Algebraic Geometry Homeworks

The grade will be based mostly on homeworks, possibly some extra project.

Do as much as you can. At this level one does not require perfection but an honest effort that should help you *learn*. The presentation should be readable, but the level of detail should be sufficient to explain the situation to yourself.

It is acceptable to state that you understand a problem on the level that it is a waste of time for you to write it down, but be sure you do not cheat yourself.

Difficulties: Office hours, we can also have homework help sessions when desired.

It is a very good idea to discuss the problems that resist reasonable effort with other students, however it is *critical* that you write solutions by yourself.

Looking into books I have or have not mentioned may help.

If a problem require expertise in something you did not learn yet (say, manifolds), it is *acceptable* to state so, and not do the problem. It is *preferable* to do the problem anyway by consulting other students or me.

Homework 1

♡[1]

1.1. Explain how the following look like (i.e., how do the obvious pieces fit together into a geometric shape):

(1) $\mathbb{P}^1_{\mathbb{R}}$,
(2) $\mathbb{P}^2_{\mathbb{R}}$,
(3) $\mathbb{P}^1(\mathbb{C})$.

♡

The blow up. The blow up of the vector space V is the set

$$\widetilde{V} \overset{\text{def}}{=} \{(L,v) \in \mathbb{P}(V)\times V;\ v \in L\} \subseteq \mathbb{P}(V)\times V.$$

Since it is a subset of $\mathbb{P}(V)\times V$ it comes with the projection maps

$$\mathbb{P}(V) \xleftarrow{\pi} \widetilde{V} \xrightarrow{\mu} V.$$

1.2. Explain the relation of $\widetilde{V}$ to $\mathbb{P}(V)$ and V, i.e.,

(1) Show that the fibers $\pi^{-1}(L),\ L \in \mathbb{P}(V)$ of $\pi : \widetilde{V} \to \mathbb{P}(V)$ are naturally vector spaces. (We say that $\widetilde{V}$ is a *vector bundle* over $\mathbb{P}(V)$.
(2) Describe the fibers $\mu^{-1}(v),\ v \in V$, of $\widetilde{V} \xrightarrow{\mu} V$.

1.3. Explain how does the blow up $\widetilde{\mathbb{R}^2}$ look like. Can you relate it to the Moebius strip?

♡

Finite Fields. For each prime number, ring $\mathbb{F}_p \overset{\text{def}}{=} \mathbb{Z}/p\mathbb{Z}$ is a field with p elements.[2] The advantage of a finite field $\mathbb{k}$ is that we can do polynomials, affine and projective spaces, affine and projective varieties, but now there is something to count since everything is finite. So we have a new way of measuring the size of algebraic varieties. For instance if we can ask ourselves how many elements there are in the circle $C(\mathbb{k}) = \{(x,y) \in \mathbb{k}^2;\ x^2+y^2=1\}$ over a finite field $\mathbb{k}$ with q elements ?

If we go a little further, and consider not one finite field $\mathbb{F}$ but *all of them*, we arrive at the question

How does the number of elements $X(\mathbb{F})$ depend on $q = |F|$?

[1] Due Thursday Feb 12

[2] Actually, if $q = p^e$ is a power of a prime, then there exists precisely one field with q elements, we denote it $\mathbb{F}_q$. Moreover, these are all finite fields. However, this is not going to be important.

It turns out that for many nice projective varieties the number $|X(\Bbbk)|$ is a polynomial in q. In general, we encode these numbers into the Zeta function[3] of X,

$$Z_X(T) \stackrel{\text{def}}{=} e^{\sum_1^\infty |X(\mathbb{F}_{q^n})| \cdot \frac{T^n}{n}}.$$

This function turns out to be a a rational function of the variable T which contains deep information about X.[4] Because of this, the passage from $\mathbb{C}$ to a finite field is often the strongest known method for studying algebraic varieties over $\mathbb{C}$!

Grassmannians. For a vector space V over a field $\Bbbk$ we denote by $Gr_p(V)$ the set of all p-dimensional vector subspaces of V. For instance, $Gr_1(V)$ is the projective space $\mathbb{P}(V)$. Since $Gr_p(V)$ really depends only on the dimension of V, we often look at the standard examples

$$Gr_p(n) \stackrel{\text{def}}{=} Gr_p(\Bbbk^n).$$

1.4. Consider a finite field $\mathbb{F}$ with q elements.

(1) Find the number of elements in the affine space $\mathbb{A}^n(\mathbb{F})$.
(2) Find the number of elements $N_1(q)$ in $\mathbb{P}^{n-1}(\mathbb{F}) = Gr_1(\mathbb{F}^n)$, i.e., the number of one-dimensional subspaces in $\mathbb{F}^n$.
(3) Find the number of elements $N_2(q)$ in $Gr_2(\mathbb{F}^n)$, i.e., the number of two-dimensional subspaces in $\mathbb{F}^n$.
(4) Find the limits $s_i = \lim_{q\to 1} N_i(q)$ for $i = 1, 2$. What does s_i count?

[3]This is a John Cullinan correction of a previous version.

[4]As conjectured by Weil and proved by Weil, Dwork and (the deepest part) by Deligne.

♡

Hints.

1.1. Since we want to understand these on the topological level, we consider the building blocks of these spaces and how do they glue.

1.3. From 1.2 we have two points of view on $\widetilde{\mathbb{R}^2}$: (1) a circle with a bunch of lines, each passing through one point of the circle, (2) replace a point in $\mathbb{R}^2$ by a circle.

1.4. Let $Fr_p(V)$ be the set of all p-tuples $(v_1, ..., v_p)$ of independent vectors in V. Then

$$|Gr_p(\mathbb{F}^n)| = \frac{|Fr_p(\mathbb{F}^n)|}{|Fr_p(\mathbb{F}^p)|}.$$

Numbers $|Fr_p(\mathbb{F}^n)|$ are easy to calculate for $p = 1, 2$.

Homework 2

♡[5]

1. (a) Give an example of a map of rings $\phi : B \to A$ and a maximal ideal $P \subseteq A$ such that $\phi^{-1}P$ is not maximal.

(b) Show that if ϕ is surjective then ϕ^{-1} of a maximal ideal is maximal.

2. (a) In the ring A find two different ideals I and J such that $V_I = V_J$, when

(1) $A = \mathbb{k}[x]$ for $\mathbb{k}$ a field,
(2) $A = \mathbb{Z}$.

(b) The radical of an ideal $I \subseteq A$ is the set

$$\sqrt{I} = \{a \in A;\ a^n \in I \text{ for some } n > 0\}.$$

Show that $\sqrt{I}$ is an ideal and $V_I = V_{\sqrt{I}}$.

♡

Complex curves.

3. Let $X \subseteq \mathbb{A}^2(\mathbb{C})$ be the affine curve

$$y^2 = (x - a_1) \cdots (x - a_{2n})$$

with the numbers a_i distinct. Let $\overline{X} \subseteq \mathbb{P}^2$ be the corresponding projective curve.

(a) What is the boundary $\partial X = \overline{X} - X$ of X in $\overline{X}$?

(b) Draw X with all the necessary explanations.

♡

Zariski topology. Consider a projective space $\mathbb{P}^n$ over a closed field $\mathbb{k}$. Zariski topology on $\mathbb{P}^n$ is defined so that the closed subsets are just the projective subvarieties. We will construct an open cover of $\mathbb{P}^n$ by $n+1$ affine spaces:

4. Let $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n;\ x_i \neq 0$, and let

$$\phi_i :\ \mathbb{A}^n \to U_i,\ \phi_i(a_1, ..., a_n) = [a_1 : \cdots : a_i : 1 : a_{i+1}; \cdots : a_n].$$

Show that

(1) $U_0, ..., U_n$ is an open cover of $\mathbb{P}^n$.
(2) ϕ_i is a bijection.

[5] Due Thursday Feb 26

Homework 3

♡[6]

Projective spaces as manifolds.

1. Prove that

(a) $\mathbb{P}^n(\mathbb{R})$ has a canonical structure of a manifold of dimension n.[7]

(b) $\mathbb{P}^n(\mathbb{C})$ has a canonical structure of a complex manifold of dimension n

(c) Manifolds $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$ are compact.

♡

Theta functions. Let $\tau \in \mathbb{H}$ (i.e., $Im(\tau) > 0$. It gives a lattice $L_\tau = \mathbb{Z}\oplus\mathbb{Z}\cdot\tau$ in $\mathbb{C}$, and an elliptic curve $E_\tau = \mathbb{C}/L_\tau$. It comes with the quotient map $\pi : \mathbb{C} \to E_\tau$.

We would like to find some holomorphic functions on E_τ, and this is the same as a holomorphic function f on $\mathbb{C}$ which is periodic in directions of 1 and τ: $f(z+1) = f(z) = f(z+\tau)$. However, there are no such functions, so we ask for the next best thing: periodic for 1 and *quasiperiodic.*

2. The theta series in $\tau \in \mathbb{H}$ and $u \in \mathbb{C}$ is

$$\theta_\tau(u) \stackrel{\text{def}}{=} \sum_{-\infty}^{+\infty} e^{\pi i(n^2\tau + 2nu)}.$$

(a) Show that it defines for any $\tau \in \mathbb{H}$ an entire function of u.

(b) Show that for any $u \in \mathbb{C}$ it defines a holomorphic function on $\mathbb{H}$.

(c) Show that for any $a \in \mathbb{R}$, $b > 0$, the series converges uniformly on the product

$$\{\tau \in \mathbb{H};\ Im(\tau) > b\} \times \{u \in \mathbb{C};\ Im(u) > a\}.$$

(d) Show that the series can be differentiated any number of times (with respect to τ and u), and the derivatives are calculated term by term.

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[6] Due Thursday March 4

[7] One has to (1) define topology on it, (1) define charts, (3) check that the charts are compatible, i.e., the transition functions are smooth (i.e., infinitely differentiable).

♡

Zariski topology on affine varieties.

3. Consider the Zariski topology on the affine line $\mathbb{A}^1$ over a closed field $\Bbbk$.

(a) What are the open sets of $\mathbb{A}^1$?

(b) Show that any two non-empty open sets in $\mathbb{A}^1$ meet.[8]

(c) Show that $\mathbb{A}^1$ is *quasi-compact*, i.e., any open cover reduces to a finite subcover.[9]

♡

Counting. Let X be a finite set.

- Let $Gr_k(X)$ be the set of all k-element subsets of X.
- For $J = \{j_1 < j_2 \cdots j_k\}$, let $Gr_J(X)$ be the set of partial flags of type J in X, i.e., the set of k-tuples of increasing subsets of correct size

$$Gr_J(X) \stackrel{\text{def}}{=} \{(A_1, ..., A_k) \in \mathcal{P}_{j_1} \times \cdots \mathcal{P}_{j_k},\ A_1 \subseteq \cdots \subseteq A_k\}.$$

Let $n = |X|$ be the number of elements of X. When $J = \{1, ..., n\}$ is the largest possible, we call $Gr_J(X)$ the set of flags $\mathcal{F}(X)$ in X.

4. (a) Find $|Gr_k(X)|$.

(b) For $K \subseteq J$ there is a canonical projection

$$\pi = \pi_{K \subseteq J} : Gr_J \to Gr_K, \qquad (A_j)_{j \in J} \mapsto (A_j)_{j \in K}.$$

Show that it is surjective.

(c) If $J = \{a = j_1 < b = j_2 < \cdots < j_k\} \subseteq \{1, ..., n-1\}$, and $K = J - \{a\}$, calculate the number of elements in any fiber of π.

(d) Find $|Gr_J(X)|$ for $J = \{j_1 < j_2 < \cdots < j_k\}$.

(d) Consider the case when $J = \{1, ..., n\}$ is the largest possible.

(1) Find a canonical identification of $\mathcal{F}(X)$ with the set of all total orders on X.
(2) Show that $\mathcal{F}(X)$ is a torsor for the group S_X of permutations of X.[10]
(3) Use this to (re)calculate $|S_X|$.

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[8]Such topological spaces are called *irreducible.*
[9]In this terminology "compact" means quasi-compact and Hausdorff.
[10]This means that S_X acts simply transitively on $\mathcal{F}(X)$ for the group $|S_X|$.

Homework 4

♡[11]

Symmetric powers. The n^{th} symmetric power of an affine scheme X over a closed field $\Bbbk$,[12] is defined as the affine variety $X^{(n)}$ such that

$$\mathcal{O}(X^{(n)}) \stackrel{\text{def}}{=} \mathcal{O}(X^n)^{S_n}.$$

This definition is a bit abstract so we want to understand $X^{(n)}$ as a set, and in particular we would like to compare the set $X^{(n)}$ with the set $X^n/S_n \stackrel{\text{def}}{=}$ all S_n-orbits in X^n. Here, we will see that these are the same when $X = \mathbb{A}^1$.[13]

1. Let $X = \mathbb{A}^1$. Show that

(1) To any S_n-orbit α in X^n one can associate a polynomial

$$\chi_\alpha(\lambda) \stackrel{def}{=} \prod_i (\lambda - a_i),\ \textit{where}\ a = (a_1, ..., a_n)\ \textit{is}\ \text{any}\ \textit{element of}\ \alpha.$$

(2) Denote the coefficients of this polynomial by

$$\chi_\alpha(\lambda) \stackrel{\text{def}}{=} \lambda^n - e_1(\alpha)\lambda^{n-1} + e_2(\alpha)\lambda^{n-2} - \cdots + (-1)^n e_n(\alpha)\lambda^0.$$

So, e_i's are functions on the set of orbits X^n/S_n, and therefore in particular on X^n. Show that e_i's are polynomials on $X^n = \mathbb{A}^n$.

(3) Show that $e = (e_1, ..., e_n) : X^n/S_n \to \mathbb{A}^n$ is a bijection.

(4) Prove that there is an isomorphism of algebraic varieties $X^{(n)} \cong \mathbb{A}^n$ and that as a set, $X^{(n)}$ consists of S_n-orbits in X^n.

♡

Theta functions.

2. (a) $\theta_\tau(u+1) = \theta_\tau(u)$.

(b) $\theta_\tau(u+\tau) = e^{-\pi i(\tau+2u)} . \theta_\tau(u)$.

(c) $\theta_\tau(-u) = \theta_\tau(u)$.

3. (a) θ_τ has precisely one zero in in the closed parallelogram $\overline{\mathcal{P}}_\tau$ generated by vectors $1, \tau$ in the real vector space $\mathbb{C}$:

$$\mathcal{P}_\tau \stackrel{\text{def}}{=} \{a + b\tau;\ 0 < a, b < 1\}.$$

(b) This zero is at $u_0 \stackrel{\text{def}}{=} \frac{\tau+1}{2}$.

♡

[11] Due Thursday March 11

[12] For simplicity we will assume that the characteristic n of $\Bbbk$ is zero.

[13] The trouble is that we know examples of Invariant Theory quotients do not work well in the sense that, as a set, $Y//G$ is not the set of orbits Y/G. For instance, $G_m = \Bbbk^*$ has two orbits on $\mathbb{A}^1$ but $\mathbb{A}^1//G_m$ is just a point.

Linear counting ("Quantum computing"). We will see that counting subspaces gives an interesting deformation of the standard combinatorics which counts subsets.

Let $\mathbb{F}$ be a finite field with q elements. For the vector space $V = \mathbb{F}^n$, we will consider the sets $Fr_k(V)$ of k-tuples of independent vectors $v = (v_1, ..., v_k)$ in V (the set of "k-frames" in V), and the related set $Gr_k(V)$ of all k-dimensional subspaces of V (this is the k^{th} *Grassmannian variety of* V).

The n^{th} q-integer is the polynomial

$$[n] \stackrel{\text{def}}{=} \frac{\boldsymbol{q}^n - 1}{\boldsymbol{q} - 1} = 1 + \boldsymbol{q} + \cdots + \boldsymbol{q}^{n-1} \in \mathbb{Z}[\boldsymbol{q}], \qquad n \in \mathbb{Z}.$$

Similarly one defines quantum factorials and quantum binomial coefficients

$$[n]! \stackrel{\text{def}}{=} [1] \cdot \cdot \cdot [n] \quad \text{and} \quad \begin{bmatrix} n \\ m \end{bmatrix} \stackrel{\text{def}}{=} \frac{[n]!}{[m]! \cdot [n-m]!}.$$

When we evaluate these polynomials at $\boldsymbol{q} = q$ we put index q (but we often forget to write it). For instance we know that

$$[n]_q \stackrel{\text{def}}{=} [n]|_{\boldsymbol{q}=q}$$

is the number of elements in $\mathbb{P}(\mathbb{F}^n)$.[14]

4. (a) Find $|Fr_k(\mathbb{F}^n)|$.

(b) Show that $|GL_n(\mathbb{F})| = |Fr_n(\mathbb{F}^n)| = q^{\binom{n}{2}} \cdot (q-1)^n \cdot [n]!$.

(c) Find $|Gr_k(\mathbb{F}^n)|$, i.e., the number of k-dimensional subspaces in $\mathbb{F}^n$.

(d) Find the limit

$$\lim_{q \to 1} |Gr_k(n, \mathbb{F})|$$

(e) What does this limit count? Complete the following intuitive claim, i.e., which *notion* do you think is the limit of the *notion* of a k-dimensional subspace of $\mathbb{F}^n$ as the number of elements of the field $\mathbb{F}$ approaches 1 :

$$\lim_{q \to 1} k\text{-dimensional subspaces of } \mathbb{F}^n \quad = \quad ....$$

[14] and this is our motivation for interest in q-integers.

Homework 5

♡[15]

Elliptic functions. Recall that each $\tau\in\mathbb{H}$ defines the function $\theta_\tau(u)$ on $\mathbb{C}$. The Weierstrass $\mathfrak{p}$-function is a meromorphic function on $\mathbb{C}$ which we will define as the second logarithmic derivative of the theta function

$$\mathfrak{p}_\tau(u) \stackrel{\text{def}}{=} (log(\theta_\tau(u))''.$$

Subgroup $L_\tau=\mathbb{Z}\oplus\tau{\cdot}\mathbb{Z}\subseteq\mathbb{C}$ is a lattice in $\mathbb{C}$.

1. (a) Explain why $\mathfrak{p}_\tau(u) \stackrel{\text{def}}{=} (log(\theta_\tau(u))''$ is a well defined holomorphic function on

$$\mathbb{C}\backslash\ (\tfrac{1+\tau}{2}+L_\tau),\ \ \textit{i.e., off the } L_\tau\textit{-translates of the point } \tfrac{1+\tau}{2}.$$

(b) Show that $\mathfrak{p}_\tau$ is L_τ invariant, i.e., $\mathfrak{p}_\tau(z+1)=\mathfrak{p}_\tau(z)=\mathfrak{p}_\tau(z+\tau)$.

(c) Show that $\mathfrak{p}_\tau$ has a pole of order two at $\frac{1+\tau}{2}$.

(d) Show that $\mathfrak{p}_\tau$ is meromorphic on $\mathbb{C}$.

♡

Tensoring of commutative algebras and fibered products of varieties. We consider commutative algebras over a closed field $\mathbb{k}$.

2. Let $B=\mathbb{k}[u,v]\xrightarrow{\phi}A=\mathbb{k}[x,y]$ by $u\mapsto x,\ v\mapsto xy$. Each $p=(a,b)\in\mathbb{A}^2$ defines $B=\mathbb{k}[u,v]\xrightarrow{\varepsilon}\mathbb{k}$ by $u\mapsto a,\ v\mapsto b$. These two maps can be used to think of A and $\mathbb{k}$ as B-algebras, and then $A\otimes_B\mathbb{k}$ is again an algebra over B (hence in particular over $\mathbb{k}$).

(a) Show that for each p the $\mathbb{k}$-algebra $A\otimes_B\mathbb{k}$ is isomorphic to one of the following algebras $\mathbb{k}[z]$ or $\mathbb{k}$ or 0.

(b) What is the geometric meaning of this?

[15] Due Thursday March 25

Homework 6

♡[16]

Sheaves. Sheaves are a machinery which addresses an essential problem – the relation between local and global information – so they appear throughout mathematics.

A. Example of a sheaf: smooth functions on $\mathbb{R}$. Let X be $\mathbb{R}$ or any smooth manifold. The notion of smooth functions on X gives the following data:

- for each open $U\subseteq X$ an algebra $C^\infty(U)$ (the smooth functions on U),
- for each inclusion of open subsets $V\subseteq U\subseteq X$ a map of algebras $C^\infty(U) \xrightarrow{\rho^U_V} C^\infty(V)$ (the restriction map);

and these data have the following properties

(1) (*transitivity of restriction*) $\rho^U_V\circ\rho^U_V = \rho^U_W$ for $W\subseteq V\subseteq U$,
(2) (*gluing*) if the functions $f_i \in C^\infty(U_i)$ on open subsets $U_i\subseteq X$, $i \in I$, are compatible in the sense that $f_i = f_j$ on the intersections $U_{ij} \overset{\text{def}}{=} U_i \cap U_j$, then they glue into a unique smooth function f on $U = \cup_{i\in I} U_i$.

The context of dealing with objects which can be restricted and glued compatible pieces is formalized in the notion of sheaves. The definition is formal (precise) way of saying that a given class $\mathcal{C}$ of objects forms a sheaf if it is *defined by local conditions*, i.e., conditions which can be checked in a neighborhood of each point:

B. Definition of sheaves on a topological space. A sheaf of sets $\mathcal{S}$ on a topological space $(X,\mathcal{T})$ consists of the following data:

- for each open $U\subseteq X$ a set $\mathcal{S}(U)$,
- for each inclusion of open subsets $V\subseteq U\subseteq X$ a map $\mathcal{S}(U) \xrightarrow{\rho^U_V} \mathcal{S}(V)$ (called the restriction map);

and these data are required to satisfy

(1) (*identity*) $\rho^U_U = id_{\mathcal{S}(U)}$.
(2) (*transitivity of restriction*) $\rho^V_W\circ\rho^U_V = \rho^U_W$ for $W\subseteq V\subseteq U$,
(3) $\mathcal{S}(\emptyset) = \emptyset$.
(4) (*gluing*) Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of an open $U\subseteq X$. For a family of elements $f_i \in \mathcal{S}(U_i)$, $i \in I$, compatible in the sense that $\rho^{U_i}_{U_{ij}} f_i = \rho^{U_i}_{U_{ij}} f_j$ in $\mathcal{S}(U_{ij})$ for $i, j \in I$; there is a unique $f \in \mathcal{S}(U)$ such that on the intersections $\rho^{U_i}_{U_{ij}} f = f_i$ in $\mathcal{S}(U_i)$, $i \in I$.

We can equally define sheaves of abelian groups, rings, modules, etc – only the least interesting requirement has to be modified, say in abelian groups we would ask that $\mathcal{S}(\emptyset)$ is the trivial group $\{0\}$.

[16]Due Thursday April 8

1.+2. Examples of sheaves. Which of the following constructions are sheaves? (In all examples bellow we are dealing with functions of some kind and the the restriction operation ρ^U_V is always taken to be the restriction of functions.)

(1) On a topological space X, $C_X(U) \stackrel{\text{def}}{=}$ continuous functions from U to $\mathbb{R}$.
(2) If X is a smooth manifold, $C^\infty_X(U) \stackrel{\text{def}}{=}$ smooth functions from U to $\mathbb{R}$.
(3) On a complex manifold, $\mathcal{O}^{an}_X \stackrel{\text{def}}{=}$ holomorphic functions from U to $\mathbb{C}$.
(4) Let X be a topological space and S a set. Let $S^X(U) \stackrel{\text{def}}{=}$ the set of constant functions from U to S.
(5) Let X be a topological space and S a set. Let $S_X(U) \stackrel{\text{def}}{=}$ the set of *locally* constant functions from U to S.
(6) Let $X = \mathbb{R}$ and $C_c(U) \stackrel{\text{def}}{=}$ continuous functions f from U to $\mathbb{R}$ such that the support is compact. (The support $supp(f)$ can be defined as $U - V$ for the largest open subset $V \subseteq U$ such that $f|V = 0$, or as the closure in U of $\{x \in U;\ f(x) \neq 0\}$).
(7) Let $Y \xrightarrow{\pi} X$ be a continuous map between two topological spaces. For U open in X let $\mathcal{Y}(U)$ be the set of all *continuous sections of the map* π *over* U, i.e., of all continuous maps $s : U \to Y$ such that $\pi \circ s = id_U$.[17]

Global sections functor $\Gamma : \mathcal{S}heaves(X) \to \mathcal{S}ets$**.** Elements of $\mathcal{S}(X)$ are called the sections of a sheaf $\mathcal{S}$ on $U \subseteq X$ [18]. By $\Gamma(X, \mathcal{S})$ we denote the set $\mathcal{S}(X)$ of global sections.[19]

The construction $\mathcal{S} \mapsto \Gamma(\mathcal{S})$ means that we are looking at global objects of a given class $\mathcal{S}$ (for some class of objects $\mathcal{S}$ which defined by local conditions).[20]

For instance, on any smooth manifold X, $\Gamma(C^\infty) = C^\infty(X)$ is huge while on a compact complex manifold M we will see that $\Gamma(M, \mathcal{O}^{an}_M) = \mathbb{C}$, so the holomorphic situation is more subtle.

3. Global functions on $\mathbb{P}^1(\mathbb{C})$**.** Show that $\Gamma(\mathbb{P}^1, \mathcal{O}^{an}_{\mathbb{P}^1}) = \mathbb{C}$, i.e., all global holomorphic functions are constant.

4. Line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(-1)$ **on** $\mathbb{P}^1$**.** Let $V = \mathbb{C}^2$. Recall that the blow up $\widetilde{V}$ of V lies in the product $\mathbb{P}^1 \times V$, so it comes with the maps $\mathbb{P}^1 \xleftarrow{\pi} \widetilde{V} \xrightarrow{\mu} V$.

(1) Describe natural structures of a complex manifold on $\mathbb{P}^1$ and $\widetilde{V}$.
(2) Show that map π is holomorphic.
(3) Show that if one associates to each open $U \subseteq \mathbb{P}^1$ the set $\mathcal{L}(U)$ of holomorphic sections of π over U, then $\mathcal{L}$ is a sheaf on $\mathbb{P}^1$.[21]

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[17]This is the same as asking that for any $a \in U$, the value $s(a)$ is in the fiber $Y_a \stackrel{\text{def}}{=} \pi^{-1}(a) \subseteq Y$.

[18]this terminology is from classical geometry

[19]The point of the new notation is psychological: we view $\mathcal{S}$ as a variable in the construction $\Gamma(X, -)$ which assigns something to each sheaf on X.

[20]We will see later that this idea has a hidden part, the cohomology $\mathcal{S} \mapsto H^\bullet(X, \mathcal{S})$ of sheaves on X.

[21]When one views the blow up $\widetilde{V}$ as a space over $\mathbb{P}(V)$, it is called *the tautological line bundle on a projective space* $\mathbb{P}(V)$, or *the tautological line subbundle on a projective space.* The specification *subbundle*

Homework 7

♡[22]

Cohomology of a sheaf $\mathcal{A}$ with respect to an open cover $\mathcal{U}$ (Čech cohomology). Cohomology of sheaves is a machinery which deals with the subtle ("hidden") part of the the relation between local and global information. The Čech cohomology is the simplest calculational tool for sheaf cohomology.

Calculation of global section via an open cover. The first idea is to find all global sections of a sheaf by examining how one can glue local sections into global sections. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of a topological space X, we will choose a complete ordering on I[23] We will use finite intersections $U_{i_0,\dots,i_p} \overset{\text{def}}{=} U_{i_0} \cap \cdots \cap U_{i_p}$ with $i_0 < \cdots < i_p$.

To a sheaf of abelian groups $\mathcal{A}$ on X we associate a map of abelian groups (e,

- $C^0(\mathcal{U},\mathcal{A}) \overset{\text{def}}{=} \prod_{i\in I} \mathcal{A}(U_i)$, its elements are systems $f = (f_i)_{i\in I}$, with one section $f_i \in \mathcal{A}(U_i)$ for each open set U_i,
- $C^1(\mathcal{U},\mathcal{A}) \overset{\text{def}}{=} \prod_{\{(i,j)\in I^2;\ i<j\}} \mathcal{A}(U_{ij})$, its elements are systems $g = (g_{ij})_{(i,j)\in I^2}$ of sections $g_{ij} \in \mathcal{A}(U_{ij})$ on all intersections U_{ij}.
- map sends $f = (f_i)_{i\in I} \in C^0$ to $df \in C^1$ with

$$(df)_{ij} \overset{\text{def}}{=} \rho^{U_j}_{U_{ij}} f_j - \rho^{U_i}_{U_{ij}} f_i.$$

Less formally, we usually state it as $(df)_{ij} = f_j|U_{ij} - f_i|U_{ij}$.

1. Show that for any sheaf of abelian groups $\mathcal{A}$ on X

$$\Gamma(\mathcal{A}) \xrightarrow{\cong} \operatorname{Ker}[C^0(\mathcal{U},\mathcal{A}) \xrightarrow{d} C^1(\mathcal{U},\mathcal{A})].$$

♡

Čech complex $C^\bullet(\mathcal{U},\mathcal{A})$. Emboldened, we try more of the same. We want to capture more of the relation between local sections by extending the construction into a sequence of maps of abelian groups

$$C^0(\mathcal{U},\mathcal{A}) \xrightarrow{d^0} C^1(\mathcal{U},\mathcal{A}) \xrightarrow{d^1} C^2(\mathcal{U},\mathcal{A}) \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} C^n(\mathcal{U},\mathcal{A}) \xrightarrow{d^n} C^{n+1}(\mathcal{U},\mathcal{A}) \xrightarrow{d^{n+1}} \cdots.$$

Here,

$$C^n(\mathcal{U},\mathcal{A}) \overset{\text{def}}{=} \prod_{i_0<\cdots<i_n} \mathcal{A}(U_{i_0,\dots,i_n})$$

---

is to remind us that $\widetilde{V} \subseteq \mathbb{P}(V)\times V$ is a vector subbundle of the trivial vector bundle $\mathbb{P}(V)\times V$ on $\mathbb{P}(V)$. The sheaf $\mathcal{L}$ of sections of the tautological line subbundle is often denoted $\mathcal{O}_{\mathbb{P}(V)}(-1)$ on $\mathbb{P}(V)$

[22] Due Thursday April 15

[23] It is not really necessary but it simplifies practical calculations.

consists of all systems of sections on $(n+1)$-tuple intersections. The map d^n (we call it the n^{th} *differential*), creates from $f=(f_{i_0,\dots,i_n})_{I^n}\in C^n$ an element $d^n(f)\in C^{n+1}$, with

$$d^n(f)_{i_0,\dots,i_{n+1}} = \sum_{s=0}^{n+1} (-1)^s f_{i_0,\dots,i_{s-1},i_{s+1},\dots,i_{n+1}}.$$

From this we construct groups of n-*cocycles* and n-*coboundaries*

$$Z^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \operatorname{Ker}(C^n \xrightarrow{d^n} C^{n+1}) \subseteq C^n \quad \text{and} \quad B^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} Im(C^{n-1} \xrightarrow{d^{n-1}} C^n) \subseteq C^n.$$

2. Čech complex $C^\bullet(\mathcal{U},\mathcal{A})$. (a) Show that d^0 is the same as before.

(b) Show that $(C^\bullet(\mathcal{U},\mathcal{A}),d^\bullet)$ is a complex, i.e., $d^n\circ d^{n-1}=0$.

(c) Show that $B^n(\mathcal{U},\mathcal{A})\subseteq Z^n(\mathcal{U},\mathcal{A})$.

♡

Čech cohomology $\check{H}^\bullet_{\mathcal{U}}(X,\mathcal{A})$. It is defined as the cohomology of the Čech complex $C^\bullet(\mathcal{U},\mathcal{A})$, i.e.,

$$\check{H}^n_{\mathcal{U}}(X,\mathcal{A}) \stackrel{\text{def}}{=} Z^n(\mathcal{U};\mathcal{A})/B^n(\mathcal{U};\mathcal{A}), \quad n=0,1,2,\dots.$$

This construction is a generalization of the global sections of a sheaf since

$$\check{H}^0_{\mathcal{U}}(X,\mathcal{A}) = Z^0(\mathcal{U},\mathcal{A})/B^0(\mathcal{U},\mathcal{A}) = Z^0(\mathcal{U},\mathcal{A}) = \Gamma(\mathcal{A}).$$

3. If the open cover $\mathcal{U}$ consists of two open sets U and V, show that

(1) $\check{H}^0_{\mathcal{U}}(X,\mathcal{A}) = \{(a,b)\in\mathcal{A}(U)\oplus\mathcal{A}(V);\ a=b \text{ on } U\cap V\} \cong \Gamma(X,\mathcal{A})$.
(2) $\check{H}^1_{\mathcal{U}}(X,\mathcal{A}) = \frac{\mathcal{A}(U\cap V)}{\rho^U_{U\cap V}\mathcal{A}(U)+\rho^V_{U\cap V}\mathcal{A}(V)}$.
(3) $\check{H}^i_{\mathcal{U}}(X,\mathcal{A}) = 0$ for $i>1$.

♡

Remark. **The True Cohomology of sheaves.** There is a general *cohomology theory for sheaves* which associates to any sheaf of abelian groups $\mathcal{A}$ a sequence of groups $H^i(X,\mathcal{A})$ (no dependence on any open cover!). The usefulness of Čech cohomology comes from the fact that *often*, the Čech cohomology $\check{H}^i_{\mathcal{U}}(X,\mathcal{A})$ computes these cohomology groups $H^i(X,\mathcal{A})$.[24] At least there is never a disagreement on the level 0 since always

$$\check{H}^0_{\mathcal{U}}(X,\mathcal{A}) = \Gamma(X,\mathcal{A}) = H^0(X,\mathcal{A}).$$

♡

[24] This means that in practice, for a specific class of sheaves $\mathcal{A}$ one can find the corresponding class of open covers $\mathcal{U}$ such that $\check{H}^i_{\mathcal{U}}(X,\mathcal{A}) = H^i(X,\mathcal{A})$. For instance in algebraic geometry one usually considers the *quasicoherent sheaves* and then it suffices if all U_i are affine.

Divisors and line bundles on a curve. Let X be a complex curve (i.e., a complex manifold of dimension one). The group $Div(X)$ of divisors on X is the free abelian group with a basis given by all points of X. So, any divisor $D \in Div(X)$ can be written as $D = \sum d_i{\cdot}\alpha_i$ for some distinct points $\alpha_1, ..., \alpha_p$ of X, and some integers $d_1, ..., d_p$.

We can use a divisor $D \in Div(X)$ to modify the sheaf $\mathcal{O}_X^{an}$ of holomorphic (=analytic) functions on X. For any open $U \subseteq X$ we define $\mathcal{O}_X(D)(U) \overset{\text{def}}{=}$ all holomorphic functions f on $U - \{\alpha_1, ..., \alpha_p\}$ such that at each α_i in U, the order of f at α_i[25] is at least $-d_i$, i.e.,

$$ord_{\alpha_i}(f) + d_i \geq 0.$$

4. (a) Show that for any divisor D on a complex curve X, construction $\mathcal{O}_X(D)$ is a sheaf on X.

(b) Let $\mathbf{0}$ be the zero element of $\mathbb{C}$. For $n \in \mathbb{Z}$ consider the sheaf $\mathcal{L} = \mathcal{O}_{\mathbb{C}}(n{\cdot}\mathbf{0})$ on $\mathbb{C}$. Show that for any open $U \subseteq \mathbb{C}$, $\mathcal{L}(U) = z^{-n}{\cdot}\mathcal{O}^{an}(U)$.

(c) We say that a divisor $D = \sum d_i{\cdot}\alpha_i \in Div(X)$ is effective if all multiplicities d_i are ≥ 0. Show that $\mathcal{O}_X(D)$ contains $\mathcal{O}_X$ iff D is effective.

♡

Cohomology of line bundles on $\mathbb{P}^1$. Let $\mathbf{0}$ be the zero in $\mathbb{C} \subseteq \mathbb{P}^1$. For each $n \in \mathbb{Z}$ consider the sheaf $\mathcal{L}_n \overset{\text{def}}{=} \mathcal{O}_{\mathbb{P}^1}^{an}(n{\cdot}\mathbf{0})$ on $\mathbb{P}^1$. We will use the open covering $\mathcal{U} = \{U, V\}$ of $\mathbb{P}^1$, with $U = \mathbb{CP}^1 - \{\infty\}$ and $V = \mathbb{P}^1 - \{\mathbf{0}\}$.

5. Find the dimensions of the Čech cohomology vector spaces $\check{H}^i_{\mathcal{U}}(\mathbb{P}^1, \mathcal{L}_n)$, $n \in \mathbb{Z}$, $i \geq 0$.

[25] Let ϕ be a function holomorphic on some open $V \subseteq X$. If $\alpha \in X$ is an isolated singularity of ϕ in the sense that V contains some punctured neighborhood of α, then we can define the order of ϕ at α, $ord_\alpha(\phi) \in \mathbb{Z}$, by using a local chart on X near α.

Homework 8

♡[26]

Calculations with complexes. Calculations with complexes are based on the ideas bellow.

1. Functoriality of cohomology. (a) Show that a cohomology is a functor, i.e., that a nap of complexes $A \xrightarrow{\alpha} B$ gives maps of cohomology groups $H^n(A) \xrightarrow{H^n(\alpha)} H^n(B)$.

(b) Show that a short exact sequence of complexes $0 \to A \to B \to C \to 0$ gives maps of cohomology groups $H^n(C) \xrightarrow{\partial^n} H^{n+1}(B)$. So, one needs for a class $\gamma \in \mathrm{H}^n(C)$ to construct a class $\partial\gamma \in \mathrm{H}^{n+1}$. To do this show that

1. For any choice of $c \in Z^n(C)$ such that the class of a cocycle c is $\gamma = [c]$, there is some $b \in B^n$ such that $\beta(b) = c$.
2. For such b, there is cocycle $a \in Z^{n+1}(A)$ such that $d(b) = \alpha(a)$.
3. The class $[a] \in H^{n+1}(A)$ depends only on γ (not on any choices we made).

2. Long exact sequence of cohomology groups. (b) Show that a short exact sequence of complexes $0 \to A \to B \to C \to 0$ gives a long exact sequence of cohomologies

$$\cdots \xrightarrow{\partial^{n-1}} \mathrm{H}^n(A) \xrightarrow{\mathrm{H}^n(\alpha)} \mathrm{H}^n(B) \xrightarrow{\mathrm{H}^n(\beta)} \mathrm{H}^n(C) \xrightarrow{\partial^n} \mathrm{H}^{n+1}(A) \xrightarrow{\mathrm{H}^{n+1}(\alpha)} \mathrm{H}^{n+1}(B) \xrightarrow{\mathrm{H}^{n+1}(\beta)} \cdots.$$

♡

Resolving singularities by blow-ups. Let $V = \mathbb{C}^n$. Recall that the blow up $\widetilde{V}$ of V lies in the product $\mathbb{P}(V) \times V$, so it comes with the maps $\mathbb{P}(V) \xleftarrow{\pi} \widetilde{V} \xrightarrow{\mu} V$. Let $X \subseteq V$ be an affine subvariety given by some polynomial equations. The *proper transform of* X is the subvariety $\widetilde{X}$ of the blow-up, obtained as the closure $\overline{\mu^{-1}X^*}$ in $\widetilde{V}$ of the inverse of $X^* = X - \{0\}$. It comes with a map $\widetilde{X} \xrightarrow{p} X$ (the restriction of μ). We call $E = \widetilde{X} \cap \mu^{-1}0$ the exceptional locus of the proper transform.

We will say that Y is a hypersurface of a complex manifold M if it is (locally) given by one equation $Y = \{z \in M;\ F(z) = 0\}$ for some holomorphic function F. In this case at a point $p \in Y$, Y is a submanifold of M iff $d_pF \neq 0$.

[26]Due Thursday April 22

3+4. Quadratic singularities in $\mathbb{A}^2$ and $\mathbb{A}^3$. Consider

(1) $X = \{(x, y) \in \mathbb{C}^2;\ xy = 0\} \subseteq \mathbb{C}^2$,
(2) $X = \{(x, y, z) \in \mathbb{C}^3;\ x^2 + y^2 + z^2 = 0\} \subseteq \mathbb{C}^3$.

In each case

- Show that X has an isolated singularity at 0.
- Describe the exceptional fiber E.
- Show that $\widetilde{X}$ is a submanifold of $\widetilde{V}$.[27]

♡

The strategy of resolving by blow-ups. If X has an *isolated singularity at* 0 (meaning that X is smooth off 0, i.e., X^* is a manifold), usually $\widetilde{X}$ is in some sense *less singular*. One can try to explain it in the following way. In the blow up $\widetilde{V}$, one replaces $0 \in V$ with $\mathbb{P}(V)$, i.e., *with all directions of approaching* 0. If we imagine that the singularity of X at 0 is created by a mess of many unusual ways of approaching this point from X, the blow-up has the effect of separating, pulling apart, these ways and decreasing a mess. This falls under standard idea that singularities are caused by forgetting some relevant data, so they are resolved by adding data, in this case the direction of the approach.

For instance let X be a curve in $V = \mathbb{C}^2$ with a singularity at the origin which come from two branches B_1, B_2 of X (i.e., little pieces of X), meeting at 0. In the proper transforms $\widetilde{B_i}$ of B_i one replaces 0 by the direction of approaching 0, i.e., the tangent lines T_i to B_i at 0. So if the tangent lines to B_i at 0 are different, the exceptional fiber will consist of two different points T_i and $\widetilde{X}$ will be smooth because the branches will no longer meet.

This is the case when B_1, B_2 agree to the 0^{th} order at the origin (i.e., they meet but they are not tangent to each other), and the proper transform kills this 0^{th} order contact: $\widehat{B_i}$'s do not meet. In general, if B_1, B_2 agree to the p^{th} order at the origin the transforms will agree to the $(p-1)^{\text{st}}$ order, so one need to blow up $p+1$ times to resolve singularity. Say, if we have first order contact then $T_1 = T_2$ (i.e., B_i's are tangent), so $\widetilde{B_i}'s$ meet but they are not tangent any more, so the next blow up will do.

Actually, by using slightly more general versions of the blow-up construction one can get rid of any singularity:

Theorem. [Hironaka] Any singularity can be resiolved by succesive blow-ups.

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[27]The problem assumes that $\widetilde{V}$ is a manifold. In dimension 2 this has been checked in a previous homework. In any dimension the proof is similar – one has to extend the standard charts for $\mathbb{P}(V)$ to $\widetilde{V}$.

Homework 9

♡[28]

Sheafification. If when playing with sheaves we get lost and find ourselves in a larger world of *presheaves* (and these are less interesting objects) we need to find the way home. This is the main technical step[29] in making sheaves useful.

By a presheaf we mean a the same structure as a sheaf, except that we do not require the *gluing* property. For instance while locally constant functions are a sheaf, constant functions are just a presheaf. Presheaves are by themselves not so interesting because lack of gluing means that they do not relate local and global information well. Unfortunately presheaves are not avoidable, for instance we will see that applying some basic constructions to sheaves results in presheaves.

Sheafification is a way to improve any presheaf of sets $\mathcal{S}$ into a sheaf of sets $\widetilde{\mathcal{S}}$. We will obtain the sections of the sheaf $\tilde{\mathcal{S}}$ associated to a presheaf $\mathcal{S}$ in two steps:[30]

(1) *add more sections* by gluing systems of local sections s_i which are compatible in the sense that they are *locally* the same, and
(2) *cut down on sections* by identifying two results of such gluing procedures when the local sections in the two families are *locally* the same.

In the first step for each open $U\subseteq X$ we replace $\mathcal{S}(U)$ by a larger set $\hat{\mathcal{S}}(U)$, and in the second by $\widetilde{\mathcal{S}}(U)$ which is a quotient of $\widehat{\mathcal{S}}(U)$ by an equivalence relation $\equiv$. The definitions are

(1) $\widehat{\mathcal{S}}(U)$ consists of all families $(U_i, s_i)_{i\in I}$ such that
   - $(U_i)_{i\in I}$ is an open cover of U and $s_i \in \mathcal{S}(U_i)$ is a section of $\mathcal{S}$ on U_i,
   - sections s_i are *weakly compatible* in the sense that they are locally the same, i.e., for any $i, j \in I$ and any point $x \in U_{ij}$. we ask that sections s_i and s_j are the same near x:

     There is neighborhood W of x in U_{ij} such that $s_i|W = s_j|W$.
(2) We say that two systems $(U_i, s_i)_{i\in I}$ and $(V_j, t_j)_{j\in J}$ are $\equiv$ if for any $i \in I$, $j \in J$, the sections s_i and t_j are weakly equivalent[31]

1. (a) The relation $\equiv$ on $\widehat{\mathcal{S}}(U)$ really says that $(U_i, s_i)_{i\in I} \equiv (V_j, t_j)_{j\in J}$ iff the disjoint union $(U_i, s_i)_{i\in I} \sqcup (V_j, t_j)_{j\in J}$ is again in $\widehat{\mathcal{S}}(U)$.

(b) $\equiv$ is an equivalence relation on $\widehat{\mathcal{S}}(U)$.

(c) $\widetilde{\mathcal{S}}(U)$ is a presheaf.

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[28]Due Thursday April 29

[29]and the most painful.

[30]You can think that we are imitating the passage from constant functions to locally constant functions.

[31]i.e., for each $x \in U_i \cap V_j$, there is an open set W with $x \in W\subseteq U_i \cap V_j$ such that "$s_i = t_j$ on W" in the sense of restrictions being the same.

2. $\widetilde{\mathcal{S}}$ is a sheaf.

♡

Adjointness of sheafification and forgetting. Fortunately, in practice we do not have to recall the specifics of the the sheafification construction. Instead we use an abstract (categorical) characterization of the sheaf $\widetilde{\mathcal{S}}$. First we define the category $preSh(X)$ of presheaves of sets on X: the objects are presheaves and a map of presheaves $\alpha \in \mathrm{Hom}_{preSh(X)}(\mathcal{A},\mathcal{B})$ is a system of maps $\alpha_U : \mathcal{A}(U) \to \mathcal{B}(U)$, one for each open $U \subseteq X$, which is compatible with restrictions, i.e., whenever $V \subseteq U \subseteq X$, the following commutes

$$\begin{array}{ccc} \mathcal{A}(U) & \xrightarrow{\alpha_U} & \mathcal{B}(U) \\ \rho_V^U \big\downarrow & & \rho_V^U \big\downarrow \\ \mathcal{A}(V) & \xrightarrow{\alpha_V} & \mathcal{B}(V) \end{array} .$$

The category $Sh(X)$ of sheaves of sets on X has sheaves for objects but the maps are defined the same: any sheaves $\mathcal{A},\mathcal{B}$ are in particular presheaves (we just *forget* that they do satisfy the gluing property), and $\mathrm{Hom}_{Sh(X)}(\mathcal{A},\mathcal{B}) \stackrel{\text{def}}{=} \mathrm{Hom}_{preSh(X)}(\mathcal{A},\mathcal{B})$.

3. (a) There is a canonical map of presheaves $\mathcal{S} \xrightarrow{\iota} \widetilde{\mathcal{S}}$.

(b) For any presheaf $\mathcal{S}$ and any sheaf $\mathcal{F}$ the map

$$\iota^* : \mathrm{Hom}_{\mathcal{S}heaves}(\widetilde{\mathcal{S}},\mathcal{F}) \to \mathrm{Hom}_{pre\mathcal{S}heaves}(\mathcal{S},\mathcal{F}), \qquad \iota_*\alpha = \alpha \circ \iota$$

is a bijection.

♡

The bijection ι^* relates two procedures (i.e. functors): *sheafification* (on the LHS) and *forgetting* (on the RHS). A relation of this type between two functors is called *adjunction*, we say that sheafification is the left adjoint of the forgetful functor (and that the forgetful functor is the right adjoint of sheafification). One can see that two adjoint functors determine each other by using the Yoneda lemma.

4. (a) Show that the adjunction property characterizes the sheafification, i.e., if $\mathcal{S}$ is a presheaf and $\iota_i : \mathcal{S} \to \mathcal{S}_i$ $(i = 1, 2)$, are maps of presheaves such that

(1) $\mathcal{S}_i$ are sheaves,
(2) For any sheaf $\mathcal{F}$ the maps

$$\iota_i^* : \mathrm{Hom}_{\mathcal{S}heaves}(\widetilde{\mathcal{S}},\mathcal{F}) \to \mathrm{Hom}_{pre\mathcal{S}heaves}(\mathcal{S},\mathcal{F}), \qquad \iota_*\alpha = \alpha \circ \iota$$

are bijections.

then there is a canonical isomorphism $\mathcal{S}_1 \to \mathcal{S}_2$.

(b) Show that the sheafification of constant functions is given by locally constant functions.

Homework 10

♡[32]

The direct and inverse image of sheaves. Any map of sets $X \xrightarrow{\pi} Y$ defines a linear operator $\mathbb{C}[Y] \xrightarrow{\pi^*} \mathbb{C}[X]$ between spaces of functions, this is the *pull-back* or *inverse image* operation $\pi^* g = \ g \circ \pi$. We can also go in the opposite direction with a *direct image* (or "*integration over fibers*) operation $\mathbb{C}[X] \xrightarrow{\pi_*} \mathbb{C}[Y]$ by $(pi_* f)(y) = \ \sum_{x \in \pi^{-1} y} \ f(x)$, provided we resolve some convergence problem, for instance it is fine if the fibers of π are finite.

One can do the same in sheaves and without any convergence problem. Any map of topological spaces $X \xrightarrow{\pi} Y$ defines two operations, the direct and inverse image operations

$$\mathcal{S}heaves(X) \xrightarrow{\pi_*} \mathcal{S}heaves(Y) \quad \text{and} \quad \mathcal{S}heaves(Y) \xrightarrow{\pi^*} \mathcal{S}heaves(X).$$

The direct image is much easier to define while the inverse image is much easier to calculate in practice.

In our thinking, we assumed some analogy between functions and sheaves. This is sound, i.e., one should think of sheaves as more subtle versions of functions. However, one should notice the increased level of subtlety: while functions on X form a vector space, sheaves on X form a category. So our new operations are not going to be linear operators but functors.

1. (a) Let $X \xrightarrow{\pi} Y$ be a map of topological spaces. Show that for any sheaf $\mathcal{M}$ on X, the formula

$$\pi_*(\mathcal{M}) \ (V) \overset{\text{def}}{=} \mathcal{M}(\pi^{-1} V),$$

defines a sheaf $\pi_* \mathcal{M}$ on Y, and this gives a functor $\mathcal{S}heaves(X) \xrightarrow{\pi_*} \mathcal{S}heaves(Y)$.

(b) If $X \xrightarrow{\pi} Y \xrightarrow{\tau} Z$ then $\tau_*(\pi_* \mathcal{A}) \cong (\tau \circ \pi)_* \mathcal{A}$.

(c) $(1_X)_* \mathcal{A} \cong \ \mathcal{A}$.

(d) (Map to a point.) If $a : X \to pt$ then $a_*(\mathcal{F}) = \Gamma(X, \mathcal{F})$.

♡

2. Examples of maps of sheaves. (a) Consider the sheaf $\mathcal{O}_M$ of holomorphic functions on a complex manifold $M = \mathbb{C}$. Which of the following are maps of sheaves of vector spaces (and why?):

(1) (Differentiation) $\mathcal{O}_M(U) \ni f \mapsto f' \in \mathcal{O}_M(U)$,
(2) (Squaring) $\mathcal{O}_M(U) \ni f \mapsto f^2 \in \mathcal{O}_M(U)$,
(3) (multiplication by a function) $\mathcal{O}_M(U) \ni f \mapsto zf \in \mathcal{O}_M(U)$,
(4) (translation by 1) $\mathcal{O}_M(U) \ni f(x) \mapsto f(x+1) \in \mathcal{O}_M(U)$.

[32]Due Thursday May 6

(b) Circle $S = \{z \in \mathbb{C};\ |z| = 1\}$ is clearly a one dimensional real manifold (a restriction of $\mathcal{E} : \mathbb{R} \to S,\ \mathcal{E}(x) = e^{ix}$ to any open interval $I \subseteq \mathbb{R}$ of length $< 2\pi$ provides a chart $I \to \mathcal{E}(I) \subseteq S$). Let C^∞_S be the sheaf of smooth functions on S. Show that

(1) One can define operators $\partial_U : C^\infty_S(U) \to C^\infty_S(U)$ by the formula[33] $\partial f(e^{ix}) \stackrel{\text{def}}{=} \frac{d}{dx} f(e^{ix})$.
(2) Show that ∂ is a map of sheaves $\partial : C^\infty_S \to C^\infty_S$.

♡

Category $\mathcal{A}b\mathcal{S}h(X)$ of sheaves of abelian groups is an abelian category. We want to see that in the category $\mathcal{A}b\mathcal{S}h(X)$ of sheaves of abelian groups on X, we can calculate very much like we calculate with abelian groups or with modules over rings, i.e., the precise meaning of this is:

Category $\mathcal{A}b\mathcal{S}h(X)$ of sheaves of abelian groups is an abelian category.

We will consider a map of sheaves of abelian groups $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$. We want to see that it has a kernel, image and cokernel. We will take care of the kernel, and indicate3 how to construct the image and the cokernel.

3. (a) Show that the formula $\mathcal{K}_\alpha(U) = Ker[\mathcal{A}(U) \xrightarrow{\alpha_U} \mathcal{B}(U)]$ defines a sheaf $\mathcal{K}_\alpha$ which is a subsheaf[34] of $\mathcal{A}$.

(b) Let $i : \mathcal{K}_\alpha \to \mathcal{A}$ be the inclusion, i.e., Show that $(\mathcal{K}_\alpha, i)$ is the kernel of α, according to the following categorical definition of the kernel

Any map of sheaves $\mathcal{F} \xrightarrow{\phi} \mathcal{A}$ such that $\beta \circ \alpha = 0$ factors uniquely through α, i.e., there is a unique map $\mathcal{F} \xrightarrow{\Phi} \mathcal{K}_\alpha$ such that $(\mathcal{F} \xrightarrow{\phi} \mathcal{A}) = (\mathcal{K}_\alpha \xrightarrow{i} \mathcal{A}) \circ (\mathcal{F} \xrightarrow{\Phi} \mathcal{K}))$.

4. (a) Show that the formula $I_\alpha(U) = Im[\mathcal{A}(U) \xrightarrow{\alpha_U} \mathcal{B}(U)]$ defines a presheaf I_α which is a subpresheaf of $\mathcal{B}$.

(b) Show that the formula $C_\alpha(U) = Coker[\mathcal{A}(U) \xrightarrow{\alpha_U} \mathcal{B}(U)]$ defines a presheaf C_α.

(c) Consider the the map of sheaves $\partial : C^\infty_S \to C^\infty_S$ defined above. Show that in this case I_∂ and C_∂ are not sheaves.

(d) What are the sheafifications $\mathcal{I}_\partial$ and $\mathcal{C}_\partial$ of presheaves I_∂ and C_∂ from part (c)?

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[33] If you do not like the formula you can identify for $V \subseteq S$ open, $C^\infty_S(V)$ with the $\mathbb{Z}$-periodic functions on $\mathcal{E}^{-1}V$, and then use the operator $\frac{d}{dx}$.

[34] For sheaves $\mathcal{S}$ and $\mathcal{S}'$ we say that $\mathcal{S}'$ is a sub(pre)sheaf of $\mathcal{S}$ if $\mathcal{S}'(U) \subseteq \mathcal{S}(U)$ for each open U and the restriction maps for $\mathcal{S}'$, $\mathcal{S}'(U) \xrightarrow{\rho'} \mathcal{S}'(V)$ are restrictions of the restriction maps for $\mathcal{S}$, $\mathcal{S}(U) \xrightarrow{\rho} \mathcal{S}(V)$. The same definition works for presheaves.

Homework X

THIS HOMEWORK IS DUE BY THE END OF THIS SEMESTER

♡

Monodromy of cycles in curves. For $\lambda \in \mathbb{C}$ consider the affine cubic curve

$$\mathcal{C}_\lambda = \{(x:y) \in \mathbb{A}^2;\ y^2 = x(x-1)(x-\lambda)\} \subseteq \mathbb{A}^2$$

and the corresponding projective cubic curve

$$C_\lambda = \{[x:y:u] \in \mathbb{P}^2;\ y^2 u = x(x-u)(x-\lambda u)\} \subseteq \mathbb{P}^2.$$

Circles $\alpha_\lambda, \beta_\lambda$ in C_λ. Let $0 < |\lambda| < 1$. We use the projection to the x-line $\pi : \mathcal{C}_\lambda \to \mathbb{A}^1$, $\pi(x,y) = x$, to define circles

$$\alpha_\lambda = \pi^{-1}[0,\lambda] \quad \text{and} \quad \beta_\lambda = \pi^{-1}[\lambda,1]$$

in $\mathcal{C}_\lambda$, as inverses of segments joining $0, \lambda$ and 1.

♡

1. The vanishing cycle. (a) Draw C_0 (with all explanations).

(b) What happens with the circles $\alpha_\lambda, \beta_\lambda$ as $\lambda \to 0$?

♡

♡ ★ ♡

♡

2. When λ goes around 0 on a small circle, what happens to the circles α_λ and β_λ in the torus C_λ?

♡

Explanation. Start with $\lambda = \frac{1}{2}$ and the circles $\alpha_{\frac{1}{2}} = \pi^{-1}[0, \frac{1}{2}]$ and $\beta_{\frac{1}{2}} = \pi^{-1}[\frac{1}{2}, 1]$ in $C_{\frac{1}{2}}$. As one rotates $\lambda = \frac{1}{2}$ around the origin by $\lambda_\theta = \frac{1}{2}e^{i\theta}$, $0 \le \theta \le 2\pi$, one needs to choose continuously circles $\alpha(\theta), \beta(\theta)$, in the curves $C_{\frac{1}{2}e^{i\theta}}$, and then the question is what are the circles $\alpha(2\pi), \beta(2\pi)$?

Since there are choices involved, this is not a completely precisely posed question. To eliminate the effect of continuous choices, we can look at circles up to homotopy (i.e., up to continuous deformations). Recall that the homotopy classes $[\alpha_\lambda], [\beta_\lambda]$ form a basis of the group of closed paths up to homotopy: $\pi_1(C_\lambda) = \mathbb{Z}\cdot[\alpha] \oplus \mathbb{Z}\cdot[\beta_\lambda]$, $\lambda \in \mathbb{C} - \{0,1\}$. Now the classes of $\alpha(\theta), \beta(\theta)$ in $\pi_1(C_{\frac{1}{2}e^{i\theta}})$ are well defined, and the question is to calculate $[\alpha(2\pi)], [\beta(2\pi)]$ in $\pi_1(C_{\frac{1}{2}e^{2\pi i}}) = \pi_1(C_{\frac{1}{2}})$, i.e., to find the integer coefficients

$$[\alpha_{2\pi}] = \mu_{11}\cdot[\alpha(0)] + \mu_{12}\cdot[\beta(0)] \quad \text{and} \quad [\beta_{2\pi}] = \mu_{21}\cdot[\alpha(0)] + \mu_{22}\cdot[\beta(0)].$$

A simple way to choose $\alpha(\theta), \beta(\theta)$ is to take the inverses $\alpha(\theta) = {\pi_{\frac{1}{2}e^{i\theta}}}^{-1}\, a(\theta)$, $\beta(\theta) = {\pi_{\frac{1}{2}e^{i\theta}}}^{-1}\, b(\theta)$ in $C_{\frac{1}{2}e^{i\theta}}$, of paths on the x-line $a(\theta), b(\theta) \subseteq \mathbb{C} - \{0,1\}$. Here $a(\theta)$ is a curve from 0 to $\frac{1}{2}e^{i\theta}$ and $b(\theta)$ from $\frac{1}{2}e^{i\theta}$ to 1. So, one starts with $a(0), b(0)$ which are segments, and then moves them continuously in the family $a(\theta), b(\theta)$.