ALGEBRAIC GEOMETRY
JAN 04

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The remaining two topics are **homological algebra** and **sheaves**. These are two general tools (not particular to geometry), that are useful for many kinds of mathematics and are standard in algebraic geometry.

We introduce homological algebra on the example of improved versions of intersections and fibers of maps. However the geometric content is mostly for fun, the real point is the homological algebra idea of uncovering a hidden part of constructions. This is then used to produce the cohomology of sheaves – a hidden part of the construction of taking global sections of sheaves.

The sheaves we are interested in are the (sheaves of sections of) line bundles on curves. So we will be calculating the cohomology of line bundles $\mathcal{O}_C(D)$ on a curve C, corresponding to various divisors D. Let $h^i(D)$ be the dimension of the i^{th} cohomology group $H^i[C, \mathcal{O}_C(D)]$. The number $h^0(D)$, has geometric content, this is the dimension of the vector space $\Gamma(X, \mathcal{O}_C(D)) = \mathcal{O}_C(D)\ (C)$, i.e., the number of global meromorphic functions on C that satisfy some restrictions on the positions of poles and zeros (which we specify by the choice of the divisor D). The reason we treat this geometric question in terms of sheaves is that it makes situation quite flexible (there are more sheaves then just the line bundles) and we can effectively do many calculations. Here, the higher cohomologies will be mostly a tool for calculating the zero$^{\text{th}}$ cohomology, i.e., the numbers $H^0(D)$.

We will first go through this basic application of homological algebra to algebraic geometry (cohomology of line bundles on curves), and then we will fill in some gaps by checking that the category of sheaves really has a structure of an abelian category, hence provides a setting for the use of homological algebra.

8. **Use homological algebra**

Homological algebra is a general tool, one can describe it as being in the business of observing the hidden part of the iceberg that is beneath the water level. That makes it useful in various areas of mathematics, and so in particular in algebraic geometry.

8.0.1. *What does the Homological algebra do?* Homological algebra is a general tool useful in various areas of mathematics. One tries to apply it to constructions that morally should contain more information then meets the eye. The homological algebra, if it applies, produces "derived" versions of the construction ("the higher cohomology"), which contain the "hidden" information. We will visit some examples of the use of homological algebra:

(1) **Cohomology of sheaves.** This is a very standard tool in algebraic geometry and we will try to understand how it works.

 Sheaves are a framework for dealing with an omnipresent problem of relating local and global information on a space. The global information is codified as the functor $\Gamma(X,-)$ of global sections of sheaves on a topological space X. When a sheaf has few global sections, more information may be contained in the derived construction – the cohomology of sheaves.

(2) **Subtle spaces.** This is a more advanced topic, so we will get just a glimpse.
   - The notion of *dg-schemes* (*differential graded schemes*) is a generalization of the notion of a scheme.[1] Formally, the difference is that the functions on a dg-scheme form a *commutative dg-algebra*, i.e., as we expect, we get a commutative algebra but it has some extra structure from homological algebra – the structure of a complex. The most obvious application is that such refined objects contain some more subtle information. However, we will only see how they appear in order to get *stable versions*[2] of calculations with ordinary algebraic varieties: the derived intersection, derived fiber etc.
   - A *D-brane* is a geometric space of a certain kind in string theory (contemporary physics). Mathematical formulation of a D-brane turns out to be a highly sophisticated constructs of homological algebra.

8.0.2. *Contents.* We will introduce the ideas of homological algebra on the example of *dg-schemes*. Actually, we only do one extremely simple illustrative computation and find one (sorry looking) example of a dg-scheme. The motivating idea is that the honored *Stability Principle* suggests that the fibers of maps should not jump in size. The reality does not comply until we change the notion of a fiber to a *dg-fiber* or *derived fiber*.

After that we go back to school and learn quickly the formalism of homological algebra in abelian categories. This will be needed for the more extensive use of homological algebra when we study sheaves in the next section.

[1] A few years old.

[2] As in *Stability Principle*.

8.1. **"Continuity" of fibers.** Consider a map $f : X \to Y$ of algebraic varieties. The fibers $f^{-1}(b) \subseteq X$ may jump in size as one varies b in Y. The class of maps for which the fiber does not jump are the so called *flat maps*, however many important maps are not flat. For instance we know that the blow up $\widetilde{V} \to V$ is not flat !

However, $f^{-1}(b) = \{a \in X;\ f(a) = b\}$ is the set of solutions of an algebraic equation (or system of such if we use coordinates). Now recall our beloved *Stability principle*: if a system of equations changes, the solutions should change continuously (i.e. no jumps in size). Clearly it does not apply to this situation. That's it.

Or maybe not. Remember that for $X = \mathbb{A}^2$, the simplest version $X^{(2)} = X^2/S_2$ of the moduli of unordered pairs of points turned out to be singular, we managed to modify it to a smooth version $X^{[2]}$ (the second Hilbert scheme of X), by remembering more data. $X^{(2)}$ remembers all unordered pairs of points, but $X^{[2]}$ in addition also remembers when two points collide the direction in which they approached each other.

So, if we just think of the fiber as a set the fibers most certainly may jump. In order to make this set theoretic jump less central we may try to add more to the notion of the fiber – it should be influenced by nearby fibers. So we should have an enhanced notion of a fiber which would remember something about the map f near b. Such notion appears naturally in algebra. To see this remember the construction of the

8.1.1. *Fibered product of varieties and tensor product of algebras.* Let X and Y are *varieties over* Z, in the sense that we remember certain maps $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ to a variety Z. The fibered product of X and Y over Z is defined as a set by

$$X \times_Z Y \stackrel{\text{def}}{=} \{(x, y) \in X \times Y;\ f(x) = g(y)\}.$$

It is actually an algebraic subvariety of $X \times Y$ since it is defined by an algebraic equation $f(x) = g(y)$.

Maps f and g give morphisms of algebras $\mathcal{O}(X) \xleftarrow{f^*} \mathcal{O}(Z) \xrightarrow{g^*} \mathcal{O}(Y)$, so one can form a tensor product of algebras

$$\mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y).$$

Now let X, Y, Z be affine (for simplicity), so that they are captured by their algebras of functions. Then there is the same information in the diagrams

$$X \xrightarrow{f} Z \xleftarrow{g} Y \quad \text{and} \quad \mathcal{O}(X) \xleftarrow{f^*} \mathcal{O}(Z) \xrightarrow{g^*} \mathcal{O}(Y),$$

so we can *hope* that the resulting constructions $X \times_Z Y$ and $\mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y)$ contain the same amount of information, i.e., that

Lemma. $\mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y) = \mathcal{O}(X \times_Z Y)$.

Proof. $X \times_Z Y$ is a subset of the affine variety $X \times Y$ given by algebraic equation $f(x) = g(y)$, so it is closed affine subvariety of $X \times Y$.

We will check that for any affine variety W, there is a canonical identification

$$Map(W, X\times_Z Y) \cong \ \mathrm{Hom}[\mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y), \mathcal{O}(W)].$$

But, an algebra map $F \in \mathrm{Hom}[\mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y), \mathcal{O}(W)]$ is the same as as a pair of algebra maps $\alpha \in \mathrm{Hom}[\mathcal{O}(X), \mathcal{O}(W)]$ and $\beta \in \mathrm{Hom}[\mathcal{O}(Y), \mathcal{O}(W)]$, such that the compositions are the same:

$$[\mathcal{O}(Z) \xrightarrow{f^*} \mathcal{O}(X) \xrightarrow{\alpha} \mathcal{O}(W)] \quad = \quad [\mathcal{O}(Z) \xrightarrow{g^*} \mathcal{O}(Y) \xrightarrow{\beta} \mathcal{O}(W)].$$

(In one direction α, β give $F(u\otimes v) = \alpha(u)\cdot\beta(v)$, in the opposite F gives $\alpha(u) = F(u\otimes 1)$.) But α and beta are the same as maps $W \xrightarrow{a} X$ and $W \xrightarrow{b} Y$, and the condition $\alpha\circ f^* = \beta\circ g^*$ is then the same as $f\circ A = g\circ B$. So the data contained in F amount to a map $W \xrightarrow{(A,B)} X\times Y$, such that the corresponding maps to Z, $f\circ A$ and $g\circ B$, are the same. But this precisely means that (A, B), maps W, to $X\times Y$.

Corollary. (a) If $X\subseteq Z\supseteq Y$ then $X\times_Z Y = X \cap Y$, so

$$\mathcal{O}(X \cap Y) = \ \mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y).$$

(b) If $X \xrightarrow{f} Z$ and $y \in Z$ we denote by $X_y = f^{-1}(y)$ the fiber of f at y. Then $X\times_Z Y = f^{-1}(y)$, so

$$\mathcal{O}(f^{-1}y) = \mathcal{O}(X_y) = \ \mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\Bbbk,$$

here the tensor product uses a homomorphism $\mathcal{O}(Z) \to \Bbbk = \mathcal{O}(\{y\})$ given by $\{y\} \to Z$.

8.1.2. *Higher fibered products.* The algebraic construction of tensor product $M\otimes_A N$ of two modules over a commutative algebra A, has a refinement which produces a sequence of A-modules called

$$M\otimes_A N = Tor_0^A(M, n),\ Tor_1^A(M, n),\ Tor_2^A(M, n),\ ...$$

Together they form and object called the derived tensor product $M\overset{L}{\otimes}_A N$. This object is a *complex* of A-modules. Complexes are standard objects in *Homological algebras.*

When applied to the situation of a fibered product of affine varieties we get $\mathcal{O}(Z)$-modules

$$\mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y) = \ Tor_0^{\mathcal{O}(Z)}(\mathcal{O}(X), \mathcal{O}(Y)),\ Tor_0^{\mathcal{O}(Z)}(\mathcal{O}(X), \mathcal{O}(Y)),\ Tor_0^{\mathcal{O}(Z)}(\mathcal{O}(X), \mathcal{O}(Y)),\ ...$$

and they glue together into the derived tensor product $\mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y)$ This object is a *complex* of $\mathcal{O}(Z)$-modules, but since $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are more then just $\mathcal{O}(Z)$-modules – they are $\mathcal{O}(Z)$-algebras – the derived tensor product $\mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y)$ is more then just a complex of $\mathcal{O}(Z)$-modules, it is an algebra type object in the world of complexes, and this is called a *differential graded* $\mathcal{O}(Z)$*-algebra*, or just a *dg-algebra over* $\mathcal{O}(Z)$.

Now this is going to produce a more refined version of the fibered product:

- *the derived fibered product of affine varieties X and Y over Z, is the space $X\overset{L}{\times}_Z Y$, such that the algebra of functions on $X\overset{L}{\times}_Z Y$ is*
$$\mathcal{O}(X)\overset{L}{\otimes}_{\mathcal{O}(Z)}\mathcal{O}(Y).$$

Now, since $\mathcal{O}(X)\overset{L}{\otimes}_{\mathcal{O}(Z)}\mathcal{O}(Y)$ is not just an algebra, $X\overset{L}{\times}_Z Y$ is not just a variety or a scheme. It is what is called a *differential graded scheme* or just a dg-scheme.

So, we get a refined version of a fibered product $X\times_Z Y$ of two varieties, and this refined version is a dg-scheme. (In particular we get refined versions of intersections and of fibers of maps: the derived intersection and the derived fiber.) So the entrance fee we pay for this game is that we have to extend the algebraic geometry to the category of dg-schemes, i.e., geometric objects such that their algebras of functions are not necessarily just commutative algebras but *commutative dg-algebras.*

8.2. **Homological algebra.** We will explain it on the example of categories $\mathfrak{m}(\Bbbk)$ of modules over a ring $\Bbbk$ (need not be commutative). The prototype is the example of the category of abelian groups $\mathcal{A}b=\mathfrak{m}(\mathbb{Z})$. However, the formalism works in other cases such as the all important category $\mathcal{S}h\mathcal{A}b(X)$ of sheaves of abelian groups on a topological space X. What is required of the category $\mathcal{A}$ in order to use homological algebra is that in $\mathcal{A}$ we have the basic notions that we use when we calculate with abelian groups: such as subobject, quotient, image of a map, addition of maps. Such categories are called *abelian categories.*

8.2.1. *Notion of a complex.* A complex of cochains is a sequence of $\Bbbk$-modules and maps
$$\cdots\xrightarrow{\partial^{-2}}C^{-1}\xrightarrow{\partial^{-1}}C^0\xrightarrow{\partial^0}C^1\xrightarrow{\partial^1}C^2\xrightarrow{\partial^2}\cdots,$$
such that the compositions of coboundary operators ∂^i are zero: $\partial^{i+1}\partial^i=0,\ i\in\mathbb{Z}$. We often omit the index on the coboundary operator, so we can write the preceding requirement as $\partial\circ\partial=0$.

From a complex of cochains we get three sequences of $\Bbbk$-modules

- i-cocycles $Z^i\overset{\text{def}}{=}\text{Ker}(\partial^i)\subseteq C^i$,
- i-coboundaries $B^i\overset{\text{def}}{=}Im(\partial^{i-1})=Im(\partial^{i-1})\subseteq C^i$,
- i-cohomologies $H^i\overset{\text{def}}{=}Z^i/B^i$,

Here we used $B^i\subseteq Z^i$ which follows from $\partial\partial=0$.

8.2.2. *Resolutions.* The basic example of complexes are resolutions. We will consider *resolutions of modules by free modules.*

Our motivational example will be a point b on a line $\mathbb{A}^1$. Then $\mathcal{O}(\{b\})=\Bbbk_b$ is a module for $\mathcal{O}(\mathbb{A}^1)=\Bbbk[x]$ (via the restriction map $\mathcal{O}(\mathbb{A}^1)\to\mathcal{O}(\{b\})$). In order to remember that we are looking at a point of $\mathbb{A}^1$ (or that "our point b is allowed to roam through $\mathbb{A}^1$), we consider the relation of $\Bbbk_b = \mathcal{O}(\{b\})$ to the algebra $\mathcal{O}(\mathbb{A}^1) = \Bbbk[x]$ functions on the *entire* $\mathbb{A}^1$. We capture this relation in the sense of a *resolution* of $\Bbbk_b$ – a way of encoding $\Bbbk_b$ in terms of several copies of $\Bbbk[x]$.

We first observe that $\Bbbk_b$ is a quotient of $\Bbbk[x]$. Inclusion $\{b\}\overset{j}{\hookrightarrow}\mathbb{A}^1$ gives the restriction map

$$\Bbbk[x] \xrightarrow{j^*} \Bbbk_b$$

which is surjective. However, since the map is not an isomorphism, by itself it does not quite capture $\Bbbk_b$. The error is in the kernel $\mathrm{Ker}(j^*) = (x-b)\Bbbk[x]$. So we try to record the kernel in terms of $\Bbbk[x]$. However, this is quite simple since we have an isomorphism $\Bbbk[x] \xrightarrow{x-b} (x-b)\Bbbk[x]$. This is a complete success, we expressed $\Bbbk_b$ via $\Bbbk[x]$, the summary of our thinking is just a way of interpreting the short exact sequence

$$0 \to (x-b)\Bbbk[x] \xrightarrow{\subseteq} \Bbbk[x] \xrightarrow{j^*} \Bbbk_b \to 0,$$

as an isomorphic short exact sequence the

$$0 \to \Bbbk[x] \xrightarrow{x-b} \Bbbk[x] \xrightarrow{j^*} \Bbbk_b \to 0.$$

This is called a resolution of $\Bbbk_b$ in terms of $\Bbbk[x]$.

In more complicated situations the exact sequence may be longer and one may need several copies of $\Bbbk[x]$ in each position (the reason is that the kernel of j^* need not be a free module of rank one, and indeed need not be free at all. Say for a point $\{(0,0)\}\overset{i}{\mathbb{A}}{}^2$ in a plane, one has a Koszul resolution of $\mathcal{O}(\{(0,0)\}) = \Bbbk_{(0,0)}$ by *free modules* for $\mathcal{O}(\mathbb{A}^2) = \Bbbk[x,y]$:

$$0 \to \Bbbk[x,y] \xrightarrow{h \to (h,-h)} \Bbbk[x,y] \oplus \Bbbk[x,y] \xrightarrow{(f,g) \mapsto xf+yg} \Bbbk[x,y] \xrightarrow{i^*} \Bbbk_{0,0} \to 0.$$

Precise definitions. A *left resolution* of a module M by free modules is an exact complex

$$\cdots \to P^{-2} \xrightarrow{\partial^{-2}} P^{-1} \xrightarrow{\partial^{-1}} P^0 \xrightarrow{q} M \to 0 \to \cdots,$$

in which all P^i's are free modules. However one also uses the term *resolution of M* for the complex (together with a map q)

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to 0 \to \cdots.$$

Notice one particular property of resolutions of M: these are complexes with $H^i = 0$ for $i \neq 0$ and $H^0 = M$. This again says that a resolution is a way of encoding M – the total information we can extract from the resolution is just M itself.

In this terminology $\Bbbk[x]$-module $\Bbbk_b$ has a resolution (we remember the map j^*)

$$\cdots \to \underset{\boxed{-2}}{0} \to \underset{\boxed{-1}}{\Bbbk[x]} \xrightarrow{x-b} \underset{\boxed{0}}{\Bbbk[x]} \to \underset{\boxed{1}}{0} \to \cdots.$$

The boxed numbers bellow indicate that we consider the guy that was next to $\Bbbk_b$ as being in position 0, the next one in position -1, etc.

8.2.3. *The derived versions of constructions.* Roughly speaking, the left derived version LF of a construction F (i.e., a functor F) obtained by replacing a module M by its free module resolution $P^\bullet$:

$$LF(M) \stackrel{\text{def}}{=} F(P^\bullet).$$

Since $P^\bullet$ is a complex, $F(P^\bullet)$ will again be a complex. Its cohomologies will be called the derived functors of F

$$L^iF(M) \stackrel{\text{def}}{=} H^i[LF(M)] = H^i[F(P^\bullet)].$$

8.2.4. *Exactness properties.* We say that functor F is right exact if it preserves exactness for sequences of type $M' \to M \to M'' \to 0$, i.e., if $M' \to M \to M'' \to 0$ is exact then $F(M') \to F(M) \to F(M'') \to 0$ is exact.

takes any

Lemma. (a) If F is right exact then $L^0F = F$.

(b) Functors $N\otimes_{\Bbbk}-$ of tensor product with N are right exact.

Remark. So we recover F on the level zero of the derived construction – the rest, i.e., the higher derived functors L^iF are a *bonus.*

8.2.5. *The size of a complex.* The main thing about the complex $C^\bullet$ are its cohomology groups. Actually, the complex is eventually thought of as a way of gluing all $H^i(C^\bullet)$ in one object. So the notion of the size will have to unchanged when one passes to cohomology. This is satisfied by the idea of *Euler characteristic.* If in a complex $C^\bullet$ all terms are finite dimensional vector spaces and only finitely many terms are non-zero, the Euler characteristic is defined by

$$\chi(C^\bullet) = \sum_i (-1)^i \dim(C^i).$$

Lemma. $\chi[H^*(C^\bullet)] = \chi[C^\bullet]$.

8.2.6. *Differential graded algebras (algebra objects in complexes).* A *differential graded algebra*[3] $(A^\bullet, d, cd)$, is a complex $(A^\bullet, d)$ with a structure of an algebra $A{\times}A \xrightarrow{cd} A$ on $A = \oplus_{\mathbb{Z}} A^n$, and the two should be compatible:

- $A_p{\cdot}A_q \subseteq A_{p+q}$, and
- for $a \in A_p,\ b \in A_q$

$$d(a{\cdot}b) = d(a){\cdot}b + (-1)^p a{\cdot}d(b).$$

[3]Usually we just say *dg-algebra.*

The last property is a version of the product rule $\partial(fg)d(a{\cdot}b) \;=\; \partial(f){\cdot}g + f{\cdot}\partial(g)$, except that it has been *enriched* by signs. The appearance of signs in homological algebra is particularly striking in the definition

$$\textit{dg-algebra } A \textit{ is commutative if for any } a\in A_p,\ b\in A_q \textit{ one has}$$
$$b{\cdot}a = \ (-1)^{pq}a{\cdot}b.$$

This property is also called *graded commutativity* and in physics *super commutativity* (this is the mathematical basis of super-symmetry).

A very basic example of commutative dg-algebras is the exterior algebra $A = \wedge^\bullet V$ of a vector space V, here $A_p = \wedge^p V$ and the differential is zero. Any basis $x_1,...,x_p$ of V generates A and since $deg(x_i) = 1$, these generators "anti-commute", i.e.,

$$x_ix_j \;=\; -x_jx_i.$$

In particular $x_i^2 = 0$ (at least if the characteristic of the field is not 2). A_p has a basis of monomials $x_{i_1}\cdots x_{i_p}$ with $1 \le i_1 < \cdots < i_p \le 1$. This dg-algebra is also called the *Grassmannian algebra* on generators $x_1,...,x_n$.[4]

8.3. **Example: intersection of points on a line.** We will calculate the simplest example of derived fiber product. It is indeed so simple that it does not impress at all. However, already here we will have to use the above machinery.

8.3.1. *The set theoretic level of the problem.* The problem is two take the derived intersection of two points $a,b \in \mathbb{A}^1$ on a line. Set theoretically, $\{a\}\cap\{b\}$ is empty if $a \neq b$ and it is a point $\{a\}$ if $a = b$. So the intersection jumps when we move b.

We can also view this as calculating the fiber at $b \in \mathbb{A}^1$ of the map $\{a\}\overset{i}{\hookrightarrow}\mathbb{A}^1$: $i^{-1}(b) \cong \{a\}\cap\{b\}$. So we are working on our original problem: set theoretically the fiber jumps when b has a special value $b = a$.

Let us remember the wish to create a refined notion of a fiber $i^{-1}b$ which will take into account the nearby fibers $i^{-1}c$. If $b \neq a$ the same is true for nearby c's, so $i^{-1}b = \emptyset = i^{-1}c$ and there is clearly nothing to refine. So we want a refined version of $i^{-1}a$ which not be $\emptyset$ but will take into account that nearby fibers are $\emptyset$.

In terms of the intersection picture $\{a\}\cap\{b\}$ we want a derived version $\{a\}\overset{L}{\cap}\{b\}$ which is in some sense continuous in b: so $\{a\}\overset{L}{\cap}\{a\}$ should take into account that $\{a\}\overset{L}{\cap}\{b\} = \{a\}\cap\{b\} = \emptyset$ for nearby b's. Another way to say this is that $\{a\}\overset{L}{\cap}\{a\}$ should take into account that the intersection is happening inside $\mathbb{A}^1$ (so it should be as continuous as possible in the variable $b\in\mathbb{A}^1$).

[4]In super mathematics one thinks of it as the algebra of functions on the super point $\mathbb{A}^{0,n}$.

8.3.2. *Algebraic recalculation of the set theoretic intersection.* Let $\mathcal{O}(\mathbb{A}^1) = \mathbb{k}[x]$ and denote by $\mathbb{k}_a = \mathcal{O}(\{a\})$ the functions on the point a, so algebra $\mathbb{k}_a$ is just the field $\mathbb{k}$ but we remember that this copy of $\mathbb{k}$ is related to the point $a \in \mathbb{A}^1 = \mathbb{k}$. Actually, $\{a\}\overset{i}{\hookrightarrow}\mathbb{A}^1$ corresponds to a map $i^* : \mathbb{k}[x] \ = \mathcal{O}(\mathbb{A}^1) \to \mathcal{O}(\{a\}) = \mathbb{k}_a \cong \mathbb{k}$ and i^* is the evaluation at a: $i^*(P(x)) = P(a)$ for a polynomial $P(x) \in \mathbb{k}[x]$. Similarly, $\mathbb{k}_b = \mathcal{O}(\{b\}) \cong \mathbb{k}$ and $\{b\}\overset{j}{\hookrightarrow}\mathbb{A}^1$ corresponds to $j^* : \mathbb{k}[x] \to \mathbb{k}_b \cong \mathbb{k}$ and $j^*(P(x)) = P(b)$. Now,

$$\mathcal{O}(\{a\}\cap\{b\}) = \ \mathcal{O}(\{a\}\times_{\mathbb{A}^1}\{b\}) = \ \mathcal{O}(\{a\})\otimes_{\mathcal{O}(\mathbb{A}^1)}\mathcal{O}(\{b\}) = \ \mathbb{k}_a\otimes_{\mathbb{k}[x]}\mathbb{k}_b.$$

Observe that $\mathbb{k}[x] \xrightarrow{i^*} \mathbb{k}_a$ is surjective and the kernel is the ideal $I_b = \langle x-b\rangle = (x-b)\mathbb{k}[x]$ of all functions that vanish at b. So, algebra $\mathbb{k}_b \cong \mathbb{k}[x]/\langle x-b\rangle$ is a quotient of $\mathbb{k}[x]$. This allows us to use tensor product identities:

$$\mathcal{O}(\{a\}\cap\{b\}) = \ \mathbb{k}_a\otimes_{\mathbb{k}[x]}\mathbb{k}[x]/\langle x-b\rangle \cong \ \mathbb{k}_a/\langle x-b\rangle\cdot\mathbb{k}_a = \ \mathbb{k}_a/(x-b)\cdot\mathbb{k}_a.$$

Here $x-b \in \mathcal{O}(\mathbb{A}^1)$ acts on $\mathbb{k}_a$ via i^*, the evaluation at a. So, $(x-b)\cdot\mathbb{k}_a = \ (a-b)\cdot\mathbb{k}_a$ and this is $\mathbb{k}_a$ if $b \neq a$ and 0 if $b = a$. So, we have calculated

$$\mathcal{O}(\{a\}\cap\{b\}) = \ \mathbb{k}_a/(x-b)\cdot\mathbb{k}_a = \begin{cases}\mathbb{k}_a & \text{if } b\neq a,\\ 0 & \text{if } b = a.\end{cases} = \begin{cases}\mathcal{O}(\{a\}) & \text{if } b\neq a,\\ \mathcal{O}(\emptyset) & \text{if } b = a.\end{cases},$$

i.e., we have recalculated the obvious claim $\{a\}\cap\{b\} = \begin{cases}\{a\} & \text{if } b\neq a,\\ \emptyset & \text{if } b = a.\end{cases}$, in algebraic language.

8.3.3. *Derived tensor product.* Remember that

The derived tensor product $\mathbb{k}_a\overset{L}{\otimes}_{\mathbb{k}[x]}\mathbb{k}_b$ is obtained by replacing in $\mathbb{k}_a\otimes_{\mathbb{k}[x]}\mathbb{k}_b$, the $\mathbb{k}[x]$-module $\mathbb{k}_b$ by its resolution.

We know such resolution

$$[\cdots \to \underset{\boxed{-2}}{0} \to \underset{\boxed{-1}}{\mathbb{k}[x]} \xrightarrow{x-b} \underset{\boxed{0}}{\mathbb{k}[x]} \to \underset{\boxed{1}}{0} \to \cdot],$$

so we get a complex

$$\begin{aligned}
\mathbb{k}_a\overset{L}{\otimes}_{\mathbb{k}[x]}\mathbb{k}_b &\overset{\text{def}}{=} \mathbb{k}_a\otimes_{\mathbb{k}[x]} \ [\cdots \to \underset{\boxed{-2}}{0} \to \underset{\boxed{-1}}{\mathbb{k}[x]} \xrightarrow{x-b} \underset{\boxed{0}}{\mathbb{k}[x]} \to \underset{\boxed{1}}{0} \to \cdots]\\
&= \ [\cdots \to \underset{\boxed{-2}}{\mathbb{k}_a\otimes_{\mathbb{k}[x]}0} \ \to \ \underset{\boxed{-1}}{\mathbb{k}_a\otimes_{\mathbb{k}[x]}\mathbb{k}[x]} \xrightarrow{id\otimes(x-b)} \underset{\boxed{0}}{\mathbb{k}_a\otimes_{\mathbb{k}[x]}\mathbb{k}[x]} \to \underset{\boxed{1}}{\mathbb{k}_a\otimes_{\mathbb{k}[x]}0} \ \to \ \cdots]\\
&\cong \ [\cdots \to \underset{\boxed{-2}}{0} \ \to \ \underset{\boxed{-1}}{\mathbb{k}_a} \xrightarrow{x-b} \underset{\boxed{0}}{\mathbb{k}_a} \to \underset{\boxed{1}}{0} \to \cdots]\\
&\cong \ [\cdots \to \underset{\boxed{-2}}{0} \ \to \ \underset{\boxed{-1}}{\mathbb{k}_a} \xrightarrow{a-b} \underset{\boxed{0}}{\mathbb{k}_a} \to \underset{\boxed{1}}{0} \to \cdots].
\end{aligned}$$

Now,

- if $a \neq b$ this is isomorphic to

$$[\cdots \to \underset{\boxed{-2}}{0} \;\to\; \underset{\boxed{-1}}{\Bbbk_a} \xrightarrow{id} \underset{\boxed{0}}{\Bbbk_a} \to \underset{\boxed{1}}{0} \to \cdots].$$

This we can think of as a resolution of 0 by free $\Bbbk[x]$-modules, so it is just a way to encode 0. Therefore, as desired

$$\Bbbk_a \overset{L}{\times}_{\Bbbk[x]} \Bbbk_b = \; 0 \;\text{ if }\; a \neq b.$$

- if $a = b$ this is isomorphic to

$$[\cdots \to \underset{\boxed{-2}}{0} \;\to\; \underset{\boxed{-1}}{\Bbbk_a} \xrightarrow{0} \underset{\boxed{0}}{\Bbbk_a} \to \underset{\boxed{1}}{0} \to \cdots].$$

Notice that this is *not* a resolution of anything. In fact as all maps are zero, the objects in various degrees are not related ("there is no cancellation"), so one should regard this object as a sum of its contributions in degrees 0 and -1, we call these constituents of $\Bbbk_a \overset{L}{\otimes}_{\Bbbk[x]} \Bbbk_b$, the 0^{th} and -1^{st} Tor functor:

$$Tor_{-1}^{\Bbbk[x]}(\Bbbk_a, \Bbbk_a) \;=\; \Bbbk_a \;=\; Tor_0^{\Bbbk[x]}(\Bbbk_a, \Bbbk_a).$$

In particular in degree 0 we get the correct value, i.e., just what the ordinary tensor product produces:

$$Tor_0^{\Bbbk[x]}(\Bbbk_a, \Bbbk_a) \;=\; \Bbbk_a \;=\; \Bbbk_a \otimes_{\Bbbk[x]} \Bbbk_a.$$

However, the derived computation gives a completely new ingredient in degree -1 :

$$Tor_{-1}^{\Bbbk[x]}(\Bbbk_a, \Bbbk_a) \;=\; \Bbbk_a.$$

This is a remainder that we were allowed to move b through $\mathbb{A}^1$, i.e., that we are taking an *ambiental intersection* of the two points– intersections which remembers the ambient $\mathbb{A}^1$.

8.3.4. *The size of* $\mathcal{O}(a \overset{L}{\cap}_{\mathbb{A}^1} b) \;=\; \Bbbk_a \overset{L}{\otimes}_{\Bbbk[x]} \Bbbk_b$ *is continuous in* b. We get different results depending on whether $a = b$, however the *size* does not change:

$$\chi(\Bbbk_a \overset{L}{\otimes}_{\Bbbk[x]} \Bbbk_b) \;=\; \dim[H^0(\Bbbk_a \overset{L}{\otimes}_{\Bbbk[x]} \Bbbk_b)] - \dim[H^0(\Bbbk_a \overset{L}{\otimes}_{\Bbbk[x]} \Bbbk_b)]$$

$$=\; \dim[Tor^0_{\Bbbk[x]}(\Bbbk_a, \Bbbk_b)] - \dim[Tor^{-1}_{\Bbbk[x]}(\Bbbk_a, \Bbbk_b)] = \begin{cases} 0 - 0 & \text{if } b \neq a, \\ 1 - 1 & \text{if } b = a. \end{cases} = \; 0.$$

8.4. **Differential graded schemes.** As in the ordinary (rather then homological) algebra, we will define *affine differential graded schemes* as geometric spaces $\mathfrak{X}$ that correspond to commutative dg-algebras $\mathcal{A}$ by $\mathcal{O}(\mathfrak{X})=\mathcal{A}$, then $\mathfrak{X}$ is called $\mathrm{Spec}(\mathcal{A})$.

As we have seen this idea provides a refined notion of fibered product of $X\xrightarrow{f}Z\xleftarrow{g}Y$, the derived fibered product

$$X\overset{L}{\times}_Z Y \overset{\text{def}}{=} \mathrm{Spec}(\mathcal{O}(X)\overset{L}{\otimes}_{\mathcal{O}(Z)}\mathcal{O}(Y)].$$

Roughly, the refined version remembers that one could vary the maps f and g, and the this makes the size of the fibered product stable under such motions of maps. In particular, we get the notion of *derived intersection* $X\overset{L}{\cap}_Z Y$ of two subspace X,Y of Z, which is an *ambiental* intersection since it remembers that we may move X and Y inside Z.

8.4.1. *Example: self intersection of a point.* Let us see what did we get as the self intersection of a point on a line.

Let $a,b\in\mathbb{A}^1$. If $a\neq b$ we got $\mathcal{O}(a\overset{L}{\cap}_{\mathbb{A}^1} b)=0$, hence $a\overset{L}{\cap}_{\mathbb{A}^1} b=\emptyset$ (an empty scheme), as we should. If $a=b$ we got $A=\mathcal{O}(a\overset{L}{\cap}_{\mathbb{A}^1} a)$ to be a complex with $\Bbbk$ in degrees -1 and 0 and zero differential. What is the algebra structure? $A_0=\Bbbk$ acts on A as multiplication by scalars. The remaining multiplication is zero since $A_1\times A_1\xrightarrow{\cdot}A_2=0$. So, we got a Grassmannian algebra on one generator, since $A\cong\Bbbk\oplus\Bbbk x=\ \wedge^\bullet\ \Bbbk x$ for a one dimensional space with a basis x.[5]

One can similarly calculate the self intersection of a point in $\mathbb{A}^n$:

Lemma. For a point $a\in\mathbb{A}^n$, $\mathcal{O}(a\overset{L}{\cap}_{\mathbb{A}^n} a)$ is the Grassmannian algebra $\wedge^\bullet\ \Bbbk^n$. So, $a\overset{L}{\cap}_{\mathbb{A}^n}$ is the spectrum of the Grassmannian algebra $\wedge^\bullet\ \Bbbk^n$.[6]

Proof. This is easily seen by using the *Koszul resolution.* We will do the calculation based on *faith* that dg-geometry exists and works reasonably, so that

$$(X_1\overset{L}{\cap}_{Z_1}Y_1)\times(X_2\overset{L}{\cap}_{Z_2}Y_2)\ \cong\ (X_1\times X_2)\overset{L}{\times}_{Z_1\times Z_2}(Y_1\times Y_2).$$

Then for $a,b\in\mathbb{A}^n$

$$a\overset{L}{\times}_{\mathbb{A}^n} b\ \cong\ (a_1\times\cdots\times a_n)\overset{L}{\times}_{\mathbb{A}^1\times\cdots\times\mathbb{A}^1}(b_1\times\cdots\times b_n)\ \cong\ (a_1\overset{L}{\cap}_{\mathbb{A}^1}b_1)\times\cdots\times(a_n\overset{L}{\cap}_{\mathbb{A}^1}b_n),$$

hence

$$\mathcal{O}(a\overset{L}{\times}_{\mathbb{A}^n} b)\ \cong\ \mathcal{O}(a_1\overset{L}{\cap}_{\mathbb{A}^1}b_1)\otimes\cdots\otimes\mathcal{O}(a_n\overset{L}{\cap}_{\mathbb{A}^1}b_n)$$
$$\cong\ (\Bbbk\oplus\Bbbk x_1)\otimes\cdots\otimes(\Bbbk\oplus\Bbbk x_n)\ \cong\ (\wedge^\bullet\ \Bbbk x_1)\otimes\cdots\otimes(\wedge^\bullet\ \Bbbk x_n)\cong\ \wedge^\bullet\ (\Bbbk x_1\oplus\cdots\oplus\Bbbk x_n).$$

[5]In terms of super mathematics we would say that $A=\mathcal{O}(a\overset{L}{\cap}_{\mathbb{A}^1} a)$ is a super point $\mathbb{A}^{0,1}$.

[6]In terms of super mathematics again, $A=\mathcal{O}(a\overset{L}{\cap}_{\mathbb{A}^n} a)$ is a super point of type $\mathbb{A}^{0,n}$.

8.4.2. *Koszul duality.* To indicate that Grassmannian algebra is not non-sense, let me mention the following fact without giving details[7]

8.4.3. *Theorem.* (Priddy) Let x_i and y_i be dual bases of two dual vector spaces. Then the categories of modules over dg-algebras[8] $\wedge^\bullet$ $(\Bbbk x_1 \oplus \cdots \oplus \Bbbk x_n)$ and $\Bbbk[y_1, ..., y_n]$, are canonically equivalent.

A fancy version of this gives an unexpected relations between familiar objects:

8.4.4. *Theorem.* Let X, Y be vector subspaces of a vector space Z. Then the dual vector space Z^* contains vector subspaces $X^\perp, Y^\perp$. The categories of modules over dg-algebras of functions on dg-schemes $X \overset{L}{\cap}_Z Y$ and $X^\perp \overset{L}{\cap}_{Z^*} Y^\perp$ are canonically equivalent. [9]

8.5. **Abelian categories.** An abelian category is a category $\mathcal{A}$ which has the formal properties of the category $\mathcal{A}b$, i.e., we can do in $\mathcal{A}$ all computations that one can do in $\mathcal{A}b$. The basic example: categories $\mathfrak{m}(\Bbbk)$ of modules over a ring $\Bbbk$. Here is a (long) list of properties that make a category $\mathcal{A}$ abelian

(1) Category $\mathcal{A}$ is additive if
   - (A0) For any $a, b \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(a, b)$ has a structure of abelian group such that then compositions are bilinear.
   - (A1) $\mathcal{A}$ has a zero object,
   - (A2) $\mathcal{A}$ has sums of two objects,
   - (A3) $\mathcal{A}$ has products of two objects,[10]

(2) Category $\mathcal{A}$ is abelian if it is additive and
   - (A4) It has kernels and cokernels (hence in particular it has images and coimages!).
   - (A5) The canonical maps $Coim(\phi) \to Im(\phi)$ are isomorphisms.

Let us recall what this means

8.5.1. *(Co)kernels and (co)images.* In module categories a map has kernel, cokernel and image. To incorporate these notions into our project of defining abelian categories we will find their abstract formulations. Two of these notions are primary and dual to each other:

(1) Kernels
   - *Intuition.* For a map of $\Bbbk$-modules $M \xrightarrow{\alpha} N$

[7] A module over a dg-algebra is of course a complex with extra structure – you should be able to cook up the definition.

[8] Since we are dealing with homological algebra, the appropriate categories are the so called *derived categories* of modules, which we will not define.

[9] The same holds if X, Y are vector subbundles of a vector bundle Z over S.

[10] In an additive category $a \oplus b$ is canonically the same as $a \times b$.

– the kernel $\mathrm{Ker}(\alpha)$ is a subobject of M,
– the restriction of α to it is zero,
– and this is the largest subobject with this property

- *Definition in an additive category $\mathcal{A}$.* k is a kernel of a map $a \xrightarrow{\alpha} b$ if
  – we have a map $k \xrightarrow{\sigma} a$ from k to a,
  – if we follow this map by α the composition is zero,
  – map $k \xrightarrow{\sigma} a$ is universal among all such maps.[11]

(2) *Cokernels*

- In $\mathfrak{m}(\Bbbk)$ the cokernel of $M \xrightarrow{\alpha} N$ is $N/\alpha(M)$. So there is a map $N \to Coker(\alpha)$, the composition with α kills it, and the cokernel is universal among all such objects.
- In additive $\mathcal{A}$, the cokernel of $a \xrightarrow{\sigma} b$ is an object c supplied with a map $b \to c$ which is universal among maps from b that kill α.

Now, the two secondary notions (they use kernels and cokernels).

(1) *Images.*

- In $\mathfrak{m}(\Bbbk)$, $Im(\alpha)$ is a subobject of N which is the kernel of $N \to \alpha(M)$.
- In additive $\mathcal{A}$, if $a \xrightarrow{\sigma} b$ has cokernel $b \to \mathrm{Coker}(\alpha)$, then the image of σ is $Im(\sigma) \stackrel{\text{def}}{=} \mathrm{Ker}[b \to \mathrm{Coker}(\sigma)]$ (if it exists).

(2) *Coimages.*

- In $\mathfrak{m}(\Bbbk)$ the coimage of α is $M/\mathrm{Ker}(\alpha)$.
- In additive $\mathcal{A}$, if $a \xrightarrow{\sigma} b$ has kernel $\mathrm{Ker}(\sigma) \to a$, then the coimage of σ is $Coim(\sigma) \stackrel{\text{def}}{=} Coker[\mathrm{Ker}(\sigma) \to a]$. (if it exists).

In $\mathfrak{m}(\Bbbk)$, the canonical map $Coim(\alpha) = M/\mathrm{Ker}(\alpha) \to Im(\alpha)$ is an isomorphism. This observation is the final ingredient (A5) in the definition of abelian categories.[12][13]

8.5.2. *Extending the rest of the vocabulary from modules to abelian categories.* Once we have the notion of kernel and cokernel everything follows:

- a map $i : a \to b$ makes a into a subobject of b if $\mathrm{Ker}(i) = 0$ (we denote it $a \hookrightarrow b$ or even informally by $a \subseteq b$, one also says that i is a monomorphism or informally that it is an inclusion),

[11]The "universality" means that all maps into a, $x \xrightarrow{\tau} a$, which are killed by α, factor uniquely through k (i.e., through $k \xrightarrow{\sigma} a$). So, all such maps τ are obtained from σ (by composing it with some map $x \to k$).

[12]This is also a reason why you never hear of coimages.

[13]For (A5) we also need:

Lemma. In additive $\mathcal{A}$, if $\sigma : a \to b$ has image and coimage then there is a canonical map $Coim(\sigma) \to Im(\sigma)$. It appears in a canonical factorization of σ into a composition

$$a \to Coim(\sigma) \to Im(\sigma) \to b.$$

- a map $q : b \to c$ makes c into a quotient of b if $\mathrm{Coker}(q) = 0$ (we denote it $b \twoheadrightarrow c$ and say that q is an epimorphism or informally that q is surjective),
- the quotient of b by a subobject $a \xrightarrow{i} b$ is $b/a \overset{\text{def}}{=} \mathrm{Coker}(i)$,
- a complex in $\mathcal{A}$ is a sequence of maps $\cdots A^n \xrightarrow{d^n} A^{n+1} \to \cdots$ such that $d^{n+1} \circ d^n = 0$, its cocycles, coboundaries and cohomologies are defined by $B^n = Im(d^n)$ is a subobject of $Z^n = \mathrm{Ker}(d^n)$ and $H^n = Z^n/B^n$;
- sequence of maps $a \xrightarrow{\mu} b \xrightarrow{\nu} c$ is exact (at b) if $\nu \circ \mu = 0$ and the canonical map $Im(\mu) \to \mathrm{Ker}(\nu)$ is an isomorphism.

Now with all these definitions we are in a familiar world, i.e., they work as we expect. For instance, sequence $0 \to a' \xrightarrow{\alpha} a \xrightarrow{\beta} a'' \to 0$ is exact iff a' is a subobject of a and a'' is the quotient of a by a', and if this is true then

$$\mathrm{Ker}(\alpha) = 0, \mathrm{Ker}(\beta) = a', \mathrm{Coker}(\alpha) = a'', \mathrm{Coker}(\beta) = 0, \mathrm{Im}(\alpha) = a', \mathrm{Im}(\beta) = a''.$$

8.5.3. *The difference between general abelian categories and module categories.* In a module category $\mathfrak{m}(\Bbbk)$ our arguments often use the fact that $\Bbbk$-modules are after all abelian groups and *sets* – so we can think in terms of their elements. A reasoning valid in any abelian category has to be done more formally: via *composing maps* and *factoring maps through intermediate objects.* However, this is mostly appearances – if we try to use set theoretic arguments we will not go wrong:

8.5.4. *Theorem.* [Mitchell] Any abelian category is equivalent to a full subcategory of some category of modules $\mathfrak{m}(\Bbbk)$.

8.6. **Category $C(\mathcal{A})$ of complexes with values in an abelian category $\mathcal{A}$.** Let $\mathcal{A}$ be the category $\mathfrak{m}(\Bbbk)$ of modules over a ring $\Bbbk$.[14] A map of complexes $f : A^\bullet \to B^\bullet$ is a system of maps f^n of the corresponding terms in complexes, which "preserves" the differential in the sense that in the diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A^{-2} & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle f^{-2}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow{\scriptstyle f^{1}} & & \downarrow \\
\cdots & \longrightarrow & B^{-2} & \longrightarrow & B^{-1} & \longrightarrow & B^0 & \longrightarrow & B^1 & \longrightarrow & \cdots
\end{array}$$

all squares commute.[15] This clearly defines a category of complexes $\mathcal{C}^\bullet[\mathcal{A}]$, objects are complexes and morphisms are maps of complexes.

We observe some of the properties of the category $C(\mathcal{A})$.

8.6.1. *Properties.* The next two lemmas give basic properties of the above structures on the category $C(\mathcal{A})$.

[14]However everything works the same in any abelian category $\mathcal{A}$.

[15]Meaning that two possible ways of following arrows give the same result: $f^n \circ d_A^{n-1} = d_B^{n-1} \circ f^{n-1}$, for all n; i.e., $f \circ d = d \circ f$.

8.6.2. *Lemma.* $C(\mathcal{A})$ is an abelian category and a sequence of complexes is exact iff it is exact on each level!

Proof. For a map of complexes $A \xrightarrow{\alpha} B$ we can define $K^n = \mathrm{Ker}(A^n \xrightarrow{\alpha^n} B^n)$ and $C^n = A^n/\alpha^n(B^n)$. This gives complexes since d_A induces a differential d_K on K and d_B a differential d_C on C. Moreover, it is easy to check that in category $C(\mathcal{A})$ one has $K = \mathrm{Ker}(\alpha)$ and $C = \mathrm{Coker}(\alpha)$. Now one finds that $Im(\alpha)^n = Im(\alpha^n) = \alpha^n(A^n)$ and $Coim(\alpha)^n = Coim(\alpha^n) = A^n/\mathrm{Ker}(\alpha^n)$, so the canonical map $Coim \to Im$ is given by isomorphisms $A^n/\mathrm{Ker}(\alpha^n) \xrightarrow{\cong} \alpha^n(A^n)$. Exactness claim follows.

8.6.3. *Lemma.* A short exact sequence of complexes $0 \to A \to B \to C \to 0$ gives a long exact sequence of cohomologies.

$$\cdots \xrightarrow{\partial^{n-1}} \mathrm{H}^n(A) \xrightarrow{\mathrm{H}^n(\alpha)} \mathrm{H}^n(B) \xrightarrow{\mathrm{H}^n(\beta)} \mathrm{H}^n(C) \xrightarrow{\partial^n} \mathrm{H}^{n+1}(A) \xrightarrow{\mathrm{H}^{n+1}(\alpha)} \mathrm{H}^{n+1}(B) \xrightarrow{\mathrm{H}^{n+1}(\beta)} \cdots$$

8.7. **Exactness of functors and the derived functors.** Remember that derived versions are suppose to improve some constructions, i.e., *functors.* How this is exactly done depends on *exactness properties* of the functor in question. We will consider *additive functors* $F : \mathcal{A} \to \mathcal{B}$ between additive categories,[16] this means that the maps $\mathrm{Hom}_{\mathcal{A}}(a', a'') \to \mathrm{Hom}_{\mathcal{B}}(Fa', Fa'')$ are required to be morphisms of abelian groups.

Lemma. If F is additive then $F(0) = 0$ and $F(a \oplus b) \cong F(a) \oplus F(b)$.

8.7.1. *Exactness properties.*

(1) *Exact functors.* We will say that F is exact if it preserves short exact sequences, i.e., for any SES $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ in $\mathcal{A}$, its F-image in $\mathcal{B}$ is exact, i.e., the sequence $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ is a SES in $\mathcal{B}$.
(2) *Left exact functors.* [17] We say, that F is left exact if for any SES its F-image $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ is exact except possibly in the A''-term, i.e., $F(\beta)$ *need not be surjective.*
(3) *Right exact functors.* F is right exact if it the F-image $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ of a SES is exact except possibly in the A'-term, i.e., $F(\alpha)$ may fail to be injective.

Proposition. Let $\mathcal{A}$ be an abelian category, for any $a \in \mathcal{A}$,

- (a) $\mathrm{Hom}_{\mathcal{A}}(a, -) : \mathcal{A} \to \mathcal{A}b$ is left exact!,
- (b) $\mathrm{Hom}_{\mathcal{A}}(-, a) : \mathcal{A}^o \to \mathcal{A}b$ is right exact!

[16]One can think of the case where $\mathcal{A} = \mathfrak{m}(\Bbbk)$ and $\mathcal{B} = \mathfrak{m}(l)$ since the general case works the same.
[17]Few interesting functors are exact so we have to relax the notion of exactness.

Proof. (a) For any exact sequence $0 \to b' \xrightarrow{\alpha} b \xrightarrow{\beta} b'' \to 0$ we consider the corresponding sequence $\mathrm{Hom}_{\mathcal{A}}(a,b') \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathcal{A}}(a,b) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathcal{A}}(a,b'')$.

(1) α_* is injective. if $a \xrightarrow{\mu} b'$ and $0 = \alpha_*(\mu) \stackrel{\text{def}}{=} \alpha\circ\mu$, then μ factors through the kernel $\mathrm{Ker}(\alpha)$ (by the definition of the kernel). However, $\mathrm{Ker}(\alpha) = 0$ (by definition of a short exact sequence), hence $\mu = 0$.

(2) $\mathrm{Ker}(\beta^*) = Im(\alpha_*)$. First, $\beta_*\circ\alpha_* = (\beta\circ\alpha)_* = 0_* = 0$, hence $Im(\alpha_*) \subseteq \mathrm{Ker}(\beta^*)$. If $a \xrightarrow{\nu} b$ and $0 = \beta_*(\nu)$, i.e., $0 = \beta\circ\nu$, then ν factors through the kernel $\mathrm{Ker}(\beta)$. But $\mathrm{Ker}(\beta) = a'$ and the factorization now means that ν is in $Im(\alpha_*)$.

Remark. $\mathrm{Hom}(a,-)$ *is not always exact.* Let $\mathcal{A} = \mathcal{A}b$ and apply $\mathrm{Hom}(a,-)$ for $a = \mathbb{Z}/2\mathbb{Z}$ to $0 \to 2\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \to 0$. Then $id_{\mathbb{Z}/2\mathbb{Z}}$ does not lift to a map from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$. So β_* need not be surjective.

Lemma. Tensoring is right exact in each argument, i.e., for any left $\Bbbk$-module M the functor $M\underset{\Bbbk}{\otimes}- : \mathfrak{m}^r(\Bbbk) \to \mathcal{A}b$ is right exact, and so is $-\underset{\Bbbk}{\otimes}N : \mathfrak{m}(\Bbbk) \to \mathcal{A}b$ for any right $\Bbbk$-module N.

8.7.2. *Projectives and the existence of projective resolutions.* Let $\mathcal{A}$ be an abelian category. We say that $p \in \mathcal{A}$ is a projective object if the functor $\mathrm{Hom}_{\mathcal{A}}(p,-) : \mathcal{A} \to \mathcal{A}b$ is exact.

Since $\mathrm{Hom}_{\mathcal{A}}(p,-)$ is known to be always left exact, what we need is that for any short exact sequence $0 \to a \xrightarrow{\alpha} b \xrightarrow{\beta} c \to 0$ map $\mathrm{Hom}(p,b) \to \mathrm{Hom}(p,c)$ is surjective. In other words, if c is a quotient of b then any map from p to the quotient $p \xrightarrow{\gamma} c$ lifts to a map to b, i.e., there is a map $p \underset{\tilde{\gamma}}{\to} b$ such that $\gamma = \beta\circ\tilde{\gamma}$ for the quotient map $b \xrightarrow{\beta} c$.

Lemma. (a) In $\mathfrak{m}(\Bbbk)$, free modules are projective. More precisely, P is projective iff P is a summand of a free module.

(b) $\oplus_{i\in I}\, p_i$ is projective iff all summands p_i are projective.

We say that abelian category $\mathcal{A}$ has enough projectives if any object is a quotient of a projective object.

Corollary. Module categories have enough projectives.

The importance of "enough projectives" comes from

Proposition. For an abelian category $\mathcal{A}$ the following is equivalent

(1) Any object of $\mathcal{A}$ has a projective resolution (i.e., a left resolution consisting of projective objects).
(2) $\mathcal{A}$ has enough projectives.

8.7.3. *Injectives and the existence of injective resolutions.* Dually, we say that $i \in \mathcal{A}$ is an injective object if the functor $\mathrm{Hom}_{\mathcal{A}}(-,i): \mathcal{A} \rightarrow \mathcal{A}b^o$ is exact.

Again, since $\mathrm{Hom}_{\mathcal{A}}(-,i)$ is always right exact, we need for any short exact sequence $0 \rightarrow a \xrightarrow{\alpha} b \xrightarrow{\beta} c \rightarrow 0$ that the map $\mathrm{Hom}(b,p) \xrightarrow{\alpha^*} \mathrm{Hom}(a,p)$, $\alpha^*(\phi) = \phi\circ\alpha$; be surjective. This means that if a is a subobject of b then any map $a \xrightarrow{\gamma} i$ from a subobject a to i extends to a map from b to i, i.e., there is a map $b \xrightarrow{\tilde{\gamma}} i$ such that $\gamma = \tilde{\gamma}\circ\alpha$. So, an object i is injective if each map from a subobject $a'\hookrightarrow a$ to i, extends to the whole object a.

Proposition. (a) A $\mathbb{Z}$-module I is injective iff I is divisible, i.e., for any $a \in I$ and $n \in \{1,2,3,...\}$ there is some $\tilde{a} \in I$ such that $a = n{\cdot}\tilde{a}$. (So, we ask that the multiplications $n: I \rightarrow I$ with $n \in \{1,2,3,...\}$, are all surjective.)[18]

(b) For any abelian group M denote $\widehat{M} = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$. Then the canonical map $M \xrightarrow{\rho} \widehat{\widehat{M}}$ is injective.

Proof. (a) For $m \in M$, $\chi \in \widehat{M}$, $\rho(m)\ (\chi) \stackrel{\mathrm{def}}{=} \chi(m)$. So, $\rho(m) = 0$ means that m is killed by each $\chi\widehat{M}$ ("each character of M"). If $m \neq 0$ then $\mathbb{Z}{\cdot}m$ is isomorphic to $\mathbb{Z}$ or to one of $\mathbb{Z}/n\mathbb{Z}$, in each case we can find a $\mathbb{Z}{\cdot}m \xrightarrow{\chi_0} \mathbb{Q}/\mathbb{Z}$ which is $\neq 0$ on the generator m. Since $\mathbb{Q}/\mathbb{Z}$ is injective we can extend χ_0 to M.

Proof. [19] For any $a \in I$ and $n > 0$ we can consider $\frac{1}{n}\mathbb{Z} \supseteq \mathbb{Z} \xrightarrow{\alpha} I$ with $\alpha(1) = a$. If I is injective then α extends to $\widetilde{\alpha}: \frac{1}{n}\mathbb{Z} \rightarrow I$ and $a = n\widetilde{\alpha}(\frac{1}{n})$.

Conversely, assume that I is divisible and let $A \supseteq B \xrightarrow{\beta} I$. Consider the set $\mathcal{E}$ of all pairs (C,γ) with $B \subseteq C \subseteq A$ and $\gamma: C \rightarrow I$ an extension of β. It is partially ordered with $(C,\gamma) \leq (C',\gamma')$ if $C \subseteq C'$ and γ' extends γ. From Zorn lemma and the following observations it follows that $\mathcal{E}$ has an element (C,γ) with $C = A$:

(1) For any totally ordered subset $\mathcal{E}' \subseteq \mathcal{E}$ there is an element $(C,\gamma) \in \mathcal{E}$ which dominates all elements of $\mathcal{E}'$ (this is clear: take $C = \cup_{(C',\gamma') \in \mathcal{E}'} C'$ and γ is then obvious).
(2) If $(C,\gamma) \in \mathcal{E}$ and $C \neq A$ then (C,γ) is not maximal:
  - choose $a \in A$ which is not in C and let $\widetilde{C} = C + \mathbb{Z}{\cdot}a$ and $C \cap \mathbb{Z}{\cdot}a = \mathbb{Z}{\cdot}na$ with $n \geq 0$. If $n = 0$ then $\widetilde{C} = C \oplus \mathbb{Z}{\cdot}a$ and one can extend γ to C by zero on $\mathbb{Z}{\cdot}a$. If $n > 0$ then $\gamma(na) \in I$ is n-divisible, i.e., $\gamma(na) = nx$ for some $x \in I$.

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[18] $\mathbb{Z}$ is projective in $\mathcal{A}b$ but it is not injective in $\mathcal{A}b$: $\mathbb{Z} \subseteq \frac{1}{n}\mathbb{Z}$ and the map $1_{\mathbb{Z}}: \mathbb{Z} \rightarrow \mathbb{Z}$ does not extend to $\frac{1}{n}\mathbb{Z} \rightarrow \mathbb{Z}$.

[19] The proof will use the Zorn lemma which is an essential part of any strict definition of set theory:

- Let $(I,\leq)$ be a (non-empty) partially ordered set such that any chain J in I (i.e., any totally ordered subset) is dominated by some element of I (i.e., there is some $i \in I$ such that $i \geq j$, $j \in J$). Then I has a maximal element.

Then one can extend γ to $\widetilde{C}$ by $\widetilde{\gamma}(a) = x$ (first define a map on $C \oplus \mathbb{Z}\cdot a$, and then descend it to the quotient $\widetilde{C}$).

Lemma. (a) For any abelian category $\mathcal{A}$ the following is equivalent

(1) Any object of $\mathcal{A}$ has an injective resolution (i.e., a right resolution consisting of injective objects).
(2) $\mathcal{A}$ has enough injectives. (We say that abelian category $\mathcal{A}$ has enough injectives if any object is a subobject of an injective object.)

(b) Product $\prod_{i\in I} J_i$ is injective iff all factors J_i are injective.

Corollary. Category of abelian groups has enough injectives.

Proof. To M we associate a huge injective abelian group $I_M = \prod_{x\in\widehat{M}} \mathbb{Q}/\mathbb{Z}\cdot x = (\mathbb{Q}/\mathbb{Z})^{\widehat{M}}$, its elements are $\widehat{M}$-families $c = (c_\chi)_{\chi\in\widehat{M}}$ of elements of $\mathbb{Q}/\mathbb{Z}$ (we denote such family also as a (possibly infinite) formal sum $\sum_{\chi\in\widehat{M}} c_\chi\cdot\chi$). By part (a), canonical map *io* is injective

$$M \xrightarrow{\iota} I_M, \quad \iota(m) = (\chi(m))_{\chi\in\widehat{M}} = \sum_{\chi\in\widehat{M}} \chi(m)\cdot\chi, \quad m \in M.$$

Theorem. Module categories $\mathfrak{m}(\Bbbk)$ have enough injectives.

Proof. The problem can be reduced to the known case $\Bbbk = \mathbb{Z}$ via the canonical map of rings $\mathbb{Z} \xrightarrow{\phi} \Bbbk$.

Remarks. (1) An injective resolution of the $\mathbb{Z}$-module $\mathbb{Z}$ is $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/Z \to 0$.

(2) Injective resolutions are often big, hence more difficult to use in specific calculations then say, the free resolutions. However, they are necessary for the functor $\Gamma(X, -)$ of global sections of sheaves.

8.7.4. *Left derived functor RF of a right exact functor F.* We observe that if F is right exact then the correct way to extend it to a functor on the derived level is the construction $LF(M) \overset{\text{def}}{=} F(P^\bullet)$, i.e., replacement of the object by a projective resolution. "Correct" means here that LF is really more then F – it contains the information of F in its zero$^{\text{th}}$ cohomology, i.e., $L^0F \cong F$ for $L^iF(M) \overset{\text{def}}{=} H^i[LF(M)]$. Letter L reminds us that we use a left resolution.

Lemma. If the functor $F : \mathcal{A} \to \mathcal{B}$ is right exact, there is a canonical isomorphism of functors $H^0(LF) \cong F$.

Proof. Let $\cdots \to P^{-2} \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{q} M \to 0$ be a projective resolution of M. Then $LF(M) = F[\cdots \to P^{-2} \to P^{-1} \xrightarrow{d^{-1}} P^0 \to 0 \to \cdots]$ equals

$$[\cdots \to F(P^{-2}) \to F(P^{-1}) \xrightarrow{F(d^{-1})} F(P^0) \to 0 \to \cdots],$$

so $H^0[LF(M)] = F(P^0)/F(d^{-1})F(P^{-1})$.

If we apply F to the exact sequence $P^{-1} \xrightarrow{d^{-1}} P^0 \to M \xrightarrow{q} 0$, the right exactness gives an exact sequence $F(P^{-1}) \xrightarrow{F(d^{-1})} F(P^0) \xrightarrow{F(q)} F(M) \to 0$. Therefore, $F(q)$ factors to a canonical map $F(P^0)/F(d^{-1})F(P^{-1}) \to F(M)$ which is an isomorphism.

8.7.5. *Right derived functor RF of a left exact functor F.* Obviously, we want to define for any left exact functor $F : \mathcal{A} \to \mathcal{B}$ a right derived functor RF by replacing an object M by its injective resolution $I^\bullet$

$$(RF)M \stackrel{\text{def}}{=} F(I^\bullet) \quad \text{and} \quad (R^iF)M \stackrel{\text{def}}{=} H^i[(RF)M] = H^i[F(I^\bullet)].$$

As above, $(R^0F)(M) \stackrel{\text{def}}{=} H^0[(RF)M] \cong F(M)$, i.e., $R^0F = F$.

8.8. **Appendix: The ideal setup for homological algebra.** We have a prescription that corrects a functor which is only half-exact. Say, if F is a left exact functor we had to correct it on the right side, so we replaced a module by its injective resolution $I^\bullet$ which is a right resolution (i.e., it is in degrees ≥ 0). This gives $R^iF(M) \stackrel{\text{def}}{=} H^i[F(I\bullet)]$. However, notice some

8.8.1. *Foundational and calculational problems.*

8.8.2. *Questions.*

(1) There may be more then one injective resolution of M, which one do I use?
(2) Does a map $\alpha : M \to N$ give something relating $R^iF(M)$ and $R^iF(N)$?
(3) If M is obtained by gluing simpler modules M', M'', i.e., if there is a short exact sequence $0 \to M' \to M \to M'' \to 0$, can I analyze $R^iF(M)$ in terms of $R^iF(M')$ and $R^iF(M'')$?

The answer is **Yes**:

8.8.3. *Lemma.*

(1) Use any injective resolution:

$R^iF(M)$ does not depend on the choice, for two injective resolutions $I^\bullet, J^\bullet$ of M, there are canonical isomorphisms $H^i[F(I^\bullet)] \cong H^i[F(J^\bullet)]$.

(2) Each R^iF is a functor:

for any map $M \xrightarrow{\alpha} N$ the following is true:

- *(i) for any injective resolutions* $I^\bullet, M^\bullet$ *of* M *and* N*, there is a lift* $\widetilde{a}l : I^\bullet \to J^\bullet$ *of* α
- *(ii) the corresponding map* $H^i(\widetilde{\alpha}) : \ H^i[F(I^\bullet)] \to H^i[F(J^\bullet)]$*? does not depend on the choices of* $I^\bullet, J^\bullet, \widetilde{\alpha}$*, so*
- *(iii) it is a well defined map* $R^iF(M) \xrightarrow{R^i(\alpha)} R^iF(N)$.

(3) A short exact sequence gives a long exact sequence of derived functors:

$$\textit{Any short exact sequence } 0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0,$$

- *(i) lifts to a short exact sequence of injective resolutions* $0 \to I'^\bullet \xrightarrow{\widetilde{\alpha}} I^\bullet \xrightarrow{\widetilde{\beta}} I''^\bullet \to 0$*,*
- *(ii) A short exact sequence of injective resolutions always splits, i.e., there is a subcomplex* $J^\bullet \subseteq I^\bullet$ *complementary to* $I'^\bullet$*. Therefore,*
- *(iii) the sequence of complexes* $0 \to F(I'^\bullet) \xrightarrow{F(\widetilde{\alpha})} F(I^\bullet) \xrightarrow{F(\widetilde{\beta})} F(I''^\bullet) \to 0$*, is again exact. So,*
- *applying cohomology to it we get a long exact sequence*

$$0 \to F(M') \to F(M) \to F(M'') \to R^1F(M') \to R^1F(M) \to R^1F(M'') \to R^2F(M') \to R^2F(M) \to R^2F(M'')\cdots$$

The direct proofs of these facts are routine and only take finite amount of time. However, there is a (calculationaly superior) conceptual approach which involves finding appropriate categories:

8.8.4. *Homotopy category of complexes.* The origin of messiness is having to *choose* a resolution for each object. Though resolutions $I^\bullet, J^\bullet$ of one object M may be very different complexes, claim (1) above suggests that – in some sense – they are the same. This is achieved by replacing the category of complexes $C^\bullet(\mathcal{A})$ with the *homotopic category of complexes* $K^\bullet(\mathcal{A})$ – the objects are again the complexes but there are more isomorphisms, and any two resolutions of M are *canonically isomorphic* in $K^\bullet(\mathcal{A})$. So, taking injective resolutions becomes a *functor* $\mathcal{I} : \mathcal{A} \to K(\mathcal{A})$. On the other hand, there are some obvious functors on homotopy categories: just by applying F to complexes we get a functor $K(F) : K(\mathcal{A}) \to K(\mathcal{B})$, and also the cohomology of complexes gives functors $H^n : K(\mathcal{B}) \to \mathcal{B}$. Therefore we get *functorial constructions of derived functors* as compositions of known functors:

$$RF \stackrel{\text{def}}{=} K(F)\circ\mathcal{I} \ : \ \mathcal{A} \to K(\mathcal{B}) \quad \text{and} \quad R^iF \stackrel{\text{def}}{=} H^i\circ K(F)\circ\mathcal{I} \ : \ \mathcal{A} \to \mathcal{B}.$$

This is pretty neat, but it turns out that there is an even cleaner point of view on homological algebra:

8.8.5. *Derived category of complexes.* One can do better then make resolutions functorial, one can get all resolutions of M to be *canonically isomorphic to* M, so that when we are using a resolution no complications are introduced. For that one passes to the *derived category* $D(\mathcal{A})$ by adding more isomorphisms to $K(\mathcal{A})$:

- Objects are (again) complexes.
- Any map of complexes $A^\bullet \xrightarrow{\alpha} B^\bullet$ which gives isomorphism of cohomology groups (i.e., all maps $H^i(A^\bullet) \xrightarrow{H^i(\alpha)} H^i(B^)\bullet$ are isomorphisms), acquires an inverse in $D(\mathcal{A})$, i.e., α becomes an isomorphism in $D(\mathcal{A})$.

This derived category is the standard set up for homological algebra. One problem it resolves is how to derive functor FG which is neither left nor right exact, but is a composition of say a left exact functor F and a right exact functor G.[20]

[20] This is not doable without derived categories, the reason is essentially that while taking cohomology of complexes forgets a lot of information, the derived categories take complexes seriously.

9. **Use sheaves**

Sheaves are a machinery which addresses an essential problem – the relation between local and global information – so they appear throughout mathematics, but sheaves are particularly useful and highly developed in algebraic geometry.

The passage from local to global is formalized here as the procedure $\Gamma(X,-)$ of taking global sections of sheaves on a space X. However, this becomes really useful only when combined with homological algebra. The derived functors of $\Gamma(X,-)$ are the sheaf cohomology functors $H^i(X,-),\ i=0,1,....$

We start on sheaf cohomology with an approximate version – the Čech cohomology $\check{H}^i_{\mathcal{U}}(X,\mathcal{A})$ of a sheaf $\mathcal{A}$ with respect to an open cover $\mathcal{U}$ of X. This is a great calculational tool because in many situations (i.e., under some conditions on the relation between the cover and the sheaf) it computes the true cohomology $H^i(X,\mathcal{A})$. However, Čech cohomology is much more down to Earth[21] then the correct version. After this introduction we define the general cohomology of sheaves.

In algebraic geometry the basic example of sheaves are the so called coherent sheaves, for instance the sheaves of sections of vector bundles. We will be most interested in line bundles on curves and the main tool will be the Riemann-Roch theorem.

9.1. **Sheaves.**

9.1.1. *Example of a sheaf: smooth functions on* $\mathbb{R}$. Let X be $\mathbb{R}$ or any smooth manifold. The notion of smooth functions on X gives the following data:

- for each open $U\subseteq X$ an algebra $C^\infty(U)$ (the smooth functions on U),
- for each inclusion of open subsets $V\subseteq U\subseteq X$ a map of algebras $C^\infty(U)\xrightarrow{\rho^U_V} C^\infty(V)$ (the restriction map);

and these data have the following properties

(1) (*transitivity of restriction*) $\rho^U_V\circ\rho^U_V = \rho^U_W$ for $W\subseteq V\subseteq U$,
(2) (*gluing*) if the functions $f_i\in C^\infty(U_i)$ on open subsets $U_i\subseteq X,\ i\in I$, are compatible in the sense that $f_i=f_j$ on the intersections $U_{ij}\overset{\text{def}}{=}U_i\cap U_j$, then they glue into a unique smooth function f on $U=\cup_{i\in I}U_i$.

The context of dealing with objects which can be restricted and glued compatible pieces is formalized in the notion of sheaves. The definition is formal (precise) way of saying that a given class $\mathcal{C}$ of objects forms a sheaf if it is *defined by local conditions*, i.e., conditions which can be checked in a neighborhood of each point:

[21] Gea.

9.1.2. *Definition of sheaves on a topological space.* A sheaf of sets $\mathcal{S}$ on a topological space $(X,\mathcal{T})$ consists of the following data:

- for each open $U\subseteq X$ a set $\mathcal{S}(U)$,
- for each inclusion of open subsets $V\subseteq U\subseteq X$ a map $\mathcal{S}(U) \xrightarrow{\rho^U_V} \mathcal{S}(V)$ (called the restriction map);

and these data are required to satisfy

(1) (*identity*) $\rho^U_U = id_{\mathcal{S}(U)}$.
(2) (*transitivity of restriction*) $\rho^V_W \circ \rho^U_V = \rho^U_W$ for $W\subseteq V\subseteq U$,
(3) (*gluing*) Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of an open $U\subseteq X$. For a family of elements $f_i \in \mathcal{S}(U_i)$, $i\in I$, compatible in the sense that $\rho^{U_i}_{U_{ij}} f_i = \rho^{U_i}_{U_{ij}} f_j$ in $\mathcal{S}(U_{ij})$ for $i,j\in I$; there is a unique $f\in\mathcal{S}(U)$ such that on the intersections $\rho^{U_i}_{U_{ij}} f = f_i$ in $\mathcal{S}(U_i)$, $i\in I$.
(4) $\mathcal{S}(\emptyset) = \emptyset$.

9.1.3. *Sheaves with values in a category $\mathcal{A}$.* We can equally define sheaves of abelian groups, rings, modules, etc – only the last, and least interesting requirement has to be modified, say in abelian groups we would ask that $\mathcal{S}(\emptyset)$ is the trivial group $\{0\}$.

9.1.4. *Examples.* (1) *Structure sheaves.* On a topological space X one has a sheaf of continuous functions C_X. If X is a smooth manifold there is a sheaf C^∞_X of smooth functions, etc., holomorphic functions $\mathcal{H}_X$ on a complex manifold, "polynomial" functions $\mathcal{O}_X$ on an algebraic variety. In each case the topology on X and the sheaf contain all information on the structure of X.

(2) The constant sheaf S_X on X associated to a set S: $S_X(U)$ is the set of *locally* constant functions from U to X.

(3) Constant functions do not form a sheaf, neither do the functions with compact support. A given class $\mathcal{C}$ of objects forms a sheaf if it is defined by local conditions. For instance, being a (i) function with values in S, (ii) non-vanishing (i.e., invertible) function, (iii) solution of a given system $(*)$ of differential equations; are all local conditions: they can be checked in a neighborhood of each point.

9.2. **Global sections functor** $\Gamma : \mathcal{S}heaves(X) \to \mathcal{S}ets$**.** Elements of $\mathcal{S}(U)$ are called the *sections* of a sheaf $\mathcal{S}$ on $U\subseteq X$ (this terminology is from classical geometry). By $\Gamma(X,\mathcal{S})$ we denote the set $\mathcal{S}(X)$ of *global sections.*

The construction $\mathcal{S}\mapsto\Gamma(X,\mathcal{S})$ means that we are looking at global objects of a given class $\mathcal{S}$ of objects, which is defined by local conditions. We will see that the procedure $\mathcal{S}\mapsto\Gamma(X,\mathcal{S})$ has a hidden part, the *cohomology* $\mathcal{S}\mapsto H^\bullet(X,\mathcal{S})$ of the sheaf $\mathcal{S}$ on X.

9.2.1. *Smooth manifolds.* On a smooth manifold X, $\Gamma(X, C^\infty) = C^\infty(X)$ is huge. The holomorphic setting will be more subtle.

9.2.2. *Example: sheaves corresponding to multivalued function.* Let $\mathcal{S}$ be the sheaf of solutions of $zy' = \lambda y$ in holomorphic functions on $X = \mathbb{C}^*$. On any disc $c \in D \subseteq X$, evaluation at the center gives $\mathcal{S}(D) \xrightarrow{\cong} \mathbb{C}$ (the solutions are multiples of functions $z^\lambda = e^{\lambda \log(z)}$ defined using a branch of logarithm on D). However, $\Gamma(X, \mathcal{S}) = 0$ if $\lambda \notin \mathbb{Z}$. So locally there is a lot, but nothing globally. This is a restatement of: multi-valued function z^λ is useful but has no single-valued meaning on $\mathbb{C}^*$.

9.2.3. *Example: global holomorphic functions on* $\mathbb{P}^1$. $\mathbb{P}^1 = \mathbb{C} \cup \infty$ can be covered by $U_1 = U = \mathbb{C}$ and $U_2 = V = \mathbb{P}^1 - \{0\}$. We think of $X = \mathbb{P}^1$ as a complex manifold by identifying U and V with $\mathbb{C}$ using coordinates u, v such that on $U \cap V$ one has $uv = 1$.

Lemma. $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$.

Proof. (1) *Proof using a cover.* A holomorphic function f on X restricts to $f|U = \sum_{n \geq 0} \alpha_n u^n$ and to $f|V = \sum_{n \geq 0} \beta_n v^n$. On $U \cap V = \mathbb{C}^*$, $\sum_{n \geq 0} \alpha_n u^n = \sum_{n \geq 0} \beta_n u^{-n}$, and therefore $\alpha_n = \beta_n = 0$ for $n \neq 0$.

(2) *Proof using maximum modulus principle.* The restriction of a holomorphic function f on X to $U = \mathbb{C}$ is a bounded holomorphic function (since X is compact), hence a constant.

9.3. **Čech cohomology of sheaves.** Cohomology of sheaves is a machinery which deals with the subtle ("hidden") part of the the relation between local and global information. The Čech cohomology is its simplest calculational tool.

9.3.1. *Cohomology of sheaves.* There is a general *cohomology theory for sheaves* which associates to any sheaf of abelian groups $\mathcal{A}$ a sequence of groups $H^i(X, \mathcal{A})$. The Čech cohomology $\check{H}^i_{\mathcal{U}}(X, \mathcal{A})$ can be viewed as an approximation of the *true cohomology* $H^i(X, \mathcal{A})$, which is calculated using an open cover $\mathcal{U}$ of X. We start with the Čech cohomology which is conceptually much simpler, however it is very useful since in practice, for a specific class of sheaves $\mathcal{A}$ one can find the corresponding class of open covers $\mathcal{U}$ such that $\check{H}^i_{\mathcal{U}}(X, \mathcal{A}) = H^i(X, \mathcal{A})$.[22]

9.3.2. *Calculation of global section via an open cover.* The first idea is to find all global sections of a sheaf by examining how one can glue local sections into global sections. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of a topological space X, we will choose a complete ordering on I[23] We will use finite intersections $U_{i_0, \ldots, i_p} \stackrel{\text{def}}{=} U_{i_0} \cap \cdots \cap U_{i_p}$ with $i_0 < \cdots < i_p$.

[22]For instance in algebraic geometry one usually considers the *quasicoherent sheaves* and then it suffices if all U_i are affine.

[23]It is not really necessary but it simplifies practical calculations.

To a sheaf of abelian groups $\mathcal{A}$ on X we associate a map of abelian groups (e,

- $C^0(\mathcal{U},\mathcal{A})\overset{\text{def}}{=} \prod_{i\in I}\ \mathcal{A}(U_i)$, its elements are systems $f = (f_i)_{i\in I}$, with one section $f_i \in \mathcal{A}(U_i)$ for each open set U_i,
- $C^1(\mathcal{U},\mathcal{A})\overset{\text{def}}{=} \prod_{\{(i,j)\in I^2;\ i<j\}}\ \mathcal{A}(U_{ij})$, its elements are systems $g = (g_{ij})_{(i,j)\in I^2}$ of sections $g_{ij} \in \mathcal{A}(U_{ij})$ on all intersections U_{ij}.
- map sends $f = (f_i)_{i\in I} \in C^0$ to $df \in C^1$ with

$$(df)_{ij} \overset{\text{def}}{=} \rho^{U_j}_{U_{ij}} f_j - \rho^{U_i}_{U_{ij}} f_i.$$

Less formally, we usually state it as $(df)_{ij} =\ f_j|U_{ij} - f_i|U_{ij}$.

Lemma. For any sheaf of abelian groups $\mathcal{A}$ on X

$$\Gamma(\mathcal{A})\overset{\cong}{\to} \text{Ker}[C^0(\mathcal{U},\mathcal{A}) \overset{d}{\to} C^1(\mathcal{U},\mathcal{A})].$$

9.3.3. *Čech complex* $C^\bullet(\mathcal{U},\mathcal{A})$. Emboldened, we try more of the same. We want to capture more of the relation between local sections by extending the construction into a sequence of maps of abelian groups

$$C^0(\mathcal{U},\mathcal{A}) \xrightarrow{d^0} C^1(\mathcal{U},\mathcal{A}) \xrightarrow{d^1} C^2(\mathcal{U},\mathcal{A}) \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} C^n(\mathcal{U},\mathcal{A}) \xrightarrow{d^n} C^{n+1}(\mathcal{U},\mathcal{A}) \xrightarrow{d^{n+1}} \cdots.$$

Here,

$$C^n(\mathcal{U},\mathcal{A})\overset{\text{def}}{=} \prod_{i_0<\cdots<i_n} \mathcal{A}(U_{i_0,\dots,i_n})$$

consists of all systems of sections on $(n+1)$-tuple intersections. The map d^n (we call it the n^{th} *differential*), creates from $f = (f_{i_0,\dots,i_n})_{I^n} \in C^n$ an element $d^n(f) \in C^{n+1}$, with

$$d^n(f)_{i_0,\dots,i_{n+1}} = \sum_{s=0}^{n+1} (-1)^s f_{i_0,\dots,i_{s-1},i_{s+1},\dots,i_{n+1}}.$$

From this we construct groups of *n-cocycles* and *n-coboundaries*

$$Z^n(\mathcal{U},\mathcal{A}) \overset{\text{def}}{=} \text{Ker}(C^n \xrightarrow{d^n} C^{n+1}) \subseteq C^n \quad \text{and} \quad B^n(\mathcal{U},\mathcal{A}) \overset{\text{def}}{=} Im(C^{n-1} \xrightarrow{d^{n-1}} C^n) \subseteq C^n.$$

Lemma. (Čech complex $C^\bullet(\mathcal{U},\mathcal{A})$.)

(a) Show that d^0 is the same as before.

(b) Show that $(C^\bullet(\mathcal{U},\mathcal{A}),d^\bullet)$ is a complex, i.e., $d^n \circ d^{n-1} = 0$.

(c) Show that $B^n(\mathcal{U},\mathcal{A})\subseteq Z^n(\mathcal{U},\mathcal{A})$.

9.3.4. *Čech cohomology* $\check{H}^\bullet_{\mathcal{U}}(X,\mathcal{A})$. It is defined as the cohomology of the Čech complex $C^\bullet(\mathcal{U},\mathcal{A})$, i.e.,

$$\check{H}^n_{\mathcal{U}}(X,\mathcal{A}) \overset{\text{def}}{=} Z^n(\mathcal{U};\mathcal{A})/B^n(\mathcal{U};\mathcal{A}), \quad n=0,1,2,....$$

This construction is a generalization of the global sections of a sheaf since

$$\check{H}^0_{\mathcal{U}}(X,\mathcal{A}) = Z^0(\mathcal{U},\mathcal{A})B^0(\mathcal{U},\mathcal{A}) = Z^0(\mathcal{U},\mathcal{A}) = \Gamma(\mathcal{A}).$$

Lemma. If the open cover $\mathcal{U}$ consists of two open sets U and V, show that

(1) $\check{H}^0_{\mathcal{U}}(X,\mathcal{A}) = \{(a,b)\in\mathcal{A}(U)\oplus\mathcal{A}(V);\ a=b \text{ on } U\cap V\} \cong \Gamma(X,\mathcal{A})$.
(2) $\check{H}^1_{\mathcal{U}}(X,\mathcal{A}) = \frac{\mathcal{A}(U\cap V)}{\rho^U_{U\cap V}\mathcal{A}(U)+\rho^V_{U\cap V}\mathcal{A}(V)}$.
(3) $\check{H}^i_{\mathcal{U}}(X,\mathcal{A}) = 0$ for $i>1$.

9.4. **Quasi-coherent sheaves on algebraic varieties.**

9.5. **True cohomology of sheaves.**

9.5.1. *Functors* $H^i(X,\mathcal{F})$. The cohomology of sheaves starts with the functor of global sections considered as a functor

$$\Gamma(X,-):\ \mathcal{S}h\mathcal{A}b(X)\to\ \mathcal{A}b$$

from the category of sheaves of abelian groups to the category of abelian groups. It turns out that

Lemma. (a) $\mathcal{S}h\mathcal{A}b(X)$ is an *abelian category* with enough injectives.

(b) Functor $\Gamma(X,-)$ is left exact.

Therefore there are right derived functors of $\Gamma(X,-)$ which one denotes

$$H^i(X,-) \overset{\text{def}}{=} R^i\Gamma(X,-):\ \mathcal{S}h\mathcal{A}b(X)\to\ \mathcal{A}b.$$

9.5.2. *Computation.* The unpleasant part here is that by definition this involves injective resolutions in the category of sheaves and injective objects in sheaves tend to be very large. So we try to minimize the use of definitions and use general properties such as

Theorem.

(1) $H^0(X,\mathcal{A}) = \Gamma(X,\mathcal{A})$ and it equals $\check{H}^0_{\mathcal{U}}(X,\mathcal{A})$ so it can be calculated using any open cover.
(2) A short exact sequence of sheaves gives a long exact sequence of cohomologies.

9.5.3. *Computation in algebraic geometry.* In any given setting (topology, analysis, complex manifolds, logic) one develops the understanding of the classes of sheaves relevant for that setting. In algebraic geometry the most relevant sheaves are the *quasi-coherent sheaves*, and here are some basic facts

Theorem. Let $\mathcal{F}$ be a quasi-coherent sheaf on an algebraic variety X.

(1) If X is affine, the higher cohomologies vanish: $H^i(X,\mathcal{F}) = 0$ for $i > 0$.
(2) If $\mathcal{U}$ is an open cover such that all finite intersections $U_{i_1,\dots,i_n}$ are affine, the cohomology is the same as the Čech cohomology:
$$H^i(X,\mathcal{F}) \;=\; \check{H}^i_{\mathcal{U}}(X,\mathcal{F}).$$
(3) Cohomologies vanish beyond dimension of X: $H^i(X,\mathcal{F}) = 0$ for $i > \dim(X)$.

Corollary. On curves we only have to worry about $H^0(C,\mathcal{F})$ and $H^1(C,\mathcal{F})$.

9.6. **Geometric representation theory: cohomology of line bundles on $\mathbb{P}^1$.**

9.6.1. *Cohomology of vector bundles.* Recall that to a vector bundle V on X we associate the sheaf $\mathcal{V}$ of sections of the vector bundle V. If V is obtained by gluing trivial vector bundles $V_i = U_i{\times}\mathbb{C}^n$ by transition functions ϕ_{ij}, then $\mathcal{V}(U)$ consists of all systems of $f_i \in \mathcal{H}(U_i\cap U,\mathbb{C}^n)$ such that on all intersections $U_{ij}\cap U$ one has $f_i = \phi_{ij}f_j$.

By *cohomology $H^*(X,V)$ of the vector bundle V* we mean the cohomology of the associated sheaf $\mathcal{V}$.

9.6.2. *Line bundles L_n on $\mathbb{P}^1$.* On $\mathbb{P}^1$ let L_n be the vector bundle obtained by gluing trivial vector bundles $U{\times}\mathbb{C}$, $V{\times}\mathbb{C}$ over $U\cap V$ by identifying $(u,\xi)\in U{\times}\mathbb{C}$ and $(v,\zeta)\in V{\times}\mathbb{C}$ if $uv = 1$ and $\zeta = u^n{\cdot}\xi$. So for $U_1 = U$ and $U_s = V$ one has $\phi_{12}(u) = u^n$, $U \in U\cap V{\subseteq}U$. Let $\mathcal{L}_n$ be the sheaf of holomorphic sections of L_n.

9.6.3. *Lemma.* (a) $\Gamma(\mathbb{P}^1,\mathcal{L}_n) \cong\ \mathbb{C}_n[x,y]\overset{\text{def}}{=}$ homogeneous polynomials of degree n. So, it is zero if $n<0$ and for $n\geq 0$ the dimension is $n+1$.

(b) $H^1_{\mathcal{U}}(\mathbb{P}^1,\mathcal{L}_n) \cong$?.

9.6.4. *Representations of $G = SL_2(\mathbb{C})$.* $SL_2(\mathbb{C})$ acts on $\mathbb{C}_2$ and therefore also on the algebra of functions $\mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x,y]$, and its homogeneous summands $\mathbb{C}_n[x,y]$. Also, action on $\mathbb{C}^2$ factors to an action on $\mathbb{P}^1$ = lines in $\mathbb{C}^2$, and this extends to an action on line bundles $\mathcal{L}_n$ over $\mathbb{P}^1$. In particular, $SL_2(\mathbb{C})$ acts on the vector space $H^i(\mathbb{P}^1,\mathcal{L}_n)$ that are naturally produced from L_n. In fact,

9.6.5. *Lemma.* $\mathbb{C}_n[x,y] = \Gamma(\mathbb{P}^1,\mathcal{L}_n), n = 0,1,2,..$ is the list of *all irreducible finite dimensional holomorphic representations of* $SL_2(\mathbb{C})$.

9.6.6. *Borel-Weil-Bott theorem.* For each semisimple (reductive) complex group G there is a space $\mathcal{B}$ (the flag variety of G) such that all irreducible finite dimensional holomorphic representations of G are obtained as global sections of line bundles on $\mathcal{B}$.

9.7. **Riemann-Roch theorem.** This is the most useful tool for calculations on curves. Classically, the Riemann-Roch theorem is a a deep geometric statement that relates the counts in two related situations, of meromorphic functions and 1-forms satisfying certain conditions on zeros and poles that are allowed. We will present it in terms of cohomology of line bundles and break it into several standard ideas that all generalize individually to higher dimensional geometry. The proof is now almost trivial because of the added flexibility in the sheaf theoretic setting.

9.7.1. *Other approaches.* A purely geometric proof of Riemann-Roch may take a semester. So sheaves are useful, but one pays a price incorporating sheaves into a standard part of our thinking. There is another, faster, approach to Riemann-Roch theorem for curves through the ring of *adels.* This is simpler then learning sheaves,[24] but this approach has a disadvantage that it has not been developed as well in higher dimensions, so it is not a standard tool in mathematics (except in one theory that specializes in the one-dimensional objects: the *Number Theory*).

9.7.2. *Riemann-Roch spaces.* To a divisor D on a curve C one associates the *Riemann-Roch* vector space

$$H(D) \stackrel{\text{def}}{=} \{f \in \mathfrak{M}(C);\ div(f) + D \geq 0\}$$

and the Riemann-Roch number $h(D) \stackrel{\text{def}}{=} \dim[H(D)]$, the number of (linearly independent) global meromorphic functions on C that satisfy some restrictions on the positions of poles and zeros (which we specify by the choice of the divisor D).

9.7.3. *Canonical divisors.* A divisor K on C is called a *canonical divisor* if $\mathcal{O}_C(K) \cong \Omega^1_C$.

9.7.4. *Theorems.*

Theorem. [A. Riemann-Roch theorem.] Let g be the genus of C and let K be any canonical divisor on C. Then, for any divisor D on C

$$h(D) - h(K - D) = deg(D) + 1 - g.$$

[24] It is not obvious, but adels really amount to learning sheaves precisely in the amount needed for line bundles on curves.

Theorem. [B. Riemann-Roch companion.]

(a) $deg(K) = 2(g-1)$.

(b) $deg(D) < 0 \;\Rightarrow\; h(D) = 0$.

The vanishing claim above leads to a special case of the Riemann-Roch theorem which is particularly satisfying:

Corollary. If $deg(D) > deg(K) = 2(g-1)$ then

$$h(D) = \; deg(D) + 1 - g.$$

9.7.5. *$h(D)$ as a function of $deg(D)$.* Notice that the dependence of $h(D)$ on $deg(D)$ is given by the line $y = x + (1-g)$ when $deg(D) > 2(g-1)$ and by the line $y = 0$ when $deg(D) < 0$. In between there is a subtlety interval $0 \le deg(D) \le 2(g-1)$ where $h(D)$ depends on D in subtle ways (in particular, it is not determined by $deg(D)$), and this is a source of a lot of nice mathematics.

9.7.6. *Examples.* There are three different kinds of behavior:

(1) $\boxed{g = 0.}$ Everything is known:

$$h(D) = max[deg(D) + 1, 0].$$

This can be used to show that any curve of genus zero is isomorphic to $\mathbb{P}^1$.

(2) $\boxed{g = 1.}$ When $deg(D) \neq 0$ then

$$h(D) = max[deg(D), 0].$$

In the only undetermined case $deg(D) = 0$ one has different possibilities since $h(0) = 1$ and $h(a-b) = 0$ when $a \neq b$.

(3) $\boxed{g > 1.}$ Here the unknown grows.

9.8. **Abel-Jacobi maps $C^{(n)} \xrightarrow{\mathcal{AJ}_n} Pic_n(C)$.** Recall the Abel-Jacobi maps

$$C^{(n)} \xrightarrow{\mathcal{AJ}_n} Pic_n(C), \qquad D \mapsto [\mathcal{O}_C(D)] \quad (n \ge 0).$$

9.8.1. *Lemma.* The fiber ${\mathcal{AJ}_n}^{-1}(L)$ of the n^{th} Abel-Jacobi map, at a line bundle $L \in Pic_n(C)$ is the projective spaces $\mathbb{P}[\Gamma(C,L)]$.

Proof. Recall that $C^{(n)}$ consists of all effective divisors of degree n. So, the fiber ${\mathcal{AJ}_n}^{-1}(L)$ consists of all effective divisors of degree n such that $\mathbb{L} \cong \mathcal{O}_C(D)$. But, $\mathbb{L} \cong \mathcal{O}_C(D)$ means that D is a divisor of a meromorphic section $s \neq 0$ of L. Moreover, since D is effective, such section has to be holomorphic. So, the fiber is

$${\mathcal{AJ}_n}^{-1}(L) \;=\; \{div(s);\; s \in \Gamma(C,L) - 0\}.$$

Therefore, we consider the map

$$\Gamma(C,L)-0 \xrightarrow{div} Div(C).$$

If s_1, s_2 have the same image: $div(s_1) = div(s_2)$ then $s_1 {s_2}^{-1}$ is a meromorphic section of $L \otimes L^* \cong T = C \times \mathbb{C}$ and $div(s_1 {s_2}^{-1}) = div^L(s_1) - div^L(s_2) = 0$. So this is a holomorphic functions which does not vanish, i.e., a non-zero constant! Therefore,

$$div(s_1) = div(s_2) \;\Leftrightarrow\; s_1 \in \mathbb{C}^* s_2 \;\Leftrightarrow\; [s_1] = [s_2] \in \mathbb{P}[\Gamma(C,L)].$$

Therefore, we have found that

$$\mathcal{AJ}_n{}^{-1}(L) \;\cong\; div[\Gamma(C,L)-0] \;\cong\; \mathbb{P}[\Gamma(C,L)-0] \;\cong\; \mathbb{P}^{\dim[\Gamma(C,L)]-1}.$$

9.8.2. *Theorem.* When $n > 2(g-1)$ the Abel-Jacobi map

$$C^n \xrightarrow{\mathcal{AJ}_n} Pic_n(C), \qquad D \mapsto [\mathcal{O}_C(D)],$$

is a bundle of projective spaces $\mathbb{P}^{n-g}$.

Proof. If $deg(L) = n$ is $> 2(g-1)$, we know that

$$\dim[\Gamma(C,L)] - 1 = \; [\deg(L) + 1 - g] - 1 = \; n - g.$$

In particular, the fiber is the same at all points of $Pic_n(C)$, and this implies that $\mathcal{AJ}_n$ is a bundle.

9.8.3. *Remarks.* (0) When $\Gamma(C,L) = 0$ the fiber is $\mathbb{P}(\{0\}) = \emptyset$.

(1) When n is not large enough, each fiber is either empty (if $\Gamma(C,L) = 0$) or a projective space $\mathbb{P}[\Gamma(C,L)]$. However, the fibers now may vary with L. For instance

(1) For $g > 0$ map $C \xrightarrow{\mathcal{AJ}_1} J_1(C)$ is an embedding (so most fibers are empty).
(2) For $1 \le d \le g$ the image of $C^{(d)} \xrightarrow{\mathcal{AJ}_d} J_d(C)$ has dimension d. So $\mathcal{AJ}_g$ is the first surjective map while
    The image of $C^{(g-1)}$ is a codimension one hypersurface called the theta divisor. $\Theta \subseteq J_{g-1}(C)$.

For instance in (1) it is clear that $C \xrightarrow{\mathcal{AJ}_1} J_1(C)$ is injective since the non-empty fibers are projective spaces and if $g > 0$ then C does not contain any $\mathbb{P}^n$ with $n > 0$. So the non-empty fibers are points. (However, for $g = 0$ one has $C \cong \mathbb{P}^1$ and this is indeed a fiber of $\mathcal{AJ}_1$.)

9.9. **Class Field Theory.** The *Class Field Theory* is a central part of Number Theory. It has the *arithmetic part* (study of Spec($\mathbb{Z}$)), and the *geometric part* (study of curves over finite fields). The two areas provide completely parallel theories but the arithmetic part is much deeper and the geometric part is often used as a source of ideas.

The above theorem

When $n > 2(g-1)$ the Abel-Jacobi map

$$C^n \xrightarrow{\mathcal{AJ}_n} Pic_n(C), \qquad D \mapsto [\mathcal{O}_C(D)],$$

is a bundle of projective spaces $\mathbb{P}^{n-g}$.

is essentially the *unramified case* of the geometric Class Field Theory. The transition to the *ramified part* means that we allow curves which are not complete – a few points may be missing. Roughly, the story is the same, except that it takes longer to tell because Jacobians get larger.

What is the use of the above theorem? It gives *linearization* for certain kind of data on C. Imagine some object $\mathcal{L}$ spread over C. The first step in its linearization is the integration over finite unordered subsets of C,[25] it gives an object $\mathcal{L}^{(n)}$ spread over $C^{(n)}$.

$$\mathcal{L}^{(n)}(D) \stackrel{\text{def}}{=} \int_D \mathcal{L}.$$

Next, for sufficiently large n (i.e., $n > 2(g-1)$, one uses the theorem to descend $\mathcal{L}^{(n)}$ to an object $\mathcal{L}_n$ spread over $J_n(C)$, in the sense that

$$\mathcal{L}^{(n)} \;\cong\; (\mathcal{AJ}_n)^* \mathcal{L}_n.$$

Finally, these $\mathcal{L}_n$'s for $n > 2(g-1)$ constitute a version of $\mathcal{L}$ that lives on an abelian group $J(C)$, and $\mathcal{L}_n$'s are in some sense compatible with the group structure on $J(C)$. This compatibility allows one to to extend the construction of objects $\mathcal{L}_n$ on $J_n(C)$, from $n > 2(g-1)$ to *all* integers $n \in \mathbb{Z}$. In the light of this compatibility of the family $\mathcal{L}_\bullet$ spread over $J(C)$ with the group structure, one can view $\mathcal{L}_\bullet$ as a linearization of $\mathcal{L}$.

9.9.1. *Langlands program.* The objects $\mathcal{L}$ that one can linearize in this way (for instance one dimensional local systems, or equivalently the abelian Galois representations), are necessarily simple enough to be understood in terms of an abelian group. In 68 Langlands proposed a program to linearize more complicated objects in terms of non-abelian groups. Ever since, it has been one of the central undertakings in pure math.

9.9.2. *Completely integrable systems.* This is another example of objects (specially nice and interesting partial differential equations), that linearize on Jacobians of curves. So one can ask what is the relation to Class Field Theory?

9.10. **Cohomology of line bundles on curves.**

[25]Remember that because of $C^{(n)} \cong C^{[n]}$ we can view *finite unordered subsets with multiplicities* as *finite subschemes* of C.

9.10.1. *Line bundles and sheaf cohomology.* By definitions, the Riemann-Roch space $H(D)$ is the space of global sections

$$H(D) \ = \ \Gamma(C, \mathcal{O}_C(D)) = \mathcal{O}_C(D)\ (C)$$

of the line bundle $\mathcal{O}_C(D)$ associated to C. So, it is possible that the entire cohomology of line bundles $\mathcal{O}_C(D)$ is relevant, so let us denote by $h^i(D)$ be the dimension of the i^{th} cohomology group $H^i[C, \mathcal{O}_C(D)]$. Then $h^0(D) = h(D)$ and it will turn out that the second ingredient of the Riemann-Roch theorem is

$$h^1(D) = h(K - D).$$

So, the Riemann-Roch theorem really is about cohomology of line bundles. Moreover, the treatment of a geometric question $h(D) =?$, in terms of sheaves makes situation quite flexible since there are more sheaves then just the line bundles.

In terms of sheaf cohomology the Riemann-Roch theorem is explained as a combination of several facts, each of which has important generalizations to higher dimensional objects.

(1) Euler characteristic. The following statement has important generalizations to higher dimensional varieties (Riemann-Roch-Hirzebruch theorem), and to schemes and maps of schemes (Grothendieck-Riemann-Roch theorem), etc. The Euler characteristic of cohomology of a line bundle is

$$\chi[H^*(X; L)] = \ \dim[H^0(X; L)] - \dim[H^1(X; L)]$$

because we know that on a curve $H^i(X; L) = 0$ for $i > \dim(C) = 1$.

Theorem. For any line bundle L on C,

$$\chi[H^*(X; L)] = \ deg(L) + 1 - g.$$

(2) Serre duality.

Theorem. For any line bundle L on C

$$H^i(C, L)^* \ \cong \ H^{1-i}(C, L^* \otimes \Omega^1_C).$$

(3) Kodaira vanishing

Theorem. (a) If $deg(L) > 2g - 2$ then $H^1(C, L) = 0$.
(b) If $deg(L) < 0$ then $H^0(C, L) = 0$.

(4) Kodaira embedding.

Theorem. If $deg(L) > 2g$ then L gives a projective embedding of C, i.e., C can viewed as a submanifold of a projective space

$$C \hookrightarrow \ \mathbb{P}[\Gamma(C, L)^*].$$

(5) Riemann-Hurwitz. $deg(\Omega^1_C) = 2(g - 1)$.

Remarks. (1) The two claims of Kodaira vanishing are equivalent by Serre duality.

(2) Serre duality for the line bundle $L = \mathcal{O}_C(D)$ says that

$$h^1(D) = \ h^0(K - D)$$

where K is any canonical divisor (a divisor such that $\mathcal{O}_C(K) \cong \Omega^1_C$). Now, the Euler characteristic statement for $L = \mathcal{O}_C(D)$ gives the Riemann-Roch theorem, while the Kodaira vanishing and Riemann-Hurwitz give the companion theorem.

In the remainder we indicate the proofs of these claims.

9.10.2. *The Euler characteristic.* In general, the Euler characteristic of a line bundle behaves much better then the individual cohomology groups

Proposition. The value of $\chi[H^*(X; L)] - deg(L)$ is the same for all line bundles L on C.

Lemma. For a divisor D any point $a \in D$ gives short exact sequence of sheaves

$$0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D + a) \to \delta_a \to 0,$$

where δ_a is the sheaf analogue of the δ-distribution at a point:

$$\delta_a(V) = \begin{cases} 0 & \text{if } a \notin V \text{, and} \\ \mathbb{C} & \quad \text{if } V \ni a. \end{cases}.$$

Proof. Point $a \in C$ gives an inclusion of sheaves $\mathcal{O}_C(D) \subseteq \mathcal{O}_C(D + a)$. The quotient sheaf is the sheafification of the presheaf $V \mapsto Q(V) = \mathcal{O}_C(D + a)(V)/\mathcal{O}_C(D)(V)$. On $U = C - a \subseteq C$, inclusion $\mathcal{O}_C(D) \subseteq \mathcal{O}_C(D+a)$ is equality, so for $V \subseteq C - a$ one has $Q(V) = 0$. On the other hand, if V is a small neighborhood of a then on V $\mathcal{O}_C(D) = \ (z - z(a))^{-ord_a(D)} \mathcal{O}_C$ and so,

$$Q(V) = \ (z - z(a))^{-ord_a(D)-1} \mathcal{O}_C(V)/(z - z(a))^{-ord_a(D)} \mathcal{O}_C(V) \ \cong \ \mathcal{O}_C(V)/(z - z(a)) \cdot \mathcal{O}_C(V)$$

$$\xrightarrow[\cong]{f \mapsto f(a)} \ \mathbb{C}.$$

The sheafification of Q is then the sheaf δ_a.

Now, any line bundle L is isomorphic to one of $\mathcal{O}_C(D)$, $D \in Div(C)$, and we check in the same way that a point $a \in C$ gives a short exact sequence of sheaves

$$0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D + a) \to \delta_a \to 0.$$

Proof of the proposition. The above short exact sequence of sheaves leads to a long exact sequence of cohomology groups

$$0 \to H^0[C, \mathcal{O}_C(D)] \to H^0[C, \mathcal{O}_C(D+a)] \to H^0[C, \delta_a] \to H^1[C, \mathcal{O}_C(D)] \to H^1[C, \mathcal{O}_C(D+a)] \to H^1[C, \delta_a] \to 0.$$

The zero at the right end appears because we know that on a curve there is only H^0 and H^1. Moreover, we know that $H^0[C, \delta_a] = \mathbb{C}$, and we believe that $H^1[C, \delta_a] = 0$ because δ_a really lives on the point a and on a point there is only H^0. This gives the following relation of Euler characteristics:

$$\chi(H^0[C, \mathcal{O}_C(D+a)]) = \chi[H^*[C, \mathcal{O}_C(D)]] + \chi(H^*[C, \delta_a]) = \chi[H^*[C, \mathcal{O}_C(D)]] + 1).$$

However, one also has

$$deg[\mathcal{O}_C(D+a)] = deg(D+a) = deg(D) + 1 = deg[\mathcal{O}_C(D)] + 1.$$

So, the validity of the theorem for $\mathcal{O}_C(D)$ and for $\mathcal{O}_C(D+a)$ is equivalent (and one connect any two line bundles by a sequence of such changes).

9.10.3. *Cohomology of differential forms.* Now, it suffices to check the Euler characteristic theorem for one line bundle, and we use Ω^1_C. Recall that for a complex curve C, there are two ways to think of the genus g of C (and therefor one needed an argument to check that these are the same).

Theorem. (a) $\dim[H^0(X, \Omega^1_C)] = \dim[\Omega^1_C(C)] = g$.

(b) There is a canonical isomorphism (called the *trace map*)

$$H^1(C, \Omega^1_C) \xrightarrow[\cong]{tr} \mathbb{C}.$$

(c) $deg(\Omega^1_C) = 2g - 2$.

Proof. (a) is the holomorphic definition of the genus $g \stackrel{\text{def}}{=} \dim[\Omega^1_C(C)]$.

(c) Any meromorphic function f on C can be viewed as a *holomorphic* map $f : C \to \mathbb{P}^1$. If f is not constant then it is surjective and then this map can be used to reduce the claim from C to $\mathbb{P}^1$. However, the $\mathbb{P}^1$ case is easy since here we know the cohomology of all line bundles.

(b) We will construct the trace map using *residues.* Recall that if U is a neighborhood of a point $a \in C$ and ω is a meromorphic one form on U which is holomorphic off a that we can define the residue $Res_a(\omega)$ either[26]

- analytically as the integral $\int_\gamma \omega$ for a curve γ that goes once around a, or
- algebraically as the $(-1)^{\text{st}}$ coefficient of the expansion $\omega = \sum_i a_i \cdot z^i \cdot dz$ of ω in a local coordinate z.

[26]The second definition requires checking that it is independent of the choice of a local coordinate. This can be done either directly, or by the comparison with the first definition.

To apply this observe that Ω^1_C has a trivialization on $C-F$ for a finite subset F (if D is the divisor of any meromorphic section of Ω^1_C then $\Omega^1_C \cong \Omega_C(D)$ so it has a trivialization off the support $F = supp(D)$ of the divisor D). Also it has trivialization on small neighborhood V_a of each $a \in F$, moreover we can choose V_a to be identified with a disc with center a by a local chart, and also small enough so that all V_a's are disjoint. So, we get an open cover $\mathcal{U}$ of C by $U = C - F$ and $V = \cup_{a\in F}$. For some general reasons, one can calculate $H^*(C, \mathbb{P}1^1_C)$ as the Čech cohomology $H^*_{\mathcal{U}}(C, \mathbb{P}1^1_C)$ for this cover. But then

$$H^1(C, \Omega^1_C) \;\cong\; H^1_{\mathcal{U}}(C, \Omega^1_C) \;=\; \frac{\Omega^1_C(U\cap V)}{\rho^U_{U\cap V}\Omega^1_C(U) + \rho^V_{U\cap V}\Omega^1_C(V)}.$$

Now, $U \cap V$ is the disjoint union of punctured neighborhoods $V^*_a = V_a - \{a\}$ of points $a \in F$, so we have a map

$$\Omega^1_C(U\cap V) \xrightarrow{\sum_{a\in F}\ Res_a} \mathbb{C}.$$

This map kills the restrictions $\rho^V_{U\cap V}\Omega^1_C(V)$ of forms holomorphic on V (they are holomorphic at each $a \in F$!), and also the restrictions $\rho^U_{U\cap V}\omega$ of all forms $\omega \in \Omega^1_C(U)$, since for such ω $\sum_{a\in F}\ Res_a\omega = \ \sum_{a\in C}\ Res_a\omega$, however, for any meromorphic form η on C

$$\sum_{a\in C}\ Res_a\eta \;=\; 0.$$

We proved this before but only in the case when $\eta = df$ is the differential of some meromorphic function f on C (then the result was that $\sum\ ord_a f = \ sum\ Res_a(df/f) = 0$), the proof however works in general. Therefore, the map $\sum_{a\in C}\ Res_a :\ \Omega^1_C(U\cap V) \to\ \mathbb{C}$ factors to $H^1_{\mathcal{U}}(C, \Omega^1_C) \to \mathbb{C}$. This is the trace map.

9.10.4. *Euler characteristic theorem (the end of the proof).* Now we know that for all line bundles L on C, the number $\chi[H^*(X; L)] - deg(L)$ is the same, so we only need to calculate it for $L = \Omega^1_C$. Here

$$\dim[H^0(C, \Omega^1_C)] - \dim[H^1(C, \Omega^1_C)] - deg(\Omega^1_C)] = \ g - 1 - 2(g-1) = \ 1 - g.$$

9.10.5. *Kodaira vanishing.* The claim that $deg(L) < 0$ implies $\Gamma(C, L) = 0$ is quite elementary.

Choose a divisor D such that $L \cong \mathcal{O}_C(D)$. Recall that a holomorphic section $f \in \mathcal{O}_C(D)(C)$ is by definition a meromorphic function on C, and the two points of view are related by :

$$ord^{\mathcal{O}_C(D)}_a(f) \;=\; ord_a(f) + ord_a(D),\ a \in C, \quad i.e., \quad div^{\mathcal{O}_C(D)}(f) \;=\; div(f) + D.$$

This gives a contradiction:

$$deg[div^{\mathcal{O}_C(D)}(f)] = \ deg[div(f)] + deg[D] = deg(D) < 0,$$

while $div^{\mathcal{O}_C(D)}(f)$ has to be effective for holomorphic sections.

9.10.6. *Serre duality.* We need the following cohomological idea. The cohomology of line bundles L and M contributes to the cohomology of their tensor product $L\otimes M$ by the canonical maps

$$H^i(C,L)\otimes H^j(C.M) \xrightarrow{m_{ij}} H^{i+j}(C.L\otimes M).$$

This gives a pairing

$$H^i(C,L)\otimes H^{1-i}(C,L^*\otimes\Omega^1_C) \xrightarrow{m_{ij}} H^1(C,L\otimes L^*\otimes\Omega^1_C) \xrightarrow{\cong} H^1(C,\Omega^1_C) \xrightarrow{tr} \mathbb{C},$$

which one checks is non-degenerate so it induces $H^{1-i}(C,L^*\otimes\Omega^1_C)\xrightarrow{\cong} H^i(C,L)^*$.

9.10.7. *Kodaira embedding.* Let us see the meaning of the embedding claim. We will examine how a line bundle L on a compact manifold C gives a natural embedding of C into a projective space, *provided* that L has *sufficiently many sections.*

Any point $c\in C$ gives the evaluation map $e_c:\Gamma(C,L)\to L_c$.

(1) If there is a section s which does not vanish at c then e_c is surjective and $\mathrm{Ker}[e_c]$ is a hyperplane in $H_c\subseteq\Gamma(C,L)$, so it corresponds to a line $\iota(c)\stackrel{\text{def}}{=} H_c^\perp \in \mathbb{P}[\Gamma(C,L)^*]$.
(2) If for each point $c\in C$ there is a section that does not vanish at c, then ι is a map $\iota: C\to\mathbb{P}[\Gamma(C,L)^*]$.
(3) If for any pair of different points $(a,b)\in C^2$ there is a section s such that $s(a)\neq 0=s(b)$, then ι is injective.
(4) If for each point $c\in C$ and any tangent vector $v\in T_c(C)$ there is a section s such that $s(c)=0$ and $d_cs\in T_{\iota(c)}(C)$ is not zero, then ι is an embedding of manifolds, i.e., it makes C into a submanifold of the projective space $\mathbb{P}[\Gamma(C,L)^*]$.

Remark. One can state (3) as

sections of L distinguish the points of C.

One can view (4) in the same way :

sections of L distinguish infinitesimally close points of C.

since (in the scheme theoretic language) one can view the tangent vector v as giving the second point $a+v$ which is infinitesimally closed point to a. Now we can summarize the discussion into

Lemma. If L has enough sections (in the sense that they distinguish points and infinitesimally close points of C), then C embeds as a submanifold of $\mathbb{P}[\Gamma(C,L)^*]$.

Remarks. (1) So, the embedding theorem claims that if the degree of L is sufficiently large (precisely, $deg(L) > deg(\Omega^1_C) = 2(g-1)$), then L has enough sections to give a projective embedding of C.

(2) Actually, more is true. Once we embed a compact complex manifold X into some projective space $\mathbb{P}^n$, one can prove that inside $\mathbb{P}^n$, X is described by homogeneous polynomial equations, so the embedding gives a structure of a projective algebraic variety, Therefore, any compact complex manifold that has a line bundle with enough sections has a structure of a projective variety!

Proof. Let L be a line bundle on C of degree $n > 2g$.

(1) If M is a line bundle of degree $\geq 2g$, all evaluation maps are $\neq 0$. At each point $a \in C$ there is the evaluation map $\Gamma(C, M) \xrightarrow{e_a} M_a$ with values in the fiber M_a, $e_a(s) \stackrel{\text{def}}{=} s(a) \in M_a$. Choose a presentation of M as $\mathcal{O}_C(D)$ The short exact sequence of sheaves

$$0 \to \mathcal{O}_C(D-a) \to \mathcal{O}_C(D) \to \delta_a \to 0,$$

gives a long exact sequence of cohomology groups

$$0 \to H^0[C, \mathcal{O}_C(D-a)] \to H^0[C, \mathcal{O}_C(D)] \to H^0[C, \delta_a] \to H^1[C, \mathcal{O}_C(D-a)] \to \cdots.$$

If $deg[\mathcal{O}_C(D-a)] > 2g-2$, i.e., $deg[\mathcal{O}_C(D)] \geq 2g$, then $H^1[C, \mathcal{O}_C(D-a)] = 0$. So, the map $H^0[C, \mathcal{O}_C(D)] \to H^0[C, \delta_a] = \mathbb{C}$ is surjective. But this is exactly the evaluation map.

Also notice (for later) that $H^0[C, \mathcal{O}_C(D-a)] \to H^0[C, \mathcal{O}_C(D)]$ is the inclusion of all sections of $\mathcal{O}_C)D)$ that vanish at a.

(2) Sections distinguish points. Let a, b be two different points of C. If $L \cong \mathcal{O}_C(D)$ then the degree of $\mathcal{O}_C(D-b)$ is $deg(L) - 1 \geq 2g$, so the evaluation e_a is nonzero on sections of $\mathcal{O}_C(D-b)$. However, these are precisely the sections of $\mathcal{O}_C(D)$ that vanish at b. So, $\cong \mathcal{O}_C(D)$ has a holomorphic section which vanishes at b but not at a.

(3) Sections distinguish infinitesimally close points. Let us think of the evaluation $e_a(s) \in L_a$, as the restriction of a section s to a point, i.e., the restriction of a section s of a line bundle L on C to a section $s(a)$ of a line bundle L_a on the point a. This is the 0^{th} order information on s at a. To get the first order information we consider the double point subscheme $a_2 = \text{Spec}(\mathcal{O}_C/\mathcal{I}_a^2)$, the restriction $L|a_2$ of L to a_2, and the restriction map $\Gamma(C, L) \xrightarrow{\rho} \Gamma(a_2, L|a_2)$. The claim that *sections of L distinguish infinitesimally close points*, means literally that the map ρ is surjective.

To check this, we again put it into a sheaf framework. This is given by a slight generalization of the above sublemma:

Sublemma. For a divisor D any point $a \in D$ gives short exact sequence of sheaves

$$0 \to \mathcal{O}_C(D-2a) \to \mathcal{O}_C(D) \to \delta_{a_2} \to 0,$$

where δ_{a_2} is the sheaf[27]

$$\delta_{a_2}(V) = \begin{cases} 0 & \text{if } a \notin V, \text{ and} \\ \mathcal{O}(a_2) & \text{if } V \ni a. \end{cases} .$$

Proof. Point $a \in C$ gives an inclusion of sheaves $\mathcal{O}_C(D-2a) \subseteq \mathcal{O}_C(D)$ and the quotient sheaf is the sheafification of the presheaf $V \mapsto Q(V) = \mathcal{O}_C(D)(V)/\mathcal{O}_C(D-2a)(V)$. On $C-a \subseteq C$, inclusion $\mathcal{O}_C(D-2a) \subseteq \mathcal{O}_C(D)$ is equality, so for $V \subseteq C-a$ one has $Q(V)=0$. On the other hand, if V is a small neighborhood of a then on V $\mathcal{O}_C(D) = (z-z(a))^{-ord_a(D)}\mathcal{O}_C$ and so,

$Q(V) = (z-z(a))^{-ord_a(D)}\mathcal{O}_C(V)/(z-z(a))^{2-ord_a(D)}\mathcal{O}_C(V) \cong \mathcal{O}_C(V)/(z-z(a))^2 \cdot \mathcal{O}_C(V)$.
The sheafification of Q is then the sheaf δ_{a_2}.

End of the proof of the embedding theorem. The sublemma puts the restriction map ρ into an exact sequence

$$0 \to H^0[C, \mathcal{O}_C(D-2a)] \to H^0[C, \mathcal{O}_C(D)] \xrightarrow{\rho} H^0[C, \delta_{a_2}] \to H^1[C, \mathcal{O}_C(D-2a)] \to \cdots.$$

If $deg[\mathcal{O}_C(D)] > 2g$, then $deg[\mathcal{O}_C(D-2a)] > 2g-2$, hence $H^1[C, \mathcal{O}_C(D-2a)] = 0$. So, ρ is surjective.

[27]The functions on a_2 are described in terms of a coordinate function z on a small neighborhood W of a, by $\mathcal{O}(a_2) \cong \mathcal{O}(W)/(z-z-z(a))^2\mathcal{O}(W)$.

10. **Abelian category of sheaves of abelian groups**

In this section we fill in some details in the construction of the cohomology of sheaves. We check that the category of sheaves of abelian groups on a given topological space, has all ingredients needed in order to use the homological algebra, i.e., it is an abelian category with enough injectives.

For a topological space X we will denote by $\mathcal{S}h(X) = \mathcal{S}heaves(X, \mathcal{A}b)$ the category of sheaves of abelian groups on X. Since a sheaf of abelian groups is something like an abelian group smeared over X, we hope that $\mathcal{S}h(X)$ is again an abelian category, i.e., that one can do the computations here the same way as one can do in the category $\mathcal{A}b$ of abelian groups. However,

10.0.8. *Presheaves and sheafification.* When we attempt to construct the cokernels of maps, we find that the first idea does not quite work – it produces something like a sheaf but without the gluing property. This forces us to

- (i) generalize the notion of sheaves to a weaker notion of a presheaf,
- (ii) find a canonical procedure that improves a presheaf to a sheaf.

We will also see another example that requires the same strategy: the *pull-back* operation on sheaves.

Now it is easy to check that we indeed have an abelian category. What allows us to compute in this abelian category is the lucky break that one can understand kernels, cokernels, images and exact sequences just by looking at the *stalks* of sheaves.

10.0.9. *Stalks of sheaves.* In order to think of sheaves as a refined notion of functions, we would like to attach to a sheaf of abelian groups $\mathcal{A}$ its "value" $\mathcal{A}_x$ at each point $x \in X$. For that one should consider the groups $\mathcal{A}(U)$ for smaller and smaller neighborhoods of a, and in fact, one can actually pass to the limit of such groups $\mathcal{A}(U)$. The limit group

$$\mathcal{A}_a \stackrel{\text{def}}{=} \lim_{\substack{\to \\ U \ni a}} \mathcal{A}(U)$$

is called the *stalk* of $\mathcal{A}$ at a. The collection of all stalks $\mathcal{A}_x$, $x \in X$, does not record all structure of a sheaf but it suffices for some purposes.

10.1. **Categories of (pre)sheaves.** A *presheaf of sets* $\mathcal{S}$ on a topological space $(X, \mathcal{T})$ consists of the following data:

- for each open $U \subseteq X$ a set $\mathcal{S}(U)$,
- for each inclusion of open subsets $V \subseteq U \subseteq X$ a map $\mathcal{S}(U) \xrightarrow{\rho^U_V} \mathcal{S}(V)$ (called the restriction map);

and these data are required to satisfy

- (Sh0)(Transitivity of restriction) $\rho^U_V \circ \rho^U_V = \rho^U_W$ for $W \subseteq V \subseteq U$

10.1.1. *Sheaves.* Now we can define sheaves as a special case of presheaves.

A *sheaf of sets* on a topological space $(X, \mathcal{T})$ is a presheaf $\mathcal{S}$ which also satisfies

- (Sh1) (Gluing) Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of an open $U \subseteq X$ (We denote $U_{ij} = U_i \cap U_j$ etc.). We ask that any family of compatible sections $f_i \in \mathcal{S}(U_i),\ i \in I$, glues uniquely. This means that if sections f_i agree on intersections in the sense that $\rho^{U_i}_{U_{ij}} f_i = \rho^{U_i}_{U_{ij}} f_j$ in $\mathcal{S}(U_{ij})$ for any $i, j \in I$; then there is a unique $f \in \mathcal{S}(U)$ such that $\rho^U_{U_i} f = f_i$ in $\mathcal{S}(U_i),\ i \in I$.
- $\mathcal{S}(\emptyset)$ is a *point*.

10.1.2. *Remarks.* (1) Presheaves of sets on X form a category $preSheaves(X, \mathcal{S}ets)$ when $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ consists of all systems $\phi = (\phi_U)_{U \subseteq X \ open}$ of maps $\phi_U : \mathcal{A}(U) \to \mathcal{B}(U)$ which are compatible with restrictions, i.e., for $V \subseteq U$

$$\begin{array}{ccc} \mathcal{A}(U) & \xrightarrow{\phi_U} & \mathcal{B}(U) \\ \rho^U_V \big\downarrow & & \rho^U_V \big\downarrow \\ \mathcal{A}(V) & \xrightarrow{\phi_V} & \mathcal{B}(V) \end{array} .$$

(One reads the diagram above as : "the diagram ... commutes".) The sheaves form a full subcategory $preSheaves(X, \mathcal{S}ets)$ of $\mathcal{S}heaves(X, \mathcal{S}ets)$.

(2) We can equally define categories of sheaves of abelian groups, rings, modules, etc. For a sheaf of abelian groups we ask that all $\mathcal{A}(U)$ are abelian groups, all restriction morphisms are maps of abelian groups, and we modify the least interesting requirement (Sh2): $\mathcal{S}(\phi)$ is the trivial group $\{0\}$. In general, for a category $\mathcal{A}$ one can define categories $preSheaves(X, \mathcal{A})$ and $\mathcal{S}heaves(X, \mathcal{A})$ similarly (the value on $\emptyset$ should be the final object of $\mathcal{A}$).

10.2. **Sheafification of presheaves.** We will use the wish to pull-back sheaves as a motivation for a procedure that improves presheaves to sheaves.

10.2.1. *Functoriality of sheaves.* Recall that for any map of topological spaces $X \xrightarrow{\pi} Y$ one wants a pull-back functor $\mathcal{S}heaves(Y) \xrightarrow{\pi^{-1}} \mathcal{S}heaves(X)$.[28] The natural formula is

$$\underline{\pi^{-1}}(\mathcal{N})\ (U) \stackrel{\text{def}}{=} \lim_{\substack{\to \\ V \supseteq \pi(U)}} \mathcal{N}(V),$$

where limit is over open $V \subseteq Y$ that contain $\pi(U)$, and we say that $V' \leq V''$ if V'' better approximates $\pi(U)$, i.e., if $V'' \subseteq V'$.

[28] A special case of this, when we pull-back to a point, will be the notion of a *stalk* of a sheaf.

10.2.2. *Lemma.* This gives a functor of presheaves $preSheaves(X) \xrightarrow{\underline{\pi}^{-1}} preSheaves(Y)$.

Proof. For $U' \subseteq U$ open, $\underline{\pi}^{-1}\mathcal{N}(U') = \lim_{\to\, V\supseteq\pi(U')} \mathcal{N}(V)$ and $\underline{\pi}^{-1}\mathcal{N}(U) = \lim_{\to\, V\supseteq\pi(U)} \mathcal{N}(V)$ are limits of inductive systems of $\mathcal{N}(V)$'s, and the second system is a subsystem of the first one, this gives a canonical map $\underline{\pi}^{-1}\mathcal{N}(U) \to \underline{\pi}^{-1}\mathcal{N}(U')$.

10.2.3. *Remarks.* Even if $\mathcal{N}$ is a sheaf, $\underline{\pi}^{-1}(\mathcal{N})$ need not be sheaf.

For that let $Y = pt$ and let $\mathcal{N} = S_Y$ be the constant sheaf of sets on Y given by a set S. So, $S_Y(\emptyset) = \emptyset$ and $S_Y(Y) = S$. Then $\underline{\pi}^{-1}(S_Y)\ (U) = \begin{cases} \emptyset & if\ U = \emptyset, \\ S & U \neq \emptyset \end{cases}$. We can say: $\underline{\pi}^{-1}(S_Y)\ (U) =$ constant functions from U to S. However, we have noticed that constant functions do not give a sheaf, so we need to correct the procedure $\underline{\pi}^{-1}$ to get sheaves from sheaves. For that remember that for the presheaf of constant functions there is a related sheaf S_X of *locally constant* functions.

Our problem is that the presheaf of constant functions is defined by a global condition (constancy) and we need to change it to a local condition (local constancy) to make it into a sheaf. So we need the procedure of

10.2.4. *Sheafification.* This is a way to improve any presheaf of sets $\mathcal{S}$ into a sheaf of sets $\widetilde{\mathcal{S}}$. We will imitate the way we passed from constant functions to locally constant functions. More precisely, we will obtained the sections of the sheaf $\tilde{\mathcal{S}}$ associated to the presheaf $\mathcal{S}$ in two steps:

(1) we glue systems of local sections s_i which are compatible in the weak sense that they are locally the same, and
(2) we identify two results of such gluing if the local sections in the two families are locally the same.

Formally these two steps are performed by replacing $\mathcal{S}(U)$ with the set $\widetilde{\mathcal{S}}(U)$, defined as the set of all equivalence classes of systems $(U_i, s_i)_{i\in I}$ where

(1) Let $\widehat{\mathcal{S}}(U)$ be the class of all systems $(U_i, s_i)_{i\in I}$ such that
   - $(U_i)_{i\in I}$ is an open cover of U and s_i is a section of $\mathcal{S}$ on U_i,
   - sections s_i are *weakly compatible* in the sense that they are locally the same, i.e., for any $i', i'' \in I$ sections $s_{i'}$ and $s_{i''}$ are the same near any point $x \in U_{i'i''}$. (Precisely, this means that there is neighborhood W such that $s_{i'}|W = s_{i''}|W$.)

(2) We say that two systems $(U_i, s_i)_{i\in I}$ and $(V_j, t_j)_{j\in J}$ are $\equiv$, iff for any $i \in I,\ j \in J$ sections s_i and t_j are weakly equivalent (i.e., for each $x \in U_i \cap V_j$, there is an open set W with $x \in W \subseteq U_i \cap V_j$ such that "$s_i = t_j$ on W" in the sense of restrictions being the same).

10.2.5. *Remark.* The relation $\equiv$ on $\widehat{\mathcal{S}}(U)$ really says that $(U_i,s_i)_{i\in I}\equiv(V_j,t_j)_{j\in J}$ iff the disjoint union $(U_i,s_i)_{i\in I}\sqcup(V_j,t_j)_{j\in J}$ is again in $\widehat{\mathcal{S}}(U)$.

10.2.6. *Lemma.* (a) $\equiv$ is an equivalence relation.

(b) $\widetilde{\mathcal{S}}(U)$ is a presheaf and there is a canonical map of presheaves $\mathcal{S}\xrightarrow{q}\widetilde{\mathcal{S}}$.

(c) $\widetilde{\mathcal{S}}$ is a sheaf.

10.2.7. *Sheafification as a left adjoint of the forgetful functor.* As usual, we have not invented something new: it was already there, hidden in the more obvious forgetful functor

10.2.8. *Lemma.* Sheafification functor $pre\mathcal{S}heaves\ni\mathcal{S}\mapsto\widetilde{\mathcal{S}}\in\mathcal{S}heaves$, is the left adjoint of the inclusion $\mathcal{S}heaves\subseteq pre\mathcal{S}heaves$, i.e, for any presheaf $\mathcal{S}$ and any sheaf $\mathcal{F}$ there is a natural identification

$$\mathrm{Hom}_{\mathcal{S}heaves}(\widetilde{\mathcal{S}},\mathcal{F})\xrightarrow{\cong}\mathrm{Hom}_{pre\mathcal{S}heaves}(\mathcal{S},\mathcal{F}).$$

Explicitly, the bijection is given by $(\iota_{\mathcal{S}})_*\alpha=\ \alpha\circ\iota_{\mathcal{S}}$, i.e., $(\widetilde{\mathcal{S}}\xrightarrow{\alpha}\mathcal{F})\mapsto(\mathcal{S}\xrightarrow{\iota_{\mathcal{S}}}\widetilde{\mathcal{S}}\xrightarrow{\alpha}\mathcal{F})$.

10.3. **Inductive limits (or "colimits") of abelian groups.** Remember that we want to define the stalk of a presheaf $\mathcal{A}$ at a point x as the limit over (diminishing) neighborhoods U of x

$$\mathcal{A}_x\stackrel{\text{def}}{=}\lim_{\rightarrow}\ \mathcal{A}(U).$$

This will mean that

(1) any $s\in\mathcal{A}(U)$ with $U\ni x$ defines an element s_x of the stalk,
(2) all elements of $\mathcal{A}_x$ arise in this way, and
(3) For $s'\in\mathcal{A}(U')$ and $s''\in\mathcal{A}(U'')$ one has $s'_x=s''_x$ iff for some neighborhood W of x in $U'\cap U''$ one has $s'=s''$ on W.

This can be achieved in the following way:

10.3.1. *Lemma.* (a) The relation $\sim$ defined on the disjoint union

$$\sqcup_{U\ni x}\ \mathcal{A}(U)\ \stackrel{\text{def}}{=}\ \cup_U\ \ \mathcal{A}(U)\times\{U\},$$

by

$$(a,U)\sim(b,V)\ \textit{(for } a\in\mathcal{A}(U),\ \ b\in\mathcal{A}(V)\textit{), if there is some } W\subseteq U\cap V\textit{ such that}\\ \textit{“}a=b\textit{ in }\mathcal{A}(W)\textit{”, i.e., if }\rho^U_W a=\rho^V_W b\ ,$$

is an equivalence relation.

(b) The quotient $\lim\limits_{\rightarrow\ U\ni x}\ \mathcal{A}(U)\ \stackrel{\text{def}}{=}\ [\sqcup_U\ \mathcal{A}(U)]/\sim$, has a canonical structure of an abelian group, and it satisfies the above properties (1-3).

10.3.2. *Inductive limits.* One can skip the remainder of this subsection. We just gibe the categorical framework of the above construction of a limit. An *inductive system* of objects in a category $\mathcal{C}$, over a partially ordered set $(I, \leq)$, consists of

- objects $a_i \in \mathcal{C},\ i \in I$; and
- maps $\phi_{ji} : a_i \longrightarrow a_j$ for all $i \leq j$ in I;

such that

$$\phi_{ii} = 1_{a_i},\ i \in I \quad \text{and} \quad \phi_{kj} \circ \phi_{ji} = \phi_{ki}, \quad i \leq j \leq k.$$

Its *limit* is a pair $(a, (\rho_i)_{i \in I})$ of $a \in \mathcal{C}$ and maps $\rho_i : a_i \longrightarrow a$ such that

(1) $\rho_j \circ \phi_{ji} = \rho_i$ for $i \leq j$, and moreover
(2) $(a, (\rho_i)_{i \in I})$ is universal with respect to this property in the sense that for any $(a', (\rho'_i)_{i \in I})$ that satisfies $\rho'_j \circ \phi_{ji} = \rho'_i$ for $i \leq j$, there is a unique map $\rho : a \longrightarrow a'$ such that $\rho'_i = \rho \circ \rho_i,\ i \in I$.

Informally, we write: $\lim\limits_{\rightarrow\, I, \leq} a_i = a$.

10.3.3. *Limits in sets, abelian groups, modules and such.* In each of the categories $\mathcal{S}ets, \mathcal{A}b, \mathfrak{m}(\Bbbk)$ inductive limits exist and are calculated in the following way

Lemma. Let $(I, \leq)$ be a partially ordered set such that for any $i, j \in I$ there is some $k \in I$ such that $i \leq k \geq j$. Let the family of sets $(A_i)_{i \in I}$ and maps $(\phi_{ji} : A_i \longrightarrow A_j)_{i \leq j}$ be an inductive system of sets.

(1) The relation $\sim$ defined on the disjoint union $\sqcup_{i \in I} A_i \overset{\text{def}}{=} \cup_{i \in I} A_i \times \{i\}$ by
$$(a, i) \sim (b, j) \text{ (for } a \in A_i,\ b \in A_j\text{), if there is some } k \geq i, j \text{ such that } \text{“}a = b \text{ in } A_k\text{”, i.e., if } \phi_{ki} a = \phi_{kj} b\ ,$$
is an equivalence relation.
(2) $\lim\limits_{\rightarrow} A_i$ is the quotient $[\sqcup_{i \in I} A_i]/\sim$ of the disjoint union by the above equivalence relation.

Corollary. (a) For an inductive system of abelian groups (or sets) A_i over $(I, \leq)$, inductive limit $\lim\limits_{\rightarrow} A_i$ can be described by

- for $i \in I$ any $a \in A_i$ defines an element $\overline{a}$ of $\lim\limits_{\rightarrow} A_i$,
- all elements of $\lim\limits_{\rightarrow} A_i$ arise in this way, and
- for $a \in A_i$ and $b \in A_j$ one has $\overline{a} = \overline{b}$ iff for some $k \in I$ with $i \leq k \geq j$ one has $a = b$ in A_k.

(b) For a subset $K \subseteq I$ one has a canonical map $\lim\limits_{\rightarrow\, i \in K} A_i \longrightarrow \lim\limits_{\rightarrow\, i \in I} A_i$.

10.4. **Stalks.**

10.4.1. *Stalks of a sheaf.* We want to restrict a sheaf of sets $\mathcal{F}$ on a topological space X to a point $a \in X$. The restriction $\mathcal{F}|a$ is a sheaf on a point, so it just one set $\mathcal{F}_a \stackrel{\text{def}}{=} (\mathcal{F}|a)(\{a\})$ called the stalk of $\mathcal{F}$ at a. What should $\mathcal{F}_a$ be? It has to be related to all $\mathcal{F}(U)$ where $U \subseteq X$ is is open and contains a, and $\mathcal{F}(U)$ should be closer to $\mathcal{F}_a$ when U is a smaller neighborhood. A formal way to say this is that

- (i) the set $\mathcal{N}_a$ of neighborhoods of a in X is partially ordered by $U \leq V$ if $V \subseteq U$,
- (ii) the values of $\mathcal{F}$ on neighborhoods $(\mathcal{F}(U))_{U \in \mathcal{N}_a}$ form an inductive system,
- (iii) we define the stalk by $\mathcal{F}_a \stackrel{\text{def}}{=} \underset{\substack{\rightarrow \\ U \in \mathcal{N}_a}}{\lim} \mathcal{F}(U)$.

Example. The stalk at the origin of a the sheaf $\mathcal{H}_{\mathbb{C}}$ of holomorphic functions on $\mathcal{C}$ is canonically identified with the ring of convergent power series. ("Convergent" means that the series converges on *some* disc around the origin.)

10.4.2. *Lemma.* For a presheaf $\mathcal{S}$, the canonical map $\mathcal{S} \to \widetilde{\mathcal{S}}$ is an isomorphism on stalks.

Proof. We consider a point $a \in X$ as a map $\text{pt} = \{a\} \xrightarrow{i} X$, so that $\mathcal{A}_x = i^{-1}\mathcal{A}$. For a sheaf B on the point

$$\begin{aligned} \text{Hom}_{\mathcal{S}h(pt)}(i^{-1}\widetilde{\mathcal{S}}, \mathcal{B}) &\cong \text{Hom}_{\mathcal{S}h(X)}(\widetilde{\mathcal{S}}, i_*\mathcal{B}) \cong \text{Hom}_{pre\mathcal{S}h(X)}(\mathcal{S}, i_*\mathcal{B}) \\ &\cong \text{Hom}_{pre\mathcal{S}h(pt)}(i^{-1}\mathcal{S}, \mathcal{B}) = \text{Hom}_{\mathcal{S}h(pt)}(i^{-1}\mathcal{S}, \mathcal{B}). \end{aligned}$$

10.4.3. *Germs of sections and stalks of maps.* For any neighborhood U of a point x we have a canonical map $\mathcal{S}(U) \to \underset{\rightarrow\, V \ni x}{\lim} \mathcal{S}(V) \stackrel{\text{def}}{=} \mathcal{S}_x$ (see lemma 10.3.3.b), and we denote the image of a section $s \in \Gamma(U, \mathcal{S})$ in the stalk $\mathcal{S}_x$ by s_x, and we call it the *germ* of the section at x. The germs of two sections are the same at x iff the sections are the same on some (possibly very small) neighborhood of x (this is again by the lemma 10.3.3.b).

A map of sheaves $\phi : \mathcal{A} \to \mathcal{B}$ defines for each $x \in M$ a map of stalks $\mathcal{A}_x \to \mathcal{B}_x$ which we denote ϕ_x. It comes from a map of inductive systems given by ϕ, i.e., from the system of maps $\phi_U : \mathcal{A}(U) \to \mathcal{B}(U)$, $U \ni x$; and on germs it is given by $\phi_x(a_x) = [\phi_U(a)]_x$, $a \in \mathcal{A}(U)$.

Example. For instance, let $\mathcal{A} = \mathcal{H}_{\mathbb{C}}$ be the sheaf of holomorphic functions on $\mathbb{C}$. Remember that the stalk at $a \in \mathbb{C}$ can be identified with all convergent power series in $z - a$. Then the germ of a holomorphic function $f \in \mathcal{H}_{\mathbb{C}}(U)$ at a can be thought of as the power series expansion of f at a. An example of a map of sheaves $\mathcal{H}_{\mathcal{C}} \xrightarrow{\Phi} \mathcal{H}_{\mathbb{C}}$ is the multiplication by an entire function $\phi \in \mathcal{H}_{\mathbb{C}}(\mathbb{C})$, its stalk at a is the multiplication of the the power series at a by the power series expansion of ϕ at a.

10.4.4. The following lemma shows how much the study of sheaves reduces to the study of their stalks.

Lemma. (a) Maps of sheaves $\phi, \psi : \mathcal{A} \to \mathcal{B}$ are the same iff the maps on stalks are the same, i.e., $\phi_x = \psi_x$ for each $x \in M$.

(b) Map of sheaves $\phi : \mathcal{A} \to \mathcal{B}$ is an isomorphism iff ϕ_x is an isomorphism for each $x \in M$.

10.5. **Inverse and direct images of sheaves.**

10.5.1. *Pull back of sheaves (finally!)* Now we can define for any map of topological spaces $X \xrightarrow{\pi} Y$ a pull-back functor

$$\mathcal{S}heaves(Y) \xrightarrow{\pi^{-1}} \mathcal{S}heaves(X), \qquad \pi^{-1}\mathcal{N} \stackrel{\text{def}}{=} \widetilde{\underline{\pi^{-1}}\mathcal{N}}.$$

10.5.2. *Examples.* (a) A point $a \in X$ can be viewed as a map $\{a\} \xrightarrow{\rho} X$. Then $\rho^{-1}\mathcal{S}$ is the stalk $\mathcal{S}_a$.

(b) Let $a : X \to \text{pt}$, for any set S one has $S_X = a^{-1}S$.

10.5.3. *Direct image of sheaves.* Besides the pull-back of sheaves which we defined in 10.5.1, there is also a much simpler procedure of the push-forward of sheaves:

10.5.4. *Lemma.* (Direct image of sheaves.) Let $X \xrightarrow{\pi} Y$ be a map of topological spaces. For a sheaf $\mathcal{M}$ on X, formula

$$\pi_*(\mathcal{M})\ (V) \stackrel{\text{def}}{=} \mathcal{M}(\pi^{-1}V),$$

defines a sheaf $\pi_*\mathcal{M}$ on Y, and this gives a functor $\mathcal{S}heaves(X) \xrightarrow{\pi_*} \mathcal{S}heaves(Y)$.

10.5.5. *Adjunction between the direct and inverse image operations.* The two basic operations on sheaves are related by adjunction:

Lemma. For sheaves $\mathcal{A}$ on X and $\mathcal{B}$ on Y one has a natural identification

$$\text{Hom}(\pi^{-1}\mathcal{B}, \mathcal{A}) \cong \text{Hom}(\mathcal{B}, \pi_*\mathcal{A}).$$

Proof. We want to compare $\beta \in \text{Hom}(\mathcal{B}, \pi_*\mathcal{A})$ with α in

$$\text{Hom}_{\mathcal{S}h(X)}(\pi^{-1}\mathcal{B}, \mathcal{A}) = \text{Hom}_{\mathcal{S}h(X)}(\widetilde{\underline{\pi^{-1}}\mathcal{B}}, \mathcal{A}) \cong \text{Hom}_{pre\mathcal{S}h(X)}(\underline{\pi^{-1}}\mathcal{B}, \mathcal{A}).$$

α is a system of maps

$$\lim_{\to\ V \supseteq \pi(U)} \mathcal{B}(V) = \underline{\pi^{-1}}\mathcal{B}(U) \xrightarrow{\alpha_U} \mathcal{A}(U), \quad \text{for } U \text{ open in } X,$$

and β is a system of maps

$$\mathcal{B}(V) \xrightarrow{\beta_V} \mathcal{A}(\pi^{-1}V), \quad \text{for } V \text{ open in } Y.$$

Clearly, any β gives some α since

$$\lim_{\to\ V \supseteq \pi(U)} \mathcal{B}(V) \xrightarrow{\lim_{\to} \beta_V} \lim_{\to\ V \supseteq \pi(U)} \mathcal{A}(\pi^{-1}V) \to \mathcal{A}(U),$$

the second map comes from the restrictions $\mathcal{A}(\pi^{-1}V) \to \mathcal{A}(U)$ defined since $V \supseteq \pi(U)$ implies $\pi^{-1}V \supseteq U$.

For the opposite direction, any α gives for each V open in Y, a map $\lim_{\to\, W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W) = \underline{\pi^{-1}}\mathcal{B}(\pi^{-1}V) \xrightarrow{\alpha_{\pi^{-1}V}} \mathcal{A}(\pi^{-1}V)$. Since $\mathcal{B}(V)$ is one of the terms in the inductive system we have a canonical map $\mathcal{B}(V) \to \lim_{\to\, W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W)$, and the composition with the first map $\mathcal{B}(V) \to \lim_{\to\, W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W) \xrightarrow{\alpha_{\pi^{-1}V}} \mathcal{A}(\pi^{-1}V)$, is the wanted map β_V.

10.5.6. *Lemma.* (a) If $X \xrightarrow{\pi} Y \xrightarrow{\tau} Z$ then

$$\tau_*(\pi_*\mathcal{A}) \cong (\tau\circ\pi)_*\mathcal{A} \quad \text{and} \quad \tau_*(\pi_*\mathcal{A}) \cong (\tau\circ\pi)_*\mathcal{A}.$$

(b) $(1_X)_*\mathcal{A} \cong \mathcal{A} \cong (1_X)^{-1}\mathcal{A}$.

Proof. The statements involving direct image are very simple and the claims for inverse image follow by adjunction.

10.5.7. *Corollary.* (Pull-back preserves the stalks) For $a \in X$ one has $(\pi^{-1}\mathcal{N})_a \cong \mathcal{N}_{\pi(a)}$.

This shows that the pull-back operation which was difficult to define is actually very simple in its effect on sheaves.

10.6. **Abelian category structure.** Let us fix a map of sheaves $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ since the nontrivial part is the construction of (co)kernels. Consider the example where the space is the circle $X = \{z \in \mathbb{C},\ |z| = 1\}$ and $\mathcal{A} = \mathcal{B}$ is the sheaf $\mathcal{C}_X^\infty$ of smooth functions on X, and the map α is the differentiation $\partial = \frac{\partial}{\partial\theta}$ with respect to the angle θ. For $U \subseteq X$ open, $\mathrm{Ker}(\partial_U) :\ \mathcal{C}_X^\infty(U) \to \mathcal{C}_X^\infty(U)$ consists of locally constant functions and the cokernel $\mathcal{C}_X^\infty(U)/\partial_U\mathcal{C}_X^\infty(U)$ is

- zero if $U \neq X$ (then any smooth function on U is the derivative of its indefinite integral defined by using the exponential chart $z = e^{i\theta}$ which identifies U with an open subset of $\mathbb{R}$),
- one dimensional if $U = X$ – for $g \in C^\infty(X)$ one has $\int_X \partial g = 0$ so say constant functions on X are not derivatives (and for functions with integral zero the first argument applies).

So by taking kernels at each level we got a sheaf but by taking cokernels we got a presheaf which is not a sheaf (local sections are zero but there are global non-zero sections, so the object is not controlled by its local properties).

10.6.1. *Subsheaves.* For (pre)sheaves $\mathcal{S}$ and $\mathcal{S}'$ we say that $\mathcal{S}'$ is a sub(pre)sheaf of $\mathcal{S}$ if $\mathcal{S}'(U) \subseteq \mathcal{S}(U)$ and the restriction maps for $\mathcal{S}'$, $\mathcal{S}'(U) \xrightarrow{\rho'} \mathcal{S}'(V)$ are restrictions of the restriction maps for $\mathcal{S}$, $\mathcal{S}(U) \xrightarrow{\rho} \mathcal{S}(V)$.

10.6.2. *Lemma.* (Kernels.) Any map $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ has a kernel and $\operatorname{Ker}(\alpha)(U) = \operatorname{Ker}[\mathcal{A}(U) \xrightarrow{\phi(U)} \mathcal{B}(U)]$ is a subsheaf of $\mathcal{A}$.

Proof. First, $\mathcal{K}(U) \stackrel{\text{def}}{=} \operatorname{Ker}[\mathcal{A}(U) \xrightarrow{\phi(U)} \mathcal{B}(U)]$ is a sheaf, and then a map $\mathcal{C} \xrightarrow{\mu} \mathcal{A}$ is killed by α iff it factors through the subsheaf $\mathcal{K}$ of $\mathcal{A}$.

Lemma. (Cokernels.) Any map $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ defines a presheaf $C(U) \stackrel{\text{def}}{=} \mathcal{B}(U)/\alpha_U(\mathcal{A}(U))$, the associated sheaf $\mathcal{C}$ is the cokernel of α.

Proof. For a sheaf $\mathcal{S}$ one has

$$\operatorname{Hom}_{\mathcal{S}heaves}(\mathcal{B}, \mathcal{S})_\alpha \cong \operatorname{Hom}_{pre\mathcal{S}heaves}(C, \mathcal{S}) \cong \operatorname{Hom}_{\mathcal{S}heaves}(\mathcal{C}, \mathcal{S}).$$

The second identification is the adjunction. For the first one, a map $\mathcal{B} \xrightarrow{\phi} \mathcal{S}$ is killed by α, i.e., $0 = \phi\circ\alpha$, if for each U one has $0 = (\phi\circ\alpha)_U\mathcal{A}(U) = \phi_U(\alpha_U\mathcal{A}(U))$; but then it gives a map $C \xrightarrow{\overline{\phi}} \mathcal{S}$, with $\overline{\phi}_U : C(U) = \mathcal{B}(U)/\alpha_U\mathcal{A}(U) \to SS(U)$ the factorization of ϕ_U. The opposite direction is really obvious, any $\psi : C \to \mathcal{S}$ can be composed with the canonical map $\mathcal{B} \to C$ (i.e., $\mathcal{B}(U) \to \mathcal{B}(U)/\alpha_U\mathcal{A}(U)$) to give map $\mathcal{B} \to \mathcal{S}$ which is clearly killed by α.

10.6.3. *Lemma.* (Images.) Consider a map $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$.

(a) It defines a presheaf $I(U) \stackrel{\text{def}}{=} \alpha_U(\mathcal{A}(U)) \subseteq \mathcal{B}(U)$ which is a subpresheaf of $\mathcal{B}$. The associated sheaf is the image of α.

(b) It defines a presheaf $c(U) \stackrel{\text{def}}{=} \mathcal{A}(U)/\operatorname{Ker}(\alpha_U)$, the associated sheaf $\mathcal{I}$ is the coimage of α.

(c) The canonical map $Coim(\alpha) \to Im(\alpha)$ is isomorphism.

Proof. (a) $Im(\alpha) \stackrel{\text{def}}{=} \operatorname{Ker}[\mathcal{B} \to \operatorname{Coker}(\alpha)]$ is a subsheaf of $\mathcal{B}$ and $b \in \mathcal{B}(U)$ is a section of $Im(\alpha)$ iff it becomes zero in $\operatorname{Coker}(\alpha)$. But a section $b + \alpha_U\mathcal{A}(U)$ of C on U is zero in $\mathcal{B}$ iff it is locally zero in C, i.e., there is a cover U_i of U such that $b|U_i \in \alpha_{U_i}\mathcal{A}(U_i)$. But this is the same as saying that b is locally in the subpresheaf I of $\mathcal{B}$, i.e., the same as asking that b is in the corresponding presheaf $\mathcal{I}$ of $\mathcal{B}$.

(b) The coimage of α is by definition $Coim(\alpha) \stackrel{\text{def}}{=} \operatorname{Coker}[\operatorname{Ker}(\alpha) \to \mathcal{A}]$, i.e., the sheaf associated to the presheaf $U \mapsto \mathcal{A}(U)/\operatorname{Ker}(\alpha)(U) = c(U)$.

(c) The map of sheaves $Coim(\alpha) \to Im(\alpha)$ is associated to the canonical map of presheaves $c \to I$, however already the map of presheaves is an isomorphism: $c(U) = \mathcal{A}(U)/\operatorname{Ker}(\alpha)(U) \cong \alpha_U \stackrel{\text{def}}{=} \mathcal{A}(U) = I(U)$.

10.6.4. *Stalks of kernels, cokernels and images; exact sequences of sheaves.*

10.6.5. *Lemma.* For a map of sheaves $\mathcal{A}\xrightarrow{\alpha}\mathcal{B}$ and $x\in X$

- (a) $\mathrm{Ker}(\mathcal{A}\xrightarrow{\alpha}\mathcal{B})_x = \mathrm{Ker}(\alpha_x:\ \mathcal{A}_x\to\mathcal{B}_x)$,
- (b) $\mathrm{Coker}(\mathcal{A}\xrightarrow{\alpha}\mathcal{B})_x = \mathrm{Coker}(\alpha_x:\ \mathcal{A}_x\to\mathcal{B}_x)$,
- (c) $Im(\mathcal{A}\xrightarrow{\alpha}\mathcal{B})_x = Im(\alpha_x:\ \mathcal{A}_x\to\mathcal{B}_x)$.

Proof. (a) Let $x\in U$ and $a\in\mathcal{A}(U)$. The germ a_x is killed by α_x if $0=\alpha_x(a_x)\overset{\text{def}}{=}(\alpha_U(a))_x$, i.e., iff $\alpha_U(a)=0$ on some neighborhood U' of x in U. But this is the same as saying that $0=\ \alpha_U(a)|U'=\ \alpha_{U'}(a|U')$, i.e., asking that some restriction of a to a smaller neighborhood of x is a section of the subsheaf $\mathrm{Ker}(\alpha)$. And this in turn, is the same as saying that the germ a_x lies in the stalk of $\mathrm{Ker}(\alpha)$.

(b) Map $\mathcal{B}\xrightarrow{q}\mathrm{Coker}(\alpha)$ is killed by composing with α, so the map of stalks $\mathcal{B}_x\xrightarrow{q_x}\mathrm{Coker}(\alpha)_x$ is killed by composing with α_x.

To see that q_x is surjective consider some element of the stalk $\mathrm{Coker}(\alpha)_x$. It comes from a section of a presheaf $U\mapsto\mathcal{B}(U)/\alpha_U\mathcal{A}(U)$, so it is of the form $[b+\alpha_U(\mathcal{A}(U))]_x$ for some section $b\in\mathcal{B}(U)$ on some neighborhood U of x. Therefore it is the image $\alpha_x(b_x)$ of an element b_x of $\mathcal{B}_x$.

To see that q_x is injective, observe that a stalk $b_x\in\mathcal{B}_x$ (of some section $b\mathcal{B}(U)$), is killed by q_x iff its image $\alpha_x(b_x)=\ [b+\alpha_U(\mathcal{A}(U))]_x$ is zero in $\mathrm{Coker}(\alpha)$, i.e., iff there is a smaller neighborhood $U'\subseteq U$ such that the restriction $[b+\alpha_U(\mathcal{A}(U))]|U'=\ b|U'+\alpha_{U'}(\mathcal{A}(U'))$ is zero, i.e., $b|U'$ is in $\alpha_{U'}\mathcal{A}(U')$. But the existence of such U' is the same as saying that b_x is in the image of α_x.

(c) follows from (a) and (b) by following how images are defined in terms of kernels and cokernels.

10.6.6. *Corollary.* A sequence of sheaves is exact iff at each point the corresponding sequence of stalks of sheaves is exact.

10.7. **Injective resolutions of sheaves.** We state the last ingredient need in order to use the homological algebra in the category of sheaves:

10.7.1. *Theorem.* The category $\mathcal{S}h\mathcal{A}b(X)$ of sheaves of abelian groups on X has enough injectives, i.e., any sheaf of abelian groups is a subsheaf of an injective sheaf of abelian groups.

10.8. **Appendix: Sheafifications via the etale space of a presheaf.** One can skip this subsection. We will once again construct the sheafification of a presheaf $\mathcal{S}$. This approach is more elegant and less explicit (it is more abstract and we use the notion of stalks). The main idea is that to a presheaf $\mathcal{S}$ over X one can attach a map of topological spaces $\dot{\mathcal{S}}\to X$. Here, $\dot{\mathcal{S}}$ is called the *etale space* of the presheaf. Then the sheafification of $\mathcal{S}$ is obtained using the following idea:

10.8.1. *Sheaf of sections of a map.* If $Y \xrightarrow{p} X$ is a continuous map, let us attach to each open $U \subseteq X$ the set

$$\mathcal{Y}(U) \stackrel{\text{def}}{=} \{s : U \longrightarrow Y,\ s \text{ is continuous and } p \circ s = 1_u\}.$$

its the elements are called the (continuous) sections of p over U.

Lemma. $\mathcal{Y}$ is a sheaf of sets (the *sheaf of sections of* p).

10.8.2. *The* etale space *of a presheaf.* To apply this construction we need a space $\overset{\bullet}{\mathcal{S}}$ that maps to X:

- Let $\overset{\bullet}{\mathcal{S}}$ be the union of all stalks $\mathcal{S}_m,\ m \in X$.
- Let $p : \overset{\bullet}{\mathcal{S}} \to X$ be the map such that the fiber at m is the stalk at m.
- For any pair (U, s) with U open in X and $s \in \mathcal{S}(U)$, define a section $\tilde{s}$ of p over U by
$$\tilde{s}(x) \stackrel{\text{def}}{=} s_x \in \mathcal{S}_x \subset \overset{\bullet}{\mathcal{S}}, \qquad x \in U.$$

10.8.3. *Lemma.* (a) If for two sections $s_i \in \mathcal{S}(U_i),\ i = 1, 2;$ of $\mathcal{S}$, the corresponding sections $\tilde{s}_1$ and $\tilde{s}_2$ of p agree at a point then they agree on some neighborhood of of this point.[29]

(b) All sets $\tilde{s}(U)$ (for $U \subseteq X$ open and $s \in \mathcal{S}(U)$), form a basis of a topology on $\overset{\bullet}{\mathcal{S}}$.

(c) Map $p : \overset{\bullet}{\mathcal{S}} \to M$ is continuous. Moreover, it is *etale*[30]

(d) Let Σ be the sheaf of continuous sections of p over U. Then there is a canonical map of presheaves $i : \mathcal{S} \to \Sigma$.

10.8.4. *Lemma.* The canonical map of presheaves $i : \mathcal{S} \to \Sigma$, is the sheafification of $\mathcal{S}$.

Proof. Sections of p over $U \subseteq X$ are the same as the equivalence classes of systems $\widehat{\mathcal{S}}/\equiv$ defined in 10.2.4.

[29]If $\tilde{s}_1(x) = \tilde{s}_2(x)$ for some $x \in U_{12} \stackrel{\text{def}}{=} U_1 \cap U_2$, we claim that there is a neighborhood $W \subseteq U_{12}$ of x, such that $\tilde{s}_1 = \tilde{s}_2$ on W.

[30]"Etale" means "locally an isomorphism", i.e., for each point $\sigma \in \overset{\bullet}{\mathcal{S}}$ there are neighborhoods $\sigma \in W \subseteq \overset{\bullet}{\mathcal{S}}$ and $p(\sigma) \subseteq U \subseteq X$ such that $p|W$ is a homeomorphism $W \xrightarrow{\cong} U$.