \mathcal{N}(V),$$

where limit is over open $V\subseteq Y$ that contain $\pi(U)$, and we say that $V'\le V''$ if V'' better approximates $\pi(U)$, i.e., if $V''\subseteq V'$.

8.2.2. *Lemma.* This gives a functor of presheaves $pre\mathcal{S}heaves(X) \xrightarrow{\underline{\pi^{-1}}} pre\mathcal{S}heaves(Y)$.

Proof. For $U'\subseteq U$ open, $\underline{\pi^{-1}}\mathcal{N}(U') = \lim_{\to\ V\supseteq\pi(U')} \mathcal{N}(V)$ and $\underline{\pi^{-1}}\mathcal{N}(U) = \lim_{\to\ V\supseteq\pi(U)} \mathcal{N}(V)$ are limits of inductive systems of $\mathcal{N}(V)$'s, and the second system is a subsystem of the first one, this gives a canonical map $\underline{\pi^{-1}}\mathcal{N}(U) \to \underline{\pi^{-1}}\mathcal{N}(U')$.

8.2.3. *Remark.* Even if $\mathcal{N}$ is a sheaf, $\underline{\pi^{-1}}(\mathcal{N})$ need not be sheaf.

For that let $Y = pt$ and let $\mathcal{N} = S_Y$ be the constant sheaf of sets on Y given by a set S. So, $S_Y(\emptyset) = \emptyset$ and $S_Y(Y) = S$. Then $\underline{\pi^{-1}}(S_Y)\ (U) = \begin{cases} \emptyset & if\ U=\emptyset, \\ S & U\ne\emptyset \end{cases}$. We can say: $\underline{\pi^{-1}}(S_Y)\ (U) =$ constant functions from U to S. However, we have noticed that constant functions do not give a sheaf, so we need to correct the procedure $\underline{\pi^{-1}}$ to get sheaves from sheaves. For that remember that for the presheaf of constant functions there is a related sheaf S_X of *locally constant* functions.

Our problem is that the presheaf of constant functions is defined by a global condition (constancy) and we need to change it to a local condition (local constancy) to make it into a sheaf. So we need the procedure of

8.2.4. *Sheafification.* This is a way to improve any presheaf of sets $\mathcal{S}$ into a sheaf of sets $\widetilde{\mathcal{S}}$. We will imitate the way we passed from constant functions to locally constant functions. More precisely, we will obtained the sections of the sheaf $\tilde{\mathcal{S}}$ associated to the presheaf $\mathcal{S}$ in two steps:

(1) we glue systems of local sections s_i which are compatible in the weak sense that they are <u>*locally*</u> the same, and
(2) we identify two results of such gluing if the local sections in the two families are <u>*locally*</u> the same.

Formally these two steps are performed by replacing $\mathcal{S}(U)$ with the set $\widetilde{\mathcal{S}}(U)$, defined as the set of all equivalence classes of systems $(U_i, s_i)_{i\in I}$ where

(1) Let $\widehat{\mathcal{S}}(U)$ be the class of all systems $(U_i, s_i)_{i\in I}$ such that
   - $(U_i)_{i\in I}$ is an open cover of U and s_i is a section of $\mathcal{S}$ on U_i,
   - sections s_i are *weakly compatible* in the sense that they are locally the same, i.e., for any $i', i'' \in I$ sections $s_{i'}$ and $s_{i''}$ are the same near any point $x \in U_{i'i''}$. (Precisely, this means that there is neighborhood W such that $s_{i'}|W = s_{i''}|W$.)
(2) We say that two systems $(U_i, s_i)_{i\in I}$ and $(V_j, t_j)_{j\in J}$ are $\equiv$, iff for any $i \in I,\ j \in J$ sections s_i and t_j are weakly equivalent (i.e., for each $x \in U_i \cap V_j$, there is an open set W with $x \in W \subseteq U_i \cap V_j$ such that "$s_i = t_j$ on W" in the sense of restrictions being the same).

8.2.5. *Remark.* The relation $\equiv$ on $\widehat{\mathcal{S}}(U)$ really says that $(U_i, s_i)_{i\in I} \equiv (V_j, t_j)_{j\in J}$ iff the disjoint union $(U_i, s_i)_{i\in I} \sqcup (V_j, t_j)_{j\in J}$ is again in $\widehat{\mathcal{S}}(U)$.

8.2.6. *Lemma.* (a) $\equiv$ is an equivalence relation.

(b) $\widetilde{\mathcal{S}}(U)$ is a presheaf and there is a canonical map of presheaves $\mathcal{S} \xrightarrow{q} \widetilde{\mathcal{S}}$.

(c) $\widetilde{\mathcal{S}}$ is a sheaf.

Proof. (a) is obvious.

(b) The restriction of a system $(U_i, s_i)_{i\in I}$ to $V \subseteq U$ is the system $(U_i \cap V, s_i|U_i \cap V)_{i\in I}$. The weak compatibility of restrictions $s_i|U \cap V$ follows from the weak compatibility of sections s_i. Finally, restriction is compatible with $\equiv$, i.e., if $(U_i', s_i')_{i\in I}$ and $(U_j'', s_j'')_{j\in J}$ are $\equiv$, then so are $(U_i' \cap V, s_i'|U_i' \cap V)_{i\in I}$ and $(U_j'' \cap V, s_j''|U_j'' \cap V)_{j\in J}$.

The map $\mathcal{S}(U) \to \widetilde{\mathcal{S}}(U)$ is given by interpreting a section $s \in \mathcal{S}(U)$ as a (small) system: open cover of $(U_i)_{i\in\{0\}}$ is given by $U_0 = U$ and $s_0 = s$.

(c') Compatible systems of sections of the presheaf $\widetilde{\mathcal{S}}$ glue. Let $V^j,\ j \in J$, be an open cover of an open $V\subseteq X$, and for each $j \in J$ let $\sigma^j = [(U_i^j, s_i^j)_{i\in\ I_j}]$ be a section of $\widetilde{\mathcal{S}}$ on V_j. So, σ^j is an equivalence class of the system $(U_i^j, s_i^j)_{i\in\ I_j}$ consisting of an open cover $U_i^j,\ i \in\ I_j$, of V_j and weakly compatible sections $s_j^i \in \mathcal{S}(U_j^i)$.

Now, if for any $j, k \in J$ sections $\sigma^j =\ [(U_p^j, s_p^j)_{p\in\ I_j}]$ and $\sigma^k =\ [(U_q^k, s_q^k)_{q\in\ I_k}]$ of $\widetilde{\mathcal{S}}$ on V^j and V^k, agree on the intersection V^{jk}. This means that for any j, k $\sigma^j|V^{jk} = \sigma^k|V^{jk}$, i.e.,

$$(U_p^j \cap V^{jk}, s_p^j|U_p^j \cap V^{jk})_{p\in\ I_j} \quad\equiv\quad (U_q^k \cap V^{jk}, s_q^k|U_q^k \cap V^{jk})_{q\in\ I_k}.$$

This in turn means that for $j, k \in J$ and any $p \in I_j,\ q \in I_k$, sections s_p^j and s_q^k are weakly compatible. Since all sections $s_p^j,\ j \in J,\ p \in I_j$ are weakly compatible, the disjoint union of all systems $(U_i^j, s_i^j)_{i\in\ I_j},\ j \in J$ is a system in $\widehat{\mathcal{S}}(V)$. Its equivalence class σ is a section of $\widetilde{\mathcal{S}}$ on V, and clearly $\sigma|V^j = \sigma^j$.

(c") Compatible systems of sections of the presheaf $\widetilde{\mathcal{S}}$ glue uniquely. If $\tau \in \widetilde{\mathcal{S}}(V)$ is the class of a system $(U_i, s^i)_{i\in I}$ and $\tau|V^j = \sigma^j$ then σ's are compatible with all s_p^j's, hence $(U_i, s^i)_{i\in I} \equiv \sqcup_{j\in J}(U_i^j, s_i^j)_{i\in\ I_j}$, hence $\tau = \sigma$.

8.2.7. *Sheafification as a left adjoint of the forgetful functor.* As usual, we have not invented something new: it was already there, hidden in the more obvious forgetful functor

8.2.8. *Lemma.* Sheafification functor $pre\mathcal{S}heaves \ni \mathcal{S}\mapsto\widetilde{\mathcal{S}} \in \mathcal{S}heaves$, is the left adjoint of the inclusion $\mathcal{S}heaves\subseteq pre\mathcal{S}heaves$, i.e, for any presheaf $\mathcal{S}$ and any sheaf $\mathcal{F}$ there is a natural identification

$$\mathrm{Hom}_{\mathcal{S}heaves}(\widetilde{\mathcal{S}}, \mathcal{F}) \xrightarrow{\cong} \mathrm{Hom}_{pre\mathcal{S}heaves}(\mathcal{S}, \mathcal{F}).$$

Explicitly, the bijection is given by $(\iota_{\mathcal{S}})_*\alpha =\ \alpha\circ\iota_{\mathcal{S}}$, i.e., $(\widetilde{\mathcal{S}} \xrightarrow{\alpha} \mathcal{F})\mapsto (\mathcal{S} \xrightarrow{\iota_{\mathcal{S}}} \widetilde{\mathcal{S}} \xrightarrow{\alpha} \mathcal{F})$.

8.3. **Inverse and direct images of sheaves.**

8.3.1. *Pull back of sheaves (finally!)* Now we can define for any map of topological spaces $X \xrightarrow{\pi} Y$ a pull-back functor

$$\mathcal{S}heaves(Y) \xrightarrow{\pi^{-1}} \mathcal{S}heaves(X), \qquad \pi^{-1}\mathcal{N} \stackrel{\text{def}}{=} \widetilde{\underline{\pi^{-1}}\mathcal{N}}.$$

8.3.2. *Examples.* (a) A point $a \in X$ can be viewed as a map $\{a\} \xrightarrow{\rho} X$. Then $\rho^{-1}\mathcal{S}$ is the stalk $\mathcal{S}_a$.

(b) Let $a : X \to$pt, for any set S one has $S_X = a^{-1}S$.

8.3.3. *Direct image of sheaves.* Besides the pull-back of sheaves which we defined in 8.3.1, there is also a much simpler procedure of the push-forward of sheaves:

8.3.4. *Lemma.* (Direct image of sheaves.) Let $X \xrightarrow{\pi} Y$ be a map of topological spaces. For a sheaf $\mathcal{M}$ on X, formula

$$\pi_*(\mathcal{M})\ (V) \stackrel{\text{def}}{=} \mathcal{M}(\pi^{-1}V),$$

defines a sheaf $\pi_*\mathcal{M}$ on Y, and this gives a functor $\mathcal{S}heaves(X) \xrightarrow{\pi_*} \mathcal{S}heaves(Y)$.

8.3.5. *Adjunction between the direct and inverse image operations.* The two basic operations on sheaves are related by adjunction:

Lemma. For sheaves $\mathcal{A}$ on X and $\mathcal{B}$ on Y one has a natural identification

$$\text{Hom}(\pi^{-1}\mathcal{B},\mathcal{A}) \cong \text{Hom}(\mathcal{B},\pi_*\mathcal{A}).$$

Proof. We want to compare $\beta \in \text{Hom}(\mathcal{B},\pi_*\mathcal{A})$ with α in

$$\text{Hom}_{\mathcal{S}h(X)}(\pi^{-1}\mathcal{B},\mathcal{A}) = \text{Hom}_{\mathcal{S}h(X)}(\widetilde{\underline{\pi^{-1}}\mathcal{B}},\mathcal{A}) \cong \text{Hom}_{pre\mathcal{S}h(X)}(\underline{\pi^{-1}}\mathcal{B},\mathcal{A}).$$

α is a system of maps

$$\lim_{\to\ V\supseteq\pi(U)} \mathcal{B}(V) = \underline{\pi^{-1}}\mathcal{B}(U) \xrightarrow{\alpha_U} \mathcal{A}(U), \quad \text{for } U \text{ open in } X,$$

and β is a system of maps

$$\mathcal{B}(V) \xrightarrow{\beta_V} \mathcal{A}(\pi^{-1}V), \quad \text{for } V \text{ open in } Y.$$

Clearly, any β gives some α since

$$\lim_{\to\ V\supseteq\pi(U)} \mathcal{B}(V) \xrightarrow{\lim_{\to} \beta_V} \lim_{\to\ V\supseteq\pi(U)} \mathcal{A}(\pi^{-1}V) \to \mathcal{A}(U),$$

the second map comes from the restrictions $\mathcal{A}(\pi^{-1}V) \to \mathcal{A}(U)$ defined since $V\supseteq\pi(U)$ implies $\pi^{-1}V\supseteq U$.

For the opposite direction, any α gives for each V open in Y, a map $\lim_{\to\ W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W) = \underline{\pi^{-1}}\mathcal{B}(\pi^{-1}V) \xrightarrow{\alpha_{\pi^{-1}V}} \mathcal{A}(\pi^{-1}V)$. Since $\mathcal{B}(V)$ is one of the terms in the inductive system we have a canonical map $\mathcal{B}(V) \to \lim_{\to\ W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W)$, and the composition with the first map $\mathcal{B}(V) \to \lim_{\to\ W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W) \xrightarrow{\alpha_{\pi^{-1}V}} \mathcal{A}(\pi^{-1}V)$, is the wanted map β_V.

8.3.6. *Lemma.* (a) If $X \xrightarrow{\pi} Y \xrightarrow{\tau} Z$ then

$$\tau_*(\pi_*\mathcal{A}) \cong (\tau\circ\pi)_*\mathcal{A} \quad \text{and} \quad \tau_*(\pi_*\mathcal{A}) \cong (\tau\circ\pi)_*\mathcal{A}.$$

(b) $(1_X)_*\mathcal{A} \cong \mathcal{A} \cong (1_X)^{-1}\mathcal{A}$.

Proof. The statements involving direct image are very simple and the claims for inverse image follow by adjunction.

corr (Pull-back preserves the stalks) For $a \in X$ one has $(\pi^{-1}\mathcal{N})_a \cong \mathcal{N}_{\pi(a)}$.

This shows that the pull-back operation which was difficult to define is actually very simple in its effect on sheaves.

8.4. **Stalks.** Part (a) of the following lemma is the recollection of the description of inductive limits of abelian groups from the remark 5.3.8.

8.4.1. *Lemma.* (Inductive limits of abelian groups.) (a) For an inductive system of abelian groups (or sets) A_i over $(I, \leq)$, inductive limit $\lim\limits_{\rightarrow} A_i$ can be described by

- for $i \in I$ any $a \in A_i$ defines an element $\overline{a}$ of $\lim\limits_{\rightarrow} A_i$,
- all elements of $\lim\limits_{\rightarrow} A_i$ arise in this way, and
- for $a \in A_i$ and $b \in A_j$ one has $\overline{a} = \overline{b}$ iff for some $k \in I$ with $i \leq k \geq j$ one has $a = b$ in A_k.

(b) For a subset $K \subseteq I$ one has a canonical map $\lim\limits_{\rightarrow\, i\in K} A_i \rightarrow \lim\limits_{\rightarrow\, i\in I} A_i$.

Proof. In general (b) is clear from the definition of $\lim\limits_{\rightarrow}$, and for abelian groups also from (a).

8.4.2. *Stalks of (pre)sheaves.* Remember that the stalk of a presheaf $\mathcal{A}$b at a point x is $\mathcal{A}_x \overset{\text{def}}{=} \lim\limits_{\rightarrow} \mathcal{A}(U)$, the limit over (diminishing) neighborhoods u of x. This means that

- any $s \in \mathcal{A}(U)$ with $U \ni x$ defines an element s_x of the stalk,
- all elements of $\mathcal{A}_x$ arise in this way, and
- For $s' \in \mathcal{A}(U')$ and $s'' \in \mathcal{A}(U'')$ one has $s'_x = s''_x$ iff for some neighborhood W of x in $U' \cap U''$ one has $s' = s''$ on W.

8.4.3. *Lemma.* For a presheaf $\mathcal{S}$, the canonical map $\mathcal{S} \rightarrow \widetilde{\mathcal{S}}$ is an isomorphism on stalks.

Proof. We consider a point $a \in X$ as a map $\mathrm{pt} = \{a\} \xrightarrow{i} X$, so that $\mathcal{A}_x = i^{-1}\mathcal{A}$. For a sheaf B on the point

$$\mathrm{Hom}_{\mathcal{S}h(pt)}(i^{-1}\widetilde{\mathcal{S}}, \mathcal{B}) \cong \mathrm{Hom}_{\mathcal{S}h(X)}(\widetilde{\mathcal{S}}, i_*\mathcal{B}) \cong \mathrm{Hom}_{pre\mathcal{S}h(X)}(\mathcal{S}, i_*\mathcal{B})$$
$$\cong \mathrm{Hom}_{pre\mathcal{S}h(pt)}(i^{-1}\mathcal{S}, \mathcal{B}) = \mathrm{Hom}_{\mathcal{S}h(pt)}(i^{-1}\mathcal{S}, \mathcal{B}).$$

8.4.4. *Germs of sections and stalks of maps.* For any neighborhood U of a point x we have a canonical map $\mathcal{S}(U) \to \lim_{\to\, V\ni x} \mathcal{S}(V) \stackrel{\text{def}}{=} \mathcal{S}_x$ (see lemma 8.4.1), and we denote the image of a section $s \in \Gamma(U, \mathcal{S})$ in the stalk $\mathcal{S}_x$ by s_x, and we call it the germ of the section at x. The germs of two sections are the same at x if the sections are the same on some (possibly very small) neighborhood of x (this is again by the lemma 8.4.1).

A map of sheaves $\phi : \mathcal{A} \to \mathcal{B}$ defines for each $x \in M$ a map of stalks $\mathcal{A}_x \to \mathcal{B}_x$ which we call ϕ_x. It comes from a map of inductive systems given by ϕ, i.e., from the system of maps $\phi_U : \mathcal{A}(U) \to \mathcal{B}(U)$, $U \ni x$ (see 5.3.5) ; and on germs it is given by $\phi_x(a_x) = [\phi_U(a)]_x$, $a \in \mathcal{A}(U)$.

For instance, let $\mathcal{A} = \mathcal{H}_{\mathbb{C}}$ be the sheaf of holomorphic functions on $\mathbb{C}$. Remember that the stalk at $a \in \mathbb{C}$ can be identified with all convergent power series in $z - a$. Then the germ of a holomorphic function $f \in \mathcal{H}_{\mathbb{C}}(U)$ at a can be thought of as the power series expansion of f at a. An example of a map of sheaves $\mathcal{H}_{\mathcal{C}} \xrightarrow{\Phi} \mathcal{H}_{\mathbb{C}}$ is the multiplication by an entire function $\phi \in \mathcal{H}_{\mathbb{C}}(\mathbb{C})$, its stalk at a is the multiplication of the the power series at a by the power series expansion of ϕ at a.

8.4.5. The following lemma from homework shows how much the study of sheaves reduces to the study of their stalks.

Lemma. (a) Maps of sheaves $\phi, \psi : \mathcal{A} \to \mathcal{B}$ are the same iff the maps on stalks are the same, i.e., $\phi_x = \psi_x$ for each $x \in M$.

(b) Map of sheaves $\phi : \mathcal{A} \to \mathcal{B}$ is an isomorphism iff ϕ_x is an isomorphism for each $x \in M$.

8.4.6. *Sheafifications via the etale space of a presheaf.* We will construct the sheafification of a presheaf $\mathcal{S}$ (once again) in an "elegant" way, using the etale space $\overset{\bullet}{\mathcal{S}}$ of the presheaf. It is based on the following example of sheaves

Example. Let $Y \xrightarrow{p} X$ be a continuous map. For any open $U \subseteq X$ the elements of

$$\Sigma(U) \stackrel{\text{def}}{=} \{s : U \to Y,\ s \text{ is continuous and } p \circ s = 1_u\}$$

are called the (continuous) sections of p over U. Σ is a sheaf of sets.

To apply this construction we need a space $\overset{\bullet}{\mathcal{S}}$ that maps to X:

- Let $\overset{\bullet}{\mathcal{S}}$ be the union of all stalks $\mathcal{S}_m$, $m \in X$.
- Let $p : \overset{\bullet}{\mathcal{S}} \to X$ be the map such that the fiber at m is the stalk at m.
- For any pair (U, s) with U open in X and $s \in \mathcal{S}(U)$, define a section $\tilde{s}$ of p over U by

$$\tilde{s}(x) \stackrel{\text{def}}{=} s_x \in \mathcal{S}_x \subset \overset{\bullet}{\mathcal{S}}, \qquad x \in U.$$

Lemma. (a) If for two sections $s_i \in \mathcal{S}(U_i),\ i = 1, 2$; of $\mathcal{S}$, the corresponding sections $\tilde{s}_1$ and $\tilde{s}_2$ of p agree at a point then they agree on some neighborhood of of this point (i.e., if $\tilde{s}_1(x) = \tilde{s}_2(x)$ for some $x \in U_{12} \stackrel{\text{def}}{=} U_1 \cap U_2$, then there is a neighborhood W of x such that $\tilde{s}_1 = \tilde{s}_2$ on W).

(b) All the sets $\tilde{s}(U)$ (for $U \subseteq X$ open and $s \in \mathcal{S}(U)$), form a basis of a topology on $\dot{\mathcal{S}}$. Map $p : \dot{\mathcal{S}} \to M$ is continuous.

(c) Let $\tilde{\mathcal{S}}(U)$ denote the set of continuous sections of p over U. Then $\tilde{\mathcal{S}}$ is a sheaf and there is a canonical map of presheaves $\iota : \mathcal{S} \to \tilde{\mathcal{S}}$.

Remark. Moreover, p is "etale" meaning "locally an isomorphism", i.e., for each point $\sigma \in \dot{\mathcal{S}}$ there are neighborhoods $\sigma \in W \subseteq \dot{\mathcal{S}}$ and $p(\sigma) \subseteq U \subseteq X$ such that $p|W$ is a homeomorphism $W \xrightarrow{\cong} U$.

Lemma. The new $\widetilde{\mathcal{S}}$ and the old $\widetilde{\mathcal{S}}$ (from 8.2.4) are the same sheaves (and the same holds for the canonical maps $\iota : \mathcal{S} \to \widetilde{\mathcal{S}}$).

Proof. Sections of p over $U \subseteq X$ are the same as the equivalence classes of systems $\widehat{\mathcal{S}}/\equiv$ defined in 8.2.4.

8.5. **Abelian category structure.** Let us fix a map of sheaves $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ since the non-trivial part is the construction of (co)kernels. Consider the example where the space is the circle $X = \{z \in \mathbb{C},\ |z| = 1\}$ and $\mathcal{A} = \mathcal{B}$ is the sheaf $\mathcal{C}_X^\infty$ of smooth functions on X, and the map α is the differentiation $\partial = \frac{\partial}{\partial\theta}$ with respect to the angle θ. For $U \subseteq X$ open, $\mathrm{Ker}(\partial_U) :\ \mathcal{C}_X^\infty(U) \to \mathcal{C}_X^\infty(U)$ consists of locally constant functions and the cokernel $\mathcal{C}_X^\infty(U)/\partial_U \mathcal{C}_X^\infty(U)$ is

- zero if $U \neq X$ (then any smooth function on U is the derivative of its indefinite integral defined by using the exponential chart $z = e^{i\theta}$ which identifies U with an open subset of $\mathbb{R}$),
- one dimensional if $U = X$ – for $g \in C^\infty(X)$ one has $\int_X\ \partial g = 0$ so say constant functions on X are not derivatives (and for functions with integral zero the first argument applies).

So by taking kernels at each level we got a sheaf but by taking cokernels we got a presheaf which is not a sheaf (local sections are zero but there are global non-zero sections, so the object is not controlled by its local properties).

8.5.1. *Subsheaves.* For (pre)sheaves $\mathcal{S}$ and $\mathcal{S}'$ we say that $\mathcal{S}'$ is a sub(pre)sheaf of $\mathcal{S}$ if $\mathcal{S}'(U) \subseteq \mathcal{S}(U)$ and the restriction maps for $\mathcal{S}'$, $\mathcal{S}'(U) \xrightarrow{\rho'} \mathcal{S}'(V)$ are restrictions of the restriction maps for $\mathcal{S}$, $\mathcal{S}(U) \xrightarrow{\rho} \mathcal{S}(V)$.

8.5.2. *Lemma.* (Kernels.) Any map $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ has a kernel and $\mathrm{Ker}(\alpha)(U) = \mathrm{Ker}[\mathcal{A}(U) \xrightarrow{\phi(U)} \mathcal{B}(U)]$ is a subsheaf of $\mathcal{A}$.

Proof. First, $\mathcal{K}(U) \stackrel{\text{def}}{=} \mathrm{Ker}[\mathcal{A}(U) \xrightarrow{\phi(U)} \mathcal{B}(U)]$ is a sheaf, and then a map $\mathcal{C} \xrightarrow{\mu} \mathcal{A}$ is killed by α iff it factors through the subsheaf $\mathcal{K}$ of $\mathcal{A}$.

Lemma. (Cokernels.) Any map $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ defines a presheaf $C(U) \stackrel{\text{def}}{=} \mathcal{B}(U)/\alpha_U(\mathcal{A}(U))$, the associated sheaf $\mathcal{C}$ is the cokernel of α.

Proof. For a sheaf $\mathcal{S}$ one has

$$\mathrm{Hom}_{\mathcal{S}heaves}(\mathcal{B},\mathcal{S})_\alpha \cong \mathrm{Hom}_{pre\mathcal{S}heaves}(C,\mathcal{S}) \cong \mathrm{Hom}_{\mathcal{S}heaves}(\mathcal{C},\mathcal{S}).$$

The second identification is the adjunction. For the first one, a map $\mathcal{B} \xrightarrow{\phi} \mathcal{S}$ is killed by α, i.e., $0 = \phi\circ\alpha$, if for each U one has $0 = (\phi\circ\alpha)_U\mathcal{A}(U) = \phi_U(\alpha_U\mathcal{A}(U))$; but then it gives a map $C \xrightarrow{\overline{\phi}} \mathcal{S}$, with $\overline{\phi}_U : C(U) = \mathcal{B}(U)/\alpha_U\mathcal{A}(U) \to \mathcal{S}(U)$ the factorization of ϕ_U. The opposite direction is really obvious, any $\psi : C \to \mathcal{S}$ can be composed with the canonical map $\mathcal{B} \to C$ (i.e., $\mathcal{B}(U) \to \mathcal{B}(U)/\alpha_U\mathcal{A}(U)$) to give map $\mathcal{B} \to \mathcal{S}$ which is clearly killed by α.

8.5.3. *Lemma.* (Images.) Consider a map $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$.

(a) It defines a presheaf $I(U) \stackrel{\text{def}}{=} \alpha_U(\mathcal{A}(U)) \subseteq \mathcal{B}(U)$ which is a subpresheaf of $\mathcal{B}$. The associated sheaf $\mathcal{I}$ is the image of α.

(b) It defines a presheaf $c(U) \stackrel{\text{def}}{=} \mathcal{A}(U)/\mathrm{Ker}(\alpha_U)$, the associated sheaf is the coimage of α.

(c) The canonical map $Coim(\alpha) \to Im(\alpha)$ is isomorphism.

Proof. (a) $Im(\alpha) \stackrel{\text{def}}{=} \mathrm{Ker}[\mathcal{B} \to \mathrm{Coker}(\alpha)]$ is a subsheaf of $\mathcal{B}$ and $b \in \mathcal{B}(U)$ is a section of $Im(\alpha)$ iff it becomes zero in $\mathrm{Coker}(\alpha)$. But a section $b + \alpha_U\mathcal{A}(U)$ of C on U is zero in $\mathcal{B}$ iff it is locally zero in C, i.e., there is a cover U_i of U such that $b|U_i \in \alpha_{U_i}\mathcal{A}(U_i)$. But this is the same as saying that b is locally in the subpresheaf I of $\mathcal{B}$, i.e., the same as asking that b is in the corresponding presheaf $\mathcal{I}$ of $\mathcal{B}$.

(b) The coimage of α is by definition $Coim(\alpha) \stackrel{\text{def}}{=} \mathrm{Coker}[\mathrm{Ker}(\alpha) \to \mathcal{A}]$, i.e., the sheaf associated to the presheaf $U \mapsto \mathcal{A}(U)/\mathrm{Ker}(\alpha)(U) = c(U)$.

(c) The map of sheaves $Coim(\alpha) \to Im(\alpha)$ is associated to the canonical map of presheaves $c \to I$, however already the map of presheaves is an isomorphism: $c(U) = \mathcal{A}(U)/\mathrm{Ker}(\alpha)(U) \cong \alpha_U \stackrel{\text{def}}{=} \mathcal{A}(U) = I(U)$.

8.5.4. *Stalks of kernels, cokernels and images; exact sequences of sheaves.*

8.5.5. *Lemma.* For a map of sheaves $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ and $x \in X$

- (a) $\mathrm{Ker}(\mathcal{A} \xrightarrow{\alpha} \mathcal{B})_x = \mathrm{Ker}(\alpha_x : \mathcal{A}_x \to \mathcal{B}_x)$,
- (b) $\mathrm{Coker}(\mathcal{A} \xrightarrow{\alpha} \mathcal{B})_x = \mathrm{Coker}(\alpha_x : \mathcal{A}_x \to \mathcal{B}_x)$,
- (c) $Im(\mathcal{A} \xrightarrow{\alpha} \mathcal{B})_x = Im(\alpha_x : \mathcal{A}_x \to \mathcal{B}_x)$.

Proof. (a) Let $x \in U$ and $a \in \mathcal{A}(U)$. The germ a_x is killed by α_x if $0 = \alpha_x(a_x) \overset{\text{def}}{=} (\alpha_U(a))_x$, i.e., iff $\alpha_U(a) = 0$ on some neighborhood U' of x in U. But this is the same as saying that $0 = \alpha_U(a)|U' = \alpha_{U'}(a|U')$, i.e., asking that some restriction of a to a smaller neighborhood of x is a section of the subsheaf $\mathrm{Ker}(\alpha)$. And this in turn, is the same as saying that the germ a_x lies in the stalk of $\mathrm{Ker}(\alpha)$.

(b) Map $\mathcal{B} \xrightarrow{q} \mathrm{Coker}(\alpha)$ is killed by composing with α, so the map of stalks $\mathcal{B}_x \xrightarrow{q_x} \mathrm{Coker}(\alpha)_x$ is killed by composing with α_x.

To see that q_x is surjective consider some element of the stalk $\mathrm{Coker}(\alpha)_x$. It comes from a section of a presheaf $U \mapsto \mathcal{B}(U)/\alpha_U\mathcal{A}(U)$, so it is of the form $[b + \alpha_U(\mathcal{A}(U))]_x$ for some section $b \in \mathcal{B}(U)$ on some neighborhood U of x. Therefore it is the image $\alpha_x(b_x)$ of an element b_x of $\mathcal{B}_x$.

To see that q_x is injective, observe that a stalk $b_x \in \mathcal{B}_x$ (of some section $b\mathcal{B}(U)$), is killed by q_x iff its image $\alpha_x(b_x) = [b + \alpha_U(\mathcal{A}(U))]_x$ is zero in $\mathrm{Coker}(\alpha)$, i.e., iff there is a smaller neighborhood $U' \subseteq U$ such that the restriction $[b + \alpha_U(\mathcal{A}(U))]|U' = b|U' + \alpha_{U'}(\mathcal{A}(U'))$ is zero, i.e., $b|U'$ is in $\alpha_{U'}\mathcal{A}(U')$. But the existence of such U' is the same as saying that b_x is in the image of α_x.

(c) follows from (a) and (b) by following how images are defined in terms of kernels and cokernels.

corr A sequence of sheaves is exact iff at each point the corresponding sequence of stalks of sheaves is exact.

Part 3. Derived categories of abelian categories

9. Homotopy category of complexes

On the way to identifying any two quasi-isomorphic complexes, i.e., to inverting all quasi-isomorphisms, in the first step we will invert a special kind of isomorphisms – the homotopy equivalences.

This is achieved by passing from the category of complexes $C(\mathcal{A})$ to the so called "homotopy category of complexes" $K(\mathcal{A})$. Category $K(\mathcal{A})$ is no more an abelian category but it has a similar if less familiar structure of a "triangulated category". The abelian structure of the category $C(\mathcal{A})$ provides the notion of short exact sequences. This is essential since one can think of putting B into a short exact sequences $0 \to A \to B \to C \to 0$, as describing the complex B in terms of simpler complexes A and C; and indeed it turns out that the cohomology groups of B are certain combinations of cohomology groups of A and B. Since $K(\mathcal{A})$ is not abelian we are forced to find an analogue of short exact sequences which works in $K(\mathcal{A})$, this is the notion of "distinguished triangles", also called "exact triangles". The properties of exact triangles in $K(\mathcal{A})$ formalize into the concept of a "triangulated category" which turns out to be the best standard framework for homological algebra.

In particular, the passage to $(\mathcal{A})$ will solve the remaining foundational problem in the definition of derived functors:

- identify any two projective resolutions of one object

(so the derived functors will be well defined since we remove the dependence on the choice of a projective resolution).

9.1. **Category $C(\mathcal{A})$ of complexes in $\mathcal{A}$.**

We observe some of the properties of the category $C(\mathcal{A})$.

9.1.1. *Structures.*

- *Shift functors.* For any integer n define a shift functor $[n] : C(\mathcal{A}) \to C(\mathcal{A})$ by $(A[n])^p \stackrel{\text{def}}{=} A^{n+p}$, and the differential $(A[n])^p \to A[n])^{p+1}$ given as $A^{p+n} \xrightarrow{(-1)^n d_A^{p+n}} A^{p+1+n}$.
- *Functors $H^i : C(\mathcal{A}) \to \mathcal{A}$.*
- *Special class of morphisms related to cohomology functors: the quasi-isomorphisms.*
- *Triangles.* These are diagrams of the form $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$.
- *Subcategories $C^b(\mathcal{A})$ etc.* If ? is one of the symbols $b, -, +$ we define a full subcategory $C^?(\mathcal{A})$ of $C^?(\mathcal{A})$, consisting respectively of bounded complexes (i.e. $A^n = 0$

for $|n| >> 0$), complexes bounded from bellow: $A^n = 0$ for $n << 0$ (hence allowed to stretch in the $+$ direction), complexes bounded from above (so they may stretch in the $-$ direction). Moreover, for a subset $\mathcal{Z}\subseteq\mathbb{Z}$ we can define $C^{\mathcal{Z}}(\mathcal{A})$ as a full subcategory consisting of all complexes A with $A^n = 0$ for $n \notin \mathcal{Z}$. In particular one has $C^{\leq 0}(\mathcal{A}) \stackrel{\text{def}}{=} C^{\{\cdots,-2,-1,0\}}$ and $C^{\geq 0}(\mathcal{A}) \stackrel{\text{def}}{=} C^{\{0,1,2,\cdots\}}$, and $C^{\{0\}}(\mathcal{A})$ is equivalent to $\mathcal{A}$.

9.1.2. *Properties.* The next two lemmas give basic properties of the above structures on the category $C(\mathcal{A})$.

9.1.3. *Lemma.* $C(\mathcal{A})$ is an abelian category and a sequence of complexes is exact iff it is exact on each level!

Proof. For a map of complexes $A \xrightarrow{\alpha} B$ we can define $K^n = \mathrm{Ker}(A^n \xrightarrow{\alpha^n} B^n)$ and $C^n = A^n/\alpha^n(B^n)$. This gives complexes since d_A induces a differential d_K on K and d_B a differential d_C on C. Moreover, it is easy to check that in category $C(\mathcal{A})$ one has $K = \mathrm{Ker}(\alpha)$ and $C = \mathrm{Coker}(\alpha)$. Now one finds that $Im(\alpha)^n = Im(\alpha^n) = \alpha^n(A^n)$ and $Coim(\alpha)^n = Coim(\alpha^n) = A^n/\mathrm{Ker}(\alpha^n)$, so the canonical map $Coim \to Im$ is given by isomorphisms $A^n/\mathrm{Ker}(\alpha^n) \xrightarrow{\cong} \alpha^n(A^n)$. Exactness claim follows.

9.1.4. *Lemma.* A short exact sequence of complexes $0 \to A \to B \to C \to 0$ gives a long exact sequence of cohomologies.

$$\cdots \xrightarrow{\partial^{n-1}} \mathrm{H}^n(A) \xrightarrow{\mathrm{H}^n(\alpha)} \mathrm{H}^n(B) \xrightarrow{\mathrm{H}^n(\beta)} \mathrm{H}^n(C) \xrightarrow{\partial^n} \mathrm{H}^{n+1}(A) \xrightarrow{\mathrm{H}^{n+1}(\alpha)} \mathrm{H}^{n+1}(B) \xrightarrow{\mathrm{H}^{n+1}(\beta)} \cdots$$

Proof. We need to construct for a class $\gamma \in \mathrm{H}^n(C)$ a class $\partial\gamma \in \mathrm{H}^{n+1}$. So if $\gamma = [c]$ is the class of a cocycle c, we need

(1) From a cocycle $c \in Z^n(C)$ a cocycle $a \in Z^{n+1}$.
(2) Independence of $[a]$ on the choice of c or any other auxiliary choices.
(3) The sequence of cohomology groups is exact.

Recall that a sequence of complexes. $0 \to A \to B \to C \to 0$ is a short exact sequence if for each integer n the sequence $0 \to A^n \to B^n \to C^n \to 0$ is exact.

The following calculation is in the set-theoretic language appropriate for module categories but can be rephrased in the language of abelian categories (and also the result for module categories implies the result for abelian categories since any abelian category is equivalent to a full subcategory of a module category).

(1) Since β^n is surjective, $c = \beta^n b$ for $b \in B^n$, Now $db = \alpha^{n+1}a$ for some $a \in A^{n+1}$ since $\beta^{n+1}(db) = d\beta^{n+1}b = dc = 0$. Moreover, a is a cocycle since $\alpha^{n+2}(d^{n+1}a) = d^{n+1}(\alpha^{n+1}a) = d^{n+1}(d^n b) = 0$.

(2) So we want to associate to $\gamma = [c]$ the class $\alpha = [a] \in \mathrm{H}^{n+1}(A)$. For that $[a]$ should be independent of the choices of c, B and a. So let $[c] = [c']$ and $c' = \beta^n b'$ with $b' \in B^n$, and $db' = \alpha^{n+1} a'$ for some $a' \in A^{n+1}$.

Since $[c] = [c']$ one has $c' = c + dz$ for some $z \in C^{n-1}$. Choose $y \in B^{n-1}$ so that $z = \beta^{n-1} y$, then

$$\beta^n b' = c' = c + dz = \beta^n b + d(\beta^{n-1} y) = \beta^n b + \beta^n dy = \beta^n (b + dy).$$

The exactness at B now shows that $b' = b + dy + \alpha^n x$ for some $x \in A^n$. So,

$$\alpha^{n+1} a' = db' = db + d\alpha^n x = \alpha^{n+1} a + \alpha^n (dx) = \alpha^{n+1}(a + dx).$$

Exactness at A implies that actually $a' = a + dx$.

(3) I omit the easier part: the compositions of any two maps are zero.

Exactness at $H^n(B)$. Let $b \in Z^n(B)$, then $\mathrm{H}^n(\beta)[b] = [\beta^n b]$ is zero iff $\beta^n b = dz$ for some $z \in C^{n-1}$. Let us lift this z to some $y \in B^{n-1}$, i.e., $z = \beta^{n-1} y$. Then $\beta^n(b - dy) = dz - dz = 0$, hence $b - dy = \alpha^n a$ for some $a \in A^n$. Now a is a cocycle since $\alpha^n(da) = d(b - dy) = 0$, and $[b] = [b - dy] = \mathrm{H}^n(\alpha)[a]$.

Exactness at $H^n(A)$. Let $a \in Z^n(A)$ be such that $\mathrm{H}^n(\alpha)[a] = [\alpha^n a]$ is zero, i.e., $\alpha^n a = db$ for some $b \in B^{n-1}$. Then $c = \beta b$ is a cocycle since $dc = \beta^n(db) = \beta^n \alpha^n a = 0$; and by the definition of the connecting morphisms (in (1)), $[a] = \delta^{n-1}[c]$.

Exactness at $H^n(C)$. Let $c \in Z^n(C)$ be such that $\partial^n[c] = 0$. Remember that this means that $c = \beta b$ and $db = \alpha a$ with $[a] = 0$, i.e., $a = dx$ with $x \in A^{n-1}$. But then $db = \alpha(dx) = d(\alpha x)$, so $b - \alpha x$ is a cocycle, and then $c = \beta(b) = \beta(b - \alpha(x))$ implies that $[c] = \mathrm{H}^n(\beta)[b - \alpha(x)]$.

9.2. **Mapping cones.** The idea is that the cone of a map of complexes $A \xrightarrow{\alpha} B$ is a complex $\boldsymbol{C}_\alpha$ which measures how far α is from being an isomorphism. However, the main role of the mapping cone here is that it partially reformulates the short exact sequences of complexes. It will turn out that the following data are the same:

(1) a mapping cone and its associated triangle
(2) a short exact sequence of complexes which splits in each degree.

This is the content of the next three lemmas. These results, together with the preceding lemmas 9.1.4 and 9.1.3, all have analogues for the homotopy category of complex $K(\mathcal{A})$ which we meet in 9.3. These analogues (theorem 9.4.1), will be more complicated and will give rise to a notion of a triangulated category.

9.2.1. *Lemma.* (a) A map $A \xrightarrow{\alpha} B$ defines a complex C called the cone of α by

- $\boldsymbol{C}^n = B^n \oplus A^{n+1}$,

- $d^n_{\boldsymbol{C}}: \ B^n \oplus A^{n+1} \to B^{n+1} \oplus A^{n+2}$ combines the differentials in A and B and the map α by:
$$d(b^n \oplus a^{n+1}) \stackrel{\text{def}}{=} (d^n_B b^n + \alpha^{n+1} a^{n+1}) \oplus \ - d^{n+1}_A a^{n+1}$$

(b) The cone appears in a canonical triangle
$$A \xrightarrow{\alpha} B \xrightarrow{\dot{\alpha}} \boldsymbol{C} \xrightarrow{\ddot{\alpha}} A[1], \quad \textit{with} \quad \dot{\alpha}^n(b^n) = \ b^n \oplus 0 \quad \text{ and } \quad \ddot{\alpha}^n(b^n \oplus a^{n+1}) \stackrel{\text{def}}{=} \ a^{n+1}.$$

Proof. (a) One has $d_{\boldsymbol{C}} = (d_B + \alpha) \oplus \ - d_A$, hence
$$(d_{\boldsymbol{C}})^2 = d_{\boldsymbol{C}} \circ [(d_B + \alpha) \oplus \ - d_A] = \ (d^2_B + d_B \circ \alpha + \alpha \circ - d_A) \oplus \ (-d_A)^2 = 0,$$
in more details
$$d^{n+1} d^n (b^n \oplus a^{n+1}) = \ d^{n+1}[(d^n_B b^n + \alpha^{n+1} a^{n+1}) \oplus \ - d^{n+1}_A a^{n+1}]$$
$$= \ (d^{n+1}_B d^n_B b^n + d^{n+1} \alpha^{n+1} a^{n+1} + \alpha^{n+1}(-d^{n+1}_A a^{n+1})) \oplus \ - d^{n+2}_A d^{n+1}_A a^{n+1} = 0.$$
(b) The claim is just that $\dot{\alpha}$ and $\ddot{\alpha}$ are maps of complexes. Clearly, $(d_{\boldsymbol{C}} \circ \dot{\alpha})(b^n) = d_{\boldsymbol{C}}(b^n \oplus 0) = \ d_B b^b \oplus 0 = \ \dot{\alpha}(d_B b^n)$ and
$$(d^n_{A[1]} \circ \ddot{\alpha})(b^n \oplus a^{n+1}) = \ (d^n_{A[1]} a^{n+1}) = \ -d^{n+1}_A a^{n+1}) = \ \ddot{\alpha}(d_B b^n + \alpha a^{n+1}) \oplus - \ d^{n+1}_A a^{n+1})$$
$$= \ (\ddot{\alpha} \circ d_{\boldsymbol{C}})(b^n \oplus a^{n+1}).$$

9.2.2. *Cone triangles and short exact sequences of complexes.* We will see that the triangles that arise from cones are reformulations of some short exact sequences of complexes.

Lemma. (a) For any map $A \xrightarrow{\alpha} B$ of complexes, the corresponding distinguished triangle $A \xrightarrow{\alpha} B \xrightarrow{\dot{\alpha}} \boldsymbol{C}_\alpha \xrightarrow{\ddot{\alpha}} A[1]$ contains an exact sequence of complexes
$$0 \to B \xrightarrow{\dot{\alpha}} \boldsymbol{C}_\alpha \xrightarrow{\ddot{\alpha}} A[1] \to 0.$$
Moreover, this exact sequence of complexes splits canonically in each degree, i..e, short exact sequences $0 \to B^n \xrightarrow{\dot{\alpha}} (\boldsymbol{C}_\alpha)^n \xrightarrow{\ddot{\alpha}} (A[1])^n \to 0$ in $\mathcal{A}$ have canonical splittings, i.e., identifications
$$(\boldsymbol{C}_\alpha)^n \xrightarrow{\cong} B^n \oplus (A[1])^n.$$
(b) Actually, one can recover the whole triangle from the short exact sequence and its splittings.

Proof. In (a), exactness is clear since $\dot{\alpha}$ is the inclusion of the first factor and $\ddot{\alpha}$ is the projection to the second factor. The splitting statement is the claim that the image of $B^n \hookrightarrow C^n_\alpha$ has a complement (which is then automatically isomorphic to $(A[1])^n$), but the image is $B^n \oplus 0$ and the complement is just $0 \oplus A^{n+1}$.

(b) Identification $(\boldsymbol{C}_\alpha)^n \xrightarrow{\cong} B^n \oplus (A[1])^n$ involves two maps $(A[1])^n \overset{\sigma^n}{\hookrightarrow} (\boldsymbol{C}_\alpha)^n \overset{\tau^n}{\twoheadrightarrow} B^n$ (actually σ^n determines τ^n and vice versa). Now we can recover the map $A^{n+1} \xrightarrow{\alpha^{n+1}} B^{n+1}$ as a part of the differential $(\boldsymbol{C}_\alpha)^n \xrightarrow{d^n_C} (\boldsymbol{C}_\alpha)^{n+1}$. Precisely, α^{n+1} is the composition

$$(A[1])^n \overset{\sigma^n}{\hookrightarrow} (\boldsymbol{C}_\alpha)^n \xrightarrow{d^n_C} (\boldsymbol{C}_\alpha)^{n+1} \overset{\tau^{n+1}}{\twoheadrightarrow} B^{n+1}.$$

For this one just recalls that d^n_C is $B^n \oplus A^{n+1} \xrightarrow{(d_B+\alpha)\oplus -d_A} B^{n+1} \oplus A^{n+2}$, i.e., $d^n_C(b^n \oplus a^{n+1}) \overset{\text{def}}{=} (d^n_B b^n + \alpha^{n+1} a^{n+1}) \oplus \; - d^{n+1}_A a^{n+1}$.

9.2.3. *From a short exact sequences to a cone triangle.* Conversely, we will see that any short exact sequence that splits on each level, produces a cone triangle. However, the two procedures of passing between cone triangles and short exact sequence that splits on each level, turn out not to be inverse to each other. This gets resolved in 9.6.1 by adopting a "correct" categorical setting – the homotopy category of complexes.

Lemma. A short exact sequence of complexes

$$0 \to P \xrightarrow{\phi} Q \xrightarrow{\psi} R \to 0$$

and a splitting $Q^n \cong P^n \oplus R^n$ for each $n \in \mathbb{Z}$, define canonically a distinguished triangle

$$R[-1] \xrightarrow{\alpha} P \xrightarrow{\phi=\dot{\alpha}} Q \xrightarrow{\psi=\ddot{\alpha}} (R[-1])[1].$$

(b) Moreover, this triangle is isomorphic in $C(\mathcal{A})$ to a cone triangle.

(c) Explicitly, if the splitting is given by maps $R^n \overset{\sigma^n}{\hookrightarrow} Q^n \cong \overset{\tau^n}{\twoheadrightarrow} P^n$, then the map $R^n = (R[-1])^{n+1} \xrightarrow{\alpha^{n+1}} P^{n+1}$ is a component of the differential $Q^n \xrightarrow{d^n_Q} Q^{n+1}$, i.e., it is the composition

$$\alpha^{n+1} = (R^n \overset{\sigma^n}{\hookrightarrow} P^n \oplus R^n \cong Q^n \xrightarrow{d^n_Q} Q^{n+1} \oplus P^{n+1} \cong P^{n+1} \oplus R^{n+1} \overset{\tau^{n+1}}{\twoheadrightarrow} P^{n+1}).$$

Proof. (a) We just need to know that $\chi = \alpha[1] : R \to P[1]$ is a map of complexes. Since $\chi^n = \tau^{n+1} \circ d^n_Q \circ \sigma^n(r^n)$ one can decompose the action of d_Q on the image of σ^n into the P and R components, by

$$d_Q \sigma^n(r^n)) = \tau^{n+1} d_Q \sigma^n(r^n)] + \sigma^{n+1} d_R(r^n), \qquad r^n \in R^n.$$

Therefore, the differential on Q is given by $(p^n \in P^n,\ r^n \in R^n)$

$$d_Q(\phi^n(p^n) + \sigma^n(r^n)) = d_Q(\phi^n(p^n)) + d_Q \sigma^n(r^n)) = \phi^{n+1}(d_P\, p^n)) + d_Q \sigma^n(r^n))$$

$= [\phi^{n+1}(d_P\, p^n)) + \tau^{n+1} d_Q \sigma^n(r^n)] + \sigma^{n+1} d_R(r^n) = [\phi^{n+1}(d_P\, p^n)) + \chi^n(r^n)] + \sigma^{n+1} d_R(r^n)$.
Now,
$0 = d^2_Q(\phi^n(p^n) + \sigma^n(r^n)) = [\phi^{n+2}(d^2_P\, p^n)) + d_Q \chi^n(r^n) + \chi^{n+1} \sigma^{n+1} d_R(r^n)] + \sigma^{n+2} d^2_R(r^n)$
shows that $\chi \circ d_r = -d_P \circ \chi = d_{P[1]} \circ \chi$.

(b) is clear: Q is isomorphic to the cone of α since $Q^n = \tau^n(P^n)\oplus\sigma^n(R^n) \cong P^n\oplus R^n = P^n\oplus(R[-1])^{n+1}$, and via these identification the differential in Q is precisely the cone differential

$$d_Q(p^n\oplus r^n) = (d_P\ p^n + \chi^n\ r^n) \oplus d_R\ r^n = (d_P\ p^n + \alpha^n\ r^n) \oplus -d_{R[-1]}\ r^n.$$

9.3. **The homotopy category $K(\mathcal{A})$ of complexes in $\mathcal{A}$.** We say that two maps of complexes $A \xrightarrow{\alpha,\beta} B$ are homotopic (we denote this $\alpha \mod \beta$), if there is a sequence h of maps $h^n :\ A^n \to B^{n-1}$, such that

$$\beta - \alpha = dh + hd, \quad i.e., \quad \beta^n - \alpha^n = d_B^{n-1}h^n + h^{n+1}d_A^n.$$

We say that h is a homotopy from α to β.

A map of complexes $A \xrightarrow{\alpha} B$ is said to be a homotopical equivalence if there is a map β in the opposite direction such that $\beta\circ\alpha \equiv 1_A$ and $\alpha\circ\beta \equiv 1_B$.

9.3.1. *Lemma.* (a) Homotopic maps are the same on cohomology.

(b) Homotopical equivalences are quasi-isomorphisms.

(c) A complex A is homotopy equivalent to the zero object iff $1_A = hd + dh$. Then the complex A is acyclic, i.e., $\mathrm{H}^*(A) = 0$.

(d) $\alpha \equiv \beta$ implies $\mu\circ\alpha \equiv \mu\circ\beta$ and $\alpha\circ\nu \equiv \beta\circ\nu$.

Proof. (a) Denote for $a \in Z^n(A)$ by $[a]$ the corresponding cohomology class in $\mathrm{H}^n(A)$. Then $\mathrm{H}(b)[a] - \mathrm{H}(\alpha)[a] = [d_B^{n-1}h^n(a) + h^{n+1}d_A^n(a)] = [d_B^{n-1}(h^n a)] = 0$.

(b) If $\beta\circ\alpha \equiv 1_A$ and $\alpha\circ\beta \equiv 1_B$ then $\mathrm{H}(\beta)\circ\mathrm{H}(\alpha) = \mathrm{H}(\beta\circ\alpha) = \mathrm{H}(1_A) = 1_{\mathrm{H}(A)}$ etc.

(c) The map $0 \xrightarrow{\alpha} A$ with $\alpha = 0$ is a homotopy equivalence if there is a map $A \xrightarrow{\beta} 0$ (then necessarily $\beta = 0$) such that $\beta\circ\alpha \equiv 1_0$ and $\alpha\circ\beta \equiv 1_A$. Since the LHS is always zero, the only condition is that $0 \equiv 1_A$.

(d) If $\beta - \alpha = d_B\circ h + h\circ d_A$ then for $X \xrightarrow{\nu} A \xrightarrow{\alpha} B \xrightarrow{\mu} Y$ one has $\mu\circ\beta - \mu\circ\alpha = d_C\circ(\mu\circ h) + (\mu\circ h)\circ d_A$ etc.

9.3.2. *Homotopy category $K(\mathcal{A})$.* The objects are again just the complexes but the maps are the homotopy classes $[\phi]$ of maps of complexes ϕ

$$\mathrm{Hom}_{K(\mathcal{A})}(A, B) \stackrel{\text{def}}{=} \mathrm{Hom}_{C(\mathcal{A})}(A, B)/\equiv .$$

Now identity on A in $K(\mathcal{A})$ is $[1_A]$ and the composition is defined by $[\beta]\circ[\alpha] \stackrel{\text{def}}{=} [\beta\circ\alpha]$, this makes sense by the part (d) of the lemma.

9.3.3. *Remarks.* (1) Observe that for a homotopy equivalence $\alpha : A \to B$ the corresponding map in $K(\mathcal{A})$, $[\alpha] : A \to B$ is an isomorphism. So we have accomplished a part of our long term goal – we have inverted some quasi-isomorphisms: the homotopy equivalences!

(2) More precisely, we know what are isomorphisms in $K(\mathcal{A})$. The homotopy class $[\alpha]$ of a map of complexes α, is an isomorphism in $K(\mathcal{A})$ iff α is a homotopy equivalence!

9.4. **The triangulated structure of $K(\mathcal{A})$.** $K(\mathcal{A})$ is not an abelian category but there are features that allow us to make similar computations:

- It is an additive category.
- It has shift functors $[n]$.
- It has a class $\mathcal{E}$ of "distinguished triangles" (or "exact triangles"), defined as all triangles isomorphic (in $K(\mathcal{A})$ to a cone of a map of complexes.
- It has cohomology functors $\mathrm{H}^i : K(\mathcal{A}) \to \mathcal{A}$.

9.4.1. *Theorem.* Distinguished triangles have the following properties

- (T0) The class of distinguished triangles is closed under isomorphisms.
- (T1) Any map α appears as the first map in some distinguished triangle.
- (T2) For any object A, triangle $A \xrightarrow{1_A} A \xrightarrow{0} 0 \to A[1]$ is distinguished.
- (T3) (Rotation) If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ is distinguished, so is
$$B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1] \xrightarrow{-\alpha[1]} B[1].$$
- (T4) Any diagram with distinguished rows
$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & A[1] \\ {\scriptstyle\mu}\downarrow & & {\scriptstyle\nu}\downarrow & & & & \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & A'[1] \end{array}$$
can be completed to a morphism of triangles
$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & A[1] \\ {\scriptstyle\mu}\downarrow & & {\scriptstyle\nu}\downarrow & & {\scriptstyle\eta}\downarrow & & {\scriptstyle\mu[1]}\downarrow \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & A'[1]. \end{array}$$
- (T5) (Octahedral axiom) If maps $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ and the composition $A \xrightarrow{\gamma=\beta\circ\alpha} C$, appear in distinguished triangles
  (1) $A \xrightarrow{\alpha} B \xrightarrow{\alpha'} C_1 \xrightarrow{\alpha''} A[1]$,
  (2) $B \xrightarrow{\beta} C \xrightarrow{\beta'} A_1 \xrightarrow{\beta''} B[1]$,
  (3) $A \xrightarrow{\gamma} C \xrightarrow{\gamma'} B_1 \xrightarrow{\gamma''} C[1]$;

then there is a distinguished triangle

$$C_1 \xrightarrow{\phi} B_1 \xrightarrow{\psi} A_1 \xrightarrow{\chi} C_1[1]$$

that fits into the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "="'] & B \arrow[r, "\alpha'"] \arrow[d, "\beta"'] & C_1 \arrow[r, "\alpha''"] \arrow[d, "\phi"'] & A[1] \arrow[d, "="'] \\
A \arrow[r, "\gamma"] \arrow[d, "\beta"'] & C \arrow[r, "\gamma'"] \arrow[d, "="'] & B_1 \arrow[r, "\gamma''"] \arrow[d, "\psi"'] & A[1] \arrow[d, "\beta[1]"'] \\
B \arrow[r, "\beta"] & C \arrow[r, "\beta'"] & C_1 \arrow[r, "\beta''"] \arrow[d, "\chi"'] & B[1] \arrow[d, "="'] \\
 & & C_1[1] & B_1[1] \arrow[l, "\alpha'[1]"']
\end{tikzcd}$$

9.4.2. *Remarks.* (1) Octahedral axiom (T5) is the most complicated and the least used part.

(2) In (T4), the map γ is not unique nor is there a canonical choice. This is a source of some subtleties in using triangulated categories.

Proof. (T0) is a part of the definition of $\mathcal{E}$.

(T1) Any map in $K(\mathcal{A})$, $\alpha \in \mathrm{Hom}_{K(\mathcal{A})}(A, B)$ is a homotopy class $[\alpha]$ of some map of complexes $\alpha \in \mathrm{Hom}_{C(\mathcal{A})}(A, B)$. The cone of α gives a triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} \boldsymbol{C} \xrightarrow{\gamma} A[1]$ in $C(\mathcal{A})$ such that its image in $K(\mathcal{A})$ is a distinguished triangle $A \xrightarrow{[\alpha]} B \xrightarrow{[\beta]} \boldsymbol{C} \xrightarrow{[\gamma]} A[1]$ in $K(\mathcal{A})$ which starts with $[\alpha]$.

(T2) means that the cone $\boldsymbol{C}_{1_A} = C$ of the identity map on A is isomorphic in $K(\mathcal{A})$ to the zero complex, i.e., that the cone $\boldsymbol{C}_{1_A}$ is homotopically equivalent to zero. The homotopy $h^n : C^n \to C^{n-1}$ is simply identity on the common summand and zero on its complement

$$A^n \oplus A^{n+1} \to A^{n-1} \oplus A^n, \quad h^n(a^n \oplus a^{n+1}) \stackrel{\text{def}}{=} 0 \oplus a^n.$$

For that we calculate

$$(d_C^{n-1} h^n + h^{n+1} d_C^n)(a^n \oplus a^{n+1}) = \ d_C^{n-1}(0 \oplus a^n) \ + \ h^{n+1}[(d_A a^n + 1_A a^{n+1}) \oplus - d_A^{n+1} a^{n+1})]$$
$$= \ (1_A a^n \oplus - d_A a^n) \ + \ 0 \oplus (d_A a^n + 1_A a^{n+1}) = \ a^n \oplus a^{n+1}.$$

(T4) The obvious strategy is to lift the first diagram in (T4) to the level of $C(\mathcal{A})$. First, one can replace the rows with isomorphic ones which are cone triangles $A \xrightarrow{\alpha_0} B \xrightarrow{\dot{\alpha}_0} \boldsymbol{C}_{\alpha_0} \xrightarrow{\ddot{\alpha}_0} A[1]$ and $A' \xrightarrow{\alpha_0'} B' \xrightarrow{\dot{\alpha}_0'} \boldsymbol{C}_{\alpha_0'} \xrightarrow{\ddot{\alpha}_0'} A'[1]$ for maps α_0, α_0' in $C(\mathcal{A})$ (these maps are some representatives of homotopy classes α, α'). So, the diagram takes form (for any representatives μ_0, ν_0 of μ, ν)

$$\begin{tikzcd} A \arrow[r, "{[\alpha_0]}"] \arrow[d, "{[\mu_0]}"'] & B \arrow[r, "{[\dot{\alpha}_0]}"] \arrow[d, "{[\nu_0]}"'] & \boldsymbol{C}_{\alpha_0} \arrow[r, "{[\ddot{\alpha}_0]}"] & A[1] \\ A' \arrow[r, "{[\alpha'_0]}"] & B' \arrow[r, "{[\dot{\alpha}'_0]}"] & \boldsymbol{C}_{\alpha'_0} \arrow[r, "{[\ddot{\alpha}'_0]}"] & A'[1] \end{tikzcd}$$

It would be nice to lift the diagram completely into $C(\mathcal{A})$ in the sense that we look for representatives μ_0, ν_0 of μ, ν so that the diagram in $C(\mathcal{A})$ still commutes, then a representative η_0 of η would simply come from the functoriality ("naturality") of the cone construction. However we will make piece with the homotopic nature of the diagram and incorporate the homotopy corrections. Homotopical commutativity $[\nu_0]\circ[\alpha_0] = [\alpha'_0]\circ[\mu_0]$ means that one has maps $h^n : B^n \to (\boldsymbol{C}_{\alpha'_0})^{n-1}$ such that

$$\nu_0\circ\alpha_0 - \alpha'_0\circ\mu_0 = dh + hd.$$

Now, we construct a map $\boldsymbol{C}_{\alpha_0} \xrightarrow{\eta_0} \boldsymbol{C}_{\alpha'_0}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha_0"] & B \arrow[r, "\beta_0"] \arrow[d, "\nu_0"'] & \boldsymbol{C}_{\alpha_0} \arrow[r, "\gamma_0"] \arrow[d, "\eta_0"'] & {[A1]} \arrow[d, "\mu_0"'] \\ A' \arrow[r, "\alpha'_0"] & B' \arrow[r, "\beta'_0"] & \boldsymbol{C}_{\alpha'_0} \arrow[r, "\gamma'_0"] & A'[1] \end{tikzcd}$$

commutes in $C(\mathcal{A})$, by

$$\eta_0 : B^n\oplus A^{n+1} \to (B')^n\oplus(A')^{n+1}, \quad b^n\oplus a^{n+1}\mapsto(\nu_0 b^n + h^{n+1}a^{n+1})\oplus\mu_0 a^{n+1}.$$

(T3) says that if one applies rotation to any cone triangle in $C(\mathcal{A})$, $A \xrightarrow{\alpha} B \xrightarrow{\beta} \boldsymbol{C}_\alpha \xrightarrow{\gamma} A[1]$, the resulting triangle

$$B \xrightarrow{\beta} \boldsymbol{C}_\alpha \xrightarrow{\gamma} A[1] \xrightarrow{-\alpha[1]} B[1]$$

is isomorphic in $K(\mathcal{A})$ to the cone triangle

$$B \xrightarrow{\beta} \boldsymbol{C}_\alpha \xrightarrow{\phi} \boldsymbol{C}_\beta \xrightarrow{\mu} B[1].$$

This requires a homotopy equivalence $A \xrightarrow{\zeta} \boldsymbol{C}_\beta$ such that the following diagram commutes in $K(\mathcal{A})$:

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[d, "="'] & \boldsymbol{C}_\alpha \arrow[r, "\gamma"] \arrow[d, "="'] & A[1] \arrow[r, "{-\alpha[1]}"] \arrow[d, "\zeta"'] & B[1] \arrow[d, "="'] \\ B \arrow[r, "\beta"] & \boldsymbol{C}_\alpha \arrow[r, "\phi"] & \boldsymbol{C}_\beta \arrow[r, "\mu"] & B[1]. \end{tikzcd}$$

We define the map

$$A^{n+1} = (A[1])^n \xrightarrow{\zeta^n} (\boldsymbol{C}_\beta)^n = (B^n\oplus A^{n+1})\oplus B^{n+1}, \quad a^{n+1}\mapsto 0\oplus a^{n+1}\oplus - \alpha(a^{n+1});$$

and in the opposite direction $\xi(b^n \oplus a^{n+1} \oplus b^{n+1}) \stackrel{\text{def}}{=} a^{n+1}$. It suffices to check that

(1) ζ and ξ are maps of complexes,
(2) $\xi \circ \zeta = 1_{A[1]}$,
(3) $\zeta \circ \xi \equiv 1_{\boldsymbol{C}_\beta}$,
(4) $\zeta \circ \gamma = \phi$
(5) $\mu \circ \zeta = -\alpha[1]$

In (3), the homotopy h such that $1_{\boldsymbol{C}_\beta} - \zeta \circ \xi = dh + hd$, is the map $h^n : (\boldsymbol{C}_\beta)^n \to (\boldsymbol{C}_\beta)^{n-1}$ given by $B^n \oplus A^{n+1} \oplus B^{n+1} \ni b^n \oplus a^{n+1} \oplus b^{n+1} \mapsto 0 \oplus 0 \oplus b^n \in B^{n-1} \oplus A^n \oplus B^n$.

(T5) is a description of a certain "complicated" relation between cones of maps and compositions of maps.

9.4.3. *Triangulated categories.* These are additive categories with a functor [1] (called shift) and a class of distinguished triangles $\mathcal{E}$, that satisfy the conditions $(T0 - T5)$.

So, $K(\mathcal{A})$ is our first triangulated category.

9.5. **Long exact sequence of cohomologies.**

9.5.1. *Lemma.* Any distinguished triangle $X \to Y \to Z \to X[1]$ gives a long exact sequence of cohomologies

$$\cdots \to \mathrm{H}^i(X) \to \mathrm{H}^i(Y) \to \mathrm{H}^i(Z) \to \mathrm{H}^{i+1}(X) \to \cdots.$$

Proof. Up to isomorphism in $K(\mathcal{A})$, we can replace the triangle $X \to Y \to Z \to X[1]$ with the homotopy image of a cone triangle $X \xrightarrow{\alpha} Y \xrightarrow{\dot{\alpha}} \boldsymbol{C}_\alpha \xrightarrow{\ddot{\alpha}} X[1]$ in $C(\mathcal{A})$. We have observed that this triangle contains a short exact sequence of complexes $0 \xrightarrow{\alpha} Y \xrightarrow{\dot{\alpha}} \boldsymbol{C}_\alpha \xrightarrow{\ddot{\alpha}} X[1] \to 0$, and the short exact sequence of complexes does indeed provide a long exact sequence of cohomologies.

The proof was based on the relation of

9.6. **Exact (distinguished) triangles and short exact sequences of complexes.**

By definition distinguished triangles in $K(\mathcal{A})$ come from maps $A \xrightarrow{\alpha} B$ in $C(\mathcal{A})$. We will just restate it as:

- distinguished triangles in $K(\mathcal{A})$ come from short exact sequences in $C(\mathcal{A})$ that split on each level.

9.6.1. *Lemma.* (a) Any short exact sequence of complexes in $C(\mathcal{A})$

$$0 \to P \to Q \to R \to 0$$

which splits on each level, defines a distinguished triangle in $K(\mathcal{A})$ of the form

$$P \to Q \to R \xrightarrow{-\chi} P[1].$$

(b) Any distinguished triangle in $K(\mathcal{A})$ is isomorphic to one that comes from a short exact sequence of complexes that splits on each level.

(c) Explicitly, in (a) the map χ in $C(\mathcal{A})$ comes from a choice of splittings $Q^n \cong\ P^n \oplus R^n$. The map $R^n \xrightarrow{\chi^n} P^{n+1}$ is the component of the differential $Q^n \xrightarrow{d_Q^n} Q^{n+1}$, i.e.,

$$\chi^n =\ (R^n \hookrightarrow P^n \oplus R^n \cong Q^n \xrightarrow{d_Q^n} Q^{n+1} P^{n+1} \cong P^{n+1} \oplus R^{n+1} \twoheadrightarrow P^{n+1}).$$

Proof. We know that the short exact sequence associated to a cone of a map splits canonically on each level (lem 9.2.2). In the opposite direction, by lemma 9.2.3 any short exact sequence of complexes

$$0 \to P \xrightarrow{\phi} Q \xrightarrow{\psi} R \to 0$$

with a splitting $Q^n \cong\ P^n \oplus R^n$ on each level, defines a canonical cone triangle in $C(\mathcal{A})$

$$R[-1] \xrightarrow{\alpha} P \xrightarrow{\phi = \dot{\alpha}} Q \xrightarrow{\psi = \ddot{\alpha}} (R[-1])[1],$$

hence an exact triangle in $K(\mathcal{A})$.

Since we are in $K(\mathcal{A})$ we can rotate this triangle backwards using the property (T3) to get an exact triangle $P \to R \to Q \to P[1]$. Now the two procedures of going between short exact sequences in $C(\mathcal{A})$ and exact triangles in $K(\mathcal{A})$ are "inverse to each other".

The formulas in (c) come from lemma 9.2.3c.

9.7. **Extension of additive functors to homotopy categories.** The following lemma is quite obvious:

9.7.1. *Lemma.* (a) There is a canonical functor $C(\mathcal{A}) \to K(\mathcal{A})$ which sends each complex A to itself and each map of complexes ϕ to its homotopy class $[\phi]$.

(b) Any additive functor between abelian categories $\mathcal{A} \xrightarrow{F} \mathcal{B}$ extends to a functor $C(\mathcal{A}) \xrightarrow{C(F)} C(\mathcal{B})$, here $[C(F)A]^n =\ F(A^n)$ and the differential $d^n_{C(F)}$ is $F(d^n)$.

(c) Moreover, $C(\mathcal{A}) \xrightarrow{C(F)} C(\mathcal{B})$, factors to a functor $K(\mathcal{A}) \xrightarrow{K(F)} K(\mathcal{B})$, i.e., there is a unique functor $K(F)$ such that

$$\begin{array}{ccc} C(\mathcal{A}) & \xrightarrow{C(F)} & C(\mathcal{B}) \\ \downarrow & & \downarrow \\ K(\mathcal{A}) & \xrightarrow{K(F)} & K(\mathcal{B}). \end{array}$$

It is the same as $C(F)$ on objects and the action of $K(F)$ on homotopy classes of maps of complexes comes from the action of $C(F)$ on maps of complexes.

9.8. **Projective resolutions and homotopy.**

9.8.1. *Lemma.* If we have two complexes

$$\cdots \to P^{-n} \to \cdots \to P^0 \to a \to 0 \to \cdots$$

$$\cdots \to B^{-n} \to \cdots \to B^0 \to b \to 0 \to \cdots$$

such that all P^k are projective and the second complex is exact, then any map $\alpha : a \to b$ lifts to a map $P \xrightarrow{\phi} B$, i.e.,

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & P^{-n} & \longrightarrow & \cdots & \longrightarrow & P^0 & \longrightarrow & a & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & {\scriptstyle\phi^{-n}}\downarrow & & & & {\scriptstyle\phi^0}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle=}\downarrow & & \\ \cdots & \longrightarrow & B^{-n} & \longrightarrow & \cdots & \longrightarrow & B^0 & \longrightarrow & b & \longrightarrow & 0 & \longrightarrow & \cdots. \end{array}$$

(b) Any two such lifts are homotopic.

Proof. (a) Since $\varepsilon_B : B^0 \to B$ is surjective and P^0 is projective, the map $P^0 \xrightarrow{\alpha\circ\varepsilon_A} B$ factors through ε_B, i.e., it lifts to $P^0 \xrightarrow{phi^0} B^0$.

Actually, $\phi^0 \circ d_A{}^{-1}$ goes to $\mathrm{Ker}(\varepsilon_B)$ since

$$\varepsilon_B(\phi^0 d_A{}^{-1}) = (\alpha\circ\varepsilon_A)d_A = \alpha\circ 0 = 0.$$

Exactness of the second complex, shows that $d_B{}^{-1}$ gives a surjective map $d_B{}^{-1} : B^{-1} \to \mathrm{Ker}(\varepsilon_B)$. So, since P^{-1} is projective $\phi^0\circ d_A{}^{-1}$ factors through $d_B{}^{-1}$, giving a map $\phi^{-1} : P^{-1} \to B^{-1}$, such that $\phi^0\circ d_A{}^{-1} = d_B{}^{-1}\circ\phi^{-1}$.

In this way we construct all ϕ^n inductively.

(b) If we have another solution ψ then

$$\varepsilon_B\phi^0 = \alpha\circ\varepsilon_A = \varepsilon_B\psi^0$$

gives $\varepsilon_B(\phi^0 - \psi^0) = 0$, hence

$$(\phi^0 - \psi^0)P^0 \subseteq \mathrm{Ker}(\varepsilon_B) = d^{-1}{}_B(B^{-1}).$$

Now, since P^0 is projective, map $\phi^0 - \psi^0 : P^0 \to Im(d_B{}^{-1})$ lifts to $h^0 : P^0 \to B^{-1}$, i.e.,

$$\phi^0 - \psi^0 = \ d_B{}^{-1} \circ h^0.$$

One continuous similarly

$$d_B{}^{-1} \circ (\phi^{-1} - \psi^{-1}) = \ (\phi^0 - \psi^0) \circ d_A{}^{-1} = \ d_B{}^{-1} \circ h^0,$$

hence

$$Im[(\phi^{-1} - \psi^{-1}] - d_B{}^{-1} \circ h^0 \subseteq \mathrm{Ker}(D_B{}^{-1}) = \ Im(d_B^{-2}).$$

So,

$$\phi^{-1} - \psi^{-1} = \ h^{-1} \circ d_A^{-2}$$

for some map $h^{-1} : P^{-1} \to B^{-2}$. Etc.

corr (a) If P and Q are projective resolutions of objects a and b in $\mathcal{A}$, then any map $a \to b$ lifts uniquely to a map $P \to Q$.

(b) Any two projective resolutions of the same object of $\mathcal{A}$ are canonically isomorphic in $K(\mathcal{A})$.

9.9. **Derived functors** $LF : \mathcal{A} \to K^-(\mathcal{B})$ **and** $RG : \mathcal{A} \to K^+(\mathcal{B})$**.**

9.9.1. *Lemma.* If $\mathcal{A}$ has enough projectives:

(1) There is a canonical projective resolution functor $\mathcal{P} : \mathcal{A} \to K^-(\mathcal{A})$.
(2) For any additive right exact functor $F : \mathcal{A} \to \mathcal{B}$ its left derived functor $LF : \mathcal{A} \to K^{\leq 0}(\mathcal{B})$ is well defined by replacing objects with their projective resolutions

$$LF(A) \overset{\text{def}}{=} \ F(\mathcal{P}(A))$$

and its zero cohomology is just the original functor F:

$$\mathrm{H}^0[(LF)(A) \ \cong \ F(A).$$

Proof. (1) is clear from the corollary and then (2) follows.

10. Bicomplexes and the extension of resolutions and derived functors to complexes

For a right exact functor $F : \mathcal{A} \to \mathcal{B}$ we have defined its left derived functor $LF : \mathcal{A} \to K^-(\mathcal{B})$ (if $\mathcal{A}$ has enough projectives) by replacing objects with their projective resolutions. However, it is necessary to extend this construction to $LF : \ K^-(\mathcal{A}) \to K^-(\mathcal{B})$. For one thing, for calculational reasons we need the property $L(F \circ G)A = LF(LG(A))$, but this means that we have to apply LF to a complex.

We will need the notion of bicomplexes (roughly "complexes that stretch in a plane rather then on a line"), as a tool for finding projective resolutions of complexes.

10.1. **Filtered and graded objects.** A graded object A of $\mathcal{A}$ is a sequence of objects $A^n \in \mathcal{A},\ n \in \mathbb{Z}$. We think of it as a sum $A = \oplus_{\mathbb{Z}} A^n$ (and this is precise if the sum exists in $\mathcal{A}$).

An increasing (resp. decreasing) filtration on an object $a \in \mathcal{A}$ is an increasing (resp. decreasing) sequence of subobjects $a_n \hookrightarrow a,\ n \in \mathbb{Z}$. When we talk of a filtration F we denote a_n by $F_n a$.

A filtration defines a graded object $Gr(a)$ with $Gr^n(a) \overset{\text{def}}{=} a_n/a_{n-1}$ (resp. $Gr^n(a) \overset{\text{def}}{=} a_n/a_{n+1}$). Also, a graded object $A = \oplus A^n$ has a canonical increasing filtration $A_n = \oplus_{i\le n} A^i$ (if the sums exist), and a canonical decreasing filtration $A_n = \oplus_{i\ge n} A^i$. (However, these grading and filtering are far from being inverse to each other.)

We will be interesting in decreasing filtrations F of complexes $(A = \oplus A^i, d)$. This is a sequence of subcomplexes $F_n A = \oplus F_n(A^i)$, i.e.,

(1) $\cdots F_{-1}(A^i) \subseteq F_0(A^i) \subseteq F_1(A^i) \supseteq \cdots \subseteq A^i$ is a filtration of A^i and
(2) $d(F_n A^i) \subseteq F_n(A^{i+1})$.

A filtration F of a complex $(A = \oplus A^i, d)$ gives a filtration F of its cohomology groups by

$$F_n[\mathrm{H}^i(A)] \overset{\text{def}}{=} Im[\mathrm{H}^i(F_n A) \to \mathrm{H}^i(A)].$$

10.2. **Bicomplexes.**

10.2.1. *Bicomplexes.* A bicomplex is a bigraded object $B = \oplus_{p.q\in\mathbb{Z}} B^{p,q}$ with differentials $B^{p,q} \xrightarrow{d'} B^{p+1,q}$ and $B^{p,q} \xrightarrow{d''} B^{p,q+1}$, such that $d = d' + d''$ is also a differential. So we ask that i.e.,

$$0 = d^2 = (d' + d'')^2 = (d')^2 + d'd'' + d''d' + (d'')^2 = d'd'' + d''d'.$$

We draw a bicomplex as a two dimensional object:

$$
\begin{tikzcd}
\cdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \cdots \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & B^{-1,2} \arrow[u, "d''"] \arrow[r, "d'"] & B^{0,2} \arrow[u, "d''"] \arrow[r, "d'"] & B^{1,2} \arrow[u, "d''"] \arrow[r, "d'"] & B^{2,2} \arrow[u, "d''"] \arrow[r, "d'"] & B^{3,2} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & B^{-1,1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{0,1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{1,1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{2,1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{3,1} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & B^{-1,0} \arrow[u, "d''"] \arrow[r, "d'"] & B^{0,0} \arrow[u, "d''"] \arrow[r, "d'"] & B^{1,0} \arrow[u, "d''"] \arrow[r, "d'"] & B^{2,0} \arrow[u, "d''"] \arrow[r, "d'"] & B^{3,0} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & B^{-1,-1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{0,-1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{1,-1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{2,-1} \arrow[u, "d''"] \arrow[r, "d'"] & B^{3,-1} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & \vdots \arrow[u, "d''"] \arrow[r, "d'"] & \vdots \arrow[u, "d''"] \arrow[r, "d'"] & \vdots \arrow[u, "d''"] \arrow[r, "d'"] & \vdots \arrow[u, "d''"] \arrow[r, "d'"] & \vdots \arrow[u, "d''"] \arrow[r, "d'"] & \cdots . \arrow[u, "d''"]
\end{tikzcd}
$$

So, B^{pq} has horizontal position p and height q, and d' is a horizontal differential while d'' is a vertical differential.

10.2.2. *Remarks.* (1) Anti-commutativity relation $d'd'' + d''d' = 0$ can be interpreted as commutativity in the correct framework: the super-mathematics.

(2) If d' and d'' would happen to commute, we would have to correct one of these (say replace d'' by $(\check{d}'')^{p,q} \stackrel{\text{def}}{=} (-1)^p (d'')^{p,q}$).

10.2.3. *The total complex* $Tot(B)$ *of a bicomplex and the cohomology of a bicomplex.* The total complex of a bicomplex is the complex $(Tot(B), d)$ with $Tot(B)^n \stackrel{\text{def}}{=} \oplus_{p+q=n} B^{p,q}$. The cohomology of B is by definition the cohomology of the complex $Tot(B)$.

10.2.4. *Decreasing filtrations* $'F$ *and* $''F$ *on a bicomplex and on the total complex.* The fact that the complex $Tot(B)$ has come from a bicomplex will be used to produce two decreasing filtrations on the complex $Tot(B)$. Actually, any complex A has a stupid decreasing filtration F where the subcomplex $F_n A$ is obtained by erasing all terms A^k with $k < n$:

$$F_n A \stackrel{\text{def}}{=} \quad (\cdots \to 0 \to 0 \to A^n \to A^{n+1} \to A^{n+2} \to \cdots).$$

In turn, any bicomplex B has two decreasing filtrations $'F$ and $''F$. The sub-bicomplex $'F_i B$ of a bicomplex B is obtained by erasing the part of B which is on the left from the

i^{th} column, and symmetrically, $''F_j B$ is obtained by erasing beneath the j^{th} row. Say, the subbicomplex $'F_i B$ is given by

$$
\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow \\
\cdots & \xrightarrow{d'} & 0 & \xrightarrow{d'} & B^{i,2} & \xrightarrow{d'} & B^{i+1,2} & \xrightarrow{d'} & B^{i+2,2} & \xrightarrow{d'} & B^{i+3,2} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow \\
\cdots & \xrightarrow{d'} & 0 & \xrightarrow{d'} & B^{i,1} & \xrightarrow{d'} & B^{i+1,1} & \xrightarrow{d'} & B^{i+2,1} & \xrightarrow{d'} & B^{i+3,1} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow \\
\cdots & \xrightarrow{d'} & 0 & \xrightarrow{d'} & B^{i,0} & \xrightarrow{d'} & B^{i+1,0} & \xrightarrow{d'} & B^{i+2,0} & \xrightarrow{d'} & B^{i+3,0} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow \\
\cdots & \xrightarrow{d'} & 0 & \xrightarrow{d'} & B^{i,-1} & \xrightarrow{d'} & B^{i+1,-1} & \xrightarrow{d'} & B^{i+2,-1} & \xrightarrow{d'} & B^{i+3,-1} & \xrightarrow{d'} & \cdots \\
{\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow & & {\scriptstyle d''}\uparrow \\
\cdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \vdots & \xrightarrow{d'} & \cdots.
\end{array}
$$

This then induces filtrations on the total complex, say

$$[{}'F_i Tot(B)]^n \stackrel{\text{def}}{=} Tot({}'F_i B)^n = \oplus_{p+q=n,\ p\geq i} B^{p,q} \subseteq Tot(B)^n \supseteq \oplus_{p+q=n,\ q\geq j} B^{p,q} = {}''F_j[Tot(B)^n].$$

Finally, ‘F and $''F$ induce filtrations on the cohomology

$$'F_i \mathrm{H}^n(Tot\ B) \stackrel{\text{def}}{=} Im[\mathrm{H}^n(Tot\ 'F_i B) \to \mathrm{H}^n(Tot\ 'F_i B)],$$

so the cohomology groups are extensions of pieces

$$'Gr_i[\mathrm{H}^n(Tot\ B)] \stackrel{\text{def}}{=} \frac{'F_i \mathrm{H}^n(Tot\ B)}{'F_{i+1}\mathrm{H}^n(Tot\ B)}.$$

These pieces can be calculated by the method of spectral sequences (see 10.5).

10.3. **Partial cohomologies.** By taking the "horizontal" cohomology we obtain a bigraded object $'\mathrm{H}(B)$ with

$$'\mathrm{H}(B)^{p.q} \stackrel{\text{def}}{=} \mathrm{H}^p(B^{\bullet,q}) = \frac{\mathrm{Ker}(B^{p,q} \xrightarrow{d'} B^{p+1.q})}{Im(B^{p-1,q} \xrightarrow{d'} B^{p.q})}.$$

The vertical differential d'' on B factors to a differential on $'\mathrm{H}(B)$ which we denote again by d'':

$$'\mathrm{H}(B)^{p,q} \xrightarrow{d''} {}'\mathrm{H}(B)^{p,q+1}.$$

Next, we take the “vertical” cohomology of $'\mathrm{H}(B)$ (i.e., with respect to the new d''), and get a bigraded object $''\mathrm{H}('\mathrm{H}(B))$ with

$$''('\mathrm{H}(B))^{p,q} \overset{\text{def}}{=} \mathrm{H}^q('\mathrm{H}(B)^{p,\bullet}) = \frac{\mathrm{Ker}['\mathrm{H}(B)^{p,q} \xrightarrow{d''} {}'\mathrm{H}(B)^{p.q+1}]}{Im['\mathrm{H}(B)^{p,q-1} \xrightarrow{d''} {}'\mathrm{H}(B)^{p.q}]}.$$

One defines $''\mathrm{H}(B)$ and $'\mathrm{H}(''\mathrm{H}(B))$ by switching the roles of the first and second coordinates.

10.3.1. *Remark.* Constructions $'\mathrm{H}(B)$ and $''\mathrm{H}(B)$ are upper bounds on the cohomology of the bicomplex, and $'\mathrm{H}(''\mathrm{H}(B))$ and $''\mathrm{H}('\mathrm{H}(B))$ are even better upper bounds. The precise relation is given via the notion of spectral sequences (see 10.5).

10.4. **Resolutions of complexes.** An injective resolution of a complex $A \in C * \mathcal{A})$ is a quasi-isomorphism $A \to I$ such that all I^n are injective objects of the abelian category $\mathcal{A}$. The next two theorems will state that injective resolutions of complexes exist and and are can be chosen compatible with short exact sequences of complexes.

10.4.1. *Theorem.* If $\mathcal{A}$ has enough injectives any $A \in C^+(\mathcal{A})$ has an injective resolution. More precisely,

(a) There is a bicomplex

$$\begin{tikzcd}
\cdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \vdots \arrow[r, "d'"] & \cdots \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & I^{-1,2} \arrow[u, "d''"] \arrow[r, "d'"] & I^{0,2} \arrow[u, "d''"] \arrow[r, "d'"] & I^{1,2} \arrow[u, "d''"] \arrow[r, "d'"] & I^{2,2} \arrow[u, "d''"] \arrow[r, "d'"] & I^{3,2} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & I^{-1,1} \arrow[u, "d''"] \arrow[r, "d'"] & I^{0,1} \arrow[u, "d''"] \arrow[r, "d'"] & I^{1,1} \arrow[u, "d''"] \arrow[r, "d'"] & I^{2,1} \arrow[u, "d''"] \arrow[r, "d'"] & I^{3,1} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & I^{-1,0} \arrow[u, "d''"] \arrow[r, "d'"] & I^{0,0} \arrow[u, "d''"] \arrow[r, "d'"] & I^{1,0} \arrow[u, "d''"] \arrow[r, "d'"] & I^{2,0} \arrow[u, "d''"] \arrow[r, "d'"] & I^{3,0} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & A^{-1} \arrow[u, "d''"] \arrow[r, "d'"] & A^{0} \arrow[u, "d''"] \arrow[r, "d'"] & A^{1} \arrow[u, "d''"] \arrow[r, "d'"] & A^{2} \arrow[u, "d''"] \arrow[r, "d'"] & A^{3} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"] \\
\cdots \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & 0 \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \arrow[u, "d''"]
\end{tikzcd}$$

such that the columns are injective resolutions of terms in the complex A.

(b) For any such bicomplex the canonical map $A \to Tot(I)$ is an injective resolution of A.

10.4.2. *Theorem.* Let P and R be projective resolutions of objects A and C that appear in an exact sequence $0A \to B \to C \to 0$. Then $Q = P \oplus R$ appears in a short exact sequence of projective resolutions

We start with the baby case of the theorem 10.4.2.

10.4.3. *Lemma.* Assume that $\mathcal{A}$ has enough injectives. A short exact sequence in $\mathcal{A}$ can always be lifted to a short exact sequence of injective resolutions. More precisely,

(a) Let I and K be injective resolutions of objects A and C that appear in an exact sequence $0 \to A \to B \to C \to 0$. Then all $J^n = I^n \oplus K^n$ appear in a short exact sequence of injective resolutions

$$\begin{tikzcd}
\vdots \arrow[r] & \vdots \arrow[r] & \vdots \arrow[r] & \vdots \arrow[r] & \vdots \\
0 \arrow[r] \arrow[u] & I^2 \arrow[r, "\alpha^2"] \arrow[u, "d_I^2"] & J^2 \arrow[r, "\beta^2"] \arrow[u, "d_J^2"] & K^2 \arrow[r] \arrow[u, "d_K^2"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & I^1 \arrow[r, "\alpha^1"] \arrow[u, "d_I^1"] & J^1 \arrow[r, "\beta^1"] \arrow[u, "d_J^1"] & K^1 \arrow[r] \arrow[u, "d_K^1"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & I^0 \arrow[r, "\alpha^0"] \arrow[u, "d_I^0"] & J^0 \arrow[r, "\beta^0"] \arrow[u, "d_J^0"] & K^0 \arrow[r] \arrow[u, "d_K^0"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & A \arrow[r, "\alpha"] \arrow[u, "\iota_A"] & B \arrow[r, "\beta"] \arrow[u, "\iota_B"] & C \arrow[r] \arrow[u, "\iota_C"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

with α^n the inclusion of the first summand and $\beta^n$the projection to the second summand.

(b) Any short exact sequence of injective resolutions

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & J \arrow[r] & K \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & A \arrow[r, "\alpha"] \arrow[u, "\iota_A"] & B \arrow[r, "\beta"] \arrow[u, "\iota_B"] & C \arrow[r] \arrow[u, "\iota_C"] & 0 \arrow[u]
\end{tikzcd}$$

necessarily splits on each level, i.e., $J^n \cong I^n \oplus K^n$. (However, complex J is not a sum of I and K.)

Proof. (a) We need to define ι_B so that the middle column is a resolution and the diagram commutes. Define $\iota_B : B \to J^0 = I^0 \oplus K^0$ by

$$\iota_B(b) \stackrel{\text{def}}{=} \widetilde{\iota}_A(b) \oplus \iota_C(\beta b)$$

where $\widetilde{\iota_A} : B \to I^0$ is any extension of $\iota_A : A \to I^0$ (it exists since I^0 is injective). This choice ensures that the two squares that contain ι_B commute. Moreover, ι_B is injective since the kernel of the second component $\iota_C \circ \beta$ is $\mathrm{Ker}(\beta) = A$ and on $A \subseteq B$ ι_B is ι_A.

To continue in this way, we denote $\widetilde{B} = \mathrm{Coker}(\iota_B) = (I^0 \oplus K^0)/\iota_B(B)$ and notice that the second projection gives a surjection from $\widetilde{B}$ to $\widetilde{C} \stackrel{\text{def}}{=} K^0/\iota_C(C)$. Its kernel is the inverse of $\iota_C(C) \subseteq K^0$ under the second projection, taken modulo $\iota_B(B)$, i.e., $(I^0 \oplus \iota_C(C))/\iota_B(B) \cong I^0/\iota_A(A) \stackrel{\text{def}}{=} \widetilde{A}$. So we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I^1 \arrow[r, "\alpha^1"] & I^1 \oplus K^1 \arrow[r, "\beta^1"] & K^1 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & \widetilde{A} \arrow[r, "\alpha^1"] \arrow[u, "\overline{d_I^0}"] & \widetilde{B} \arrow[r, "\beta^1"] & \widetilde{C} \arrow[r] \arrow[u, "\overline{d_K^0}"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & I^0 \arrow[r, "\alpha^0"] \arrow[u, "q'"] & I^0 \oplus K^0 \arrow[r, "\beta^0"] \arrow[u, "q"] & K^0 \arrow[r] \arrow[u, "q''"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & A \arrow[r, "\alpha"] \arrow[u, "\iota_A"] & B \arrow[r, "\beta"] \arrow[u, "\iota_B"] & C \arrow[r] \arrow[u, "\iota_C"] & 0 \arrow[u]
\end{tikzcd}$$

in which $\overline{d_I^1}$ and $\overline{d_K^1}$ are factorizations of d_I^1 and d_K^1 through the the canonical quotient maps q' and q''. Maps $\overline{d_I^1}$ and $\overline{d_K^1}$ are embeddings, and we need to supply an embedding $\widetilde{B} \xrightarrow{\overline{d_J^1}} I^1 \oplus K^1$ which would give two more commuting squares, then d_J^1 is defined as a composition of $\overline{d_J^1}$ and the quotient map q. However, this is precisely the problem we solved in the first step.

(b) Since I^n is injective, one can extend the identity map on I^n to $J^n \xrightarrow{\phi^n} I^n$, and gives a splitting i.e., a complement $\mathrm{Ker}(\phi^n)$ to I^n in J^n.

10.4.4. *Existence of injective resolutions of complexes: proof of the theorem 10.4.2.* .
(0) *About the proof.* We choose injective resolutions of coboundaries and cohomologies of A: $B^n(A) \to J^n = J^{\bullet,n}$, $\mathrm{H}^n(A) \to K^n = K^{\bullet,n}$. Then the n^{th} column of I is $I^{\nu},n = J^n \oplus K^n \oplus J^{n+1}$ and the horizontal differentials are the compositions

$$I^{p,q} = J^{p,q} \oplus K^{p,q} \oplus J^{p,q+1} \twoheadrightarrow J^{p,q+1} \subseteq J^{p,q+1} \oplus K^{p,q+1} \oplus J^{p,q+2} = I^{p,q+1}.$$

The vertical differential make the n^{th} column into a complex $I^{\bullet,n}$ such that

- $J^{\bullet,n} \subseteq I^{\bullet,n}$ is a subcomplex, and
- $K^{\bullet,n} \subseteq I^{\bullet,n}/J^{\bullet,n}$ is a subcomplex.

The fist task is to choose suitable injective resolutions of everything in site. We start by choosing injective resolutions of coboundaries and cohomologies

$$B^n(A) \to \mathcal{B}^n = \mathcal{B}^{n,\bullet},\ \mathrm{H}^n(A) \to \mathcal{H}^n = \mathcal{H}^{n,\bullet}.$$

Now, cocycles are an extension of cohomologies and coboundaries, i.e., there is an exact sequence, and then A^n is an extension of $Z^n(A)$ and B^{n+1}, i.e., there are exact sequences

$$0 \to B^n(A) \to Z^n(A) \to \mathrm{H}^n(A) \to 0 \qquad \text{and} \qquad 0 \to Z^n(A) \to A^n \to B^{n+1}(A) \to 0.$$

By the preceding lemma 10.4.3 we can combine $\mathcal{H}$ and $\mathcal{B}$ to get injective resolutions of these short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{B}^n & \longrightarrow & \mathcal{Z}^n & \longrightarrow & \mathcal{H}^n & \longrightarrow & 0 \\
\Big\uparrow & & {\scriptstyle\iota}\Big\uparrow & & {\scriptstyle\iota}\Big\uparrow & & {\scriptstyle\iota}\Big\uparrow & & \Big\uparrow \\
0 & \longrightarrow & B^n(A) & \longrightarrow & Z^n(A) & \longrightarrow & \mathrm{H}^n(A) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{Z}^n & \longrightarrow & \mathcal{A}^n & \longrightarrow & \mathcal{B}^{n+1} & \longrightarrow & 0 \\
\Big\uparrow & & {\scriptstyle\iota}\Big\uparrow & & {\scriptstyle\iota}\Big\uparrow & & {\scriptstyle\iota}\Big\uparrow & & \Big\uparrow \\
0 & \longrightarrow & Z^n(A) & \longrightarrow & A^n & \longrightarrow & B^{n+1}(A) & \longrightarrow & 0
\end{array}$$

Since $\mathcal{A}^n$ is an injective resolution of A^n, we can use them to build a bicomplex as in the the part (a) of the theorem, with the vertical differentials d'' the differentials in $\mathcal{A}^n$'s. Now we need the horizontal differentials $\mathcal{A}^n \xrightarrow{d'^n} \mathcal{A}^{n+1}$), these are the compositions

$$(\mathcal{A}^n \xrightarrow{d'^n} \mathcal{A}^{n+1}) \stackrel{\text{def}}{=} [\mathcal{A}^n \to \mathcal{B}^{n+1} \xrightarrow{\subseteq} \mathcal{Z}^{n+1} \xrightarrow{\subseteq} \mathcal{A}^{n+1}].$$

Since these are morphisms of complexes vertical and horizontal differentials commute, however this is easily corrected to $d'd'' + d''d' = 0$ by 10.2.2(2).

(b) Let I be a bicomplex from the part (a). A canonical map $A \xrightarrow{\iota_A} Tot(I)$ comes from $A^n \xrightarrow{\iota_{A^n}} I^{n,0} \subseteq Tort(I)^n$. Moreover, ι_A is a quasi isomorphism since all maps are quasi isomorphisms. This is easy to see directly and there is an elegant tool for such problems – the concept of spectral sequences (see 10.5).

10.4.5. *Existence of injective resolutions of exact sequences of complexes: proof of the theorem 10.4.2.* This is a combination of ideas in proofs of the lemma 10.4.3 and the theorem 10.4.1.

corr If $\mathcal{A}$ has enough injectives any short exact sequence in $C^+(\mathcal{A})$ defines a distinguished triangle in $K(\mathcal{A})$.

Proof. By the theorem any short exact sequence in $C^+(\mathcal{A})$ is isomorphic in $K(\mathcal{A})$ to a short exact sequence of complexes with injective objects. Since such sequence splits on each level it defines a distinguished triangle in $K(\mathcal{A})$.

10.5. **Spectral sequences.** In general, spectral sequences are associated to filtered complexes but we will be happy with the special case of spectral sequences associated to bicomplexes. The idea of a spectral sequence is to relate the cohomology of a complex with the cohomology of a simplified complex which may be more accessible.

10.5.1. *Abstract definition.* A spectral sequence is a sequence of complexes $(E_r, d_r),\ r \geq 0$ such that

(1) E_r is bigraded, $E_r = \oplus\ E_r^{p,q}$ and the differential d_r has type $(r, 1-r)$, i.e.,
$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q+1-r}.$$
(2) $E_{r+1} = \mathrm{H}(E_r, d_r)$, i.e.,
$$E_{r+1}^{p,q} = \ \mathrm{Ker}(E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q+1-r})/Im(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q}).$$

10.5.2. *The limit of a spectral sequence.* The limit can be defined for any spectral sequence but the most interesting case is when the spectral sequence stabilizes. We will say that the (p,q)-term stabilizes in the r^{th} term if for $s \geq r$ the differentials d_s from the (p,q)-term and into the (p,q)-term are zero. Then clearly
$$E_r^{(p,q)} \cong\ E_{r+1}^{(p,q)} \cong\ E_{r+2}^{(p,q)} \cong \cdots.$$
Then we say that
$$E_\infty^{(p,q)} \stackrel{\text{def}}{=}\ E_r^{(p,q)}.$$
We will say that the spectral sequence stabilizes (degenerates) in the r^{th} term if $d_s = 0,\ s \geq r$. Then E_∞ is by definition E_r.

10.5.3. *Theorem.* To any bicomplex B one associates two ("symmetric") spectral sequences $'E$ and $''E$. The first one satisfies

(1) $'E_0^{p,q} =\ B^{p,q}$
(2) $'E_1^{p,q} =\ ''\mathrm{H}^{p,q}(B)$
(3) $'E_2^{p,q} =\ '\mathrm{H}^p[''\mathrm{H}^{\bullet,q}(B)$
(4) $'E_\infty^{p,q} =\ 'Gr_p[\mathrm{H}^{p+q}(Tot\ B)]$

10.5.4. *Remark.* The basic consequence is that the piece $'Gr_p[\mathrm{H}^n(Tot\ B)]$ of $\mathrm{H}^n(Tot\ B)$ is a subquotient of $'\mathrm{H}^{n-p}[''\mathrm{H}^{\bullet,q}(B)]$, hence in particular, of $''\mathrm{H}^{p,n-p}(B)$ (since $E_{r+1}^{i,j}$ is always a subquotient of $E_r^{i,j}$). This gives upper bounds on the dimension of $\mathrm{H}^n(Tot\ B)$.

10.5.5. *Degeneration criteria.* One general result that guarantees the stabilization of each term in some E_r (with r depending on p and q):

Theorem. If for each n the sum $(Tot\ B)^n =\ \oplus_{i+j=n}\ B^{i,j}$ is finite, then each term stabilizes.

This applies for instance if B lies in first quadrant.

Remarks. (a) The simplest way to prove that a spectral sequence degenerates at E_r is if one can see that for $s \geq r$ and any p,q one of objects $E_r^{p,q}$ or $E_r^{p+s,q+1-s}$ is zero.

(2) Since d_s is of type $(s, 1-s)$ it changes the total parity $p+q$. So one way to satisfy the requirement in (1) is to know that all $p+q$ with $E_r^{pq} \neq 0$ are of the same parity.

(3) We are fond of bicomplexes such that the first spectral sequence degenerates at E_2, then we recover the constituents of $\mathrm{H}^n(Tot\ B)$ from partial cohomology

$$Gr_\bullet[\mathrm{H}^n(Tot\ B)] \cong \ \oplus_{p+q=n}\ {}'\mathrm{H}^p[{}''\mathrm{H}^{\bullet,n-p}(B).$$

11. **Derived categories of abelian categories**

Historically the notion of triangulated categories has been discovered independently in

- Algebra: derived category of an abelian category $\mathcal{A}$ is a convenient setting for doing homological algebra – i.e., the calculus of complexes in $\mathcal{A}$ (and much more).
- Topology: the stable homotopy theory deals with the category whose objects are topological spaces (rather then complexes!), but the total structure is the same as for a derived category $D(\mathcal{A})$ (shift is giving by the operation of suspension of topological spaces, exact triangles come from the topological construction of the mapping cone C_f corresponding to a continuous map $f : X \to Y$ of topological spaces ...)

Here we will only be concerned with the (more popular) appearance of derived categories in algebra.

11.1. **Derived category** $D(\mathcal{A})$ **of an abelian category** $\mathcal{A}$**.** The objects of $D(\mathcal{A})$ are again the complexes in $\mathcal{A}$, however $\mathrm{Hom}_{D(\mathcal{A})}(A, B)$ is an equivalence class of diagrams in $K(\mathcal{A})$:

(1) Let $\widetilde{\mathrm{Hom}}_{D(\mathcal{A})}(A, B)$ be the class of all diagrams in $K(\mathcal{A})$ Of the form

$$A \xrightarrow{\phi} X \xleftarrow{s} B, \qquad \text{with } s \text{ a quasi-isomorphism.}$$

(2) Two diagrams $A \xrightarrow{\phi_i} X_i \xleftarrow{s_i} B,\ i = 1, 2$; are equivalent iff they are quasi-isomorphic to a third diagram, in the sense that there are quasi-isomorphisms $X_i \xrightarrow{u_i} X,\ i = 1, 2$; such that

$$\begin{matrix} u_1\circ\phi_1 = & u_2\circ\phi_2 \\ u_1\circ s_1 = & u_2\circ s_2 \end{matrix}, \qquad i.e., \qquad \begin{tikzcd} A \arrow[r, "\phi_1"] \arrow[d, "="'] & X_1 \arrow[d, "u_1"'] & B \arrow[l, "s_1"'] \arrow[d, "="'] \\ A \arrow[r, "\phi"] & X & B \arrow[l, "s"'] \\ A \arrow[r, "\phi_2"] \arrow[u, "="] & X_2 \arrow[u, "u_2"] & B. \arrow[l, "s_2"'] \arrow[u, "="] \end{tikzcd}$$

11.1.1. *Symmetry.* It follows from the next lemma that one can equivalently represent morphisms by diagrams $A \xleftarrow{s} C \xrightarrow{\phi} B$ with s a quasi-isomorphism.

11.1.2. *Lemma.* If s is a quasi-isomorphism a diagram $A \xleftarrow{s} B$ in $K(\mathcal{A})$ can be canonically completed to a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & B \\ {\scriptstyle u}\downarrow & & {\scriptstyle s}\downarrow \\ A & \xrightarrow{\phi} & X \end{array}$$

with u a quasi-isomorphism. (Conversely, one can also complete u, ψ to s, ϕ.)

11.1.3. *Remarks.* (0) We can denote the map represented by the diagram $A \xrightarrow{\phi} X \xleftarrow{s} B$ by $[s, \phi]$, its intuitive meaning is that it the morphism $s^{-1} \circ \phi$ once s is inverted. Identifications of diagrams should correspond to equalities ${s_1}^{-1} \circ \phi_1 = {s_2}^{-1} \circ \phi_2$, to compare it with the requirement (2) rewrite it as first as $(u_1 \circ s_1)^{-1} \circ (u_1 \circ \phi_1) = (u_2 \circ s_2)^{-1} \circ (u_2 \circ \phi_2)$, and then as $(u_1 \circ \phi_1) \circ (u_2 \circ \phi_2)^{-1} = (u_1 \circ s_1) \circ (u_2 \circ s_2)^{-1}$; then (2) actually says that both sides are equal to the identity 1_X.

(1) The composition of (equivalence classes of) diagrams is based on lemma 11.1.2. Putting together two diagrams $[s, \phi]$ and $[u, \psi]$ gives $A \xrightarrow{\phi} X \xleftarrow{s} B \xrightarrow{\psi} Y \xleftarrow{u} C$. Now lemma 11.1.2 allows us to replace the inner part $X \xleftarrow{s} B \xrightarrow{\psi} Y$ by $X \xleftarrow{\psi'} Z \xrightarrow{s'} Y$ with s' a quasi-isomorphism. This gives a diagram $A \xrightarrow{\phi} X \xleftarrow{\psi'} Z \xrightarrow{s'} Y \xleftarrow{u} C$ and we define

$$\stackrel{\text{def}}{=} [u, \psi] \circ [s, \phi] \circ [s' \circ u, \psi' \circ \phi].$$

(2) The above procedure inverts quasi-isomorphisms in $K(\mathcal{A})$. We can actually invert quasi-isomorphisms directly in $C(\mathcal{A})$, however lemma 11.1.2 does not hold in $C(\mathcal{A})$, so one is forced to take a more complicated definition of maps as coming from long diagrams

$$A \xrightarrow{\phi_0} X_0 \xleftarrow{s_0} B_0 \xrightarrow{\phi_1} X_0 \xleftarrow{s_1} B_1 \xrightarrow{\phi_2} X_2 \xleftarrow{s_2} \cdots \xrightarrow{\phi_{n-1}} X_{n-1} \xleftarrow{s_{n-1}} B_{n-1} \xrightarrow{\phi_n} X_n \xleftarrow{s_n} B_n = B$$

which are composed in the obvious way.

11.1.4. *Theorem.* $D(\mathcal{A})$ is a triangulated category if we define the exact triangles as images of exact triangles in $K(\mathcal{A})$.

One can also define triangulated subcategories $D^?(\mathcal{A})$ for $? \in \{+, b, -\}$, these are full subcategories of all complexes A such that $H^\bullet(A) \in C^?(\mathcal{A})$ (with zero differential). For $\mathcal{Z} \subseteq \mathcal{Z}$ one can also define a full subcategory $D^{\mathcal{Z}}(\mathcal{A})$ by again requiring a condition on cohomology: that $H^\bullet(A) \in C^{\mathcal{Z}}(\mathcal{A})$.

11.1.5. *The origin of exact triangles in $D(\mathcal{A})$.* They can be associated to either of the following:

(1) a map of complexes,
(2) a short exact sequence of complexes that splits on each level,
(3) if $\mathcal{A}$ has enough injectives, any short exact sequence of complexes in $C^+(\mathcal{A})$, and if $\mathcal{A}$ has enough projectives, any short exact sequence of complexes in $C^-(\mathcal{A})$.

11.1.6. *Cohomology functors $H^i : D(\mathcal{A}) \to \mathcal{A}$.* H^i is clearly defined on objects, on morphisms it is well defined by $\mathrm{H}^i([s,\phi]) \overset{\text{def}}{=} \mathrm{H}^i(s)^{-1} \circ \mathrm{H}^i(\phi)$.

11.2. **Truncations.** These are functors

$$D^{\leq n}(\mathcal{A}) \xleftarrow{\tau_{\leq n}} D(\mathcal{A}) \xrightarrow{\tau_{\leq n}} D^{\geq n},$$

they come with canonical maps $\tau_{\leq n} A \to A \to \tau_{\geq n} A$ defined by

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \longrightarrow & Z^n(A) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & (\tau_{\leq n}(A)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\
\cdots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & A^{n+2} & \longrightarrow & \cdots & (\tau_{\leq n}(A)) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \uparrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A^n/B^n(A) & \longrightarrow & A^{n+1} & \longrightarrow & A^{n+2} & \longrightarrow & \cdots & (\tau_{\geq n}(A))
\end{array}$$

11.2.1. *Lemma.* (a) $\tau_{\leq n}$ is the left adjoint to the inclusion $D^{\leq n}(\mathcal{A}) \subseteq D(\mathcal{A})$, and $\tau_{\geq n}$ is the right adjoint to the inclusion $D^{\geq n}(\mathcal{A}) \subseteq D(\mathcal{A})$.

(b) $\mathrm{H}^i(\tau_{\leq n} A) = \left\{ \begin{smallmatrix} \mathrm{H}^i(A) & i \leq n \\ 0 & i > n \end{smallmatrix} \right\}$, and $\mathrm{H}^i(\tau_{\geq n} A) = \left\{ \begin{smallmatrix} \mathrm{H}^i(A) & i \geq n \\ 0 & i < n \end{smallmatrix} \right\}$.

11.3. **Inclusion $\mathcal{A} \hookrightarrow D(\mathcal{A})$.**

Lemma. (a) By interpreting each $A \in \mathcal{A}$ as a complex concentrated in degree zero, one identifies $\mathcal{A}$ with a full subcategory of $C(\mathcal{A})$, $K(\mathcal{A})$ or $D(\mathcal{A})$.

(b) The inclusion of $\mathcal{A}$ into the full subcategory $D^0(\mathcal{A})$ (all complexes A with $H^i(A) = 0$ for $i \neq 0$) is an equivalence of categories. (The difference is that $D^0(\mathcal{A})$ is closed in $D(\mathcal{A})$ under isomorphisms and $\mathcal{A}$ is not).

Proof. In (a) we need to see that for A, B in $\mathcal{A}$ the canonical map $\mathrm{Hom}_{\mathcal{A}}(A,B) \to \mathrm{Hom}_{\mathcal{X}}(A,B$ is an isomorphism:

- when $\mathcal{X} = C(\mathcal{A})$ since any homotopy between two complexes concentrated in degree zero is clearly 0 (recall that $h^n : A^N \to B^{n-1}$),
- when $\mathcal{X} = D(\mathcal{A})$ one shows that

(1) any diagram in $K(\mathcal{A})$ of the form $A \xrightarrow{\phi} X \xleftarrow{s} B$ with s a quasi-isomorphism, is equivalent to a diagram in $\mathcal{A}$ $A \xrightarrow{\mathrm{H}^0(\phi)} \mathrm{H}^0(X) \xleftarrow{\mathrm{H}^0(s)} B$, hence to $A \xrightarrow{\mathrm{H}^0(s)^{-1}\mathrm{H}^0(\phi)} B \xleftarrow{1_B} B$, hence it comes from a map $A \xrightarrow{\mathrm{H}^0(s)^{-1}\mathrm{H}^0(\phi)} B$ in $\mathcal{A}$,

(2) Two diagrams of the form $A \xrightarrow{\alpha_i} B \xleftarrow{1_B} B$, are equivalent iff $\alpha_1 = \alpha_2$.

Remark. Part (b) of the lemma describes $\mathcal{A}$ inside $D(\mathcal{A})$ (up to equivalence) by only using the functors H^i on $D(\mathcal{A})$.

11.4. **Homotopy description of the derived category.** The following provides a down to earth description of the derived category and gives us a way to calculate in the derived category.

11.4.1. *Theorem.* Let $\mathcal{I}_\mathcal{A}$ be the full subcategory of $\mathcal{A}$ consisting of all injective objects.

(a) $\mathcal{I}_\mathcal{A}$ is an abelian subcategory.

(b) If $\mathcal{A}$ has enough injectives the canonical functors

$$K^+(\mathcal{I}_\mathcal{A}) \xrightarrow{\sigma} D^+(\mathcal{I}_\mathcal{A}) \xrightarrow{\tau} D^+(\mathcal{A})$$

are equivalences of categories.

Proof. (A sketch.) One first observes that

- Quasi-isomorphism between complexes in $C^+(\mathcal{I})$ are always homotopical equivalences.

This tells us that quasi-isomorphisms in $K^+(\mathcal{I})$ are actually isomorphisms in $K^+(\mathcal{I})$. Since quasi-isomorphisms in $K^+(\mathcal{A})$ are already invertible the passage to $D^+(\mathcal{I})$ obviously gives an equivalence $K^+(\mathcal{I}) \to D^+(\mathcal{I})$.

We know that any complex A in $C^+(\mathcal{A})$ is quasi-isomorphic to its injective resolution I, and also any map $A' \to A''$ is quasi-isomorphic to a map of injective resolutions $I' \to I''$. This is the surjectivity of τ on objects and morphisms.

The remaining observation is that

- For $I, J \in D^+(\mathcal{I})$ map $\mathrm{Hom}_{D(\mathcal{I})}(I,J) \to \mathrm{Hom}_{D(\mathcal{A})}(I,J)$ is injective.

12. **Derived functors**

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.

12.1. **Derived functors $R^pF : \mathcal{A} \to \mathcal{A}$.** Suppose that $\mathcal{A}$ has enough injectives and set

$$R^pF(A) = \mathrm{H}^p(FI) \quad \text{for any injective resolution } I \text{ of } A.$$

We call these the (right) derived functors of F.

12.1.1. *Theorem.* (a) Functors R^p are well defined.

(b) $R^0F \cong\ F$.

(c) Any short exact sequence $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ in $\mathcal{A}$ defines a long exact sequence of derived functors

$$0 \to R^0F(A') \xrightarrow{R^0F(\alpha)} R^0F(A) \xrightarrow{RF^n(\beta)} R^0F(A'') \xrightarrow{\partial^0} \cdots \xrightarrow{\partial^{n-1}} R^nF(A') \xrightarrow{R^nF(\alpha)} R^nF(A) \xrightarrow{RF^n(\beta)} R^nF(A'')$$

Proof. (a) and (b) follow from 9.9. (c) follows from the lemma 10.4.3. First we can choose an injective resolution $0 \to I' \xrightarrow{\alpha} I \xrightarrow{\beta} I'' \to 0$ of the short exact sequence $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$, and then we apply F to it. The sequence of complexes $0 \to F(I') \xrightarrow{\alpha} F(I) \xrightarrow{\beta} F(I'') \to 0$ is exact since the short exact sequence of resolutions splits level-wise (lemma 10.4.3b), and F is additive.

12.1.2. *Remark.* , even if F is not left exact the above construction produces a left exact functor R^F.

12.2. **Derived functors $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$.** We start with the definition of its right derived functor RF as the universal one among all extensions of $\mathcal{A} \xrightarrow{F} \mathcal{B}$ to $D^+(\mathcal{A}) \to D^+(\mathcal{B})$. Then we see that the "replacement by injective resolution" construction satisfies the universality property.

12.2.1. *Notion of the derived functor of F.* A functor between two triangulated categories is said to be a morphism of triangulated categories (a triangulated or ∂-functor) if it preserves all structure of these categories:

(1) it is additive,
(2) it preserves shifts
(3) it preserves exact triangles

The right derived functor of an additive functor $F : \mathcal{A} \to \mathcal{B}$ is the universal one among all extensions of F to $D^+(\mathcal{A}) \to D^+(\mathcal{B})$. Precisely, it consists of the following data

- A triangulated functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$,
- a morphism of functors $i_{\mathcal{B}} \circ F \xrightarrow{\xi} RF \circ i_{\mathcal{A}}$;

and these data should satisfy the universality property:

- for any other such pair $(\widetilde{RF}, \widetilde{\xi})$, morphism $\widetilde{\xi}$ factors uniquely through ξ, i.e., there is a unique morphism of functors $RF \xrightarrow{\mu} \widetilde{RF}$ such that $\widetilde{\xi} = \mu \circ \xi$, i.e., $i_{\mathcal{B}} \circ F \xrightarrow{\widetilde{\xi}} \widetilde{R}F \circ i_{\mathcal{A}}$ is the composition $i_{\mathcal{B}} \circ F \xrightarrow{\xi} RF \circ i_{\mathcal{A}} \xrightarrow{\mu \circ 1_{\mathcal{A}}} \widetilde{R}F \circ i_{\mathcal{A}}$.

12.2.2. *Remark.* (0) For $A \in \mathcal{A}$, $i_{\mathcal{A}}A$ is A viewed as a complex, and ξ_A relates objects that should be the same if RF extends F: $RF(i_{\mathcal{A}}A) \xrightarrow{\xi_A} i_{\mathcal{B}}(FA)$. So, intuitively ξ is the "commutativity constraint" for the diagram,

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ {\scriptstyle i_{\mathcal{A}}}\downarrow & & {\scriptstyle i_{\mathcal{B}}}\downarrow \\ D^{+}(\mathcal{A}) & \xrightarrow{RF} & D^{+}(\mathcal{B}) \end{array}$$

, it takes care of the fact that the functors $i_{\mathcal{B}}\circ F$ and $RF\circ i_{\mathcal{A}}$ are not literally the same but only canonically isomorphic (though the universality property does not require ξ to be an isomorphism, in practice it will be an isomorphism).

(1) The problem is when does the universal extension exist? The simplest case is when F is exact, then then one can define RF simply as F acting on complexes. In general the most useful criterion is

12.2.3. *Theorem.* Suppose that $\mathcal{A}$ has enough injectives. Then for any additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$:

(a) $RF : D^{+}(\mathcal{A}) \to D(\mathcal{B})$ exists.

(b) For any complex $A \in D^{+}(\mathcal{A})$, there is a canonical isomorphism $(RF)A \cong\ F(I)$ for any injective resolution I of A.

(c) In particular, for $A \in \mathcal{A}$ the cohomologies of $(RF)A$ are the derived functors $(R^{i}F)A$ introduced above.

Proof. Recall from 11.4.1 that one has equivalences $K^{+}(\mathcal{I}_{\mathcal{A}}) \xrightarrow{\sigma} D^{+}(\mathcal{I}_{\mathcal{A}}) \xrightarrow{\tau} D^{+}(\mathcal{A})$, and from recall that an additive functor $F_{\mathcal{I}} \stackrel{\text{def}}{=} (\mathcal{I}_{\mathcal{A}}\subseteq\mathcal{A} \xrightarrow{F} \mathcal{B})$ has a canonical extension $K(F_{\mathcal{I}}) : K(\mathcal{A}) \to K(\mathcal{B})$. So we can define RF as a composition

$$RF \stackrel{\text{def}}{=} \ [D^{+}(\mathcal{A}) \xrightarrow{(\tau\circ\sigma)^{-1}} K^{+}(\mathcal{I}_{\mathcal{A}}) \xrightarrow{K(F_{\mathcal{I}})} K^{+}(\mathcal{B}) \to D^{+}(\mathcal{B})].$$

Our construction satisfies the description of RF in (b). We have $(\tau\sigma)^{-1}I = I$ and a quasi-isomorphism $A \to I$. Therefore, $RF(A) \stackrel{\text{def}}{=} [K(F_{\mathcal{I}})\circ(\tau\circ\sigma)^{-1}]\ A = K(F_{\mathcal{I}})\ I = F(I)$. (c) follows from (b).

It remains to check that our RF is the universal extension.

RF preserves shifts by its definition. Any exact triangle in $D^(\mathcal{A})$ is isomorphic to a triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ that comes from an exact sequence $0 \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\gamma} C \to 0$ in $C(\mathcal{A})$ that splits on each level. Moreover, we can replace the exact sequence $0 \xrightarrow{\alpha} A' \xrightarrow{\beta} A \xrightarrow{\gamma} A'' \to 0$ with an isomorphic (in $D(\mathcal{A})$) short exact sequence of injective resolutions $0 \to I' \xrightarrow{a} I \xrightarrow{b} I'' \to 0$. Since it also splits on each level its F-image $0 \to F(I') \xrightarrow{F(a)} F(I) \xrightarrow{F(b)} F(I'') \to 0$ is an exact sequence in $C(\mathcal{B})$. Therefore it defines an exact triangle $F(I') \xrightarrow{F(a)} F(I) \xrightarrow{F(b)} F(I'') \xrightarrow{\widetilde{\gamma}} F(I')[1]$ in $D(\mathcal{A})$. To see that this is the triangle we observe that by definition $F(I') = RF(A')$ $F(I) = RF(A)$ $F(I'') = RF(A'')$, and also $F(a) = RF(\alpha)$ and $F(b) = RF(\beta)$. It remains to see that $\widetilde{\gamma} = RF(\gamma)$. For this recall that γ and $\widetilde{\gamma}$ are defined using splittings of the first and second row of

$$\begin{array}{ccccccccc}
0 & \xrightarrow{\alpha} & A' & \xrightarrow{\beta} & A & \xrightarrow{\gamma} & A'' & \longrightarrow & 0 \\
 & & {\scriptstyle\iota'}\downarrow & & {\scriptstyle\iota}\downarrow & & {\scriptstyle\iota''}\downarrow & & \\
0 & \longrightarrow & I' & \xrightarrow{a} & I & \xrightarrow{b} & I'' & \longrightarrow & 0.
\end{array}$$

So, we need to be able to choose the splitting of the second row compatible with the one in the first row. However, this is clearly possible by the construction of the second row.

12.3. **Usefulness of the derived category.** We list a few more reasons to rejoice in derived categories.

12.3.1. *Some historical reasons for the introduction of derived categories.* (1) For a left exact functor F we have derived functors R^iF and and then concept of derived category allows us to glue them into one functor RF. Does this make a difference?

It is easy to check from definitions that for two left exact functors F and G one has $R(G\circ F) \cong RG\circ RF$. If we instead work with R^iF, R^jG and $R^n(G\circ F)$ the above formula degenerates to a weak and complicated relation in terms of spectral sequences. So, seemingly more complicated construction RF is more natural and has better properties then a bunch of functors R^iF.

(2) If $G : \mathcal{B} \to \mathcal{C}$ is right exact and $\mathcal{B}$ has enough projectives then one similarly has functors $L^iG : \mathcal{B} \to \mathcal{C}$ and they glue to $LG : D^-(\mathcal{B}) \to D^-(\mathcal{C})$.

However, if one has a combination of two functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ and F is left exact while G is right exact, we have a functor $D^b(\mathcal{A}) \xrightarrow{LG\circ RF} D^b(\mathcal{C})$ which is often very useful but has no obvious analogue as a family of functors from $\mathcal{A}$ to $\mathcal{C}$ since the composition $G\circ F$ need not be neither left nor right exact.

(3) Some of essential objects and tools exist only on the level of derived categories: the dualizing sheaves in topology and algebraic geometry, the perverse sheaves.

12.3.2. *Some unexpected gains.* It turns out that in practice many deep relations between abelian categories $\mathcal{A}$ and $\mathcal{B}$ become understandable only on the level of derived categories, for instance $D(\mathcal{A})$ and $D(\mathcal{B})$ are sometimes equivalent though the abelian categories $\mathcal{A}$ and $\mathcal{B}$ are very different.

Part 4. Homeworks

Homework 1

1.1. Homology of a torus. Calculate the dimensions of the homology groups of the torus T^2 using a triangulation into a simplicial complex.

Homework policy.

(1) The grade will be based on homeworks.
(2) It is a good idea to discuss the problems that resist reasonable effort with others, but the actual writing should be done in isolation.
(3) It is not necessary to turn out perfect homeworks, rather it is expected that your work shows that that you are acquiring the material of the course.
(4) The amount of detail in your expositions (i) should be based on what is most useful to you, i.e., you need to make it perfectly clear to yourself, (ii) but it should be also written so that another person can follow the line of thought.
(5) If you are quite sure that you understand certain problem (or a part of), to the extent that writing the solution would be a waste of time, just say so and formulate what is happening in a few words. I reserve the right to ask about the missing ideas. Homeworks are due next week on Thursday.

Homework 2

2.1. Homology of spheres. S^n is the boundary of σ_n, hence has a triangulation by all facets of σ_n except σ_n itself.

- (a) Use this to compute the dimension of homology groups $H_*(S^3, \Bbbk)$ for a field $\Bbbk$.
- (b) Compute $H_*(S^n, \Bbbk)$ by an intuitive argument, using a covering by *two* cells.

2.2. Euler characteristic. The Euler characteristic of a finite sequence of finite dimensional vector spaces $A^\bullet$ is $e(A^\bullet) \overset{\text{def}}{=} \sum_i (-1)^i \dim(A^i)$. For a finite complex of vector spaces $C^\bullet$ show that $e(C^\bullet) = e(H^\bullet(C^\bullet))$.

2.3. Duality for $\Bbbk$-modules. Prove lemma 2.2.1, i.e. show that for $M \in \mathfrak{m}^l(\Bbbk)$, M^* is really in $\mathfrak{m}^r(\Bbbk)$ and for $N \in \mathfrak{m}^r(\Bbbk)$, N^* is in $\mathfrak{m}^l(\Bbbk)$.

2.4. Biduality for $\Bbbk$-modules. Prove lemma 2.2.2, i.e., show that

- (1) for $m \in M$ $\iota(m)$ is really in $(M^*)^*$, and
- (2) ι_M is a map of $\Bbbk$-modules.

2.5 Dualizing maps. For a module $M = \Bbbk \in \mathfrak{m}^l(\Bbbk)$ show that

(a) The map that assigns to $a \in \Bbbk$ the operator of right multiplication $R_a x \overset{\text{def}}{=} x{\cdot}a$, gives an isomorphism of right $\Bbbk$-modules $\Bbbk \xrightarrow{R} \Bbbk^*$.

(b) $\iota_{\Bbbk}$ is an isomorphism.

Homework 3

3.1. The dual of the structure sheaf of a point in a plane. For the point Y in a plane $\mathbb{A}^2$ show that

$$L\mathbb{D}[\mathcal{O}(Y)] \cong \mathcal{O}(Y)[-2]$$

(i.e., calculate the cohomology of $\mathbb{D}P^\bullet$ for the resolution $P^\bullet$ from 2.7.2).

Homework 4

SHEAVES

4.0. The sheaf of solutions is constant on an interval. Let X be an interval in $\mathcal{R}$ and $*$ be the differential equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_0(t) = 0$$

with $a_i \in C^\infty(X)$. Find *some* identification of the sheaf $\mathcal{S}ol_*$ of solutions of $*$, with a constant sheaf V_X for some vector space $V \cong \mathbb{C}^n$.

MULTI-LINEAR ALGEBRA: Tensor product of modules over a ring
If needed see S.Lang's *Algebra* or N.Bourbaki's *Algebra*.

Let A be a ring with a unit 1.

4.1. Tensor product of A-modules. *For a left A-module U and a right A-module V, we define a free abelian group $F = F_{U,V}$, with a basis $U{\times}V$: $F = \oplus_{u\in U,\ v\in V}\ \mathbb{Z}{\cdot}(u,v)$. The tensor product of U and V is the abelian group $U{\otimes}_A V$ defined as a quotient $U{\otimes}_A V \overset{def}{=} F/S$ of F by the subgroup R generated by the elements of one of the following forms (here $u, u_i \in U,\ v, v_i \in V,\ a \in A$):*

(1) $(u_1{+}u_2, v){-}(u_1,v){-}(u_2,v)$, (2) $(u, v_1{+}v_2){-}(u,v_1){-}(u,v_2)$, (3) $(u{\cdot}a, v) - (u, a{\cdot}v)$, .

The image of $(u,v) \in \mathbb{F}$ in $U{\otimes}_A V$ is denoted $u{\otimes}v$. Let $\pi : U{\times}V{\to}U{\otimes}_A V$ be the composition of maps $U{\times}V{\hookrightarrow}F{\to}U{\otimes}_A V$, so that $\pi(u,v) = u{\otimes}v$.

(a) Show that

(a_1) $(u_1 + u_2){\otimes}v = u_1{\otimes}v + u_2{\otimes}v$,

(a_2) $u{\otimes}(v_1 + v_2) = u{\otimes}v_1 + u{\otimes}v_2$,

(a_3) $(u{\cdot}a){\otimes}v) = u{\otimes}(a{\cdot}v)$.

(b) Show that each element of $U \otimes_A V$ is a finite sum of the form $\sum_{i=1}^n \ u_i \otimes v_i$, for some $u_i \in U,\ v_i \in V$.

4.2. The universal property of the tensor product $\otimes_A$. *We say that a map $\phi : U \times V \to H$ with values in an abelian group H, is A-balanced if it satisfies the conditions $\phi(u_1+u_2, v) = \phi(u_1, v)+\phi(u_2, v)$, $\phi(u, v_1+v_2) = \phi(u, v_1)+\phi(u, v_2)$, $\phi(u \cdot a, v) = \phi(u, a \cdot v)$, for $u, u_i \in U,\ v, v_i \in V,\ a \in A$.*

Show that the balanced maps $\phi : U \times V \to H$ are in a one-to-one correspondence with the morphisms of abelian groups $\psi : U \otimes_A V \to H$, by $\psi \mapsto \phi \overset{\text{def}}{=} \psi \circ \pi$.

[*The above notion of "balanced maps" is a version of the notion of bilinear maps which makes sense even for non-commutative rings A. In one direction of the above one-to-one correspondence above says that the tensor product reformulates balanced maps in terms of linear maps. In the opposite direction, the universal property tells us* <u>*how to constructs maps from a tensor product $U \otimes_A V$*</u> *– one needs to construct a balanced map from the product $U \times V$.* (This direction is used all the time!)]

4.3. Cancellation, Quotient as tensoring, Additivity, Free Modules. .

(a) **Cancellation.** Show that the map

$$U \otimes_A A \xrightarrow{\alpha} A,\ \alpha(\sum \ u_i \otimes a_i) = \sum u_i \cdot a_i \ ,$$

is (i) well defined, (ii) an isomorphism of abelian groups.

(b) **Quotient by relations as tensoring.** For each left ideal I in A, the map

$$U \otimes_A A/I \xrightarrow{\alpha} U/U \cdot I,\ \alpha[\sum \ u_i \otimes (a_i + I)] \ = \ (\sum u_i \cdot a_i) + U \cdot I \ ,$$

is (i) well defined, (ii) an isomorphism of abelian groups.

(c) **Additivity of tensoring.** Let $V = \oplus_{i \in I} V_i$ be a direct sum of left A-modules, then $U \otimes_A V \ = \ \oplus_{i \in I} U \otimes_A V_i$.

(d) **Free Modules.** If V is a free left A-module with a basis $v_i,\ i \in I$, then $U \otimes_A V \cong U^I$, i.e., it is a sum of I copies of U.

4.4. Tensoring of bimodules. If U is a bimodule for a pair of rings (R, A) and V is a bimodule for a pair of rings (A, S), show that $U \otimes_A V$ is a bimodule for (R, S).

4.5. Tensoring over a commutative ring. (a) If A is a commutative ring then the left and right modules coincide, say a right A-module U becomes a left A-module with the action defined by $a \cdot u \overset{\text{def}}{=} u \cdot a$.

(b) In that case the above general construction allows us to tensor two left modules U, V: $U \otimes_A V \overset{\text{def}}{=} \oplus_{u \in U,\ v \in V} \ \mathbb{Z} \cdot (u, v)/R$ where R is the subgroup generated by the elements of the form $(1), (2)$ and $(3')$ $(a{\cdot}u, v) - (u, a{\cdot}v)$. One has again properties $(a_1), (a_2)$ and (a'_3) $(a{\cdot}u)\otimes v) = u\otimes(a{\cdot}v)$, and the analogous universal property.

(c) Moreover, for a commutative ring A, $U\otimes_A V$ is again an A-module with the action $a{\cdot}\ (u\otimes v) \stackrel{\text{def}}{=} (a{\cdot}\ u)\otimes v = u\otimes(a{\cdot}v)$.

4.6. Tensoring over a field. If A is a field then A-modules are vector spaces over A. Let $u_i, i \in I$, $v_j, j \in J$, be bases of U and V, show that $u_i\otimes v_j,\ i \in I, j \in J$; is a basis of $U\otimes_A V$ and $\dim(U\otimes_A V) = \dim(U){\cdot}\dim(V)$.

4.7. Tensoring of finite abelian groups over $\mathbb{Z}$. Show that $\mathbb{Z}_n \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_m \cong \mathbb{Z}_k$ for some k and calculate k.

Homework 5

D-modules

5.0. Inverse image and the direct image of D-modules. Let $Y = \{0\} \stackrel{i}{\subseteq} X = \mathbb{A}^1 \stackrel{q}{\to} Z = pt$. Calculate the dimensions of cohomology groups $(L^s i^o)M \stackrel{\text{def}}{=} H^i[Li^o(M)]$ and $(L^s q_*)M \stackrel{\text{def}}{=} H^i[Lq_*(M)]$ for

(1) $M = D_X$,
(2) $M = \mathcal{O}(X)$,
(3) $M = \delta$ for $\delta \stackrel{\text{def}}{=}\ D_X/D_X z$,
(4) $M = M_*$ for $*$ given by $zy' = \lambda y$ (the answer will depend somewhat on λ).

(Recall that $i^0 M = M/zM$ and $q_* M = M/\partial M$.)

MULTI-LINEAR ALGEBRA: Multiple tensor products and Tensor algebras of modules

5.1. Multiple tensor products. *Let M_i be an (A_{i-1}, A_i)-bimodule for $i = 1, ..., n$.*

The tensor product $M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$ is defined as a quotient of a free abelian group F with a basis $M_1\times\cdots\times M_n$, by the subgroup $\mathcal{A}$ generated by the elements of one of the following forms ($m_i \in M_i,\ a_i \in A_i$):

$$(1)\quad (m_1, ..., m'_i + m''_i, ..., m_n) - (m_1, ..., m'_i, ..., m_n) - (m_1, ..., m''_i, ..., m_n),$$

$$(2)\quad (m_1, ..., m_{i-1}{\cdot}a_i, ..., m_n) - (m_1, ..., a_i{\cdot}m_i, ..., m_n).$$

The image of $(m_1,...,m_n) \in \mathbb{F}$ *in* $U \otimes_A V$ *is denoted* $m_1 \otimes \cdots \otimes m_n$. *Let* $\pi : M_1 \times \cdots \times M_n \to M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$ *be the composition* $M_1 \times \cdots \times M_n \hookrightarrow F \to M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$, *so that* $\pi(m_1,...,m_n) = m_1 \otimes \cdots \otimes m_n$.

(a) Show that

$$(1) \quad m_1 \otimes \cdots \otimes m_i' + m_i'' \otimes \cdots \otimes m_n = m_1 \otimes \cdots \otimes m_i' \otimes \cdots \otimes m_n + m_1 \otimes \cdots \otimes m_i'' \otimes \cdots \otimes m_n,$$

$$(2) \quad m_1 \otimes \cdots \otimes m_{i-1} \cdot a_i \otimes m_i \otimes \cdots \otimes m_n = m_1 \otimes \cdots \otimes m_{i-1} \otimes a_i \cdot m_i \otimes \cdots \otimes m_n.$$

(b) Each element of $M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$ is a finite sum of the form $\sum_{k=1}^{p} m_{k,1} \otimes \cdots \otimes m_{k,n}$.

(c) Formulate and prove the universal property of tensor products.

(e) Show that $M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$ is an (A_0, A_n)-bimodule.

(f) This definition is associative in the sense that there are canonical isomorphisms

$$(M_1 \underset{A_1}{\otimes} \cdots \underset{A_{p_1-1}}{\otimes} M_{p_1}) \underset{A_{p_1}}{\otimes} (M_{p_1+1} \underset{A_{p_1+1}}{\otimes} \cdots \underset{A_{p_1+p_2-1}}{\otimes} M_{p_1+p_2}) \underset{A_{p_1+p_2}}{\otimes} \cdots \otimes (M_{p_1+\cdots+p_{k-1}+1} \otimes \cdots \otimes M_{p_1+\cdots+p_k})$$

$$\cong M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{p_1+\cdots+p_k-1}}{\otimes} M_{p_1+\cdots+p_k}.$$

(e) For $n = 2$ this notion of a tensor product agrees with the one introduced previously.

5.2. Tensor algebra of an A-module. *Let* M *be a module for a commutative ring* A *with a unit. We will denote the* n*-tuple tensor product* $M \otimes_A \cdots \otimes_A M$ *by* $T_A^n(M) = \overset{n}{\otimes}_A M \overset{def}{=} M^{\otimes n}$. *For* $n = 0$ *this is - by definition -* A *itself (so it does not depend on* M*). For* $n = 1$ *this is the module* M.

(a) Show that $T(M) \overset{\text{def}}{=} \sum_{n \geq 0} T^n(M)$ has a unique structure of an associative A-algebra, such that for all $p, q \geq 0$ and $m_i, n_j \in M$,

$$(m_1 \otimes \cdots \otimes m_p) \cdot (n_1 \otimes \cdots \otimes n_q) = m_1 \otimes \cdots \otimes m_p \otimes n_1 \otimes \cdots \otimes n_q.$$

For this algebra structure structure $m_1 \otimes \cdots \otimes m_p$ is the product $m_1 \cdots m_p$ of $m_i \in M^{\otimes 1} = M$.

(b) Universal property of the tensor algebra. For each A-algebra B restriction

$$\mathrm{Hom}_{assoc.\ A-alg.\ with\ 1}(TM, B) \ni \phi \mapsto \phi|M \in \mathrm{Hom}_{A-modules}(M, B),$$

is a bijection.
[*We say that* $T(M)$ *is that* A*-algebra defined by the* A*-module* M*, or that* $T(M)$ *is universal among* A*-algebras* B *endowed with a map of* A*-modules* $M \to B$*.*]

(c) Algebra $A(M, \mathcal{R})$ defined by an A-module M and the relations $\mathcal{R}$. What we mean by algebraic relations between elements of an A-module M are intuitively the conditions of type

$$(*)\quad \sum_{i=1}^{k} a_i{\cdot}m_{i,1}{\cdot}\cdots{\cdot}m_{i,n_i}=0$$

for some $a_i\in A,\ m_{i,j}\in M$. The precise meaning of that is that the expression on the left hand side defines an element $r=\sum_{i=1}^k\ a_i{\cdot}m_{i,1}{\otimes}\cdots{\otimes}m_{i,n_i}$ of the tensor algebra. So any set of such relations defines (i) a subset $\mathcal{R}\subseteq T(M)$, (ii) an A-algebra $A(M,\mathcal{R})\stackrel{\text{def}}{=}T(M)/<\mathcal{R}>$ where $<\mathcal{R}>$ denotes the 2sided ideal in $T(M)$ generated by $\mathcal{R}$, with (iii) a canonical map of A-modules $\iota\stackrel{\text{def}}{=}[M\subseteq T(M)\twoheadrightarrow A(M,\mathcal{R})]$.

Show that for each A-algebra B, $\mathrm{Hom}_{assoc.\ A-alg.\ with\ 1}[A(M,\mathcal{R}),B]$ is naturally identified with the set of all $\beta\in\mathrm{Hom}_{A-modules}(M,B)$, such that for all $r=\sum_{i=1}^k\ a_i{\cdot}m_{i,1}{\otimes}\cdots{\otimes}m_{i,n_i}$ in $\mathcal{R}$, the following relation between (images of) elements of M holds in B: $\sum_{i=1}^k\ a_i{\cdot}\beta(m_{i,1}){\cdot}\cdots{\cdot}\beta(m_{i,n_i})=0$.
[*Therefore, an algebraic relation of type* $(*)$ *between elements of* M *acquires meaning in any algebra* B *supplied with a map of* A*-modules* $M{\to}B$. *Algebra* $A(M,\mathcal{R})$ *is universal among all such* A*-algebras* B *that satisfy relations from* $\mathcal{R}$.]

5.3. Right exactness of tensor products. (a) Let A be a ring, L a right A-module and $0{\to}M'{\to}M{\to}M''{\to}0$ a short exact sequence of left A-modules. Show that there is a short exact sequence of abelian groups $L{\otimes}_AM'{\to}L{\otimes}_AM{\to}L{\otimes}_AM''{\to}0$. Find an example when the sequence $0{\to}L{\otimes}_AM'{\to}L{\otimes}_AM{\to}L{\otimes}_AM''{\to}0$ is not exact, i.e., $L{\otimes}_AM'\not\subseteq L{\otimes}_AM$.

(b) Let A be a ring, L a right A-module and M a left A-module. Show that any algebra morphism $\phi:B{\to}A$, gives a surjective map $L\underset{B}{\otimes}M{\to}L{\otimes}_AM$, with the kernel generated by elements of the form $x{\cdot}a{\otimes}y-x{\otimes}a{\cdot}y,\ x\in L,\ y\in M,\ a\in A$.

MULTI-LINEAR ALGEBRA: Exterior and Symmetric algebras of a module

5.4. Exterior algebra of an A-module. *Let* M *be a module for a commutative ring* A *with a unit. The exterior algebra* $\overset{\bullet}{\wedge}M=\overset{\bullet}{\underset{A}{\wedge}}M$ *is the associative* A*-algebra with* 1 *generated by* M *and by anti-commutativity relations* $\mathcal{R}=\{x{\otimes}y+y{\otimes}x,\ x,y\in M\}$. *The multiplication operation in* $\overset{\bullet}{\wedge}M$ *is denoted* $\wedge$, *so that the image of* $m_1{\otimes}\cdots{\otimes}m_n\in T(M)$ *in* $\overset{\bullet}{\wedge}M$ *is denoted* $m_1{\wedge}\cdots{\wedge}m_n\in\overset{n}{\wedge}M$.

(a) $\overset{\bullet}{\wedge}M$ is a graded algebra. Show that (a_0) the ideal $I=<\mathcal{R}>$ in $T(M)$ is homogeneous, i.e., $I=\oplus_{n\ge0}\ I^n$ for $I^n\stackrel{\text{def}}{=}\ I\cap T^nM$. Show that the quotient algebra $\overset{\bullet}{\wedge}M=T(M)/I$ satisfies (a_1) $\overset{\bullet}{\wedge}M\cong\oplus_{n\ge0}\overset{n}{\wedge}M$ for $\overset{n}{\wedge}M\stackrel{\text{def}}{=}T^n(M)/I^n$; and (a_2) $\overset{\bullet}{\wedge}M$ is a graded algebra, i.e., $\overset{n}{\wedge}M\cdot\overset{m}{\wedge}M\subseteq\overset{n+m}{\wedge}M,\ n,m\ge0$.

5.5. Universal property of the exterior algebra. Show that for each A-algebra B, $\mathrm{Hom}_{assoc.\ A-alg.\ with\ 1}[\overset{\bullet}{\wedge}M,B]$ can be identified with a set of all

$\phi : \mathrm{Hom}_{A-moduli}(M,B)$, such that the ϕ-images of elements of M anti-commute in B, i.e., $\phi(y)\phi(x) = -\phi(x)\phi(y),\ x,y \in M$.

5.6. Basic properties of exterior algebras. (a) Low degrees. $\overset{0}{\wedge} M = T^0(M) = A$ and $\overset{1}{\wedge} M = T^1(M) = M$.

(b) Bilinear forms extend to exterior algebras. For any A-modules L and M, and any A-bilinear map $< \ , \ >: L{\times}M{\rightarrow}A$ (i.e., linear in each variable); there is a unique A-bilinear map $< \ , \ >: \overset{n}{\wedge} L{\times}\overset{n}{\wedge} M{\rightarrow}A$, such that $< l_1{\wedge}\cdots\langle_n, m_1{\wedge}\cdots{\wedge}m_n >= \det(< l_i, m_j >)$.

(c) Free modules If M is a free A-module with a basis $e_1, ..., e_d$, then $\overset{k}{\wedge} M$ is a free A-module with a basis $e^J = e_{j_1}{\wedge}\cdots{\wedge}e_{j_k}$, indexed by all subsets $J = \{j_1 < \cdots < j_n\}{\subseteq}I$ with k elements.

(d) Dimension. $\dim(\overset{\bullet}{\wedge}\ \mathbb{C}^n) = 2^n$.

5.7. Symmetric algebra of an A-module. *Let M be a module for a commutative ring A with a unit. The symmetric algebra $S(M) = S_A(M)$ of M is the algebra generated by M and the relations $\mathcal{R} = \{x{\otimes}y - y{\otimes}x,\ x,y \in M\}$.*

(a) $S(M)$ is a graded algebra. Show that

- (a_0) the ideal $I =< \mathcal{R} >$ in $T(M)$ is homogeneous and $S(M) = T(M)/I$ satisfies
- (a_1) $S(M) \cong \oplus_{n\geq 0}\ S^n(M)$ for $S^n(M) \overset{\text{def}}{=} T^n(M)/I^n$; and
- (a_2) $S(M)$ is a graded algebra.

5.8. Universal property of the symmetric algebra. (a) Show that (i) $S(A)$ is commutative, (ii) for any commutative A-algebra B, $\mathrm{Hom}_{assoc.\ A-alg.\ with\ 1}[S(M), B]$ can be identified with $\mathrm{Hom}_{A-modules}(M,B)$.

5.9. Basic properties of symmetric algebras. (a) $T^0(M) = S^0(M) = A$ and $T^1(M) = S^1(M) = M$.

(b) If M is a free A-module with a basis $e_i,\ i \in I$, then $S^n(M)$ is a free A-module with a basis $e^J = \prod_{i\in I}\ e_i^{J_i}$, indexed by all maps $J : I{\rightarrow}\mathbb{N}$ with the integral n.

(c) The algebra of polynomials $\mathbb{C}[x_1, ..., x_n]$ is isomorphic to the symmetric algebra $S(\mathbb{C}x_1{\oplus}\cdots{\oplus}\mathbb{C}x_n)$.

The next topic (Koszul resolution) will use both symmetric and exterior algebras.

Homework 6

6.0. Koszul complex. Koszul complex of the vector space V is the complex

$$\cdots \xrightarrow{d^{-k-1}} \overset{k}{\wedge} V \otimes S(V) \xrightarrow{d^{-k}} \cdots \xrightarrow{d^{-3}} \overset{-2}{\wedge} V \otimes S(V) \xrightarrow{d^{-2}} \overset{k}{\wedge} V \otimes S(V) \xrightarrow{d^{-1}} \overset{0}{\wedge} V \otimes S(V) \to 0 \to \cdots$$

If $\dim(V) = n$ it is finite

$$\cdots \to 0 \to \overset{n}{\wedge} V \otimes S(V) \xrightarrow{d^{-n}} \cdots \xrightarrow{d^{-3}} \overset{-2}{\wedge} V \otimes S(V) \xrightarrow{d^{-2}} V \otimes S(V) \xrightarrow{d^{-1}} V \otimes S(V) \to 0 \to \cdots.$$

(a) Show that the maps d^{-k} are well defined by the formula (here $v_i \in V,\ f \in S(V)$ and $\widehat{\ }$ means that we are omitting this term)

$$d^{-k}(v_1 \wedge \cdots \wedge v_k \otimes f) = \sum_1^k (-1)^{k-i} v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k \otimes v_i f.$$

(b) Show that this is a complex.

(c) Consider the Koszul complex for $V = \mathbb{C}$ and $V = \mathbb{C}^2$, where have you used it before?

6.1. An example of inductive limit. Let $(I, \leq)$ be $\{1, 2, 3, ...\}$ with the order $i \leq j$ if i divides j. In $\mathcal{A}b$ let $A_i = \mathbb{Q}/\mathbb{Z}$ for all $i \in I$, and let ϕ_{ji} be the multiplication by j/i when i divides j. This is an inductive system and $\lim\limits_{\rightarrow} A_i =?$.

6.2. An example of projective limit. Let $(I, \leq)$ be $\mathbb{N} = \{0, 1, ...\}$ with the standard order. In $\mathcal{R}ings$ let $\Bbbk_n = \mathbb{C}[x]/x^{n+1}$ and for $i \leq j$ let ϕ_{ij} be the obvious quotient map. This is a projective system and $\lim\limits_{\leftarrow} \Bbbk_i =?$.

6.3. Passage to a final subset in an inductive limit. A subset K of a partially ordered set is called final if for each $i \in I$ there is some $k \in K$ with $i \leq k$. Let $(c_i,\ i \in I; \phi_{ji},\ i \leq j)$ be an inductive system in a category $\mathcal{C}$. Show that if

- $K \subseteq I$ is final and
- for any $i', i'' \in I$ there is some $i \in I$ that dominates both,

the canonical map

$$\lim_{\substack{\rightarrow \\ k \in K}} c_k \xrightarrow{\iota} \lim_{\substack{\rightarrow \\ i \in I}} c_i$$

is an isomorphism. (More precisely, if one of these objects exists so does the other and there is a canonical isomorphism.)

6.4. Fibered products of sets. (a) If in the diagram of sets $A \xrightarrow{\subseteq} S \xleftarrow{\supseteq} B$, S is a point, then the fibered product $A \underset{S}{\times} B$ is just the usual product $A \times B$.

(b) If $A \xrightarrow{\subseteq} S \xleftarrow{\supseteq} B$ are inclusions of subsets of S the fibered product $A \underset{S}{\times} B$ is just the intersection $A \cap B$.

(c) In $\mathcal{S}ets$ fibered products exist and the fibered product of $A \xrightarrow{p} S \xleftarrow{q} B$ is the set $A \underset{S}{\times} B =$?.

6.5. Use of coordinates in Linear Algebra. Let $\Bbbk$ be a field and $\mathcal{V}_\Bbbk$ the category such that $Ob(\mathcal{V}_k) = \ \mathbb{N} = \{0, 1, ...\}$ and $\mathrm{Hom}(n, m) = M_{mn}$ (matrices with m rows and n columns. Let ι be the functor $\mathcal{V}_\Bbbk \xrightarrow{\iota} \mathcal{V}ect_\Bbbk^{fg}$ into finite dimensional vector spaces, given by $\iota(n) = \Bbbk^n$ and for a matrix $\alpha \in M_{mn}$, $\iota_\alpha : \ \Bbbk^m \to \Bbbk^n$ is the multiplication by α.

(a) Show that ι is an equivalence of categories.

(b) Let us choose for each finite dimensional vector space V a basis $e_1^V, ..., e_{dim(V)}^V$. Use this to construct a functor G which is both left and right adjoint to ι. (It plays the role of the inverse of ι.)

6.6. Construction of projective limits of sets. Let $(I, \leq)$ be a partially ordered set such that for any $i, j \in I$ there is some $k \in I$ such that $i \leq k \geq j$. Let $(A_i)_{i \in I}$ and maps $(\phi_{ij} : A_j \to A_i)_{i \leq j}$ be a projective system of sets. Then

$\lim\limits_{\leftarrow} A_i$ is the subset of $\prod\limits_{i \in I} A_i$ consisting of all families $a = (a_i)_{i \in I}$ such that $\phi_{ij} a_j = a_i$.]

6.7. Construction of inductive limits of sets. Let $(I, \leq)$ be a partially ordered set such that for any $i, j \in I$ there is some $k \in I$ such that $i \leq k \geq j$. Let $(A_i)_{i \in I}$ and maps $(\phi_{ji} : A_i \to A_j)_{i \leq j}$ be an inductive system of sets.

(a) Show that the relation $\sim$ defined on the disjoint union $\sqcup_{i \in I} A_i \stackrel{\text{def}}{=} \cup_{i \in I} \ A_i \times \{i\}$ by

- $(a, i) \sim (b, j)$ (for $a \in A_i, \ b \in A_j$), if there is some $k \geq i, j$ such that "$a = b$ in A_k", i.e., if $\phi_{ki} a = \phi_{kj} a$,

is an equivalence relation.

(b) Show that $\lim\limits_{\rightarrow} A_i$ is the quotient $[\sqcup_{i \in I} \ A_i] / \sim$ of the disjoint union by the above equivalence relation.

Homework 7

7.1. Functoriality of modules under the change of rings. For a map of rings $\phi : \Bbbk \to \langle$ show that the direct image of modules $\phi_* M = \langle \otimes_\Bbbk M$ has a right adjoint.

7.2. Representable functors. Find functors which are (co)represented by an initial (resp. final) object.

7.3. Direct image of sheaves. Let $X \xrightarrow{\pi} Y$ be a map of topological spaces.

(a) Show that for a sheaf $\mathcal{M}$ on X, formula

$$\pi_*(\mathcal{M})\ (V) \stackrel{\text{def}}{=} \mathcal{M}(\pi^{-1}V),$$

defines a sheaf $\pi_*\mathcal{M}$ on Y.

(b) Show that this gives a functor $\mathcal{S}heaves(X) \xrightarrow{\pi_*} \mathcal{S}heaves(Y)$.

7.4. Examples of sheaves. (a) Let $\phi : Y{\to}X$ be a continuous map of topological spaces. Denote $\mathcal{S}(U)$ the set of all sections over U i.e. all continuous maps $s : U{\to}Y$ such that $s(x) \in Y_x \stackrel{\text{def}}{=} \phi^{-1}x,\ x \in U$. Show that $\mathcal{S}$ is a sheaf of sets.

(b) Which of the following classes of functions on $\mathbb{R}$ form a presheaf, a sheaf: (0) all functions, (1) continuous functions, (2) smooth functions, (3) even functions, (4) L^2 functions, (5) compactly supported continuous functions, (6) constant functions. Why?

(c) Let $p : V{\to}M$ be a holomorphic vector bundle and denote the set of holomorphic sections over an open set $U \subset M$ by $\mathcal{V}(U)$. Show that $\mathcal{V}$ is a sheaf of complex vector spaces.

(d) Let M be a complex manifold and $\mathcal{O}_M(U)$ the holomorphic functions on U. Show that $\mathcal{O}_M$ is a sheaf of algebras on M.

7.5. Examples of maps of sheaves. Consider the following sheaves of vector spaces on M: constant sheaf $\mathbb{C}_M$ and the zero sheaf $0 = \{0\}_M$ (if M is any topological space), sheaf $\mathcal{C}^\infty_M$ of smooth functions on M (if M is open in $\mathbb{R}^n$), sheaf of holomorphic functions $\mathcal{H}_M$ (if M is open in $\mathbb{C}$).

(1) If M is open in $\mathbb{C}$ define canonical maps of sheaves of vector spaces: $0_M \xrightarrow{a} \mathbb{C}_M \xrightarrow{b} \mathcal{C}^\infty_M \xrightarrow{c} \mathcal{O}_M$.
(2) Which of the following are maps of sheaves of vector spaces (and why?):
   (a) (Differentiation) $\mathcal{C}^\infty_M(U) \ni f \mapsto f' \in \mathcal{C}^\infty_M(U)$,
   (b) (Squaring) $\mathcal{C}^\infty_M(U) \ni f \mapsto f^2 \in \mathcal{C}^\infty_M(U)$,
   (c) (multiplication by a function) $\mathcal{O}_M(U) \ni f \mapsto zf \in \mathcal{O}_M(U)$,
   (d) (translation by 1) $\mathcal{C}^\infty_M(U) \ni f(x) \mapsto f(x+1) \in \mathcal{C}^\infty_M(U)$.

7.6. Kernels, images and quotients of maps of sheaves. Let $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ be one of the following maps of sheaves of vector spaces:

- (a) $M = \mathbb{R}$ and $\alpha = \frac{\partial}{\partial x}$, i.e., $\alpha_U : \mathcal{C}^\infty_M(U){\to}\mathcal{C}^\infty_M(U)$ is given by $f \mapsto f'$;
- (b) $M = \mathbb{C}$ and $\alpha = z$, i.e., $\alpha_U : \mathcal{O}_M(U){\to}\mathcal{O}_M(U)$ by $f \mapsto zf$.

In both cases

(1) Show that $\mathcal{K}(U) = Ker[\mathcal{A}(U) \xrightarrow{\alpha_U} \mathcal{B}(U)]$ is a sheaf and a subsheaf of $\mathcal{A}$, and it is the kernel of α,
(2) Show that $I(U) = Im[\mathcal{A}(U) \xrightarrow{\alpha_U} \mathcal{B}(U)]$ is a sub-presheaf of $\mathcal{B}$.
(3) Show that $C(U) = \mathcal{B}(U)/I(U)$ is a presheaf.
(4) Describe the sheaf $\mathcal{K}$.
(5) Calculate the sheaf $\mathcal{C}$ associated to the presheaf C, and show that this is the cokernel of α.
(6) Calculate the subsheaf $\mathcal{I}$ of $\mathcal{B}$ associated to I, and show that this is the image of α.

(Describe the sections $\mathcal{I}(U), \mathcal{Q}(U)$ and the stalks $\mathcal{I}_x, \mathcal{Q}_x$, as well as you can, for instance find the dimensions of these vector spaces.)

7.7. Stalks. (a) Show that any map of sheaves $\phi : \mathcal{A} \to \mathcal{B}$ defines for each $x \in M$ a map of stalks $\mathcal{A}_x \to \mathcal{B}_x$.

(b) Show that maps of sheaves $\phi, \psi : \mathcal{A} \to \mathcal{B}$ are the same iff the maps on stalks are the same, i.e., $\phi_x = \psi_x$ for each $x \in M$.

(c) Map of sheaves $\phi : \mathcal{A} \to \mathcal{B}$ is an isomorphism iff ϕ_x is an isomorphism for each $x \in M$.