ALGEBRAIC GEOMETRY
JAN 04

Contents

Appendix A. **Categories**

A.0.1. *Why categories?* The notion of a category is misleadingly elementary. It formalizes the idea that we study certain kind of objects (i.e., endowed with some specified structures) and that it makes sense to go from one such object to another via something (a "morphism") that preserves the relevant structures. Since this is indeed what we usually do, the *language* of categories is convenient.

However, soon one finds that familiar notions and constructions (such as (i) empty set, (ii) union of sets, (iii) product of sets, (iv) abelian group, ...) **categorify**, i.e., have analogues (and often more then one) in general categories (respectively: (i) initial object, final object, zero object; (ii) sum of objects or more generally a direct (inductive) limit of objects; (iii) product of objects or more generally the inverse (projective) limit of objects; (iv) additive category, abelian category; ...). This enriched language of categories was recognized as fundamental for describing various complicated phenomena, and the study of *special kinds of categories* mushroomed to the level of study of functions with various properties in analysis.

A.1. **Categories.** A category $\mathcal{C}$ consists of

1. a class $Ob(\mathcal{C})$ whose elements are called objects of $\mathcal{C}$,
2. for any $a, b \in Ob(\mathcal{C})$ a set $\mathrm{Hom}_{\mathcal{C}}(a, b)$ whose elements are called morphisms ("maps") from a to b,
3. for any $a, b, c \in Ob(\mathcal{C})$ a function $\mathrm{Hom}_{\mathcal{C}}(b, c) \times \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}}(a, c)$, called composition,
4. for any $a \in Ob(\mathcal{C})$ an element $1_a \in \mathrm{Hom}_{\mathcal{C}}(a, a)$,

such that the composition is associative and 1_a is a neutral element for composition.

Instead of $a \in \mathrm{Ob}(\mathcal{C})$ we will just say $a \in \mathcal{C}$.

A.1.1. *Examples.*

1. Categories of *sets with additional structures*: $\mathcal{S}ets$, $\mathcal{A}b$, $\mathfrak{m}(\Bbbk)$ for a ring $\Bbbk$ (denoted also $\mathcal{V}ect(\Bbbk)$ if $\Bbbk$ is a field), $\mathcal{G}roups$, $\mathcal{R}ings$, $\mathcal{T}op$, $\mathcal{O}rd\mathcal{S}ets \stackrel{\text{def}}{=}$ category of ordered sets, ...
2. To a category $\mathcal{C}$ one attaches the opposite category $\mathcal{C}^o$ so that objects are the same but the "direction of arrows reverses":
$$\mathrm{Hom}_{\mathcal{C}^o}(a, b) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(b, a).$$
3. Any partially ordered set $(I, \leq)$ defines a category with $Ob = I$ and $\mathrm{Hom}(a, b) =$ point (call this point (a, b)) if $a \leq b$ and $\emptyset$ otherwise.
4. Sheaves of sets on a topological space X, Sheaves of abelian groups on X,...

A.2. **Functors.** The analogue on the level of categories of a function between two sets is a *functor between two categories.*

A functor F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of

- for each object $a\in\mathcal{A}$ an object $F(a)\in\mathcal{B}$,
- for each map $\alpha\in \mathrm{Hom}_{\mathcal{A}}(a',a'')$ in $\mathcal{A}$ a map $F(\alpha)\in\mathrm{Hom}_{\mathcal{B}}(Fa',Fa'')$

such that F preserves compositions and units, i.e., $F(\beta\circ\alpha)=F(\beta)\circ F(\alpha)$ and $F(1_a)=1_{Fa}$.

A.2.1. *Examples.* (1) A *functor* means some construction, say a map of rings $\Bbbk\xrightarrow{\phi} l$ gives

- a pull-back functor $\phi^*:\mathfrak{m}(l)\to\mathfrak{m}(\Bbbk)$ where $\phi^*N=N$ as an abelian group, but now it is considered as module for $\Bbbk$ via ϕ.
- a push-forward functor $\phi_*:\mathfrak{m}(\Bbbk)\to\mathfrak{m}(l)$ where $\phi_*M\overset{\text{def}}{=}l\otimes_{\Bbbk}M$. This is called "change of coefficients".

To see that these are functors, we need to define them also on maps. So, a map $\beta:N'\to N''$ in $\mathfrak{m}(l)$ gives a map $\phi^*(\beta):\phi^*(N')\to\phi^*(N'')$ in $\mathfrak{m}(\Bbbk)$ which as a function between sets is really just $\beta:N'\to N''$. On the other hand, $\alpha:M'\to M''$ in $\mathfrak{m}(\Bbbk)$ gives $\phi_*(\alpha):\phi_*(M')\to\phi_*(M'')$ in $\mathfrak{m}(l)$, this is just the map $1_l\otimes\alpha:\ l\otimes_{\Bbbk}M'\to l\otimes_{\Bbbk}M''$, $c\otimes x\mapsto c\otimes\alpha(x)$.

Here we see a general feature:

functors often come in pairs ("adjoint pairs of functors") and usually one of them is stupid and the other one an interesting construction.

(2) For any category $\mathcal{A}$ there is the identity functor $1_{\mathcal{A}}:\mathcal{A}\to\mathcal{A}$. Two functors $\mathcal{A}\xrightarrow{F}\mathcal{B}$ and $\mathcal{B}\xrightarrow{G}\mathcal{C}$ can be composed to a functor $\mathcal{A}\xrightarrow{G\circ F}\mathcal{C}$.

(3) An object $a\in\mathcal{A}$ defines two functors, $\mathrm{Hom}_{\mathcal{A}}(a,-):\ \mathcal{A}\to\mathcal{S}ets$, and $\mathrm{Hom}_{\mathcal{A}}(-,a):\mathcal{A}^o\to\mathcal{S}ets$. Moreover, $\mathrm{Hom}_{\mathcal{A}}(-,-)$ is a functor from $\mathcal{A}^o\times\mathcal{A}$ to sets!

(4) For a ring $\Bbbk$, tensoring is a functor $-\otimes_{\Bbbk}-:\ \mathfrak{m}^r(\Bbbk)\times\mathfrak{m}^l(\Bbbk)\to\mathcal{A}b$.

A.2.2. *Contravariant functors.* We say that a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is given by assigning to any $a\in\mathcal{A}$ some $F(a)\in\mathcal{B}$, and for each map $\alpha\in\mathrm{Hom}_{\mathcal{A}}(a',a'')$ in $\mathcal{A}$ a map $F(\alpha)\in\mathrm{Hom}_{\mathcal{B}}(Fa'',Fa')$ – notice that we have changed the direction of the map so now we have to require $F(\beta\circ\alpha)=F(\alpha)\circ F(\beta)$ (and $F(1_a)=1_{Fa}$).

This is just a way of talking, not a new notion since a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is the same as a functor F from $\mathcal{A}$ to $\mathcal{B}^o$ (or a functor F from $\mathcal{A}^o$ to $\mathcal{B}$).

A.3. **Natural transformations of functors ("morphisms of functors").** A natural transformation η of a functor $F:\mathcal{A}\to\mathcal{B}$ into a functor $G:\mathcal{A}\to\mathcal{B}$ consists of maps

$\eta_a \in \mathrm{Hom}_{\mathcal{B}}(Fa, Ga),\ a \in \mathcal{A}$ such that for any map $\alpha : a' \to a''$ in $\mathcal{A}$ the following diagram commutes

$$\begin{array}{ccc} F(a') & \xrightarrow{F(\alpha)} & F(a'') \\ \eta_{a'}\downarrow & & \eta_{a''}\downarrow \\ G(a') & \xrightarrow{G(\alpha)} & G(a'') \end{array}, \quad i.e., \quad \eta_{a''}\circ F(\alpha) = G(\alpha)\circ\eta_{a'}.$$

So, η relates values of functors on objects in a way compatible with the values of functors on maps. In practice, any "natural" choice of maps η_a will have the compatibility property.

A.3.1. *Example.* For the functors $\phi_* M = l\otimes_{\Bbbk} M$ and $\phi^* N = N$ from A.2.1(1), there are canonical morphisms of functors

$$\alpha:\ \phi_*\circ\phi^* \to 1_{\mathsf{m}(l)}, \qquad \phi_*\circ\phi^*(N) = l\otimes_{\Bbbk} N \xrightarrow{\alpha_N} N = 1_{\mathsf{m}(l)}(N)$$

is the action of l on N and

$$\beta:\ 1_{\mathsf{m}(\Bbbk)} \to \phi^*\circ\phi_*, \qquad \phi^*\circ\phi_*(M) = l\otimes_{\Bbbk} M \xleftarrow{\beta_M} M = 1_{\mathsf{m}(\mathbb{M})}(M)$$

is the map $m \mapsto 1_l \otimes m$.

For any functor $F : \mathcal{A} \to \mathcal{B}$ one has $1_F : F \to F$ with $(1_F)_a = 1_{Fa} : Fa \to Fa$. For three functors F, G, H from $\mathcal{A}$ to $\mathcal{B}$ one can compose morphisms $\mu : F \to G$ and $\nu : G \to H$ to $\nu\circ\mu : F \to H$

A.3.2. *Lemma.* For two categories $\mathcal{A}, \mathcal{B}$, the functors from $\mathcal{A}$ to $\mathcal{B}$ form a category $Funct(\mathcal{A}, \mathcal{B})$.

Proof. For $F, G : \mathcal{A} \to \mathcal{B}$ one defines $\mathrm{Hom}(F, G)$ as the set of natural transforms from F to G, then all the structure is routine.

A.4. **Construction (description) of objects via representable functors.** Yoneda lemma bellow says that passing from an object $a \in \mathcal{A}$ to the corresponding functor $\mathrm{Hom}_{\mathcal{A}}(-, a)$ does not loose any information – a can be recovered from the functor $\mathrm{Hom}_{\mathcal{A}}(-, a)$.[1] This has the following applications:

(1) One can describe an object a by describing the corresponding functor $\mathrm{Hom}_{\mathcal{A}}(-, a)$. This turns out to be **the** *most natural* description of a.
(2) One can start with a functor $F : \mathcal{A}^o \to \mathcal{S}ets$ and ask whether it comes from some objects of a. (Then we say that a represents F and that F is representable).
(3) Functors $F : \mathcal{A}^o \to \mathcal{S}ets$ behave somewhat alike the objects of $\mathcal{A}$, and we can think of their totality as a natural enlargement of $\mathcal{A}$ (like one completes $\mathbb{Q}$ to $\mathbb{R}$).

[1]This is the precise form of the *Interaction Principle* on the level of categories that we used to pass from varieties to spaces and stacks. The interactions of a with all objects of the same kind are encoded in the functor $\mathrm{Hom}_{\mathcal{A}}(-, a)$, so Yoneda says that if you know the interactions of a you know a.

A.4.1. *Category* $\widehat{\mathcal{A}}$. To a category $\mathcal{A}$ one can associate a category

$$\widehat{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{F}unct(\mathcal{A}^o, \mathcal{S}ets)$$

of contravariant functors from $\mathcal{A}$ to sets. Observe that each object $a \in \mathcal{A}$ defines a functor

$$\iota_a = \ \mathrm{Hom}_{\mathcal{A}}(-, a) \in \ \widehat{\mathcal{A}}.$$

The following statement essentially says that one can recover a form the functor $\mathrm{Hom}_{\mathcal{A}}(-, a)$, i.e., that this functor contains all information about a.

A.4.2. *Theorem.* (Yoneda lemma)
(a) Construction ι is a functor $\iota : \mathcal{A} \to \widehat{\mathcal{A}}$.

(b) For any functor $F \in \widehat{\mathcal{A}} = \mathcal{F}unct(\mathcal{A}^o, \mathcal{S}ets)$ and any $a \in \mathcal{A}$ there is a canonical identification

$$\mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, F) \cong \ F(a).$$

Proof. (b) Recall that a map of functors $\eta : \iota_a \to F$ (functors from $\mathcal{A}^o$ to $\mathcal{S}ets$), means for each $x \in \mathcal{A}$ one map of sets $\eta_x : \iota_a(x) = \mathrm{Hom}_{\mathcal{A}}(x, a) \to F(x)$, and this system of maps should be such that for each morphism $y \xrightarrow{\alpha} x$ in $\mathcal{A}$ (i.e., $x \xrightarrow{\alpha} y$ in $\mathcal{A}^o$), the following diagram commutes

$$\begin{array}{ccc} F(x) & \xrightarrow{F(\alpha)} & F(y) \\ {\scriptstyle\eta_x}\uparrow & & {\scriptstyle\eta_y}\uparrow \\ \iota_a(x) & \xrightarrow{\iota_a(\alpha)} & \iota_a(y) \end{array} \quad , \quad \textit{i.e.,} \quad F(\alpha)\circ\eta_x \ = \ \eta_y\circ\iota_a(\alpha).$$

Such η in particular gives $\eta_a : \iota_a \to F(a)$, and since $\iota_a = \mathrm{Hom}_{\mathcal{A}}(a, a) \ni 1_a$ we get an element $\overline{\eta} \stackrel{\text{def}}{=} \eta_a(1_a)$ of $F(a)$.

In the opposite direction, a choice of $f \in F(a)$, gives for any $x \in \mathcal{A}$ the composition of functions

$$\widetilde{f}_x \stackrel{\text{def}}{=} [\iota_a(x) = \mathrm{Hom}_{\mathcal{A}}(x, a) = \mathrm{Hom}_{\mathcal{A}^o}(a, x) \xrightarrow{F}= \mathrm{Hom}_{\mathcal{S}ets}[F(a), F(x)] \xrightarrow{ev_f} F(x)].$$

Now one checks that

- (i) $\widetilde{f}$ is a map of functors $\iota_a \to F$, and
- (ii) procedures $\eta \mapsto \overline{\eta}$ and $f \mapsto \widetilde{f}$ are inverse functions between $\mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, F)$ and $F(a)$.

Corollary. (a) Yoneda functor $\iota : \mathcal{A} \to \widehat{\mathcal{A}}$ is a full embedding of categories, i.e., for any $a, b \in \mathcal{A}$ the map

$$\iota : \mathrm{Hom}_{\mathcal{A}}(a, b) \ \to \ \mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, \iota_b),$$

given by the functoriality of ι, is an isomorphism.

(b) Functor $\mathrm{Hom}_{\mathcal{A}}(-,a)=\iota_a$ determines a up to a unique isomorphism, i.e., if $\iota_a\cong\iota_b$ in $\widehat{\mathcal{A}}$ then $a\cong b$ in $\mathcal{A}$.

Proof. (a) follows the part (b) of the Yoneda lemma (take $F=\iota_b$). (b) follows from (a).

Remark. We say that a functor $F:\mathcal{B}\to\mathcal{C}$ is a full embedding of categories if for any $a,b\in\mathcal{B}$ the map $\mathrm{Hom}_{\mathcal{A}}(a,b)\xrightarrow{F_{a,b}}\mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a,\iota_b)$ given by the functoriality of F, is an isomorphism. The meaning of this is we put $\mathcal{B}$ into a larger category which has objects from $\mathcal{B}$ and maybe also some new objects, but the old objects (from $\mathcal{B}$) relate to each other in $\mathcal{C}$ the same as they used to in $\mathcal{B}$. We also say that F makes $\mathcal{B}$ into a full subcategory of $\mathcal{C}$.

A.5. **Yoneda completion $\widehat{\mathcal{A}}$ of a category $\mathcal{A}$.** Yoneda lemma says that $\mathcal{A}$ lies in a larger category $\widehat{\mathcal{A}}$. The hope is that the category $\widehat{\mathcal{A}}$ may contain many *beauties* that should morally be in $\mathcal{A}$ (but are not). One example will be a way of treating inductive systems in $\widehat{\mathcal{A}}$. In particular we will see inductive systems of infinitesimal geometric objects that underlie the differential calculus.

A.5.1. *Distributions.* This Yoneda completion is a categorical analogue of one of the basic tricks in analysis:

since among functions one can not find beauties like the δ-functions, we extend the notion of of functions by adding distributions.

Remember that the distributions on an open $U\subseteq\mathbb{R}^n$ are the (nice) linear functionals on the vector space of of (nice) functions: $\mathcal{D}(U,\mathbb{C})\subseteq C_c^\infty(U,\mathbb{C})^*=\mathrm{Hom}_{\mathbb{C}}[C_c^\infty(U),\mathbb{C}]$.

A.5.2. *Representable functors.* First we get a feeling for how objects of $\mathcal{A}$ are viewed inside $\widehat{\mathcal{A}}$, i.e., the relation between thinking of $a\in\mathcal{A}$ and the functor ι_a.

We will say that a functor $F\in\widehat{\mathcal{A}}$, i.e., $F:\mathcal{A}^o\to\mathcal{S}ets$, is *representable* if there is some $a\in\mathcal{A}$ and an isomorphism of functors $\eta:\ \mathrm{Hom}_{\mathcal{A}}(-,a)\to F$. Then we say that a represents F. This is the *basic categorical trick for describing an object a up to a canonical isomorphism:* :

instead of describing a directly we describe a functor F isomorphic to $\mathrm{Hom}_{\mathcal{A}}(-,a)$.

A.5.3. *Examples.* (1) *Products.* A product of a and b is an object that represents the functor

$$\mathcal{A}\ni x\mapsto \mathrm{Hom}(x,a)\times\mathrm{Hom}_{\mathcal{A}}(x,b)\in\ \mathcal{S}ets.$$

(2) In the category of $\Bbbk$-varieties, functor

$$X\mapsto\{(f_1,...,f_n);\ f_i\in\mathcal{O}(X)\}\ =\ \mathcal{O}(X)^n$$

represents $\mathbb{A}^n$.

(3) In the category of schemes,

$$X \mapsto \{f \in \mathcal{O}(X);\ f^2 = 0\}$$

represents the double point scheme $\mathrm{Spec}(\mathbb{Z}[x]/x^2)$.

(4) If $\mathbb{A}^n = \oplus_1^n\ \Bbbk{\cdot}e_i$, then the set

$$\mathbb{A}^\infty \overset{\text{def}}{=} \cup_0^\infty\ \mathbb{A}^n = \oplus_1^\infty\ \Bbbk{\cdot}e_i$$

is an increasing union of $\Bbbk$-varieties. In analogy with (2), we see that the functor corresponding to this construction should be given by all infinite sequences of functions

$$X \mapsto \{(f_0, f_1, ..., f_n, ...);\ f_i \in \mathcal{O}(X)\} \ = \ Map(\mathbb{N}, \mathcal{O}(X)).$$

However, this functor is *not* representable in $\Bbbk$-varieties, i.e., $\mathbb{A}^\infty$ is not a $\Bbbk$-variety. We may expect that it lives in the larger world of schemes, but even this fails. So, its natural ambient is the category the Yoneda completion $\Bbbk\text{-}\widehat{Varieties}$ of the category $\Bbbk\text{-}Varieties$.

A.5.4. *Limits.* One can describe the completion of $\mathcal{A}$ to $\widehat{\mathcal{A}}$ as *adding to $\mathcal{A}$ all limits of* inductive systems *in $\mathcal{A}$*, just as one constructs $\mathbb{R}$ from $\mathbb{Q}$. The simplest kinds of inductive systems in $\mathcal{A}$ are the diagrams $\boldsymbol{a} = (a_0 \to a_1 \to \cdots)$ in $\mathcal{A}$.[2] The limit $\underset{\to}{\lim}\,\boldsymbol{a}$ is roughly speaking the object that should naturally appear at the end: $(a_0 \to a_1 \to \cdots \to \underset{\to\ n}{\lim}\ a_n)$. It need not exist in $\mathcal{A}$ at least it is easy to see that if $\mathcal{A} = \mathcal{S}ets$ then all inductive limits always exist!

A consequence of this good situation in the category $\mathcal{S}ets$ is that:

even if $\underset{\to\ n}{\lim}\ a_n$ *does not exist in $\mathcal{A}$, it always exists in the larger category* $\widehat{\mathcal{A}}$.

An inductive system $\boldsymbol{a}$ defines an object in $\mathcal{A}$ *if* the limit $\underset{\to}{\lim}\ a_n$ exists in $\mathcal{A}$, however it *always* defines a functor $\iota_{\boldsymbol{a}} = \underset{\substack{\to \\ n}}{\mathbf{lim}}\ \iota_{a_n} \in\ \widehat{\mathcal{A}}$, by

$$\iota_{\boldsymbol{a}}(c) \overset{\text{def}}{=} \underset{\to\ n}{\lim}\ \iota_{a_n}(c) = \underset{\to\ n}{\lim}\ \mathrm{Hom}_{\mathcal{A}}(c, a_n) \in\ \mathcal{S}ets.$$

(This definition uses the existence of inductive limits in the category $\mathcal{S}ets$!)

This allows us to <u>*think*</u> of the functor $\iota_{\boldsymbol{a}}$ as the limit of the inductive system $\boldsymbol{a}$ that exists in the larger category $\widehat{\mathcal{A}}$. All together, we can think of any inductive system *as if* it were an object $\underset{\to}{\mathbf{lim}}\ a_i$ in $\mathcal{A}$ (since we can identify it with $\boldsymbol{a} \in \widehat{\mathcal{A}}$). For this reason an inductive system in $\mathcal{A}$ is called an *ind-object of $\mathcal{A}$* (while it really gives an object of $\widehat{\mathcal{A}}$). [3]

[2] Here *inductive* means that it stretches to the right, while for instance $(\cdots \leftarrow b_n \leftarrow b_1 \leftarrow b_0)$ would be called a *projective system.*

[3] Similarly one calls projective systems pro-objects of $\mathcal{A}$.

Examples. The basic example of inductive system is an increasing union. Some infinite increasing unions of $\Bbbk$-schemes are not $\Bbbk$-schemes but they are objects of the category of $\Bbbk(\stackrel{\text{def}}{=})\widehat{\Bbbk\text{-}Schemes}$. The most obvious examples are $\mathbb{A}^\infty$ (above) which should be a $\Bbbk$-variety but it is not, and the formal neighborhood of a closed subscheme (bellow).

A.6. **Category of $\Bbbk$-spaces (Yoneda completion of the category of $\Bbbk$-schemes).** This will be our main example of a Yoneda completion of a category. For examples of non-representable functors, i.e., functors which are in $\widehat{\mathcal{A}}$ but not in $\mathcal{A}$.

This is a geometric example. The geometry we use here is the algebraic geometry. Its geometric objects are called schemes and they are obtained by gluing schemes of a somewhat special type, which are called affine schemes (like manifolds are all obtained by gluing open pieces of $\mathbb{R}^n$'s). We start with a brief review.

A.6.1. *Affine $\Bbbk$-schemes.* Fix a commutative ring $\Bbbk$.

An affine scheme S over $\Bbbk$ is determined by its algebra of functions $\mathcal{O}(S)$, which is a $\Bbbk$-algebra. Moreover, any commutative $\Bbbk$-algebra A is the algebra of functions on some $\Bbbk$-scheme – the scheme is called the spectrum of A and denoted $\mathrm{Spec}(A)$. So, affine $\Bbbk$-schemes are really the same as commutative $\Bbbk$-algebras, except that a map of affine schemes $X \xrightarrow{\phi} Y$ defines a map of functions $\mathcal{O}(Y) \xrightarrow{\phi^*} \mathcal{O}(X)$ in the opposite direction (the pull-back $\phi^*(f) = f\circ\phi$). The statement

information contained in two kinds of objects is the same but the directions reverse when one passes from geometry to algebra

is stated in categorical terms:

$$\text{categories } \mathcal{A}ff\mathcal{S}ch_{\Bbbk} \text{ and } (\mathcal{C}om\mathcal{A}lg_{\Bbbk})^o \text{ are equivalent.}$$

[4]

The basic strategy. Our intuition is often geometric. So, one starts by translating geometric ideas into precise statements in algebra. These are then proved in algebra. Once sufficiently many geometric statements are verified in algebra, one can build up on these and do more purely in geometry.

A.6.2. *Formal neighborhood of* $0 \in \mathbb{A}^1$. Consider the contravariant functor on $\Bbbk$-*Schemes*

$$\Bbbk\text{-}Schemes \ni X \mapsto F(X) = \{f \in \mathcal{O}(X);\ f \text{ is nilpotent}\} \in \mathcal{S}ets.$$

It is an increasing union of subfunctors

$$\Bbbk\text{-}Schemes \ni X \mapsto F_n(X) = \{f \in \mathcal{O}(X);\ f^{n+1} = 0\} \in \mathcal{S}ets.$$

[4]One can simplify this kind of thinking and *define* the category of affine schemes over $\mathbb{C}$ as the the opposite of the category of commutative $\mathbb{C}$-algebras. The part that would be skipped in this approach is how one develops a geometric point of view on affine schemes defined in this way.

Looking for geometric interpretation of these functor we start with the n^{th} infinitesimal neighborhood $IN^n_{\mathbb{A}^1_{\Bbbk}}(0)$ of the point 0 in the line $\mathbb{A}^1_{\Bbbk} = \text{Spec}(\Bbbk[x])$. This is the $\Bbbk$-scheme defined by the algebra

$$\mathcal{O}\big(IN^n_{\mathbb{A}^1_{\Bbbk}}(0)\big) \stackrel{\text{def}}{=} \Bbbk[x]/x^{n+1}, \quad \text{i.e.,} \quad IN^n_{\mathbb{A}^1}(0) \stackrel{\text{def}}{=} \text{Spec}(\Bbbk[x]/x^{n+1}).$$

For instance, $IN^0_{\mathbb{A}^1}(0) = \{0\}$ is a point while $IN^1_{\mathbb{A}^1}(0)$ is a *double point*, etc.

We see that the functor F_n is representable – it is represented by the scheme $IN^n_{\mathbb{A}^1_{\Bbbk}}(0)$. Therefore, one should think of the functor F as the increasing union of infinitesimal neighborhoods of $0 \in \mathbb{A}^1$. For that reason we call F the *formal neighborhood of* $0 \in \mathbb{A}^1$.

A.6.3. *Formal neighborhood of a closed subscheme.* In general if Y is a closed subscheme of a scheme X given by the ideal $I_Y = \{f \in \mathcal{O}(X);\ f|Y = 0\}$, one can again define the n^{th} infinitesimal neighborhood of Y in X as an affine scheme

$$IN^n_X(Y) \stackrel{\text{def}}{=} \text{Spec}(\mathcal{O}(X)/I^{n+1}_Y),$$

and then one defines the formal neighborhood $FN_X(Y)$ as a $\Bbbk$-space which is the union of infinitesimal neighborhoods, i.e., as the functor

$$\Bbbk\text{-}\mathit{Schemes} \ni Z \mapsto \cup_n \ \ \mathit{Map}[Z, IN^n_X(Y)].$$

A.7. **Groupoids (groupoid categories).** We consider a special class of categories, the *groupoid categories* . We get a new respect for categories when we notice that this special case of categories, is a common generalization of both groups and equivalence relations.

A.7.1. A groupoid category is a category such that all morphisms are invertible (i.e., isomorphisms).

A.7.2. *Example: Group actions and groupoids.* An action of a group G on a set X, produces a category X_G with $Ob(X_G) = X$ and

$$\text{Hom}_{X_G}(a, b) \stackrel{\text{def}}{=} \{(b, g, a);\ g \in G \quad \text{and} \quad b = \ ga\}.$$

Here $1_a = (a, 1, a)$ and the composition is given by multiplication in G: $(c, h, b)\circ(b, g, a) \stackrel{\text{def}}{=} (c, hg, a)$. This is a groupoid category: $(b, g^{-1}, a)\circ(b, g, a) \stackrel{\text{def}}{=} (a, 1, a)$.

A.7.3. *Example: Equivalence relations.* Any equivalence relation $\cong$ on a set X defines a category $X_{\cong}$ with $Ob(X_{\cong}) = X$ and $\text{Hom}_{X_{\cong}}(a, b)$ is a point $\{b, a)\}$ if $a \cong b$ and otherwise $\text{Hom}_{X_{\cong}}(a, b) = \emptyset$. The composition is $(c, b)\circ(b, a) \stackrel{\text{def}}{=} (c, a)$ and $1_a = (a, a)$. This is a groupoid category: $(a, b)\circ(b, a) = 1_a$.

A.7.4. *Lemma.* Let $\mathcal{C}$ be a groupoid category.

(a) A groupoid category $\mathcal{C}$ gives: a set $\pi_0(\mathcal{C})$ of isomorphism classes of objects of $\mathcal{C}$, and
(b) for each object $a \in \mathcal{G}$ a group $\pi_1(\mathcal{C}, a) \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{C}}(a, a)$.

(b) If $a, b \in \mathcal{C}$ are isomorphic then $\text{Hom}_{\mathcal{C}}(a, b)$ is a *bitorsor* for $(\text{Hom}_{\mathcal{C}}(a, a), \text{Hom}_{\mathcal{C}}(b, b))$, i.e., a torsor for each of the groups $\text{Hom}_{\mathcal{C}}(a, a)$ and $\text{Hom}_{\mathcal{C}}(a, a)$, and the actions of the two groups commute.

(c) A groupoid category on one object is the same as a group.

A.7.5. *Examples.* (1) For the *action groupoid* associated to an action of G on X

$$\pi_0(X_G) = X/G \quad \text{and} \quad \pi_1(X_G, a) = \ G_a.$$

(2) If $X_{\cong}$ is the groupoid given by an equivalence relation $\cong$ on X then

$$\pi_0(X_{\cong}) = X/\cong \quad \text{and} \quad \pi_1(X_{\cong}, a) = \ \{1\}.$$

A.7.6. *Remarks.* Passing from a groupoid category $\mathcal{C}$ to the set $\pi_0(\mathcal{C})$ of isomorphism classes in $\mathcal{C}$, the main information we forget is the *automorphism groups* $\text{Hom}_{\mathcal{C}}(a, a) = \text{Aut}_{\mathcal{C}}(a)$ of objects.

To see the importance of this loss, we will blame the formation of singularities in the invariant theory quotients on passing from a groupoid category to the set of isomorphism classes. Remember that when $G = \{\pm 1\}$ acting on $X = \mathbb{A}^2$, one can organize the set theoretic quotient X/G into algebraic variety $X//G$ which has one singular point – the image of $\mathbf{0} = (0, 0$. Recall that $\mathbf{0}$ is the only point in X which has a non-trivial stabilizer, i.e., which has a non-trivial automorphism group $\text{Aut}_{X_{\cong}}(\mathbf{0})$ when we encode the action of G on X as a category structure $X_{\cong}$ on X.

So, the hint we get from this example is:

> *One may be able to remove some singularities in sets of isomorphism classes by remembering the automorphisms, i.e., remembering the corresponding groupoid category rather then just the set of isomorphism classes of objects.*

This is the principle behind the introduction of *stack quotients.*

Appendix B. **Manifolds**

B.1. **Real manifolds.**

B.1.1. *Charts, atlases, manifolds.* A homeomorphism $U \xrightarrow{\phi} V$ with $M \overset{open}{\supseteq} U$ and $V \overset{open}{\subseteq} \mathbb{R}^n$ for some n, is called a *local chart* on the topological space M. Two charts $(U_k \xrightarrow{\phi} V_k)$ on

M $(k \in \{i,j\})$, are said to be *compatible* if (for $U_{ij} = U_i \cap U_j$), the *comparison function* (or *transition function*),

$$V_j \supseteq \phi_j(U_{ij}) \xrightarrow{\phi_{ij} \overset{\text{def}}{=} \phi \circ \phi_j{}^{-1}} \phi_i(U_{ij}) \subseteq V_i$$

is a C^∞-map between two open subsets of $\mathbb{R}^n$. An atlas on M is a family of compatible charts on M that cover M.

We say that any atlas defines on M a *structure of a manifold*, and two atlases define the same manifold structure if they are compatible, i.e., if their union is again an atlas.

So, "compatible" is an equivalence relation on atlases, and a structure of a manifold on a topological space M is precisely an equivalence class of compatible atlases on M. On the other hand, if $\mathcal{A}$ is an atlas on M the set $\widetilde{\mathcal{A}}$ of all charts on M that are compatible with the charts in $\mathcal{A}$ is a maximal atlas on M. So, any equivalence class of atlases contains the largest element and we can think of manifold structures on M as *maximal atlases* on M.[5]

B.1.2. *Once again.* A real manifold M of dimension n is a topological space M which is locally isomorphic to $\mathbb{R}^n$ in a smooth way and without contradictions. Here,

- *Locally isomorphic to* $\mathbb{R}^n$ means that we are given an open cover U_i, $i \in I$, of M, and for each $i \in I$ a topological identification (*homeomorphism*), $\phi : U_i \xrightarrow{\cong} V_i$ with V_i open in $\mathbb{R}^n$.
- *Smooth way without contradictions* means that for any $i,j \in I$ (and $U_{ij} = U_i \cap U_j$), the transition function

$$V_j \supseteq \phi_j(U_{ij}) \xrightarrow{\phi_{ij} \overset{\text{def}}{=} \phi \circ \phi_j{}^{-1}} \phi_i(U_{ij}) \subseteq V_i$$

  is a C^∞-map between two open subsets of $\mathbb{R}^n$.

B.1.3. *The sheaf* C^∞_M *of smooth functions on a manifold* M. For any open $U \subseteq M$ we define $C^\infty(U,\mathbb{R})$ to consist of all functions $f : U \to \mathbb{R}$ such that for any chart $(U_i \xrightarrow{\phi} V_i)$ the function $f \circ \phi^{-1} : \ \phi_i(U \cap U_i) \to \mathbb{R}$ is C^∞ on the open subset $\phi_i(U \cap U_i) \subseteq V_i \subseteq \mathbb{R}^n$.

Because of the *no-contradiction* policy one does not have to check all charts, but only sufficiently many to cover U.

Lemma. (a) Though the definition of C^∞_M is complicated, locally we get just the usual smooth functions on $\mathbb{R}^n$. If U lies in some chart (U_i, ϕ, V_i) (i.e., in $U \subseteq U_i$), then ϕ_i gives identification $C^\infty(U) \cong C^\infty(\phi_i(U))$ of smooth fonctions on U with smooth functions on an open part of $\mathbb{R}^n$.

(b) C^∞_M is a sheaf of $\mathbb{R}$-algebras on M,, i.e.,

- (0) for each open $U \subseteq X$ $C^\infty(U)$ is an $\mathbb{R}$-algebra,

[5] Later, we will find a nicer way to describe the manifold structure in terms of *ringed spaces*.

- (1) for each inclusion of open subsets $V{\subseteq}U{\subseteq}X$ the restriction map $C^\infty(U) \xrightarrow{\rho^U_V} C^\infty(V)$ is map of $\mathbb{R}$-algebras

and these data satisfy

- (Sh0) $\rho^U_U = id$
- (Sh1) (Transitivity of restriction) $\rho^U_V \circ \rho^U_V = \rho^U_W$ for $W{\subseteq}V{\subseteq}U$
- (Sh2) (Gluing) If $(W_j)_{j\in J}$ is an open cover of an open $U{\subseteq}M$[6], we ask that any family of compatible $f_j \in C^\infty(W_j)$, $j \in J$, glues uniquely. This means that if all f_j agree on intersections in the sense that $\rho^{W_i}_{W_{ij}} f_i = \rho^{W_i}_{W_{ij}} f_j$ in $C^\infty(W_{ij})$ for any $i, j \in J$; then there is a unique $f \in C^\infty(U)$ such that $\rho^U_{W_j} f = f_j$ in $C^\infty(W_j)$, $j \in J$.
- (Sh3) $C^\infty(\emptyset)$ is $\{0\}$.

Proof. (a) is clear from definitions. The notion of $\mathcal{F}$ is a sheaf", that appears in (b), is really a shorthand for "$\mathcal{F}$ is of local nature", i.e., "$\mathcal{F}$ is defined by some local property". Now C^∞_M is a sheaf because to check that a function f on U is smooth, one only has to check locally, i.e., one has to consider f on a small neighborhood of each point.

B.1.4. *Examples.* The following are real manifolds

(1) $M = \mathbb{R}^n$
(2) M an open subset of $\mathbb{R}^n$
(3) $M = S^1$ or $M = S^n$.
(4) $M = \mathbb{RP}^1$ or $M = \mathbb{RP}^n$.

B.1.5. *Category of real manifolds.* For two real manifolds M', M'' we define the set $\mathrm{Hom}(M', M'') = Map(M', M'')$ of *smooth maps* or *morphisms of manifolds* , to consist of all maps $F : M' \to M''$ which are smooth when checked in local charts.

This means that for each $x \in M'$ there are charts $M'{\supseteq}U_i \xrightarrow{\phi} V_i{\subseteq}\mathbb{R}^{m'}$ and $M''{\supseteq}U''_j \xrightarrow{\phi} V''_j{\subseteq}\mathbb{R}^{m''}$, such that $x \in U'_i$ and $F(x) \in U''_j$, and the map

$$\begin{array}{ccc} U'_i \cap F^{-1}U''_j & \xrightarrow{F} & U''_j \\ {\scriptstyle \phi'_i}\downarrow & & {\scriptstyle \phi''_j}\downarrow \\ V'_i \supseteq \phi'_i(U'_i \cap F^{-1}U''_j) & \xrightarrow{F_{ij}} & \phi''_j(U''_j) \subseteq V''_j \end{array}$$

is a smooth map between open subsets of $\mathbb{R}^{m'}$ and $\mathbb{R}^{m''}$.

Again, no-contradiction policy implies that if the above is true for one pair of charts at x and $F(x)$, it is true for any pair of charts.

[6]We denote $W_{ij} = W_i \cap W_j$ etc.!

B.1.6. *Examples.*

(1) For any manifold M, $\mathrm{Hom}(M,\mathbb{R}^n)=C^\infty(M,\mathbb{R})^n$.
(2) A smooth map $F\in\mathrm{Hom}(M,N)$ defines for any pair of open subsets $U\subseteq M$ and $V\subseteq N$ the pull-back map $C^\infty_N(V)\xrightarrow{F^*}C^\infty_M(U),\ g\mapsto F^*g=g\circ F|U$.

B.2. **The (co)tangent bundles.**

B.2.1. *Cotangent spaces $T^*_a(M)$.* The cotangent space at a point $m\in M$ is defined by

$$T^*_m(M)\overset{\text{def}}{=}\mathfrak{m}_a/\mathfrak{m}_a^2\quad\text{for}\ \ \mathfrak{m}_a\overset{\text{def}}{=}\{g\in C^\infty(M);\ g(a)=0\}.$$

For any open $U\subseteq M$ and $f\in C^\infty(U)$, the differential at a of f is defined as the image

$$d_af\overset{\text{def}}{=}(f-f(a))+\mathfrak{m}_a^2\in\ T^*_a(M)$$

of $f-f(a)$ in T^*_aM.

B.2.2. *Tangent spaces $T_a(M)$.* The *tangent vectors* at $a\in M$ are the "derivatives at a", i.e., all linear functionals in the tangent space

$$T_m(M)\overset{\text{def}}{=}\{\xi\in\mathrm{Hom}_{\mathbb{R}}[C^\infty(M),\mathbb{R}];\ \xi(fg)=\xi(f)\cdot g(a)+f(a)\cdot\xi(g)\}.$$

The *vector fields* on M are all "derivatives on M", i.e., all linear operators in

$$X(M)\overset{\text{def}}{=}\{\Xi\in\mathrm{Hom}_{\mathbb{R}}[C^\infty(M),C^\infty(M)];\ \Xi(fg)=\Xi(f)\cdot g+f\cdot\Xi(g)\}.$$

A vector field Ξ defines a tangent vector $\Xi_a\in T_a(M)$ at each point $a\in M$

$$\Xi_a(f)\overset{\text{def}}{=}(\Xi f)(a),\qquad f\in C^\infty(M).$$

B.2.3. *Local coordinates.* For any open $U\subseteq M$, we say that functions $x_1,...,x_n\in C^\infty(U)$ form a coordinate system on U if $\phi=(x_1,...,x_n):U\to\mathbb{R}^n$ gives a chart, i.e.,

- $\phi(U)$ is open in $\mathbb{R}^n$,
- $\phi:U\to\phi(U)$ is a bijection, and
- the inverse function is a map of manifolds.

By the Implicit Function Theorem the last condition is equivalent to

For each $a\in U$, the differentials d_ax_i form a basis of T^*_aM.

B.2.4. *Vector bundles.* Now we consider how to organize all tangent spaces $T_aM,\ a\in M$, into one manifold TM and what is the natural level of organization (structure) on TM.

Vector bundle is the relative version of the notion of a vector space. First, if M is a set a vector bundle V over M consists of a map $V\xrightarrow{p}M$ and a structure of a vector space on each fiber $V_m=p^{-1}(m),\ m\in M$. Next, if M is a *topological space*, we also ask that V is a topological space, the map $V\xrightarrow{p}M$ is continuous and the vector space structure of the fibers does not change wildly in the sense that

each $m\in M$ has a neighborhood U such that there exists a homeomorphism $\phi:V|U\to U\times\mathbb{R}^n$ which

- *maps each fiber to a fiber, i.e., the diagram*

$$\begin{array}{ccc} V|U & \xrightarrow{\ \phi\ } & U\times\mathbb{R}^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle pr_U} \\ U & \xrightarrow{\ =\ } & U \end{array}$$

  commutes,
- *The restriction of ϕ to fibers is an isomorphism of vector spaces.*

Finally, if M is a *manifold*, we ask that V is a manifold, the map $V\xrightarrow{p}M$ is a map of manifolds and the vector space structure on fibers changes smoothly in the sense that

each $m\in M$ has a neighborhood U such that there exists an isomorphism of manifolds $\phi:V|U\to U\times\mathbb{R}^n$, which preserves fibers and the restrictions of ϕ to fibers are isomorphisms of vector spaces.

Examples. (a) For any manifold M,

$$TM\overset{\text{def}}{=}\cup_{a\in M}\ T_aM \quad\text{and}\quad T^*M\overset{\text{def}}{=}\cup_{a\in M}\ T_a^*M$$

are naturally vector bundles over the manifold M.

(b) For a vector bundle V on M, any map of manifolds $f:N\to M$ can be used to pull-back the vector bundle V to a vector bundle

$$f^*V\overset{\text{def}}{=}\ \cup_{n\in N}\ V_{f(n)}$$

on N. So, by definition $(f^*V)_n=V_{f(n)}$, i.e., the fiber of f^*V at $n\in N$ is the same as the fiber of V at $f(n)\in M$.

B.3. **Constructions of manifolds.**

B.3.1. *The differential of manifold maps.* A map of manifolds $f:M\to N$, produces for any open $V\subseteq N$ and $U=\subseteq M$ such that $f(U)\subseteq V$, the pull-back of functions

$$f^*:C_N^\infty(V)\to C_M^\infty(U),\quad \phi\mapsto f^*\phi\overset{\text{def}}{=}\phi\circ f|U.$$

For each $a\in M$, $f^*I_{f(a)}^N\subseteq I_a^M$, so we get a linear map

$$d_a^*f:\ T_{f(a)}^*(N)=I_{f(a)}^N/(I_{f(a)}^N)^2\to I_a^M/(I_a^M)^2=T_a^*(M),\quad d_af(d_{f(a)}\phi)\overset{\text{def}}{=}d_a(\phi\circ f).$$

In other words,

$$d_af(\ [\phi-\phi(f(a))]+(I_{f(a)}^N)^2\ )\ =\ [\phi\circ f-(\phi\circ f)(a))]+(I_a^M)^2.$$

In the opposite (covariant) direction one has the map called the differential of f

$$d_af:\ T_a(M)\to T_{f(a)}(N),\ (d_af\xi)\phi\overset{\text{def}}{=}\xi(f^*\phi)\ =\ \xi(\phi\circ f)$$

which is the adjoint of d_a^f. In terms of the local coordinates x_i around $a\in M$ and y_j around $f(a)\in N$,

$$(d_af)\partial_{i,a}\ =\ \sum_j\ \partial_{i,a}(y_j\circ f)\cdot\partial_{j,f(a)}$$

and the matrix $(\partial_{i,a}(y_j\circ f))_{i,j}$ of d_af in the bases $\partial_{i,a}$, $\partial_{j,f(a)}$ is called the Jacobian of f at a.

B.3.2. *Theorem.* Let $f:M\to N$ be a map of manifolds which is of *constant rank* (i.e., all differentials $d_af:T_a(M)\to T_{f(a)}(N)$ have the same rank). Then the fibers $M_b\overset{\text{def}}{=}f^{-1}b,\ b\in N$, are naturally manifolds.

This is again a consequence of the Implicit Function Theorem.

B.3.3. *Examples.* Let $f\in C^\infty(M)$ and $b\in\mathbb{R}$ be such that $d_af\neq 0$ for any $a\in M_b$. Then M_b is a submanifold.

Proof. $d_af\neq 0$ for any $a\in M_b$, so the same is true for a in some neighborhood U of M_b. Now, $M_b=f^{-1}b=\ (f|U)^{-1}b$ and on U the rank is 1.

B.4. **Complex manifolds.** A complex manifold M of dimension n is a topological space M which is locally isomorphic to $\mathbb{C}^n$ in a holomorphic way and without contradictions. Here,

- *Locally isomorphic to* $\mathbb{C}^n$ means that we are given an open cover $U_i,\ i\in I$, of M, and for each $i\in I$ a topological identification (*homeomorphism*), $\phi:U_i\xrightarrow{\cong}V_i$ with V_i open in $\mathbb{C}^n$.
- *In a holomorphic way* means that for any $i,j\in I$, the transition function ϕ_{ij} is a holomorphic map between two open subsets of $\mathbb{C}^n$.[7]

[7] ...

- *No contradictions* means that for any $i, j, k \in I$, the two identifications of $\phi_k(U_{ijk}) \subseteq V_k$ and $\phi_i(U_{ijk}) \subseteq V_i$, are the same: $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$.

We call each $(U_i \xrightarrow{\phi} V_i)$ a local chart on the manifold. A collection $(U_i \xrightarrow{\phi} V_i)_{i \in I}$, of charts on a topological space is said to be compatible if it satisfies the conditions *smooth way* and *no-contradictions*. A collection of compatible charts that cover M is called an *atlas* on M. We say that any atlas defines on M a structure of a manifold, and two atlases define the same manifold structure if they are compatible, i.e., if their union is again an atlas.

So a structure of a manifold on a topological space M can be viewed is an equivalence class of compatible atlases on M. On the other hand, if $\mathcal{A}$ is an atlas on M the set $\widetilde{\mathcal{A}}$ of all charts on M that are compatible with the charts in $\mathcal{A}$ is a maximal atlas on M. So, any equivalence class of atlases contains the largest element.

B.4.1. *The sheaf $\mathcal{O}_M^{an}$ of holomorphic functions on a manifold M.* For any open $U \subseteq M$ we define $\mathcal{O}^{an}(U, \mathbb{R})$ to consist of all functions $f : U \to \mathbb{R}$ such that for any chart $(U_i \xrightarrow{\phi} V_i)$ the function $f \circ \phi^{-1} : \ \phi_i(U \cap U_i) \to \mathbb{R}$ is $\mathcal{O}^{an}$ on the open subset $\phi_i(U \cap U_i) \subseteq V_i \subseteq \mathbb{R}^n$.

Because of the *no-contradiction* policy one does not have to check all charts, but only sufficiently many to cover U.

Lemma. (a) If U lies in some chart U_i then ϕ gives identification $\mathcal{O}^{an}(U) \cong \mathcal{O}^{an}(\phi_i(U))$ of holomorphic fonctions on U with holomorphic functions on an open part of $\mathbb{C}^n$.

(b) $\mathcal{O}_M^{an}$ is a sheaf of $\mathbb{C}$-algebras on M,, i.e.,

- (0) for each open $U \subseteq X$ $\mathcal{O}^{an}(U)$ is a $\mathbb{C}$-algebra,
- for each inclusion of open subsets $V \subseteq U \subseteq X$ the restriction map $\mathcal{O}^{an}(U) \xrightarrow{\rho_V^U} \mathcal{O}^{an}(V)$ is map of $\mathbb{C}$-algebras

and these data satisfy

- (Sh0) $\rho_U^U = id$
- (Sh1) (Transitivity of restriction) $\rho_V^U \circ \rho_V^U = \rho_W^U$ for $W \subseteq V \subseteq U$
- (Sh2) (Gluing) If $(W_j)_{j \in J}$ is an open cover of an open $U \subseteq M$ we ask that any family of compatible $f_j \in \mathcal{O}^{an}(W_j)$, $j \in J$, glues uniquely.
- (Sh3) $\mathcal{O}^{an}(\emptyset)$ is $\{0\}$.

B.4.2. *Examples.*

(1) $M = \mathbb{C}^n$
(2) M an open subset of $\mathbb{C}^n$
(3) $M = \mathbb{CP}^1$ or $M = \mathbb{CP}^n$.

B.4.3. *Category of complex manifolds.* For two complex manifolds M', M'' we define the set $\mathrm{Hom}(M', M'') = Map(M', M'')$ of *holomorphic maps* or *morphisms of complex manifolds* to consist of all maps $F : M' \to M''$ which are holomorphic when checked in local charts.

B.4.4. *Examples.*

(1) For any manifold M, $\mathrm{Hom}(M, \mathbb{C}^n) = \mathcal{O}^{an}(M, \mathbb{C})^n$.
(2) A holomorphic map $F \in \mathrm{Hom}(M, N)$ defines for any pair of open subsets $U \subseteq M$ and $V \subseteq N$ the pull-back map $\mathcal{O}^{an}_N(V) \xrightarrow{F^*} \mathcal{O}^{an}_M(U)$, $g \mapsto F^*g = g \circ F|U$.

B.5. **Manifolds as ringed spaces.** We will see that a geometric space (for instance a manifold of a certain type) can naturally be thought of as a topological space with a sheaf of rings.

B.5.1. *Ringed spaces.* A ringed space consists of a topological space X and a sheaf of rings $\mathcal{O}$ on X. Usually we call $\mathcal{O}$ the structure sheaf of X and we denote it $\mathcal{O}_X$.

B.5.2. *Real manifolds as ringed spaces.* As we have seen, any real manifold M defines a ringed space (M, C^∞_M). Actually,

Lemma. (a) For a manifold M one can recover the manifold structure on M from the sheaf of rings C^∞_M[8]

(b) Manifolds are the same as ringed spaces $(X, \mathcal{O}_X)$ that are locally isomorphic to $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$.[9]

B.5.3. *Complex manifolds as ringed spaces.* The story is the same. Any complex manifold M defines a locally ringed space $(M, \mathcal{O}^{an}_M)$. Actually, complex manifolds are the same as ringed spaces $(X, \mathcal{O}_X)$ that are locally isomorphic to $(\mathbb{C}^n, \mathcal{O}^{an}_{\mathbb{C}^n})$.

B.5.4. *Terminology.* We will speak of a $\Bbbk$-manifold $(M, \mathcal{O}_M)$ where $\Bbbk$ is either $\mathbb{R}$ or $\mathbb{C}$, and we will mean the above notion of a real manifold with $\mathcal{O}_M = C^\infty_M$ if $\Bbbk = \mathbb{R}$, or the above notion of a complex manifold with $\mathcal{O}_M = \mathcal{O}^{an}_M$ if $\Bbbk = \mathbb{C}$.

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[8]The largest atlas for the manifold M consists of all data $M \overset{open}{\supseteq} U \xrightarrow[homeomorphism]{\phi} V \overset{open}{\subseteq} \mathbb{R}^n$, such that for any $g \in C^\infty(V)$ the pull-back $g \circ \phi$ is in $C^\infty_M(U)$.

[9]This means that X can be covered by open sets U such that

(1) there is a homeomorphism $\phi : U \xrightarrow{\cong} V$ with V open in some $\mathbb{R}^n$, with the property that
(2) for any U' open in U, the restriction of ϕ to $U' \to \pi(U') = V'$ identifies $\mathcal{O}_X(U')$ and $C^\infty_{\mathbb{R}^n}(V')$.

B.5.5. *Use of sheaves.* Sheaves are more fundamental for $\mathbb{C}$-manifolds then for $\mathbb{R}$-manifolds because for an $\mathbb{R}$-manifold M, all information is contained in one ring $C^\infty(M)$, while for a $\mathbb{C}$-manifold the global functions need not contain enough information – for instance $\mathcal{O}^{an}(\mathbb{CP}^n) = \mathbb{C}$. This forces one to control all local function rather then just the global functions (i.e., the sheaf $\mathcal{O}_M$ rather then just $\mathcal{O}_M(M)$).

However, the general role of sheaves is that they control the relation between local and global objects, and this make them useful in many a context.

B.6. **Manifolds as locally ringed spaces.** We saw that geometric space can naturally be thought of as a ringed spaces, actually their geometric nature will be reflected in a special property of the corresponding ringed spaces – these are the *locally ringed spaces.*

B.6.1. *Stalks.* The stalk of the sheaf $\mathcal{O}$ at $a \in X$ is intuitively $\mathcal{O}(\mathcal{U})$ for a "very small neighborhood $\underline{U}$ of a". More precisely, if $a \in V \subseteq U$ then $\mathcal{O}(U)$ and $\mathcal{O}(V)$ are related by the restriction map $\mathcal{O}(U) \xrightarrow{\rho^U_V} \mathcal{O}(V)$, and the stalk at a is a certain limit of these restriction maps (called *inductive limit* or *colimit*), i.e.,

$$\mathcal{O}_a \overset{\text{def}}{=} \lim_{\substack{\rightarrow \\ U \ni a}} \mathcal{O}(U)$$

of $\mathcal{O}(U)$ over smaller and smaller neighborhoods U of a in X.

The elements of $\mathcal{O}_a$ are called the *germs* of $\mathcal{O}$-functions at a, and $\mathcal{O}_a$ can be described in en elementary way

(1) For any neighborhood U of a point a any $f \in \mathcal{O}(U)$ defines a germ $\underline{f}_a = \underline{(U, f)}_a \in \mathcal{O}_a$, and any germ is obtained in this way.
(2) Two germs $\underline{(U, f)}_a$ and $\underline{(V, g)}_a$ at a, are the same if there is neighborhood $W \subseteq U \cap V$ such that $f = g$ on W.

Then one defines the structure of a ring on $\mathcal{O}_a$ by

$$\underline{(U, f)}_a + \underline{(V, g)}_a \overset{\text{def}}{=} \underline{(U \cap V, f + g)}_a \quad \text{and} \quad \underline{(U, f)}_a \cdot \underline{(V, g)}_a \overset{\text{def}}{=} \underline{(U \cap V, f \cdot g)}_a.$$

B.6.2. *Local rings.* A commutative ring A is said to be a *local ring* if it has the *largest proper ideal.*

Examples.

(1) Any field is local, the largest ideal is 0.
(2) The ring of formal power series $\Bbbk[[X_1, ..., X_n]]$ over a field $\Bbbk$ is local, the largest ideal $\mathfrak{m}$ consists of series that vanish at 0 (i.e, the constant term is 0).
(3) $\mathbb{C}[x]$ is not at all local.

A commutative ring is local iff it has precisely one maximal ideal (then this is the largest ideal). Remember that maximal ideals correspond to the naive notion of "ordinary" points of a space. So, uniqueness of a maximal ideal in a ring A intuitively means that this ring corresponds to a *space with one ordinary point.*

B.6.3. *Locally ringed spaces.* We say that a ringed space $(X, \mathcal{O})$ is *locally ringed* if all $\mathcal{O}(U)$ are commutative rings and each stalk $\mathcal{O}_a$, $a \in X$, is a *local ring*, i.e., it has the *largest proper ideal.* This ideal is then denoted $\mathfrak{m}_a \subseteq \mathcal{O}_a$.

Example. The stalk of the sheaf of analytic functions $\mathcal{O}^{an}_{\mathbb{C}^n,0}$ consists of all formal series in n variables $f(Z_1, ..., Z_n) = \ sum_I \ f_I \cdot Z^I$ which converge on some ball around $0 \in \mathbb{C}^n$ (think of $\underline{(}U, f)_0$ as the expansion of f at 0). This is a local ring, and the largest ideal is

$$\mathfrak{m}_a \overset{\text{def}}{=} \mathcal{O}_a \cap \sum Z_i \cdot \mathbb{C}[[Z_1, ..., Z_n]] = \text{ all germs at } a \text{ of functions that vanish at } a.$$

Remark. Remember that a local ring intuitively corresponds to a space with one ordinary point. Therefore, it makes sense that the stalk $\mathcal{O}_{X,a}$ should be a local ring since $\mathcal{O}_{X,a}$ should only see one ordinary point – the point a.

B.6.4. *Manifolds as locally ringed spaces.* As we have seen, any manifold M (real or complex) defines a ringed space. Actually,

Lemma. The ringed space of any manifold M is a locally ringed space. The largest ideal $\mathfrak{m}_a$ of the stalk at a consists of germs of functions that vanish at a.

Proof. Let $\mathcal{O}$ be the structure sheaf (i.e., C^∞_M or $\mathcal{O}^{an}_M$) and let $\phi \in \mathcal{O}_a$ be the germ $\phi = \underline{(U, f)}_a$ of a function at a. If $\phi \notin \mathfrak{m}_a$, i.e., $f(a) \neq 0$ then the restriction of f to the neighborhood $V = f^{-1}\mathbb{k}^*$ (for $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$) of a, is invertible. Therefore ϕ is invertible (so ϕ can not lie in a proper ideal!).

Appendix C. **Abelian categories**

An abelian category is a category $\mathcal{A}$ which has the formal properties of the category $\mathcal{A}b$, i.e., we can do in $\mathcal{A}$ all computations that one can do in $\mathcal{A}b$.

C.1. **Additive categories.** Category $\mathcal{A}$ is additive if

- (A0) For any $a, b \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(a, b)$ has a structure of abelian group such that then compositions are bilinear.
- (A1) $\mathcal{A}$ has a zero object,
- (A2) $\mathcal{A}$ has sums of two objects,
- (A3) $\mathcal{A}$ has products of two objects,

C.1.1. *Lemma.* (a) Under the conditions (A0),(A1) one has (A3) $\Leftrightarrow$ (A4).

(b) In an additive category $a \oplus b$ is canonically the same as $a \times b$,

For additive categories $\mathcal{A}, \mathcal{B}$ a functor $F : \mathcal{A} \to \mathcal{B}$ is additive if the maps $\mathrm{Hom}_{\mathcal{A}}(a', a'') \to \mathrm{Hom}_{\mathcal{B}}(Fa', Fa'')$ are always morphisms of abelian groups.

C.1.2. *Examples.* (1) $\mathfrak{m}(\Bbbk)$, (2) $\mathcal{F}ree(\Bbbk)$, (3) $\mathcal{F}ilt\mathcal{V}ect_{\Bbbk} \overset{\mathrm{def}}{=}$ filtered vector spaces over $\Bbbk$.

C.2. **(Co)kernels and (co)images.** In module categories a map has kernel, cokernel and image. To incorporate these notions into our project of defining abelian categories we will find their abstract formulations.

C.2.1. *Kernels: Intuition.* Our intuition is based on the category of type $\mathfrak{m}(\Bbbk)$. For a map of $\Bbbk$-modules $M \xrightarrow{\alpha} N$

- the kernel $\mathrm{Ker}(\alpha)$ is a subobject of M,
- the restriction of α to it is zero,
- and this is the largest subobject with this property

C.2.2. *Categorical formulation.* Based on this, our general definition (in an additive category $\mathcal{A}$), of "k is a kernel of the map $a \xrightarrow{\alpha} b$", is

- we have a map $k \xrightarrow{\sigma} M$ from k to M,
- if we follow this map by α the composition is zero,
- map $k \xrightarrow{\sigma} M$ is universal among all such maps, in the sense that
  - all maps into a, $x \xrightarrow{\tau} a$, which are killed by α,
  - factor uniquely through k (i.e., through $k \xrightarrow{\sigma} a$).

  So, all maps from x to a which are killed by α are obtained from σ (by composing it with some map $x \to k$). This is the "universality" property of the kernel.

C.2.3. *Reformulation in terms of representability of a functor.* A compact way to restate the above definition is:

- The kernel of $a \xrightarrow{\alpha} b$ is any object that represents the functor

$$\mathcal{A} \ni x \mapsto {}_\alpha\mathrm{Hom}_{\mathcal{A}}(x,a) \stackrel{\text{def}}{=} \{\gamma \in \mathrm{Hom}_{\mathcal{A}}(x,a);\ \alpha\circ\gamma = 0\ \}.$$

One should check that this is the same as the original definition.

We denote the kernel by $\mathrm{Ker}(\alpha)$, but as usual, remember that

- this is not one specific object – it is only determined up to a canonical isomorphism,
- it is not only an object but a pair of an object and a map into a

.

C.2.4. *Cokernels.* In $\mathfrak{m}(\Bbbk)$ the cokernel of $M \xrightarrow{\alpha} N$ is $N/\alpha(M)$. So N maps into it, composition with α kills it, and the cokernel is universal among all such objects. When stated in categorical terms we see that we are interested in the functor

$$x \mapsto \mathrm{Hom}_{\mathcal{A}}(b,x)_\alpha \stackrel{\text{def}}{=} \{\tau \in \mathrm{Hom}_{\mathcal{A}}(b,x);\ \tau\circ\alpha = 0\ \},$$

and the formal definition is symmetric to the definition of a kernel:

- The cokernel of f is any object that represents the functor $\mathcal{A} \ni x \mapsto \mathrm{Hom}_{\mathcal{A}}(b,x)_\alpha$.

So this object $\mathrm{Coker}(\alpha)$ is supplied with a map $b \to \mathrm{Coker}(\alpha)$ which is universal among maps from b that kill α.

C.2.5. *Images and coimages.* In order to define the image of α we need to use kernels and cokernels. In $\mathfrak{m}(\Bbbk)$, $Im(\alpha)$ is a subobject of N which is the kernel of $N \to \alpha(M)$. We will see that the categorical translation obviously has a symmetrical version which we call coimage. Back in $\mathfrak{m}(\Bbbk)$ the coimage is $M/\mathrm{Ker}(\alpha)$, hence there is a canonical map $Coim(\alpha) = M/\mathrm{Ker}(\alpha) \to Im(\alpha)$, and it is an isomorphism. This observation will be the final ingredient in the definition of abelian categories. Now we define

- Assume that α has cokernel $b \to \mathrm{Coker}(\alpha)$, the image of α is $Im(\alpha) \stackrel{\text{def}}{=} \mathrm{Ker}[b \to \mathrm{Coker}(\alpha)]$ (if it exists).
- Assume that α has kernel $\mathrm{Ker}(\alpha) \to a$, the coimage of α is $Coim(\alpha) \stackrel{\text{def}}{=} \mathrm{Coker}[\mathrm{Ker}(\alpha) \to a]$. (if it exists).

C.2.6. *Lemma.* If α has image and coimage, there is a canonical map $Coim(\alpha) \to Im(\alpha)$, and it appears in a canonical factorization of α into a composition

$$a \to Coim(\alpha) \to Im(\alpha) \to b.$$

C.2.7. *Examples.* (1) In $\mathfrak{m}(\Bbbk)$ the categorical notions of a (co)kernel and image have the usual meaning, and coimages coincide with images.

(2) In $\mathcal{F}ree(\Bbbk)$ kernels and cokernels need not exist.

(3) In $\mathcal{FV}\overset{\text{def}}{=} \mathcal{F}ilt\mathcal{V}ect_{\Bbbk}$ for $\phi \in \text{Hom}_{\mathcal{FV}}(M_*, N_*)$ (i.e., $\phi : M \to N$ such that $\phi(M_k)\subseteq N_k,\ k \in \mathbb{Z}$), one has

- $\text{Ker}_{\mathcal{FV}}(\phi) = \text{Ker}_{\mathcal{V}ect}(\phi)$ with the induced filtration $\text{Ker}_{\mathcal{FV}}(\phi)_n = \text{Ker}_{\mathcal{V}ect}(\phi) \cap M_n$,
- $\text{Coker}_{\mathcal{FV}}(\phi) = N/\phi(M)$ with the induced filtration $\text{Coker}_{\mathcal{FV}}(\phi)_n =$ image of N_n in $N/\phi(M) = [N_n + \phi(M)]/\phi(M) \cong N_n/\phi(M) \cap N_n$.
- $Coim_{\mathcal{FV}}(\phi) = M/\text{Ker}(\phi)$ with the induced filtration $Coim_{\mathcal{FV}}(\phi)_n =$ image of M_n in $M/\text{Ker}(\phi) = M_n + \text{Ker}(\phi)/\text{Ker}(\phi) \cong = M_n/M_n \cap \text{Ker}(\phi)$,
- $Im_{\mathcal{FV}}(\phi) = Im_{\mathcal{V}ect}(\phi)\subseteq N$, with the induced filtration $Im_{\mathcal{FV}}(\phi)_n = Im_{\mathcal{V}ect}(\phi) \cap N_n$.

Observe that the canonical map $Coim_{\mathcal{FV}}(\phi) \to Im_{\mathcal{FV}}(\phi)$ is an isomorphism of vector spaces $M/\text{Ker}(\phi) \to Im_{\mathcal{V}ect}(\phi)$, however the two spaces have filtrations induced from filtrations on M and N respectively, and these need not coincide.

For instance one may have M and N be two filtrations on the same space V, if $M_k\subseteq N_k$ then $\phi = 1_V$ is a map of filtered spaces $M \to N$ and Ker = 0Coker so that $Coim_{\mathcal{FV}}(\phi) = M$ and $Im_{\mathcal{FV}}(\phi) = N$ and the map $Coim_{\mathcal{FV}}(\phi) \to Im_{\mathcal{FV}}(\phi)$ is the same as ϕ, but ϕ is an isomorphism iff the filtrations coincide: $M_k = N_k$.

C.3. **Abelian categories.** Category $\mathcal{A}$ is abelian if

- (A0-3) It is additive,
- It has kernels and cokernels (hence in particular it has images and coimages!),
- The canonical maps $Coim(\phi) \to Im(\phi)$ are isomorphisms

C.3.1. *Examples.* Some of the following are abelian categories: (1) $\mathfrak{m}(\Bbbk)$ including $\mathcal{A}b = \mathfrak{m}(\mathbb{Z})$. (2) $\mathfrak{m}_{fg}(\Bbbk)$ if $\Bbbk$ is noetherian. (3) $\mathcal{F}ree(\Bbbk)\subseteq \mathcal{P}roj(\Bbbk)\subseteq \mathfrak{m}(\Bbbk)$. (4) $\mathcal{C}^\bullet(\mathcal{A})$. (5) Filtered vector spaces.

C.4. **Abelian categories and categories of modules.**

C.4.1. *Exact sequences in abelian categories.* Once we have the notion of kernel and cokernel (hence also of image), we can carry over from module categories $\mathfrak{m}(\Bbbk)$ to general abelian categories our homological train of thought. For instance we say that

- a map $i : a \to b$ makes a into a subobject of b if $\text{Ker}(i) = 0$ (we denote it $a \hookrightarrow b$ or even informally by $a\subseteq b$, one also says that i is a monomorphism or informally that it is an inclusion),
- a map $q : b \to c$ makes c into a quotient of b if $\text{Coker}(q) = 0$ (we denote it $b \twoheadrightarrow c$ and say that q is an epimorphism or informally that q is surjective),

- the quotient of b by a subobject $a \xrightarrow{i} b$ is $b/a \stackrel{\text{def}}{=} \text{Coker}(i)$,
- a complex in $\mathcal{A}$ is a sequence of maps $\cdots A^n \xrightarrow{d^n} A^{n+1} \to \cdots$ such that $d^{n+1} \circ d^n = 0$, its cocycles, coboundaries and cohomologies are defined by $B^n = Im(d^n)$ is a subobject of $Z^n = \text{Ker}(d^n)$ and $H^n = Z^n/B^n$;
- sequence of maps $a \xrightarrow{\mu} b \xrightarrow{\nu} c$ is exact (at b) if $\nu \circ \mu = 0$ and the canonical map $Im(\mu) \to \text{Ker}(\nu)$ is an isomorphism.

Now with all these definitions we are in a familiar world, i.e., they work as we expect. For instance, sequence $0 \to a' \xrightarrow{\alpha} a \xrightarrow{\beta} a'' \to 0$ is exact iff a' is a subobject of a and a'' is the quotient of a by a', and if this is true then

$$\text{Ker}(\alpha) = 0, \text{Ker}(\beta) = a', \text{Coker}(\alpha) = a'', \text{Coker}(\beta) = 0, \text{Im}(\alpha) = a', \text{Im}(\beta) = a''.$$

The difference between general abelian categories and module categories is that while in a module category $\mathfrak{m}(\mathbb{k})$ our arguments often use the fact that $\mathbb{k}$-modules are after all abelian groups and sets (so we can think in terms of their elements), the reasoning valid in any abelian category has to be done more formally (via composing maps and factoring maps through intermediate objects). However, this is mostly appearances – if we try to use set theoretic arguments we will not go wrong:

C.4.2. *Theorem.* [Mitchell] Any abelian category is equivalent to a full subcategory of some category of modules $\mathfrak{m}(\mathbb{k})$.

Appendix D. **Abelian category of sheaves of abelian groups**

For a topological space X we will denote by $\mathcal{S}h(X) = \mathcal{S}heaves(X, \mathcal{A}b)$ the category of sheaves of abelian groups on X. Since a sheaf of abelian groups is something like an abelian group smeared over X, we hope to $\mathcal{S}h(X)$ is again an abelian category, i.e., that one can do the computations here the same way as one can do in the category $\mathcal{A}b$ of abelian groups. However, when we attempt to construct the cokernels of maps, we find that the first idea does not quite work – it produces something like a sheaf but without the gluing property. This forces us to

- (i) generalize the notion of sheaves to a weaker notion of a presheaf,
- (ii) find a canonical procedure that improves a presheaf to a sheaf.

We will also see another example that requires the same strategy: is the *pull-back* operation on sheaves.

Now it is easy to check that we indeed have an abelian category. What allows us to compute in this abelian category is the lucky break that one can understand kernels, cokernels, images and exact sequences just by looking at the stalks of sheaves.

D.1. **Categories of sheaves.** A presheaf of sets $\mathcal{S}$ on a topological space $(X, \mathcal{T})$ consists of the following data:

- for each open $U \subseteq X$ a set $\mathcal{S}(U)$,
- for each inclusion of open subsets $V \subseteq U \subseteq X$ a map $\mathcal{S}(U) \xrightarrow{\rho^U_V} \mathcal{S}(V)$ (called the restriction map);

and these data are required to satisfy

- (Sh0)(Transitivity of restriction) $\rho^U_V \circ \rho^U_V = \rho^U_W$ for $W \subseteq V \subseteq U$

A sheaf of sets on a topological space $(X, \mathcal{T})$ is a presheaf $\mathcal{S}$ which also satisfies

- (Sh1) (Gluing) Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of an open $U \subseteq X$ (We denote $U_{ij} = U_i \cap U_j$ etc.). We ask that any family of compatible sections $f_i \in \mathcal{S}(U_i)$, $i \in I$, glues uniquely. This means that if sections f_i agree on intersections in the sense that $\rho^{U_i}_{U_{ij}} f_i = \rho^{U_i}_{U_{ij}} f_j$ in $\mathcal{S}(U_{ij})$ for any $i, j \in I$; then there is a unique $f \in \mathcal{S}(U)$ such that $\rho^U_{U_i} f = f_i$ in $\mathcal{S}(U_i)$, $i \in I$.
- $\mathcal{S}(\emptyset)$ is a *point*.

D.1.1. *Remarks.* (1) Presheaves of sets on X form a category $pre\mathcal{S}heaves(X, \mathcal{S}ets)$ when $\text{Hom}(\mathcal{A}, \mathcal{B})$ consists of all systems $\phi = (\phi_U)_{U \subseteq X \ open}$ of maps $\phi_U : \mathcal{A}(U) \to \mathcal{B}(U)$ which are compatible with restrictions, i.e., for $V \subseteq U$

$$\begin{array}{ccc} \mathcal{A}(U) & \xrightarrow{\phi_U} & \mathcal{B}(U) \\ {\scriptstyle \rho^U_V}\downarrow & & {\scriptstyle \rho^U_V}\downarrow \\ \mathcal{A}(V) & \xrightarrow{\phi_V} & \mathcal{B}(V) \end{array} .$$

(One reads the diagram above as : "the diagram ... commutes".) The sheaves form a full subcategory $preSheaves(X, \mathcal{S}ets)$ of $\mathcal{S}heaves(X, \mathcal{S}ets)$.

(2) We can equally define categories of sheaves of abelian groups, rings, modules, etc. For a sheaf of abelian groups we ask that all $\mathcal{A}(U)$ are abelian groups, all restriction morphisms are maps of abelian groups, and we modify the least interesting requirement (Sh2): $\mathcal{S}(\phi)$ is the trivial group $\{0\}$. In general, for a category $\mathcal{A}$ one can define categories $pre\mathcal{S}heaves(X, \mathcal{A})$ and $\mathcal{S}heaves(X, \mathcal{A})$ similarly (the value on $\emptyset$ should be the final object of $\mathcal{A}$).

D.2. **Sheafification of presheaves.** We will use the wish to pull-back sheaves as a motivation for a procedure that improves presheaves to sheaves.

D.2.1. *Functoriality of sheaves.* Recall that for any map of topological spaces $X \xrightarrow{\pi} Y$ one wants a pull-back functor $\mathcal{S}heaves(Y) \xrightarrow{\pi^{-1}} \mathcal{S}heaves(X)$. As we have seen in the definition of a stalk of a sheaf (pull ,back to a point), the natural formula is

$$\underline{\pi^{-1}}(\mathcal{N})\ (U) \stackrel{\text{def}}{=} \lim_{\substack{\to \\ V \supseteq \pi(U)}} \mathcal{N}(V),$$

where limit is over open $V \subseteq Y$ that contain $\pi(U)$, and we say that $V' \leq V''$ if V'' better approximates $\pi(U)$, i.e., if $V'' \subseteq V'$.

D.2.2. *Lemma.* This gives a functor of presheaves $pre\mathcal{S}heaves(X) \xrightarrow{\underline{\pi^{-1}}} pre\mathcal{S}heaves(Y)$.

Proof. For $U' \subseteq U$ open, $\underline{\pi^{-1}}\mathcal{N}(U') = \lim_{\to\, V \supseteq \pi(U')} \mathcal{N}(V)$ and $\underline{\pi^{-1}}\mathcal{N}(U) = \lim_{\to\, V \supseteq \pi(U)} \mathcal{N}(V)$ are limits of inductive systems of $\mathcal{N}(V)$'s, and the second system is a subsystem of the first one, this gives a canonical map $\underline{\pi^{-1}}\mathcal{N}(U) \to \underline{\pi^{-1}}\mathcal{N}(U')$.

D.2.3. *Remarks.* Even if $\mathcal{N}$ is a sheaf, $\underline{\pi^{-1}}(\mathcal{N})$ need not be sheaf.

For that let $Y = pt$ and let $\mathcal{N} = S_Y$ be the constant sheaf of sets on Y given by a set S. So, $S_Y(\emptyset) = \emptyset$ and $S_Y(Y) = S$. Then $\underline{\pi^{-1}}(S_Y)\ (U) = \begin{cases} \emptyset & if\ U = \emptyset, \\ S & U \neq \emptyset \end{cases}$. We can say: $\underline{\pi^{-1}}(S_Y)\ (U) =$ constant functions from U to S. However, we have noticed that constant functions do not give a sheaf, so we need to correct the procedure $\underline{\pi^{-1}}$ to get sheaves from sheaves. For that remember that for the presheaf of constant functions there is a related sheaf S_X of *locally constant* functions.

Our problem is that the presheaf of constant functions is defined by a global condition (constancy) and we need to change it to a local condition (local constancy) to make it into a sheaf. So we need the procedure of

D.2.4. *Sheafification.* This is a way to improve any presheaf of sets $\mathcal{S}$ into a sheaf of sets $\widetilde{\mathcal{S}}$. We will imitate the way we passed from constant functions to locally constant functions. More precisely, we will obtained the sections of the sheaf $\tilde{\mathcal{S}}$ associated to the presheaf $\mathcal{S}$ in two steps:

(1) we glue systems of local sections s_i which are compatible in the weak sense that they are <u>*locally*</u> the same, and
(2) we identify two results of such gluing if the local sections in the two families are <u>*locally*</u> the same.

Formally these two steps are performed by replacing $\mathcal{S}(U)$ with the set $\widetilde{\mathcal{S}}(U)$, defined as the set of all equivalence classes of systems $(U_i, s_i)_{i\in I}$ where

(1) Let $\widehat{\mathcal{S}}(U)$ be the class of all systems $(U_i, s_i)_{i\in I}$ such that
   - $(U_i)_{i\in I}$ is an open cover of U and s_i is a section of $\mathcal{S}$ on U_i,
   - sections s_i are *weakly compatible* in the sense that they are locally the same, i.e., for any $i', i'' \in I$ sections $s_{i'}$ and $s_{i''}$ are the same near any point $x \in U_{i'i''}$. (Precisely, this means that there is neighborhood W such that $s_{i'}|W = s_{i''}|W$.)

(2) We say that two systems $(U_i, s_i)_{i\in I}$ and $(V_j, t_j)_{j\in J}$ are $\equiv$, iff for any $i \in I,\ j \in J$ sections s_i and t_j are weakly equivalent (i.e., for each $x \in U_i \cap V_j$, there is an open set W with $x \in W \subseteq U_i \cap V_j$ such that "$s_i = t_j$ on W" in the sense of restrictions being the same).

D.2.5. *Remark.* The relation $\equiv$ on $\widehat{\mathcal{S}}(U)$ really says that $(U_i, s_i)_{i\in I} \equiv (V_j, t_j)_{j\in J}$ iff the disjoint union $(U_i, s_i)_{i\in I} \sqcup (V_j, t_j)_{j\in J}$ is again in $\widehat{\mathcal{S}}(U)$.

D.2.6. *Lemma.* (a) $\equiv$ is an equivalence relation.

(b) $\widetilde{\mathcal{S}}(U)$ is a presheaf and there is a canonical map of presheaves $\mathcal{S} \xrightarrow{q} \widetilde{\mathcal{S}}$.

(c) $\widetilde{\mathcal{S}}$ is a sheaf.

D.2.7. *Sheafification as a left adjoint of the forgetful functor.* As usual, we have not invented something new: it was already there, hidden in the more obvious forgetful functor

D.2.8. *Lemma.* Sheafification functor $preSheaves \ni \mathcal{S} \mapsto \widetilde{\mathcal{S}} \in Sheaves$, is the left adjoint of the inclusion $Sheaves \subseteq preSheaves$, i.e, for any presheaf $\mathcal{S}$ and any sheaf $\mathcal{F}$ there is a natural identification

$$\mathrm{Hom}_{Sheaves}(\widetilde{\mathcal{S}}, \mathcal{F}) \xrightarrow{\cong} \mathrm{Hom}_{preSheaves}(\mathcal{S}, \mathcal{F}).$$

Explicitly, the bijection is given by $(\iota_{\mathcal{S}})_* \alpha = \alpha \circ \iota_{\mathcal{S}}$, i.e., $(\widetilde{\mathcal{S}} \xrightarrow{\alpha} \mathcal{F}) \mapsto (\mathcal{S} \xrightarrow{\iota_{\mathcal{S}}} \widetilde{\mathcal{S}} \xrightarrow{\alpha} \mathcal{F})$.

D.3. **Inverse and direct images of sheaves.**

D.3.1. *Pull back of sheaves (finally!)* Now we can define for any map of topological spaces $X \xrightarrow{\pi} Y$ a pull-back functor

$$Sheaves(Y) \xrightarrow{\pi^{-1}} Sheaves(X), \qquad \pi^{-1}\mathcal{N} \stackrel{\text{def}}{=} \widetilde{\underline{\pi^{-1}}\mathcal{N}}.$$

D.3.2. *Examples.* (a) A point $a \in X$ can be viewed as a map $\{a\} \xrightarrow{\rho} X$. Then $\rho^{-1}\mathcal{S}$ is the stalk $\mathcal{S}_a$.

(b) Let $a : X \to \mathrm{pt}$, for any set S one has $S_X = a^{-1}S$.

D.3.3. *Direct image of sheaves.* Besides the pull-back of sheaves which we defined in D.3.1, there is also a much simpler procedure of the push-forward of sheaves:

D.3.4. *Lemma.* (Direct image of sheaves.) Let $X \xrightarrow{\pi} Y$ be a map of topological spaces. For a sheaf $\mathcal{M}$ on X, formula

$$\pi_*(\mathcal{M})\ (V) \stackrel{\text{def}}{=} \mathcal{M}(\pi^{-1}V),$$

defines a sheaf $\pi_*\mathcal{M}$ on Y, and this gives a functor $Sheaves(X) \xrightarrow{\pi_*} Sheaves(Y)$.

D.3.5. *Adjunction between the direct and inverse image operations.* The two basic operations on sheaves are related by adjunction:

Lemma. For sheaves $\mathcal{A}$ on X and $\mathcal{B}$ on Y one has a natural identification

$$\mathrm{Hom}(\pi^{-1}\mathcal{B}, \mathcal{A}) \cong \mathrm{Hom}(\mathcal{B}, \pi_*\mathcal{A}).$$

Proof. We want to compare $\beta \in \mathrm{Hom}(\mathcal{B}, \pi_*\mathcal{A})$ with α in

$$\mathrm{Hom}_{\mathcal{S}h(X)}(\pi^{-1}\mathcal{B}, \mathcal{A}) = \mathrm{Hom}_{\mathcal{S}h(X)}(\widetilde{\underline{\pi^{-1}}\mathcal{B}}, \mathcal{A}) \cong \mathrm{Hom}_{pre\mathcal{S}h(X)}(\underline{\pi^{-1}}\mathcal{B}, \mathcal{A}).$$

α is a system of maps

$$\lim_{\to\ V \supseteq \pi(U)} \mathcal{B}(V) = \underline{\pi^{-1}}\mathcal{B}(U) \xrightarrow{\alpha_U} \mathcal{A}(U), \quad \text{for } U \text{ open in } X,$$

and β is a system of maps

$$\mathcal{B}(V) \xrightarrow{\beta_V} \mathcal{A}(\pi^{-1}V), \quad \text{for } V \text{ open in } Y.$$

Clearly, any β gives some α since

$$\lim_{\to\ V\supseteq\pi(U)} \mathcal{B}(V) \xrightarrow{\lim_{\to} \beta_V} \lim_{\to\ V\supseteq\pi(U)} \mathcal{A}(\pi^{-1}V) \to \mathcal{A}(U),$$

the second map comes from the restrictions $\mathcal{A}(\pi^{-1}V) \to \mathcal{A}(U)$ defined since $V\supseteq\pi(U)$ implies $\pi^{-1}V\supseteq U$.

For the opposite direction, any α gives for each V open in Y, a map $\lim_{\to\ W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W) = \underline{\pi^{-1}}\mathcal{B}(\pi^{-1}V) \xrightarrow{\alpha_{\pi^{-1}V}} \mathcal{A}(\pi^{-1}V)$. Since $\mathcal{B}(V)$ is one of the terms in the inductive system we have a canonical map $\mathcal{B}(V) \to \lim_{\to\ W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W)$, and the composition with the first map $\mathcal{B}(V) \to \lim_{\to\ W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W) \xrightarrow{\alpha_{\pi^{-1}V}} \mathcal{A}(\pi^{-1}V)$, is the wanted map β_V.

D.3.6. *Lemma.* (a) If $X \xrightarrow{\pi} Y \xrightarrow{\tau} Z$ then

$$\tau_*(\pi_*\mathcal{A}) \cong (\tau\circ\pi)_*\mathcal{A} \quad \text{and} \quad \tau_*(\pi_*\mathcal{A}) \cong (\tau\circ\pi)_*\mathcal{A}.$$

(b) $(1_X)_*\mathcal{A} \cong \mathcal{A} \cong (1_X)^{-1}\mathcal{A}$.

Proof. The statements involving direct image are very simple and the claims for inverse image follow by adjunction.

D.3.7. *Corollary.* (Pull-back preserves the stalks) For $a \in X$ one has $(\pi^{-1}\mathcal{N})_a \cong \mathcal{N}_{\pi(a)}$.

This shows that the pull-back operation which was difficult to define is actually very simple in its effect on sheaves.