ALGEBRAIC GEOMETRY
JAN 04

CONTENTS

6. Transcendental methods in algebraic geometry – the complex algebraic geometry (Cubics and Elliptic curves)

6.0.1. *Opportunism Principle.* Suppose that we are interested in algebraic varieties over the field $\mathbb{C}$ of complex numbers. Applying the *Opportunism Principle*

Special situations allow special tools,

we recall the complex analysis and notice that we may use non-algebraic ("transcendental") methods to study algebraic varieties over $\mathbb{C}$.

Another important application of this strategy is the use of the counting methods when we work over the (algebraic closure of) a *finite field.*[1]

6.0.2. *This is even useful for general Algebraic Geometry.* Moreover, this is a great idea even if one is interested in algebraic varieties over some other field $\Bbbk$. If by using the complex analysis we discover or prove in this way a claim about complex algebraic varieties that does not explicitly use the fact that the ground field is $\mathbb{C}$, we can hope that the same may be true over any closed field, and start looking for an algebraic proof.

As the basic example of the use of transcendental methods we will use complex analysis to study the simplest non-trivial curves, the cubics in $\mathbb{P}^2$. Then we will sketch the extension to more complicated curves.

6.0.3. *Remark.* A standard (transcendental) tool in complex algebraic geometry is the full use of *differential geometry*, which one relates to holomorphic geometry by statements such as:

If a line bundle L has positive the curvature, then L has many sections.

However we will not cover these methods, rather we just use of our standard proficiency in holomorphic functions.

6.1. Cubics in $\mathbb{P}^2$.

Recall that we have found 3 quadratic curves in $\mathbb{A}^2$ (and two were degenerate versions of the third) in **??**. Now we classify and study the cubic curves.

The most interesting ones will be the affine cubics of a special form

$$\mathcal{C}_\lambda = \{(x,y)\in\mathbb{A}^2;\ y^2 = x(x-1)(x-\lambda)\} \subseteq \mathbb{A}^2$$

for some $\lambda \in \mathbb{A}^1 = \Bbbk$, and the corresponding projective cubic curves are

$$C_\lambda \stackrel{\text{def}}{=} \overline{\mathcal{C}}_\lambda = \{[x:y:z]\in\mathbb{P}^2;\ y^2z = x(x-z)(x-\lambda z)\} \subseteq \mathbb{P}^2.$$

[1] An example of this appears in homeworks. It illustrates the idea that the counting the number of elements of a variety over a finite field is related to the cohomology of the same variety over $\mathbb{C}$ (the precise relation is given by *Weil conjectures* proved by Deligne). It also suggests the existence of a non-trivial notion of a field with one element.

The reason we look at these is[2]

6.1.1. *Theorem.* Any cubic is isomorphic to one of C_λ.[3]

6.1.2. *Lemma.* The boundary of $\mathcal{C}_\lambda$ is a (triple) point.

Proof. The boundary of $\mathcal{C}_\lambda$ is obtained by requiring that $z = 0$, so we get all $[x : y : 0] \in \mathbb{P}^2$ with $0 = x^3$. So, it is one point $[0 : 1 : 0]$ (but it should really be regarded as a triple point).

6.2. **Drawing cubics over $\mathbb{C}$.** We will view $\mathcal{C}_\lambda$ in terms of the projection to the x-line ${\mathbb{A}^1}_x$[4]

$$\pi : \mathcal{C}_\lambda \to \mathbb{A}^1, \quad \pi(x, y) = x.$$

Notice that we can extend it continuously to $\pi : C_\lambda \to \mathbb{P}^1 = \mathbb{A}^1 \cup \infty$.

The fiber at x consists of two values $\pm\sqrt{x(x-1)(x-\lambda)}$ of the square root, except that we just get one point when $x = 0, 1, \lambda$ and ∞. We will say that C_λ is a *branched double cover of* $\mathbb{P}^1$ with branching at $0, 1, \lambda, \infty$.

Lemma. Projective cubics C_λ, $\lambda \neq 0, 1$; are all car tires, i.e., on the level of a topological space, C_λ is homeomorphic to a *torus.*

Let us write this lemma again in more details:

Lemma. (a) If D is a disc on the x-line ${\mathbb{A}^1}_x$, that does not contain $0, 1, \lambda$, the restriction $\mathcal{C}_\lambda|D \stackrel{\text{def}}{=} \pi^{-1}D$ is the disjoint union of two discs D_i, such that $\pi|D_i \to D$ is a holomorphic isomorphism.

(b) On the x-line $\mathbb{A}^1_x$ we choose curves a_λ from 0 to λ, b_λ from λ to 1 and c_λ from 1 to ∞, so that they do not intersect.[5] Their inverses $\alpha_\lambda, \beta_\lambda, \gamma_\lambda$ are circles in C_λ.

(c) Above $U = \mathbb{P}^1 - (a_\lambda \cup \gamma_\lambda)$, C_λ consists of two disjoint copies U_1, U_2 of U.

(d) The boundary $\partial U_i \stackrel{\text{def}}{=} \overline{U_i} - U_i$ is the union of two circles $\alpha_\lambda \cup \gamma_\lambda$.

(e) Therefore, $\overline{U_i}$ is homeomorphic to the sphere with two holes which are bounded by circles $\alpha_\lambda, \gamma_\lambda$. So, C_λ is obtained by gluing two "spheres with two holes" $\overline{U_1}$ and $\overline{U_2}$, and the gluing is performed by identifying the boundary circles.

(f) The result is a torus.

[2]Proof postponed.

[3]There are exceptions if the characteristic p of $\Bbbk$ is 2 or 3. We will not be interested in this ($p = 0$ is our main interest), and I will often forget to mention when things get more complicated if the characteristic is too small.

[4]This is the affine line with $\mathcal{O}({\mathbb{A}^1}_x) = \mathbb{C}[x]$.

[5]The best if we do it in some simple way. For instance, for most λ we can choose these curves as straight line segments.

Proof. (a) On D the function $x(x-1)(x-\lambda)$ has no zeros so there are two holomorphic functions $y_i(x) = \sqrt{x(x-1)(x-\lambda)}$, related by $y_2 = -y_1$. Therefore $\mathcal{C}_\lambda|D$ is the union of two discs D_i which are the graphs of two functions.

(b) follows.

(c) The argument is of the same kind as for (a), i.e., one *can* define two holomorphic functions $y_i(x)$ on U that are the two versions of $\sqrt{x(x-1)(x-\lambda)}$. The reason is that

- when one goes around 0 in the expression $\sqrt{x(x-1)(x-\lambda)} = \sqrt{x}\sqrt{x-1}\sqrt{x-\lambda}$ the first factor changes by -1 and the the other two factors do not change.
- when one goes around both 0 and 1, two factors change by -1, so the product does not change!

6.3. **Complex manifold structure.** If X is an affine or projective variety over $\Bbbk = \mathbb{C}$, then it is a subset of $\mathbb{C}^n$ (or $\mathbb{P}^n(\mathbb{C})$). We will see later that if X is *smooth*, i.e, if X has no *singularities*, then X has a canonical structure of a complex manifold.

6.3.1. *Lemma.* If $\lambda \neq 0, 1$, then (a) C_λ is a smooth (non-singular) algebraic variety, and (b) it has a natural structure of a one-dimensional complex manifold.

Proof. (a) We do not yet even know what this means.

(b) Let me check this for $\mathcal{C}_\lambda = C_\lambda \cap \mathbb{A}^2$, Similar calculation works near the infinite point of C_λ once you choose the appropriate local coordinates on $\mathbb{P}^2$ near this point.

So, consider a curve $\mathcal{C} \subseteq \mathbb{A}^2$ given by a polynomial equation $F(x, y) = 0$. If $\mathcal{C}$ has a tangent line at a point $p = (a, b)$ of $\mathcal{C}$, then near p one can use the x-projection (if the tangent line is not vertical), or the y-projection (if the tangent line is not horizontal), to identify a piece of $\mathcal{C}$ with with a piece of $\mathbb{A}^1$. The tangent line is defined if the differential d_pF (i.e., the gradient (F_x, F_y)), is not zero at p – then the equation is $F_x(p)(x-a) + F_y(p)(y-b) = 0$.

So we get a manifold structure on $\mathcal{C}$ except at the points p such that $0 = F(p) = F_x(p) = F_y(p)$.

Now consider F of the form $F = y^p - P(x)$. The above system of equations means now that $y^p = P(x)$ and $py^{p-1} = 0$ and $P'(x) = 0$. It implies that $y = 0 = P(x) = P'(x)$. So the only bad points are $(a, 0)$ with a a double root of $P(x)$. In our case, $P(x) = x(x-1)(x-\lambda)$ has no double roots for $\lambda \neq 0, 1$.

6.3.2. *Complex manifold view on cubics?* The first level of this question is to see whether there is some simple construction of C_λ as a complex manifold. The next level is to use complex analysis to study C_λ.

6.4. **Elliptic curves.** First, on a topological level, a torus can be constructed as $\mathbb{R}^2/\mathbb{Z}^2$. This however works on the level of complex manifolds: for any lattice L in $\mathbb{C}$, the quotient group $\mathbb{C}/L$ is a complex manifold in a shape of a torus. So we can hope that these are related to cubics and they will turn out to be the same thing.

6.4.1. *Lattices.* A lattice in a real vector space V is a subgroup L such that there is an $\mathbb{R}$-basis $v_1, ..., v_n$ of V such that $L = \oplus\ \mathbb{Z}{\cdot}v_i$. We consider the quotient group V/L.

Lemma. (a) The family of all subsets $U{\subseteq}V/L$ such that $\pi^{-1}U{\subseteq}V$ is open forms a topology on V/L.[6]

(b) The open box $B = \{\sum_i\ c_i v_i;\ 0 < c_i < 1\}\subseteq V$ has the property that for each $v \in V$

(1) $\pi|(\overline{B} + v)$ is surjective,
(2) $\pi(B+v){\subseteq}V/L$ is open and dense and
(3) $B + v \to \pi(B+v)$ is a homeomorphism.

(c) Group V/L is a compact topological group, i.e., group operations are continuous.

(d) As a topological space (and a topological group)

$$V/L \cong\ \oplus\ \mathbb{R}{\cdot}v_i/\mathbb{Z}{\cdot}v_i \cong\ \mathbb{R}^n/\mathbb{Z}^n =\ (\mathbb{R}/\mathbb{Z})^n \cong \mathbb{T}^n$$

for the circle group $\mathbb{T} \overset{\text{def}}{=} \{z \in \mathbb{C};\ |z| = 1\}{\subseteq}(\mathbb{C}^*, \cdot)$.

Corollary. V/L has a canonical structure of a real manifold.

Proof. We use the open cover of V/L by all $U_v \overset{\text{def}}{=} \pi(B+v),\ v \in V$. Each of these comes with a chart $V \overset{open}{\ } B\ + v \xrightarrow{\pi_v} U_v$. These charts are compatible since the transition functions are translations in V, so they form an atlas on V/L.

6.4.2. *Complex tori E_L.*

Lemma. (a) If L is a lattice in $\mathbb{C}^n$ (viewed as a $2n$-dimensional real vector space), then $E_L \overset{\text{def}}{=} \mathbb{C}^n/L$ has a unique structure of a complex manifold. such that the map $\mathbb{C}^n \xrightarrow{\pi} E_L$ is holomorphic.

(b) E_L is a compact holomorphic Lie group.[7] In particular for each $e \in E_L$ the translation $x{\mapsto}x{+}e$ is a holomorphic map (actually a an automorphism of the complex manifold E_L).

Proof. The same as above, except that now we observe that the transition functions are holomorphic, not only differentiable.

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[6]Called the quotient topology on V/L induced from the topology V by the surjective map $V \xrightarrow{\pi} V/L$. So, $U{\subseteq}V/L$ is open iff $\pi^{-1}U{\subseteq}V$ is open.

[7]holomorphic Lie group means a complex manifold with a group structure such that operations are holomorphic functions.

Remark. (0) We call these the *complex tori.*

(1) From now on we consider only the one-dimensional case $E_L = \mathbb{C}/L$ for a lattice L in $\mathbb{C}$. This is a 1-dimensional complex manifold in a shape of a torus $S^1 \times S^1$.

(2) The standard examples are the lattices $L_\tau \stackrel{\text{def}}{=} \mathbb{Z}\oplus\mathbb{Z}\tau\subseteq\mathbb{C}$ for τ in the upper hyperplane $\mathbb{H} = \{z \in \mathbb{C};\ Im(\tau) > 0\}$. We denote

$$E_\tau \stackrel{\text{def}}{=} \mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z}).$$

6.5. **The moduli of elliptic curves.** We will find that two elliptic curves E_L, E_M are isomorphic precisely when $M = c{\cdot}L$ for some $c \in \mathbb{C}^*$. Therefore,

$$\textit{Moduli of elliptic curves} = \textit{(Moduli of lattices)}/\mathbb{C}^*.$$

However, we do not understand the RHS. So we work in stages: first we see that each elliptic curve E_L is isomorphic to one of the standard ones $E_\tau,\ \tau \in \mathbb{H}$; and then two standard elliptic curves are isomorphic iff the parameters in $\mathbb{H}$ are in the same orbit of $SL_2(\mathbb{Z})$. So,

$$\textit{Moduli of elliptic curves} = \mathbb{H}/SL_2(\mathbb{Z}).$$

This turns out to be understandable and beautiful.

6.5.1. *Isomorphisms of elliptic curves E_L.* We are interested in the classification of elliptic curves E_L up to holomorphic isomorphisms (i.e., up to isomorphisms of complex manifolds).

If two lattices L and M are related by $M = c{\cdot}L$ for some $c \in \mathbb{C}^*$, then the multiplication by c descends from a holomorphic isomorphism $\mathbb{C} \to \mathbb{C}$ to a a holomorphic isomorphism $\mu_c : E_L \to E_M$. So, E_L and E_M are isomorphic. We will see that the converse is also true, if E_L and E_M are isomorphic then M is a multiple of L.

6.5.2. *Sublemma.* (Lifting.) A holomorphic map $\phi : \mathbb{C} \to E_M$, is always of the form $\pi_M \circ f$ for some holomorphic function $f : \mathbb{C} \to \mathbb{C}$. Moreover, all lifts f of ϕ are in bijection with all lifts $a \in \mathbb{C}$ of $\phi(0) \in E_M$, i.e., for any choice of $a \in \mathbb{C}$ with $\pi_M(a) = \phi(0)$ there is a unique lift f of ϕ such that $f(0) = a$.

Proof. First notice that $\phi'(z) : \mathbb{C} \to \mathbb{C}$ is well defined and holomorphic, by using *any* local chart near $\phi(z) \in C/M$. Then

$$f(z) = a + \int_0^z \phi'(u)\ du$$

is well defined because $\mathbb{C}$ is simply connected (so it does not matter which path I take from 0 to z!).

A topological proof. If we consider a continuous $\phi : \mathbb{C} \to E_M$ that for any a we get a continuous lift f through a. It works like this: a chart identifies $B{+}a$ with a neighborhood

$\pi(B+a)$ of $\pi(a)=\phi(0)$. Since ϕ is continuous, there is a disc D around 0 that ϕ maps to $\pi(B+a)$. Now on D one can define f as a composition of ϕ and the inverse of the chart $B+a\to\pi(B+a)$. Now one replaces $z_0=0\in C$ with $z_1\in D=D_0$ which lies near the boundary of D, and one extends f from $D=D_0$ to $D_0\cup D_1$ for a disc D_1 around z_1, etc. Since π is a local homeomorphism we can extend now from D_{i-1} to D_i forever. However, usually there is a problem: in principle when our sequence of connected discs comes back to itself, the newly obtained value of f need not coincide with what we found earlier. The reason such contradictions do not appear (again!) that the source $\mathbb{C}$ of the map ϕ is simply connected.

6.5.3. *Lemma.* $E_L\cong E_M$ iff $M=c{\cdot}L$ for some $c\in G_m(\mathbb{C})=\mathbb{C}^*$.

Proof. (A) Lifts to $\mathbb{C}$. For lattices L,M let $\mathcal{H}(E_L,E_M)$ be the set of all holomorphic maps $\sigma:E_L\to E_M$ such that $\sigma(0)=0$.

Such σ gives a holomorphic map $\mathbb{C}\xrightarrow{\pi_L}E_L\xrightarrow{\sigma}E_M$ that sends $0\in\mathbb{C}$ to $0\in E_M$. Notice that among the lifts of $0\in E_M$ to $\mathbb{C}$ there is a canonical choice: $0\in\mathbb{C}$. Therefore, according to the sublemma $\pi\circ\sigma$ lifts *uniquely* to an entire function $\widetilde{\sigma}:\mathbb{C}\to\mathbb{C}$ (i.e., $\sigma\circ\pi_L=\pi_M\circ\widetilde{\sigma}$) such that $\widetilde{\sigma}(0)=0$. This implies that

(1) If $\sigma\in\mathcal{H}(E_L,E_M)$, $\tau\in\mathcal{H}(E_M,E_N)$ then $\widetilde{\tau\circ\sigma}=\widetilde{\tau}\circ\widetilde{\sigma}$.
(2) $\widetilde{id_{E_L}}=id_L$.
(3) If $\sigma\in\mathcal{H}(E_L,E_M)$ is a holomorphic isomorphism then so is $\widetilde{\sigma}:\mathbb{C}\to\mathbb{C}$.

(B) $\widetilde{\sigma}(L)\subseteq M$. For any $z\in\mathbb{C}$ and $l\in L$,

$$\pi_M(\widetilde{\sigma}(z+l))=\sigma(\pi_L(z+l))=\sigma(\pi_L(z))=\pi_M(\widetilde{\sigma}(z)),$$

hence $\widetilde{\sigma}(z+l)-\widetilde{\sigma}(z)\in M$. As a function of z this is constant (the image of a non-constant holomorphic function is open!). So, M contains $\widetilde{\sigma}(z+l)-\widetilde{\sigma}(z)=\widetilde{\sigma}(0+l)-\widetilde{\sigma}(0)=\widetilde{\sigma}(l)$. Therefore $\widetilde{\sigma}(L)\subseteq M$ and for $l\in L$, $z\in\mathbb{C}$ we have

$$\widetilde{\sigma}(z+l)=\widetilde{\sigma}(z)+\widetilde{\sigma}(l).$$

The end. If E_L and E_M are isomorphic, choose some holomorphic isomorphism $S:E_L\to E_M$. Then $\sigma:E_L\to E_M$, $\sigma(x)=S(x)-S(0)$ is also an isomorphism and it lies in $\mathcal{H}(E_L,E_M)$. Therefore $\widetilde{\sigma}:\mathbb{C}\to\mathbb{C}$ is a holomorphic automorphism. This means that $\widetilde{\sigma}(z)=az+b$ for some a,b, since we know that all holomorphic automorphism are linear functions! Now $b=\widetilde{\sigma}(0)=0$ and $\widetilde{\sigma}(z)=az$ for some $a\in\mathbb{C}^*$.

Remember that $\widetilde{\sigma}(L)\subseteq M$ and $\widetilde{\sigma^{-1}}(M)\subseteq L$. Since $\widetilde{\sigma^{-1}}=\widetilde{\sigma}^{-1}$ we find that actually $\widetilde{\sigma}(L)=M$, i.e., $M=a{\cdot}L$.

Corollary. Any holomorphic isomorphism $\sigma:E_L\to E_M$ is a composition $T_e\circ\mu_c$ of some homothety μ_c, $c\in\mathbb{C}^*$, and a translation T_e by an element $e\in E_M$.

Proof. The proof of the lemma actually gives precisely this statement.

6.5.4. *Automorphisms of elliptic curves.* E_L embeds into $\mathrm{Aut}(E_L)$ by translations. The stabilizer $\mathcal{A}_L = \{c \in \mathbb{C}^*;\ cL = L\}$ of a lattice L in $\mathbb{C}^*$ also embeds into $\mathrm{Aut}(E_L)$.

always contains $\{\pm 1\}$.

Lemma. $\mathrm{Aut}(E_L) \cong\ E_L \ltimes \mathcal{A}_L$.

Proof. We know that the multiplication $E_L {\times} \mathcal{A}_L \to \mathrm{Aut}(E_L)$ is surjective. It is also injective since E_L and $\mathcal{A}_L$ do not intersect. Finally, $\mathcal{A}_L$ normalizes E_L: $\mu_c \circ \tau_x \circ {\mu_c}^{-1} = \tau_{\mu_c(x)}$.

Remark. $\mathcal{A}_L$ certainly contains $\{\pm 1\}$. It is easy to see that $\mathcal{A}_L$ is larger then they only for some special lattices. However, these special cases are very interesting. The theory of theses cases is called *complex multiplication.*

6.5.5. *Elliptic curves E_τ.* Now we that $\mathbb{H}$ parameterizes all elliptic curves (but with repetitions!).

Corollary. Any elliptic curve E_L is isomorphic to some E_τ with τ in the upper half-plane $\mathbb{H}$.

Proof. Pick a $\mathbb{Z}$-basis u_1, u_2 of L. Then $\tau = u_1/{u_2}^{-1} \in \mathbb{C} - \mathbb{R}$ and (up to reordering the basis) we can suppose that $\tau \in \mathbb{H}$ (the signs of $Im(\tau)$ and $Im(\tau^{-1})$ are opposite!). Now $L = u_2 {\cdot} L_\tau$ since multiplication by u_2 takes $\tau, 1$ to u_1, u_2.

6.5.6. *The action of $SL_2(\mathbb{R})$ on the upper half-plane.* This will be helpful in understanding which repetitions occur when we use $\mathbb{H} \ni \tau {\mapsto} E_\tau$ to parameterize all elliptic curves.

Lemma. (a) $GL_2(\mathbb{C})$ acts naturally on $\mathbb{C}^2$ and therefore also on the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$. In terms of the identification $\mathbb{C} \cup \infty \cong \mathbb{P}^1(\mathbb{C})$ by $\mathbb{C} \ni \tau {\mapsto} [\tau : 1] = \mathbb{C}{\cdot} \begin{pmatrix} \tau \\ 1 \end{pmatrix} \in \mathbb{P}^1$, the action on $\mathbb{C} \cup \infty$ is by fractional linear transforms

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \bullet \tau = \frac{\alpha + \beta\tau}{\gamma + \delta\tau}.$$

(b) Let $GL_2(\mathbb{R})_\pm {\subseteq} GL_2(\mathbb{R})$ consist of matrices g such that $det(g) > 0$ (resp. $det(g) < 0$). Then the subgroup $GL_2(\mathbb{R})_+$ preserves $\mathbb{H} {\subseteq} \mathbb{C}$ while $GL_2(\mathbb{R})_-$ takes $\mathbb{H}$ to the lower half plane $-\mathbb{H}$.

(c) $SL_2(\mathbb{R})$ acts transitively on $\mathbb{H}$ and the stabilizer of $i \in \mathbb{H}$ is the rotation group

$$K = \{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};\ \theta \in \mathbb{R}\}.$$

The subgroup $B_+ = \{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix};\ \alpha > 0,\ \beta \in \mathbb{R}\} {\subseteq} SL_2(\mathbb{R})$ acts simply transitively on $\mathbb{H}$.

Proof. (a) $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\cdot\mathbb{C}\begin{pmatrix}\tau\\ 1\end{pmatrix} = \mathbb{C}\begin{pmatrix}\alpha\tau+\beta\\ \gamma\tau+\delta\end{pmatrix} = \mathbb{C}\begin{pmatrix}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\\ 1\end{pmatrix}$.

(b) For $g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in GL_2(\mathbb{R})$ and $\tau \in \mathbb{C} - \mathbb{R}$,

$$Im(g \bullet \tau) = Im[\frac{\alpha\tau+\beta}{\gamma\tau+\delta}] = Im[\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\cdot\frac{\gamma\overline{\tau}+\beta}{\gamma\overline{\tau}+\delta}]$$

$$= Im\frac{\alpha\gamma|\tau|^2+\beta\delta+[\alpha\delta\tau+\gamma\beta\overline{\tau}]}{|\gamma\tau+\delta|^2} = \frac{Im(\tau)\cdot(\alpha\delta-\gamma\beta)}{|\gamma\tau+\delta|^2} = \det(g)\cdot\frac{Im(\tau)}{|\gamma\tau+\delta|^2}.$$

(c) First, $SL_2(\mathbb{R})\subseteq GL_2(\mathbb{R})_+$ preserves $\mathbb{H}$. Next, if $g \in SL_2(\mathbb{R})$, the above calculation gives

$$g \bullet i = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \bullet i = \frac{\alpha\gamma|i|^2+\beta\delta+i[\alpha\delta-\gamma\beta]}{|\gamma i+\delta|^2} = \frac{\alpha\gamma+\beta\delta+i}{\gamma^2+\delta^2}.$$

Now if $\gamma \bullet i = i$ then $\gamma^2+\delta^2 = 1$ (so $\gamma = -\sin\theta$ and $\delta = \cos\theta$ for some θ), and $\alpha\delta+\beta\gamma = 0$, i.e., the rows are orthogonal, hence $(\alpha,\beta) = c(\cos\theta,\sin\theta)$ for some $c \in \mathbb{R}$. Finally, $c = \det(g) = 1$, hence $g \in K$.

Finally, for $g = \begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix} \in B_+$ we see that $g \bullet i = \alpha^2 i + \alpha\beta$. So any $x+iy \in \mathbb{H}$ is $g \bullet i$ for unique $g \in B_+$.

Corollary. (Iwasava decomposition of $SL_2(\mathbb{R})$.) The product map $B_+\times K \xrightarrow{\cdot} SL_2(\mathbb{R})$ is a bijection.

6.5.7. *The moduli of elliptic curves E_τ.* For this moduli problem the "moduli with repetitions" $\widetilde{\mathcal{M}}$ that we start with is chosen as the upper half-plane $\mathbb{H}$, since it gives a complete family E_τ, $\tau \in \mathbb{H}$, of elliptic curves. Then the true moduli will be

$$\mathcal{M} \cong \mathbb{H}/SL_2(\mathbb{Z}).$$

Lemma. For $\tau_i \in \mathbb{H}$, $E_{\tau_1} \cong E_{\tau_2}$ iff $\tau_2 \in SL_2(\mathbb{Z})\cdot\tau_1$.

Proof. We know that $E_{\tau'} \cong E_\tau$ iff there is some $c \in \mathbb{C}^*$ such that $c\cdot L_{\tau'} = L_\tau$. The last condition is equivalent to: $\{c\tau', c\} = c\{\tau', 1\}$ is a basis of L_τ, i.e., to: the transition matrix $g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in GL_2(\mathbb{C})$ given by

$$c\tau' = \alpha\cdot\tau + \beta\cdot 1 \quad\text{and}\quad c = \gamma\cdot\tau + \delta\cdot 1,$$

is actually in $GL_2(\mathbb{Z})$.

Therefore, $E_{\tau'} \cong E_\tau$ iff there is some $g \in GL_2(\mathbb{Z})$ such that the following equivalent conditions hold:

- $g \cdot \binom{\tau}{1} \in \mathbb{C}^*\cdot\binom{\tau'}{1}$.

- $g \cdot \mathbb{C}\binom{\tau}{1} = \mathbb{C}\binom{\tau'}{1}$ in $\mathbb{P}^1$.
- $\tau' = g \bullet \tau$.

It remains to notice that (since $\tau, \tau' \in \mathbb{H}$), the last condition can be satisfied only when g is in the subgroup $GL_2(\mathbb{Z}) \cap GL_2(\mathbb{R})_+ = SL_2(\mathbb{Z})$!

Theorem. $\mathcal{M} \cong \mathbb{H}/SL_2(\mathbb{Z})$ is a (set theoretic) moduli of elliptic curves.

Proof. This is all known by now. The theorem says that $\mathbb{H}$ parameterizes all elliptic curves and that the repetitions come exactly from the orbits of $SL_2(\mathbb{Z})$ in $\mathcal{H}$.

6.6. **Space $\mathbb{H}/SL_2(\mathbb{Z})$.** We are interested in the geometric structure on the moduli $\mathcal{M}$ of elliptic curves which we have so far constructed on the level of sets.

6.6.1. *The double coset formulation and modular forms (automorphic forms).* From the point of view of groups the moduli can be interpreted as

$$\mathcal{M} \cong SL_2(\mathbb{Z})\backslash\mathbb{H} \cong SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/K.$$

This leads to a far reaching group theoretic generalization – one considers double coset spaces

$$\Gamma\backslash G/K$$

where G is a semisimple real Lie group, K is a compact subgroup (often the maximal compact subgroup) and Γ is a discrete subgroup of arithmetic nature.

The functions on such spaces (*automorphic functions*) are key objects of *number theory* and *representation theory.*[8]

However, I'm telling the story upside-down in the sense that (1) the case $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/K$ is interesting enough (without generalizations), and (2) the beautiful theory of *modular forms* that has been developed in the case $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/K$ is based on the above moduli interpretation of the space (and also, in the general theory the best understood and most interesting cases are the ones where $\Gamma\mathbb{G}/K$ has a nice moduli interpretation in complex geometry).

Anyway, this is the subject of Paul's course on Modular Forms so I stop here.

6.6.2. *Fundamental domains.* Let $\Gamma = SL_2(\mathbb{Z})$. We will approximated the *quotient* $\mathbb{H}/\Gamma$ by a subset D of $\mathcal{M}$, such that (i) D is nice, (ii) $D \to \mathbb{H}/\Gamma$ is close to a bijection. Such D will be called a fundamental domain for the action of Γ on $\mathbb{H}$. More precisely, we ask that (i) D be a closed region in $\mathbb{H}$ bounded by finitely many curves, and that (ii) $D \to \mathbb{H}/\Gamma$ is surjective and injectivity only fails on the boundary ∂D.

We start with two elements $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $\Gamma = SL_2(\mathbb{Z})$ with simple geometric meaning: $T(z) = z+1$ is a translation and $S(z) = -1/z$ is the inversion (minus

[8]More generally one studies the (*automorphic forms*) which are sections of line bundles on these spaces.

is need to stay in the upper half plane). Because of the translation T, $\mathbb{H} \xrightarrow{\pi} \mathbb{H}/\Gamma$ will be surjective when restricted to any vertical strip of length one, say to $\mathcal{S} = \{Z \in \mathbb{H};\ |Re(z) \leq \frac{1}{2}\}$.

Next, S sends the lines $L_\pm = \{x = \pm\frac{1}{2}\}$ to the semicircles $C_\mp$ of radius one with centers at ∓ 1 (it sends $\infty \in \mathbb{L}_\pm$ to 0 and $\pm\frac{1}{2} \in L_\pm$ to ∓ 2, ...). So, it identifies the strip $\mathcal{S}$ to the outside of two semi-discs bounded by $C_\pm$. Moreover, S clearly exchanges the inside and the outside of the circle of the $C = \{|z| = 1\}$, and on C it acts as the symmetry with respect to the imaginary axis $e^{i\phi} \mapsto e^{-i\phi+\pi i} = e^{i(\pi-\phi)}$).

We consider a part of $\mathcal{S}$ outside C:[9]

$$\mathcal{D} \stackrel{\text{def}}{=} \{z \in \mathbb{H};\ |Re(z)| \leq \frac{1}{2} \quad \text{and} \quad |z| \geq 1\}.$$

It is bounded by parts of lines $L_\pm$ and the semicircle C. It has two vertices at two sixth roots of unity $L_+ \cap C = \ e^{\pi/3} = \rho$ and $L_- \cap C = \ e^{2\pi/3} = \rho^2$, and one infinite point $\infty \in \mathbb{P}^1$. Its S-image $S(\mathcal{D})$ is then the region (still inside the strip $\mathcal{S}$), bounded by C above and $C_\pm$ bellow. It has two vertices $\rho^2 = S(\rho)$ and $\rho = S(\rho^2)$ and one infinite point $S(\infty) = 0$.

The meaning of $\mathcal{D}$ is explained in

Lemma. (a) For any $\tau \in \mathbb{H}$ the size of the imaginary part has a maximum in the orbit Γz, and this maximum is achieved in $\mathcal{D}$.

(b) The subgroup $\Gamma' \subseteq \Gamma = SL_2(\mathbb{Z})$ generated by S and T satisfies: $\Gamma' \cdot \mathcal{D} = \mathbb{H}$.

Proof. (a) For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the imaginary part of γz is $\frac{Im(z)}{|cz+d|^2}$. For a fixed $z \in \mathbb{H}$, the set of all numbers $|cz+d|$, $\gamma \in \Gamma'$, contains the smallest value (since $c, d \in \mathbb{Z}$), therefore there is a $\gamma \in \Gamma$ such that $Im(\gamma z)$ is the largest possible. Now let $n \in \mathbb{Z}$ be such that $T^n(\gamma z)$ is in the strip $\mathcal{S}$ and the imaginary part is still the largest possible. Then $(T^n\gamma)z \in \mathcal{D}$ – otherwise $|T^n\gamma)z| < 1$, but then its S image would have a larger imaginary part[10]

(b) The proof of (a) used only S and T, so it also applies here. We notice that there is some $\gamma \in \Gamma'$ such that $Im(\gamma z)$ is the largest possible in $\Gamma' \cdot z$, and then, as above, we pass to $T^n(\gamma z) \in \mathcal{S}$ and observe that it really lies in $\mathcal{D}$.

Theorem. (a) $\mathcal{D}$ is a fundamental domain (and so is $S(\mathcal{D})$).

(b) The only pairs of different $\tau, \tau' \in \mathcal{D}$ which are in the same orbit are either: (i) in different boundary lines $L_\pm$ and exchanged by T, or (ii) in the boundary semicircle C and exchanged by S.

[9] Draw $\mathcal{D}$ and $S(\mathcal{D})$!

[10] If $\mathbb{H} \in w$ and $|w| < 1$ then $|Im(w) < Im(-1/w)$ since $0, w, -1/w$ are on the same line.

(c) Since $\{\pm 1\} \subseteq SL_2(\mathbb{Z})$ acts trivially on $\mathbb{H}$, the action factors to the quotient group $\overline{\Gamma} = PSL_2(\mathbb{Z}) \stackrel{\text{def}}{=} SL_2(\mathbb{Z})/\{\pm 1\}$. The only points z in $\mathcal{D}$ with stabilizers Γ_z larger then $\{\pm 1\}$ (i.e., with nontrivial stabilizers $\overline{\Gamma}_z$ in $\overline{\Gamma}$), are

- (i) $\Gamma_i = \{\pm 1, \pm S\} \cong \mathbb{Z}_4$ (hence $\overline{\Gamma}_i \cong \mathbb{Z}_2$), and
- (ii) $\Gamma_\rho = \{\pm 1\} \cdot \{1, TS, (TS)^2\}$ and $\Gamma_{\rho^2} = \{\pm 1\} \cdot \{1, ST, (ST)^2\}$ (hence $\overline{\Gamma}_z \cong \mathbb{Z}_3$ in both cases).

(d) S, T generate $\Gamma = SL_2(\mathbb{Z})$.

Proof. Claims (b) and (c). The coincidences listed in (b) and (c) really happen. First, $T : L_- \xrightarrow{\cong} L_+$ and S acts on $\mathcal{D} \cap C$ as the reflection in the y-axis. Also, there are some points with obvious stabilizers:

- $z = i$ is fixed by the subgroup $\{\pm 1, \pm S\} \subseteq \Gamma$ generated by S.
- $z = \rho$ is fixed by $\pm TS$ and therefore by the subgroup $\{\pm 1, \pm TS, \pm (TS)^2\} = \{\pm 1\} \cdot \{1, TS, (TS)^2\} \subseteq \Gamma$.[11]
- $z = \rho^2$ is fixed by the subgroup $\{\pm 1, \pm ST, \pm (ST)^2\} = \{\pm 1\} \cdot \{1, ST, (ST)^2\} \subseteq \Gamma$.[12]

Now, consider $z, z' \in \mathcal{D}$ be in the same Γ-orbit and with $Im(z') \geq Im(z)$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ take z to z'. It remains to show that in any such case if $z' \neq z$ then z, z' appear in the list given by (b), and if $z' = z$ in the list given by (c).

Since $Im(z) \leq Im(z') = \frac{Im(z)}{|cz+d|^2}$, we have $|cz + d| \leq 1$. This implies $|c| \leq 1$ since for $z \in \mathbb{H}$, one has $|cz + d| \geq |c| \cdot Im(z) \geq |c| \cdot \frac{\sqrt{3}}{2}$. Now we discuss the possibilities $c = -1, 0, 1$.

(1) <u>If $c = 0$</u> then $d = \pm 1$ and $d \cdot \gamma = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for some $n \in \mathbb{Z}$. Since $z' = (d\gamma)z$ we see that $n \neq 0$ implies that $n = \pm 1$ and $\gamma = \pm T^{\pm 1}$. Then z, z' are have to be in different walls $L_\pm$, and this is the case (b).i with $z' \neq z$.

(2) <u>If $c = 1$</u> then $1 \geq |z + d|$, so in terms of $z = x + iy$, we have $1 \geq (d + x)^2 + y^2$. However, $y^2 \geq (\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$ and if $d \neq 0$ then $|d + x| \geq \frac{1}{2}$. So, either (i) $d = \pm 1$ and $z = \mp \frac{1}{2} + i\frac{\sqrt{3}}{2} \in \{\rho, \rho^2\}$, or (ii) $d = 0$.

- (i) $d = 0$ implies that $\gamma = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right)$, hence $\gamma \cdot S = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) = T^a$, and $\gamma = T^a S^{-1}$. Also, $1 \geq |z + d| = |z| \geq 1$ gives $z \in \mathcal{D} \cap C$. In particular $S^{-1}z$ lies in $\mathcal{D} \cap S$.

  ($\bullet$) If $a = 0$ then $\gamma = S^{-1} = -S$ and we are in the case (b).ii, and $z = z'$ iff we are in (c).i.

  ($\bullet$) If $a \neq 0$ then the facts $S^{-1}z \in \mathcal{D} \cap S$ and $T^a(S^{-1}z)w = \gamma z = z' \in \mathcal{D}$, imply that $a = \pm 1$ and $S^{-1}z = -\frac{a}{2} + i\frac{\sqrt{3}}{2}$. Therefore, $z = \frac{a}{2} + i\frac{\sqrt{3}}{2} = z'$ and $\gamma = T^a S^{-1}$ is a known stabilizer element from (c).ii.

[11]Notice that $TS = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ has characteristic polynomial $\lambda^2 - \lambda \cdot Tr + Det = \lambda^2 + \lambda + 1 = \frac{\lambda^3 - 1}{\lambda - 1}$. So, the eigenvalues are the two primitive third roots of 1 and $(TS)^3 = 1$.

[12]Notice that $ST = S(TS)S^{-1}$ is a conjugate of TS.

- (ii) If $d=1$ then $\gamma=\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} b+1 & b \\ 1 & 1 \end{pmatrix}$. Since $z=\rho^2$, $\mathcal{D}$ contains

$$z'=\frac{(b+1)z+b}{z+1}=b+\frac{z}{z+1}=b+\frac{\rho^2}{\rho}=b+\rho\ ,$$

  and therefore $b=0$ or $b=-1$. When $b=0$ then $\gamma=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}=STS^{-1}$ takes $z=\rho^2$ to $z'=\rho\neq z$, but this is already done by T (case (b).i). When $b=-1$ then $\gamma=\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}=T{\cdot}STS^{-1}$ fixes $z=\rho^2$.
  $\gamma=\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}=T{\cdot}STS^{-1}$ fixes $z=\rho^2$. However, $ST=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}=-\gamma$, so we are in (c.ii).
- If $d=-1$ and $z=\rho$ the argument is symmetric.

(3) <u>If $c=-1$</u> we pass to $-\gamma$, hence to $c=1$.

Claims (a) and (d). In the preceding lemma, part (a) says that $\Gamma{\cdot}\mathcal{D}=\mathbb{H}$, so (a) follows from (b). For (d) let $\gamma\in\Gamma$, to show that it is in the subgroup Γ' generated by S and T we choose an *interior* point τ of $\mathcal{D}$. Since $\gamma\tau\in\mathbb{H}$ part (b) of the preceding lemma there is some $\gamma'\in\Gamma'$ with $\gamma'(\gamma\tau)\in\mathcal{D}$. So, $\gamma'\gamma\in\Gamma$ sends an interior point of $\mathcal{D}$ to$\mathcal{D}$, but (b) and (c) then imply that $\gamma'\gamma=1$, hence $\gamma\in\Gamma'$.

Corollary. For any $\tau\in\mathbb{H}$ the intersection of the orbit $\Gamma{\cdot}\tau$ with $\mathcal{D}$ consists of all $w\in\Gamma{\cdot}\tau$ such that

- the imaginary part is maximal and
- w is in the strip $\mathcal{S}$, i.e., $|Re(w)|\le\frac{1}{2}$.

Proof. In view of the lemma, it suffices to see that if $\tau,\tau'\in\mathcal{D}$ are in the same Γ-orbit then $Im(\tau')=Im(\tau)$, but this is clear from the part (b) of the theorem.

6.6.3. *The topological and holomorphic structure of* $\mathbb{H}/\Gamma$. $\mathbb{H}/\Gamma$ is the quotient of $\mathcal{D}$, obtained by making identifications from the theorem 6.6.2.b. When we identify the boundary pieces on the lines $L_\pm$, then $\mathcal{D}$ gives a tube, and the piece of C on the boundary: $e^{\phi i}$, $\pi/3\le\phi\le 2\pi/3$, becomes a boundary circle on the tube since we identify the two ends. The remaining identification happens on this circle: $e^{\phi i}\leftrightarrow e^{(\pi-\phi)i}$. So, the circle gives a segment and this simply closes the bottom of the tube. So, topologically, $\mathcal{M}=\mathbb{H}/\Gamma$ is the plane.

So, if $\mathcal{M}=\mathbb{H}/\Gamma$ has a structure of a complex manifold, by Riemann's uniformization theorem it is isomorphic to either $\mathbb{C}$ or the unit disc.

To find the holomorphic structure on $\mathbb{H}/\Gamma$ and decide its nature, we will study the Γ-invariant holomorphic functions on $\mathbb{H}$. Certainly, such functions factor to functions on $\mathbb{H}/\Gamma$, and we hope that we will be able to put a complex manifold structure on $\mathbb{H}/\Gamma$ such that the holomorphic functions on $\mathbb{H}/\Gamma$ are precisely these factorizations of Γ-invariant holomorphic functions on $\mathbb{H}$.

6.6.4. *Modular functions and modular forms.* We will define *weak modular functions* as Γ-invariant holomorphic functions on $\mathbb{H}$. We are interested in these, but in practice they seem hard so we look into a larger class of *weak modular forms.* We will say that a *weak modular form of weight* $2k$ is a holomorphic functions on $\mathbb{H}$ which transforms under Γ in the following way:

$$f(\gamma{\cdot}z) = \ (cz+d)^{2k}{\cdot}f(z) \ \text{ for } \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

The idea is that if we can construct a few modular forms we can combine them to cancel the additional factor, so we get a modular function.

We get a really important mathematical object when we impose an additional condition. The *modular functions* are defined as the *weak modular functions* which are meromorphic at ∞ (and the same for modular forms). To make sense of this behavior at ∞, we first notice that the manifold $\mathbb{C}/\mathbb{Z}$ is identified with $\mathbb{C}^*$ using $\mathbb{C} \to \mathbb{C}^*$ by $z{\mapsto}e^{2\pi i z}$. This in particular identifies $\mathbb{H}/\mathbb{Z}$ with the punctured unit disc D^*. So, the T-invariant holomorphic functions f on $\mathbb{H}$ (i.e., $f(z+1) = f(z)$), are the same as holomorphic functions ϕ on D^*, via $f(z) = \phi(e^{2\pi i z})$. Notice that the infinity of the strip $\mathcal{S}$ corresponds to $0 \in D$. So, the behavior of f at ∞ is the same as the behavior of $\phi(q)$ ate $q=0$. If f is weakly modular, the requirement that it be modular is that the expansion $\phi(q) = \sum_{-\infty}^{\infty} \ \phi_n{\cdot}q^n$ has finitely many negative terms.

If ϕ is regular at 0 we can say that f is regular at put $f(\infty) = \phi(0)$. We say that a modular form f is a *cusp form* if it is regular at ∞ and $f(\infty) = 0$. The origin of the terminology *cusp* or *cuspidal* in mathematics (in particular in representation theory), is that the infinity of $\mathbb{H}/\Gamma$ can be viewed as the infinity of the fundamental domain $\mathcal{D}$, but also as the infinite point 0 of $S(\mathcal{D})$, and $S(\mathcal{D})$ has a cuspidal shape at 0.

Remarks. (1) The geometric meaning of modular forms: $f(z)$ is modular of weight $2k$ iff $f(z)(dz)^k$ is Γ-invariant:

$$d(\gamma z) = \ d\frac{az+b}{cz+d} = \ \frac{ad-bc}{(cz+d)^2}dz = \ (cz+d)^{-2} \ dz.$$

In other words, these are differentials[13] on the moduli $\mathcal{M} = \mathbb{H}/\Gamma$ of elliptic curves.

(2) A weight can not be odd[14] since $f(\gamma{\cdot}z) = \ (cz+d)^l{\cdot}f(z)$ implies for $\gamma = -I$ that $f(z) = (-1)^l{\cdot}f(z)$.

(3) Modular forms are essential in number theory and representation theory. recently they have become important in algebraic topology (computations of homotopy groups of spheres). One of the mathematically most attractive features of string theory (particle physics) is the ease with which it constructs modular forms).

[13]Here, by *differential* we mean a section of a tensor power of the cotangent bundle.

[14]Actually, odd weights appear but the definition is more complicated.

6.6.5. *Eisenstein series and the j-invariant.* A function f on the moduli of elliptic curves, means that to each (isomorphism class of) elliptic curve we attach a number. Since elliptic curves E_L come from lattices $L\subseteq\mathbb{C}$, in particular we want to attach a number to each lattice L. The obvious idea is to take some kind of average of all lattice elements. This gives Eisenstein series

$$G_k(L) \stackrel{\text{def}}{=} \sum_{0\neq l\in L} \frac{1}{l^{2k}}, \quad k = 2, 3, \ldots.$$

We will restrict the Eisenstein series to $\mathbb{H}$ by

$$G_k(\tau) \stackrel{\text{def}}{=} G_k(L_\tau) \stackrel{\text{def}}{=} \sum_{0\neq l\in L_\tau} \frac{1}{l^{2k}}.$$

The negative power $-2k$ is needed for

Lemma. $G_k(L)$ converges absolutely for $k > 1$.

Proof. For absolute convergence we consider $\sum_{0\neq l\in L} \frac{1}{|l|^{2k}}$. We find that it is comparable with the integral

$$\int_{\mathbb{C}-?} \frac{1}{(x^2+y^2)^k}\, dx\ dy,$$

where ? is any union of finitely many L-boxes that contains 0. Here, *comparable* means that, the series and the integral converge for the same k. The reason is that in any L-box B that does not contain 0, $\frac{area(B)}{|p|^2} \leq \int_B \frac{1}{(x^2+y^2)^k}\, dx\ dy \frac{area(B)}{|q|^2}$ for the points p and q in B that are closest to 0 (resp. most distant from 0).

So the question is for which k does $\int_{|(x,y)|\geq 1} \frac{1}{(x^2+y^2)^k}\, dx\ dy$ converge. In polar coordinates this is $\int_0^{2\pi} d\theta \int_1^\infty r{\cdot}dr\ \frac{1}{r^{2k}} = 2\pi \int_1^\infty \frac{dr}{r^{2k-1}}$, so we need $2k-1 > 1$, i.e., $k > 1$.

Lemma. $\Gamma_k(\tau)$ is a weak modular form of weight $2k$.

Proof. Clearly $Ga_k(L)$ is homogeneous of degree $-2k$, i.e., $\Gamma_k(c{\cdot}L) = c^{-2k}{\cdot}\Gamma_k(L)$. So, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$,

$$\Gamma_k(\gamma\tau) = \Gamma_k(L_{\gamma\tau}) = \Gamma_k(\mathbb{Z}\oplus\mathbb{Z}{\cdot}\gamma\tau) = \Gamma_k(\mathbb{Z}\oplus\mathbb{Z}{\cdot}\frac{a\tau+b}{c\tau+d}) = (cz+d)^{-2k}{\cdot}\Gamma_k(\mathbb{Z}{\cdot}(c\tau+d)\oplus\mathbb{Z}{\cdot}(a\tau+b))$$

$$\stackrel{(*)}{=} (cz+d)^{-2k}{\cdot}\Gamma_k(L_\tau) = (cz+d)^{-2k}{\cdot}\Gamma_k(\tau).$$

Here, the meaning of $(*)$ is that since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, the lattice with the basis $a\tau+b, c\tau+d$ is the same as the lattice with a basis $\tau, 1$.

We also need to know that $G_k(\tau)$ is holomorphic in τ. However,

$$G_k(\tau) = \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{m+n\tau}$$

and (i) each summand is holomorphic in τ, (ii) locally in τ, the sum converges uniformly.

Here (ii) is proved by the same argument we used above for the pointwise convergence.

Proposition. (a) G_k is regular at ∞, actually

$$G_k(\infty) \;=\; 2{\cdot}\zeta(2k),$$

for the Riemann zeta function

$$\zeta(s) \stackrel{\text{def}}{=} \sum_1^\infty \frac{1}{n^2} = \prod_{p \text{ a prime}} \frac{1}{1-p^{-s}}.$$

In particular, this is a modular form of weight $2k$.

(b) $\Delta \stackrel{\text{def}}{=} (60G_2)^3 - 27(140G_3)^2$ is a cusp form of weight 12 with an elegant q-expansion

$$(2\pi)^1 2 \cdot q{\cdot}\prod_1^\infty (1-q^n)^{24}.$$

(c) $j \stackrel{\text{def}}{=} 1728\frac{\Delta}{(60G_2)^3}$ is a modular function.

Theorem. Γ-invariant function $j : \mathbb{H} \to \mathbb{C}$ factors to a bijection $\mathbb{H}/\Gamma \xrightarrow{\cong} \mathbb{C}$.

Corollary. We can use j to make $\mathbb{H}/\Gamma$ into a complex manifold.

Remark. Caution! $\mathbb{H}/\Gamma$ is "set-theoretic" moduli in the sense that it is a complex manifold and as a set it is the set of isomorphism classes of elliptic curves. However, there is a finer version of the moduli which is a stack – we get it if we do not forget the automorphisms of elliptic curves!

6.7. **Integrals of algebraic functions.** We look at the general problem of making sense of integrals of *algebraic functions*, i.e., functions $y(x)$ defined by solving for each x a polynomial equation $a_0(x)y^n(x) + a_1(x)y^{n-1}(x) + \cdots + a_{n-1}y(x) + a_n(x) = 0$. This is clearly a multi-valued function, so in integrals of the form

$$\int_\alpha^\beta y(x)\; dx$$

we need to specify

(1) which path we use from α to β and
(2) which branch of $y(x)$ we use on this path.

The confusion results in a subgroup $\mathcal{P}eriods{\subseteq}\mathbb{C}$ of *periods* of the integral, such that

$\int_\alpha^\beta y(x)\; dx$ is not defined as a number in $\mathbb{C}$,
but only as an element of the group $\mathbb{C}/\mathcal{P}eriods$.

Our main interest is in the algebraic function $y\sqrt{x(x-1)(x-\lambda)}$ related to the cubic C_λ. In this case the integrals $\int_\alpha^\beta\ y\sqrt{x(x-1)(x-\lambda)}\ dx$ lead to a natural isomorphism of the cubic C_λ and a certain elliptic curve $E\tau$.

6.7.1. *Algebraic functions as branched covers of a line.* By an algebraic function on $\mathbb{C}$ I will mean a multivalued function $y(x)$ determined by a polynomial equation

$$a_0(x)y^n(x)+a_1(x)y^{n-1}(x)+\cdots+a_{n-1}y(x)+a_n(x)=0$$

where a_i's are polynomial functions on $\mathbb{C}$.

Examples. $\sqrt[n]{x}$, $\sqrt{x(x-1)(x-\lambda)}$, $\frac{1}{\sqrt{x(x-1)(x-\lambda)}}$ are all algebraic functions.

For a generic x there will be n different roots of the equation, so y will have n possible values. A geometric home for this non-standard mathematical object (a multi-valued function) is the *algebraic curve* $\mathcal{C}\subseteq\mathbb{A}^2$ defined by

$$\mathcal{C} \stackrel{\text{def}}{=} \{(x,y)\in\mathbb{A}^2;\ a_0(x)y^n+a_1(x)y^{n-1}+\cdots+a_{n-1}y+a_n(x)=0\}.$$

The projection $\mathcal{C}\subseteq\mathbb{A}^2_{x,y}\to\mathbb{A}^1_x$ is an n-fold branched cover of $\mathbb{A}^1$, and the branching happens over a finite subset $F\subseteq\mathbb{C}$ consisting of all $x\in\mathbb{C}$ such that the solutions y of the equation of $\mathcal{C}$ acquire multiplicities or $a_0(x)=0$.

Remark. Notice that to a multivalued function $y(x)$ over $U\subseteq\mathbb{C}$ we have associated a complex curve (a one dimensional complex manifold) which is a branched cover of U, by

$$\mathcal{C}=\{(a,b)\in\mathbb{A}^2;\ b \textit{ is one of the values of } y(a)\}.$$

This works for all *multivalued holomorphic functions.* For instance $y=\log(x)$ on $\mathbb{C}^*$ gives $\mathcal{C}\to\mathbb{C}^*$ which can be identified with $\mathbb{C}\xrightarrow{e^z}\mathbb{C}^*$ (the *universal cover* of $\mathbb{C}^*$).

We think of a multivalued function $y(x)$ in terms of "branches", i.e., pairs $(V,Y(x)$ where $V\subseteq U$ is open and a holomorphic function Y on V is a version of $y(x)$ In terms of $\mathcal{C}$ these branches of $y(x)$ correspond to open pieces $W\subseteq\mathcal{C}$ such that $W\xrightarrow{pr_x}pr_x(W)$ is a bijection. Such W gives open $V=pr_x(W)\subseteq U$ and a branch Y on V by $Y=\ pr_y\circ(W\xrightarrow{pr_x}V)^{-1}$. In the opposite direction, $W=\{(a,Y(a));\ a\in V\}$.

6.7.2. *Lifting of paths (branches of $y(x)$ along paths).* Over the open set $U=\mathbb{C}-F$, $\mathcal{C}$ is an n-fold covering. So, on each open disc $D\subseteq U$, the multivalued function $y(x)$ breaks into n holomorphic functions $y_i(x)$, and the restriction $\mathcal{C}|D$ of $\mathcal{C}$ over the discs D, is a union of n disjoint discs D_i, the graphs of y_i, such that the map $D_i\to D$ is a holomorphic isomorphism.

A consequence of this is that for any path $\gamma:[0,1]\to U$, and any lift $p\in\mathcal{C}$ of $\gamma(0)\in U$,[15] there is a unique lift $\widetilde{\gamma}:[0,1]\to\mathcal{C}$ of γ, that starts at p.

[15]A lift of $\gamma(0)$ is a point $p\in\pi^{-1}(\gamma(0))\subseteq\mathcal{C}$ that lies above $\gamma(0)$.

Such lift can be viewed as a description of a continuous choice $\widetilde{y}(x)$ of the value of y along the path γ, since $\widetilde{\gamma}(t) = (\gamma(t), \widetilde{y}(t)$ for some $\widetilde{y} : [0,1] \to \mathbb{C}$, i.e., a *choice of a branch of* $y(x)$ *along* γ.

6.7.3. *Monodromy of algebraic functions.* This is a side remark on topological aspects of the multivalued nature of the algebraic function $y(x)$.

Let γ be path in $U = \mathbb{C} - F$, and let $\widetilde{\gamma}$ be a lift of γ to a path in $\mathcal{C}$. If γ is closed, i.e., $\gamma(1) = \gamma(0)$, it does not mean that $\widetilde{\gamma}$ is closed. We only know that $\pi(\widetilde{\gamma}(1)) = \pi(\widetilde{\gamma}(0))$. We say that $\widetilde{\gamma}(1) = M_\gamma(p)$ is obtained by applying to p the γ-monodromy.[16]

Example. If $y(x)$ is given by the equation $y^n - x = 0$, i.e., $y = \sqrt[n]{x}$ the branching happens at $F = \{0\}$ and above $\mathbb{C}^*$ curve $\mathcal{C}$ is the n-fold cover. If γ is a circle around 0 (counterclockwise), the monodromy is $e^{2\pi i/n}$.

6.7.4. *Integrals of algebraic functions.* In order to make sense of integrals

$$\int_\alpha^\beta y(x)\ dx$$

of a multi-valued function $y(x)$ we need to specify

(1) which path γ we use from α to β and
(2) which branch of $y(x)$ we use on this path.

However, we saw that a path γ in $\mathbb{C}$, from α to β, and a branch of $y(x)$ along γ, together amount to a choice of a path $\widetilde{\gamma}$ in $\mathcal{C}$ (a lift of γ), which goes from some lift $\widetilde{\alpha}$ of α to some lift $\widetilde{\beta}$ of β.

So our problem is really to calculate integrals of $y(x)\ dx$ over paths $\widetilde{\gamma}$ in $\mathcal{C}$:

$$\int_{\widetilde{\gamma}} y(x)\ dx \stackrel{\text{def}}{=} \int_0^1 pr_y(\widetilde{\gamma}(t)) \cdot (pr_x \circ \widetilde{\gamma})'(t)\ dt.$$

Moreover, if we allow path γ to pass through ∞ (i.e. paths on the Riemann sphere $\mathbb{P}^1$), then we have to allow the lift $\widetilde{\gamma}$ to pass through infinite points of $\mathcal{C}$, i.e., $\widetilde{\gamma}$ should be a path in the projective closure $C \subseteq \mathbb{P}^2$ of $\mathcal{C} \subseteq \mathbb{A}^2$.

6.8. **Periods of integrals.**

[16]So, γ-monodromy acts on the fibers of $\mathcal{C} \to \mathbb{A}^1_x$. Notice the analogy with the Galois theory: monodromy permutes solutions of a polynomial equation.

6.8.1. *Periods of integrals.* We consider the problem of defining for any $\alpha, \beta \in C$ the integral $\int_\alpha^\beta \ y\ dx$ of y from α to β. For this we need a path γ from α to β in C. This can be done in many ways, however for any two choices the integrals differ by an integral over a closed path:

$$\int_{\gamma_1} y(x)\ dx - \int_{\gamma_2} y(x)\ dx = \int_\gamma y(x)\ dx$$

for the closed path $\gamma = \gamma_1 - \gamma_2$ from β to β. The integrals

$$P_\gamma \overset{\text{def}}{=} \int_\gamma y(x)\ dx$$

over closed paths γ are called the *periods* of the integral. So, we found that

$$\boxed{\int_\alpha^\beta \ y\ dx \ \ \textit{is well defined up to periods.}}$$

This raises the question of finding all periods.

6.8.2. *Periods depend on closed paths up to homotopy.* How much does the period $P_\gamma = \int_\gamma \ y(x)\ dx$ depend on a choice of a close path γ? One of the basic tricks in complex analysis is the observation that the integrals are homotopy invariant, i.e., integral does not change as long as we move the path continuously.[17] So the basic question in this direction is

$$(\star) \qquad \textit{How many closed paths are there in } C \ \ \underline{\textit{up to homotopy}}?$$

To have a standard formulation let us pick a point $a \in C$ and let $\pi_1(C, a)$ be the set of *homotopy classes of closed paths* $\gamma : [0, 1] \to C$ such that $\gamma(0) = a = \gamma(1)$. All periods come from $\pi_1(C, c)$ since any closed path can be continuously moved to one that passes through a. So, we are interested in a version of $(\star)$: what is $\pi_1(C, a)$?

Lemma. (a) $\pi_1(C, a)$ is a group.

(b) The map $\pi_1(C) \xrightarrow{\int \ y\ dx} \mathbb{C}$, given by $\pi_1(C) \ni \gamma \mapsto \int_\gamma \ y(x)\ dx \in \ \mathbb{C}$, is a morphism of groups.

Proof. (a) The operation is concatenation (composition) of paths: $\gamma_2 \circ \gamma_1$ is the path obtained by first following γ_1 and then γ_2. (b) is now clear from definitions: $\int_{\gamma_2 \circ \gamma_1} = \int_{\gamma_2} + \int_{\gamma_1}$.

Corollary. The set of periods $\mathcal{P}eriods = \ (\int \ y\ dx)\pi_1(C, c) \subseteq \mathbb{C}$ is a subgroup.

[17]Recall that there is a stronger version which gives sharper versions of statements bellow: integral depends on path only up to homology.

6.9. **Cubics are elliptic curves (periods of elliptic integrals).** The study of integrals $\int \frac{dx}{\sqrt{x(x-1)(x-la)}}$ appeared a classical mathematical question through its relation to the arc length of ellipses. We will use the above ideas on integration of algebraic functions for $y(x) = \frac{1}{\sqrt{x(x-1)(x-la)}}$. The corresponding algebraic curves that capture the multi-valued nature of $y(x)$ (i.e., of $\sqrt{x(x-1)(x-la)}$) are isomorphic to $\mathcal{C}_\lambda$ and C_λ[18]

6.9.1. *Lemma.* In a cubic C_λ ($\lambda \neq 0,1$), closed paths up to homotopy form a free abelian group with a basis $\alpha_\lambda, \beta_\lambda$, i.e.,

$$\pi_1(C) \;\cong\; \mathbb{Z}\alpha_\lambda \oplus \mathbb{Z}\beta_\lambda.$$

Proof. This is a topological question so let us consider a torus $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/L \stackrel{\text{def}}{=} E_L$. First any closed path can be deformed continuously so that it passe through $\mathbf{0} = \pi(0)$. Now, any parameterization $\gamma : [0,1] \to E_L$ with $\gamma(0) = \mathbf{0}$, of the path lifts in a unique way to a path $\widetilde{\gamma} : [0,1] \to \mathbb{C}$ with $\widetilde{\gamma}(0) = 0$. Now $l = \widetilde{\gamma}(1) \in L$ and $\widetilde{\gamma}$ deforms to a straight line path from 0 to l. For any basis u, v of L, if $l = p{\cdot}u + q{\cdot}v$ we can further deform this strait line segment into a composition of straight line segments from 0 to $p{\cdot}u$ and from $p{\cdot}u$ to $p{\cdot}u + q{\cdot}v$. With appropriate choices of u, v the image of this deformed path is $p{\cdot}\alpha_\lambda + q{\cdot}\beta_\lambda$.

6.9.2. *Corollary.* $\int_\alpha^\beta \frac{dx}{\sqrt{x(x-1)(x-la)}}$ is well defined with values in the group

$$\mathbb{C}/[\mathbb{Z}{\cdot}P_{\alpha_\lambda} \oplus \mathbb{Z}{\cdot}P_{\beta_\lambda}].$$

Proof. We know that the difference of values of any two versions of $\int_\alpha^\beta y(x)\, dx$ is $\int_\gamma y$ for some closed path γ on C_λ. If γ is homotopic to $p{\cdot}\alpha_\lambda + q{\cdot}\beta_\lambda$, then

$$\int_\gamma y = \; p{\cdot}\int_{\alpha_\lambda} y \; + \; q{\cdot}\int_{\beta_\lambda} y = \; p{\cdot}P_{\alpha_\lambda} + q{\cdot}P_{\beta_\lambda}.$$

6.9.3. *Independence of periods.* In order to be able to claim that the values of the integral are in an elliptic curve, we need

Theorem. The set $\mathcal{P}$ of all periods of the integral $\int y(x)\, dx$ is a lattice with a basis $P_{\alpha_\lambda}, P_{\beta_\lambda}$.

Proof. This will be based on the study of a differential equation that the periods satisfy as functions of λ, the *Picard-Fuchs* equation.

6.9.4. *Cubics are elliptic curves.*

[18]The values of $y(x)$ and $1/y(x)$ are related by a change of coordinates.

Theorem. Choose $\alpha \in C$ and consider the map

$$C \to \mathcal{C}/\mathcal{P}eriods \text{ by } \beta \mapsto \int_\alpha^\beta \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

This is an isomorphism of complex manifolds.

Proof. Map is holomorphic since in local coordinates we know that the derivative of $\int_a^z f(u)\ du$ is $f(z)$. Next, the map is a local isomorphism since the derivative does not vanish. Surjectivity is easy: the image is open (map is local isomorphism!) and closed (source is compact). Injectivity requires additional thinking.

Corollary. Any cubic C_λ is isomorphic to one of the elliptic curves E_τ, $\tau \in \mathbb{H}$, as a *complex manifold.*

6.10. **Theta functions on complex tori.** Any $\tau \in \mathbb{H}$ (i.e., $Im(\tau) > 0$ gives a lattice $L_\tau = \mathbb{Z}\oplus\mathbb{Z}\cdot\tau$ in $\mathbb{C}$, and an elliptic curve $E_\tau = \mathbb{C}/L_\tau$ which comes with the quotient map $\pi : \mathbb{C} \to E_\tau$.

We would like to find some holomorphic functions on E_τ, and this is the same as a holomorphic function f on $\mathbb{C}$ which is periodic in directions of 1 and τ: $f(u+1) = f(u) = f(u+\tau)$. However, there are no such functions, so we ask for the "next best thing": periodic for 1 and *quasiperiodic* for τ in the sense that $f(u+\tau)$ differs from $f(u)$ by a simple factor.

We will construct such functions as *theta series* $\theta_\tau(u)$, given by a formula. Philosophically, being periodic in one direction and *quasiperiodic* in another, means that θ_τ does not really descend to a function on E_τ, but to something close to a function: a section of a line bundle on E_τ.

However, the root of our interest in theta functions is more elementary. We will actually manage to produce a real life function $\mathfrak{p}_\tau$ on E_τ using θ_τ – a combination of θ and θ' in which the quasi-periodicity factor cancels!). It will have a defect: a pole at one point, but this turns out to be obligatory.

6.10.1. *Theta series.* The theta series[19] in $\tau \in \mathbb{H}$ and $u \in \mathbb{C}$ is

$$\theta_\tau(u) \stackrel{\text{def}}{=} \sum_{-\infty}^{+\infty} e^{\pi i(n^2\tau + 2nu)}.$$

Lemma. (a) For any $\tau \in \mathbb{H}$ it defines an entire function of u.

(b) For any $u \in \mathbb{C}$ it defines a holomorphic function on $\mathbb{H}$.

(c) For any $a \in \mathbb{R}$, $b > 0$, the series converges uniformly on the product

$$\{\tau \in \mathbb{H};\ Im(\tau) > b\} \times \{u \in \mathbb{C};\ Im(u) > a\}.$$

[19]The basic facts bellow are proved in homeworks.

(d) The series can be differentiated any number of times (with respect to τ and u), and the derivatives are calculated term by term.

6.10.2. *Transformation properties of theta functions.* We find that θ_τ is periodic in one direction, quasi-periodic in another direction, and even.

Lemma. (a) $\theta_\tau(u+1) = \theta_\tau(u)$.

(b) $\theta_\tau(u+\tau) = e^{-\pi i(\tau+2u)}\cdot\theta_\tau(u)$.

(c) $\theta_\tau(-u) = \theta_\tau(u)$.

6.10.3. *Zeros of theta functions.*

Lemma. (a) θ_τ has a zero at $u_0 \overset{\text{def}}{=} \frac{\tau+1}{2}$.

(b) This is the only zero of θ_τ in the closed parallelogram $\overline{\mathcal{P}}_\tau$ generated by vectors $1,\tau$ in the real vector space $\mathbb{C}$:

$$\mathcal{P}_\tau \overset{\text{def}}{=} \{a+b\tau;\ 0<a,b<1\}.$$

Proof. (a)

$$\theta_\tau(\frac{\tau+1}{2}) = \theta_\tau(\frac{1-\tau}{2}+\tau) = \theta_\tau(\frac{1-\tau}{2})\cdot e^{-\pi i(\tau+2\frac{1-\tau}{2})} = -\theta_\tau(\frac{1-\tau}{2})$$
$$= -\theta_\tau(\frac{\tau-1}{2}) = -\theta_\tau(\frac{1-\tau}{2}+1) = -\theta_\tau(\frac{\tau+1}{2}).$$

6.11. **Weierstrass $\mathfrak{p}$-function (elliptic curves are cubics).** By *elliptic functions* we mean meromorphic functions on elliptic curves E_τ. Our goal is to find some such, since one can not do better:

Lemma. Any holomorphic function on a (connected) compact complex curve is constant.

Proof. The image $f(C)\subseteq\mathbb{C}$ is compact. However, if f were not constant its image would have to be open (*Open mapping theorem*).

6.11.1. *Weierstrass $\mathfrak{p}$-function $\mathfrak{p}_\tau$ on E_τ.* Recall that each $\tau\in\mathbb{H}$ defines the function $\theta_\tau(u)$ on $\mathbb{C}$. The Weierstrass $\mathfrak{p}$-function is a meromorphic function on $\mathbb{C}$ which we will define as the second logarithmic derivative of the theta function

$$\mathfrak{p}_\tau(u) \overset{\text{def}}{=} (log(\theta_\tau(u))'' = (\frac{\theta_\tau'(u)}{\theta_\tau(u)})'.$$

Lemma. (a) $\mathfrak{p}_\tau(u) \stackrel{\text{def}}{=} (log(\theta_\tau(u))''$ is a well defined holomorphic function on

$$\mathbb{C}\backslash\ (\tfrac{1+\tau}{2} + L_\tau),\ \ i.e.,\ \ off\ the\ L_\tau\text{-}translates\ of\ the\ point\ \tfrac{1+\tau}{2}.$$

(b) $\mathfrak{p}_\tau$ is L_τ invariant, i.e., $\mathfrak{p}_\tau(z+1) = \mathfrak{p}_\tau(z) = \mathfrak{p}_\tau(z+\tau)$.

(c) $\mathfrak{p}_\tau$ has a pole of order two at $\frac{1+\tau}{2}$.

(d) $\mathfrak{p}_\tau$ is meromorphic on $\mathbb{C}$.

Corollary. $\mathfrak{p}_\tau$ factors to a meromorphic function $\mathfrak{p}_\tau$ on E_τ. Its only pole is at $\zeta \stackrel{\text{def}}{=} \pi(\frac{\tau+1}{2}) \in E_\tau$, and it is a double pole.

Remark. In group theoretic terms, point ζ is one of three points of order 2 in E_τ.

6.11.2. *Elliptic curves are cubics.*

Theorem. (a) Map $f = (\mathfrak{p}_\tau, \mathfrak{p}'_\tau) : E_\tau - \{\zeta\} \to \mathbb{C}^2$ has image in a cubic $\mathcal{C}$ of the form $y^2 = 4x^3 + Ax^2 + Bx + C$.

(b) f extends to a holomorphic isomorphism of E_τ and the projective closure $C = \overline{\mathcal{C}}$ of $\mathcal{C}$.

Proof. $\boxed{\text{(a)}}$ Let us dispense with the index τ, so $\theta_\tau = \theta$ etc.

1. Reduction to killing the pole at ζ. f is a holomorphic map on $E_\tau - \{\zeta\}$, and its component functions $\mathfrak{p} = (\frac{\theta'_\tau(u)}{\theta_\tau(u)})'$, $\mathfrak{p}'$ have poles at ζ of orders 2 and 3. The claim is that for some $A, B \in \mathbb{C}$

$$(\mathfrak{p}')^2 - 4\mathfrak{p}^3 - A\mathfrak{p}^2 - B\mathfrak{p}\ \ is\ a\ constant.$$

However, it suffices that $(\mathfrak{p}')^2 + 4\mathfrak{p}^3 - A\mathfrak{p}^2 - B\mathfrak{p}$ be holomorphic at ζ (holomorphic functions on E_τ are constant!).

2. Strategy. We will study the polar parts of Laurent expansions of $(\mathfrak{p}')^2, \mathfrak{p}^3, \mathfrak{p}^2, \mathfrak{p}$ at ζ and we will see that an appropriate combination has no pole. The expansions will be in the variable $v = u - \frac{1+\tau}{2}$.

3. Expansion of $\mathfrak{p}(u)$ is $-v^{-2} + a + bv^2 + O(4)$. Let us denote by $\mathcal{O}(k)$ anything of order $\geq k$ at $v = 0$, say $O(0)$ means "holomorphic at $\frac{\tau+1}{2}$". At $u = \frac{\tau+1}{2}$ holomorphic function $\mathfrak{p}(u)$ has a first order zero, so $\mathfrak{p}'(\frac{\tau+1}{2}) \neq 0$ and $\frac{\theta'(u)}{\theta(u)}$ has a first order pole at $\frac{1+\tau}{2}$,

$$\frac{\theta'(u)}{\theta(u)} \ = \ c_{-1}v^{-1} + \mathcal{O}(v).$$

Moreover,

$$c_{-1} \ = \ Res_{\frac{1+\tau}{2}}(\frac{\theta'_\tau(u)}{\theta_\tau(u)}) \ = \ ord_{\frac{1+\tau}{2}}\theta_\tau(u) \ = \ 1.$$

So, $\frac{\theta'(u)}{\theta(u)} = v^{-1} + \mathcal{O}(v)$ and therefore

$$\mathfrak{p}(u) = (\frac{\theta'(u)}{\theta(u)})' = -v^{-2} + \mathcal{O}(v).$$

Notice the absence of v^{-1}. We will see that more is true: the expansion of $\mathfrak{p}(u)$ in v's only has even terms. First, θ is even, hence θ' and θ'/θ are odd, and therefore its derivative $\mathfrak{p}$ is even: $\mathfrak{p}(-u) = \mathfrak{p}(u)$. Then, since $2{\cdot}\mathfrak{p}(\frac{1+\tau}{2} + v)$ is in the lattice,

$$\mathfrak{p}(\frac{1+\tau}{2} + v) = \mathfrak{p}(-\frac{1+\tau}{2} - v) = \mathfrak{p}(\frac{1+\tau}{2} - v).$$

4. Polar parts of $(\mathfrak{p}')^2, \mathfrak{p}^3, \mathfrak{p}^2, \mathfrak{p}$ at ζ. Now we know the first statement in the following series, and then the rest follows:

(1) $\mathfrak{p}(u) = -v^{-2} + a + bv^2 + O(4)$,
(2) $\mathfrak{p}^2(u) = v^{-4} + v^{-2}(-2a) + (a^2 - 2b) + O(2)$,
(3) $\mathfrak{p}^3(u) = -v^{-6} + v^{-4}(3a) + v^{-2}[(2b - a^2) - 2a^2 + b] + O(0)$
$= -v^{-6} + v^{-4}(3a) + v^{-2}[3(b - a^2)] + O(0)$,
(4) $\mathfrak{p}'(u) = 2v^{-3} + 2bv + O(3)$,
(5) $(\mathfrak{p}')^2(u) = 4v^{-6} + 8bv^{-2} + O(0)$.

All together, we have

four functions whose polar parts are combinations of v^{-6}, v^{-4}, v^{-2};
so an appropriate combination of these will be holomorphic!

This combination can be written explicitly:[20]

(1) $(\mathfrak{p}')^2 + 4\mathfrak{p}^3 = 12v^{-4} + (20b - 12a^2)v^{-2} + O(0)$.
(2) $(\mathfrak{p}')^2 + 4\mathfrak{p}^3 - 12\mathfrak{p}^2 = (20b - 12a^2 + 24a)v^{-2} + O(0)$.
(3) $(\mathfrak{p}')^2 + 4\mathfrak{p}^3 - 12\mathfrak{p}^2 + (20b - 12a^2 + 24a)\mathfrak{p} = O(0)$.

(b) We want to extend $f: E_\tau - \zeta \to \mathcal{C}\subseteq\mathbb{A}^2$ holomorphically to $E_\tau \to C\subseteq\mathbb{P}^2$, i.e., to check that the (possible) isolated singularity of $f: E_\tau - \zeta \to C$ at ζ, is removable. It suffices for instance to calculate $lim_{u\to(\tau+1)/2}\ f(u)$ in $\mathbb{P}^2$ in suitable coordinates near the infinite point of $\mathcal{C}$.

It is easy to see that f is locally an isomorphism – it suffices to check that the differential of f does not vanish. Similarly, surjectivity follows from abstract reasons:

- since f is a local isomorphism, $f(E_\tau)\subseteq C$ is open, and
- since E_τ is compact the image is also closed. Now,
- since C is connected $f(E_\tau)$ is everything.

Injectivity requires a little thought.

[20] My numbers may be wrong.

6.11.3. *Corollary.* Any elliptic curve is isomorphic to one of the cubics C_λ.

Proof. The theorem identifies E_τ with a cubic of the form $y^2 = -4x^3 + Ax^2 + Bx + C$. However, after an affine change of coordinates it becomes one of C_λ's. First divide by -4 and change y to get it in the form $y^2 = x^3 + Ax^2 + Bx + C = (x-\alpha)(x-\beta)(x-\gamma)$. Now an affine change $x \mapsto ax + b$ takes α, β to $0, 1$ and γ to λ (for this we need $\alpha \neq \beta$, but for $\alpha = \beta$ C_λ is not a manifold so it can not be isomorphic to E_τ).

7. **Linearization: the Jacobian of a curve**

The basic invariant of a compact connected complex curve C is its *genus* $g_C \in \mathbb{N}$. It is important both from the topological and from the holomorphic point of view. There are three very different situations:

- $g = 0$ iff $C \cong \mathbb{P}^1 = 0$.
- $g = 1$ iff C is a cubic (i.e., an elliptic curve).
- $g \geq 2$. These we understand less explicitly.

The basic object we will associate to a curve C will be its Jacobian J_C. The passage from C to J_C can be viewed from a number of points of view and the chapter is organized around explaining the meaning of these approaches and indicating some relation between them.

7.1. **Jacobian of a curve: points of view.** To a smooth complex complete[21] connected curve C we will associate an abelian complex Lie group J called the Jacobian of C. Though C is connected, the Jacobian still comes with connected components $J_n,\ n \in \mathbb{Z}$. Moreover, it comes with a canonical map $C \xrightarrow{\iota} J_1 \subseteq J$.

As all important mathematical ideas, Jacobians can be viewed from a number of points of view

(1) J is the abelian complex Lie group freely generated by C.
(2) J is built from symmetric powers of C, via the Abel-Jacobi maps $C^{(n)} = C^{[n]} \to J_n$.
(3) J is the moduli of complex line bundles on C,
(4) J appears as the universal target of integrals on C.
(5) Topologically, J_0 is the quotient of the holomorphic part $H^{1,0}$ of the first cohomology with complex coefficients $H^1(C, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$, by the image of the integral cohomology $H^1(C, \mathbb{Z}) \subseteq H^1(C, \mathbb{C})$ (when one projects $H^1(C, \mathbb{C})$ to the summand $H^{1,0}$).
(6) As an abelian group, J is the divisor class group $Cl(C)$.

There is a lot of classical mathematics hidden in the identification of these approaches. If one takes (1) as the most natural definition of the Jacobian, i.e., the most intimate relation to curve, then the other statements are all calculation of the abelian group generated by C.

7.1.1. *Formulation (6): divisors.* The group $Div(C)$ of divisors on C is the abelian group freely generated by the <u>set</u> C. It has a $\mathbb{Z}$ basis given by points of C. We can also characterize it categorically:

[21] *Complete* is used in the meaning of *compact*. The idea is that "a curve can fail to be compact only if something is missing".

Any map $C \xrightarrow{f} A$ from C to a commutative group A, canonically factors through $Div(C)$, i.e., there is precisely one map of groups $Div(C) \xrightarrow{\bar{f}} A$ such that

$$(C \xrightarrow{f} A) \;=\; (C \xrightarrow{\iota} J \xrightarrow{\bar{f}} A).$$

7.1.2. *Formulation (1): Jacobian as a "linearization" of a curve.* The disadvantage of $Div(C)$ is that we have forgotten the structure of a complex manifold on C. Jacobian $J(C)$ is the analogue of $Div(C)$ in the category of manifolds (rather then just sets). This is the meaning of the characterization (1) whose precise form is

Any holomorphic map $C \xrightarrow{f} A$ from C to a commutative complex Lie group A, canonically factors through J, i.e., there is precisely one map of complex Lie groups $J \xrightarrow{\bar{f}} A$ such that

$$(C \xrightarrow{f} A) \;=\; (C \xrightarrow{\iota} J \xrightarrow{\bar{f}} A).$$

This is the most natural way to think of the Jacobian, it relates it most intimately to C.

In mathematics we often approach problems by passing to linear algebra settings. Sometimes this is just an approximation (manifold $\mapsto$ tangent space, differential equation $\mapsto$ its linear approximation), and sometimes we pass to a larger linear setting in which we keep all information (manifold $\mapsto$ vectors space of functions, nonlinear KP-differential equation $\mapsto$ KP-hierarchy of linear differential equations). One can think of the Jacobian in this way – to a geometric object, a curve we associate a "more linear" (usually) larger, geometric object.

7.1.3. *Relation of (1) and (6).* The above universal properties of $Div(C)$ and J provide a map of groups $Div(C) \to J$. More is true – J is a quotient of $Div(C)$, so one can imagine that a manifold structure was imposed on $Div(C)$ by pushing points together. Actually this quotient can be explicitly described as

$$Cl(C) \stackrel{\text{def}}{=} Div(C)/div(\mathfrak{M}^*(C))$$

for the subgroup of *principal divisors* $div(\mathfrak{M}^*(C))$, i.e., divisors of (non-zero) meromorphic functions, where $div(f) = \sum_{a\in C} ord_a(f)\cdot a$. This is called the *divisor class group* $Cl(C)$. Since $Cl(C)$ has less structure then J it is easier to think of, and we use it as a bridge between different approaches to J.

7.1.4. *Formulation (2): symmetric powers as a "semilinearization" of a curve.* The disjoint union $\sqcup_{n\geq 0} C^{(n)}$ of all symmetric powers of C is an *abelian Lie semigroup.* The semigroup structure comes from maps $C^{(p)}\times C^{(q)} \to C^{(p+q)}$ which on the level of sets mean that if one adds p unordered points to q unordered points, one now has $p+q$ unordered points. Lie semigroup refers to a manifold structure on $C^{(p)}$'s and the fact that the operation $C^{(p)}\times C^{(q)} \to C^{(p+q)}$ is a map of manifolds.[22]

This turns out to be the abelian Lie semigroup freely generated by C. Then the free abelian Lie group J generated by C, will be the group associated to the semigroup $\sqcup_n C^{(n)}$.[23] Geometrically, this relation is of the free semigroup and free group is given by the *Abel Jacobi maps* $\mathcal{AJ}_n : C^{(n)} \to J_n$. For $g>0$, $\mathcal{AJ}_1 : C^{(1)} \to J_1$ is an embedding, and for sufficiently large n maps $\mathcal{AJ}_n : C^{(n)} \to J_n$ are bundles whose fibers are projective spaces.

7.1.5. *Formulations (4-5): integration.* These we can think of as the down to earth approach to Jacobians (less abstract), however it only makes sense over complex numbers (integration requires manifold over $\mathbb{R}$). We will adopt the approach (4) through integrals, and (5) is just its topological interpretation.

(4) will be a generalization of the idea of integrals of algebraic functions. Integrals of algebraic functions were formulated as integrals on curves associated to algebraic functions. These are curves with a special structure: a map to $\mathbb{P}^1$ which was a branched cover, and the interesting results were obtained only for the elliptic curves. Now we consider all compact complex curves C.

Let g be the genus of C. We will choose a basis $a_1,...,a_g,b_1,...,b_g$ of $\mathcal{P}aths(C)$ and a basis $\omega_1,...,\omega_g$ of $\Omega^1(C)$, the global holomorphic 1-forms on C.

The integrals from $\alpha\in C$ to $\beta\in C$ of 1-forms ω_i should produce a vector

$$\left(\int_\alpha^\beta \omega_1,...,\int_\alpha^\beta \omega_g\right) \in \mathbb{C}^g.$$

However, $\int_\alpha^\beta \omega_i$ is well defined only up to periods $\int_{a_j} \omega_i$, $\int_{b_j} \omega_i$, $1\le j\le g$. The result is that the periods form the *period lattice* $\mathcal{P}eriods$ in $\mathbb{C}^g$, so $(\int_\alpha^\beta \omega_1,...,\int_\alpha^\beta \omega_g)$ is defined as an element of the complex torus

$$J_0 \overset{\text{def}}{=} \mathbb{C}^g/\mathcal{P}eriods,$$

of complex dimension g. So, the connected component J_0 appears as the universal place where integrals have values.

[22]To see that it is a map of algebraic varieties one can describe it as a factorization of the identity map $C^p\times C^q \to C^{p+q}$ to quotients by the group S_{p+q} and its subgroup $S_p\times S_q$:

$$[C^{(p)}\times C^{(q)} \to C^{(p+q)}] = [C^p//S_p\times C^q//S_q \to C^{p+q}//S_{p+q}] = [(C^p\times C^q)//(S_p\times S_q) \to C^{p+q}//S_{p+q}].$$

[23]To any semigroup S one can naturally attach a group G with a map $S\to G$.

7.1.6. *Abel-Jacobi map $C \to J_1$ in terms of integrals.* If we choose a base point $\alpha \in C$ we get the Abel-Jacobi map

$$C \xrightarrow{\mathcal{AJ}} J_0, \quad \beta \mapsto \left(\int_\alpha^\beta \omega_1, ..., \int_\alpha^\beta \omega_g\right) + \mathcal{P}eriods.$$

Theorem. (a) Abel-Jacobi map $\mathcal{AJ} : C \to J_1$ is an embedding for $g > 0$.

(b) $\mathcal{AJ}(C) \subseteq J$ generates group J.

Example: cubics. Recall now that the integrals on a cubic C_λ had values in an elliptic curve E_τ, and this gave an isomorphism $C_\lambda \to E_\tau$. Now we can restate it as:

- *The connected component J_0 of the Jacobian $J = J(C_\lambda)$ of of a cubic C_λ. is an elliptic curve E_τ.*
- *The Abel-Jacobi map $C_\lambda \xrightarrow{\mathcal{AJ}} J_0(C_\lambda)$ is an isomorphism.*

Since in general $\dim(C) = 1$ and $\dim(J(C)) = g$, only a part of this generalizes:

7.1.7. *Formulation (3): line bundles.* The identification of (1) and (3) is the geometric part of *Class Field Theory* which is the central part of *Number Theory*. The content is that the group satisfying (1) really exists and it is the group $Pic(C)$ of all line bundles on C. This is *completely geometric* and works over *any field* and in even *larger generality.*

7.2. **Divisor class group $Cl(C)$: divisors on a curve.** The group $Div(C)$ of divisors on C is the free abelian group with a basis given by all points of C. So, any divisor $D \in Div(C)$ can be written as $D = \sum d_i \cdot \alpha_i$ for some distinct points $\alpha_1, ..., \alpha_p$ of C, and some integers $d_1, ..., d_p$. We sometimes denote $D = \sum_{a \in C} ord_a(D) \cdot D$.

7.2.1. *Principal divisors and degree.* The simplest interesting way to produce a divisor is from a meromorphic function. Let $\mathfrak{M}(C)$ be the field of meromorphic functions on C and $\mathfrak{M}^*(C)$ the multiplicative group of non-zero meromorphic functions. The divisor of $f \in \mathfrak{M}^*(C)$ is

$$div(f) \stackrel{\text{def}}{=} \sum_{a \in C} ord_a(f) \cdot a,$$

such divisor s are called *principal divisors.*

The *degree* of a divisor is defined by $deg(\sum d_i \cdot \alpha_i) \stackrel{\text{def}}{=} \sum d_i \in \mathbb{Z}$.

Lemma. $\mathfrak{M}^*(C) \xrightarrow{div} Div(C) \xrightarrow{degree} \mathbb{Z}$ are maps of abelian groups and the composition is 0.

7.2.2. *Divisor class group* $Cl(C)$. It is defined by $Cl(C) \stackrel{\text{def}}{=} Div(C)/div[\mathfrak{M}^*(C)]$. Since principal divisors live in the subgroup $Div_0(C)$ of degree 0, the degree is well defined on $Cl(C)$ and $0 \to Cl_0(C) \stackrel{\subseteq}{\to} Cl(C) \stackrel{div}{\longrightarrow} \mathbb{Z} \to 0$.

7.2.3. *Effective divisors.* We say that a divisor $D = \sum d_i{\cdot}\alpha_i$ is *effective* if all *multiplicities* d_i are ≥ 0. Notice that the $Div(C)$ contains symmetric powers of C:

Effective divisors of degree n are the same as elements of $C^{(n)}$ i.e., unordered n-tuples of points.[24]

Remark. The group of divisors $Div(C)$ is the *abelian group freely generated by the* **set** C. If we compare this with the formulation (1) of the Jacobian we *expect* that J will be obtained from $Div(C)$ by imposing identifications such that the quotient has a structure of a complex manifold !

7.2.4. *Lemma.* The degree zero part of the divisor class group of $\mathbb{P}^1$ is trivial:

$$Cl_0(\mathbb{P}^1) = 0.$$

Proof. For $a, b \in \mathbb{P}^1$ there is a meromorphic function f such that $\div(f) = a - b$. What works most of the time is $f = \frac{z-a}{z-b}$ (if none of the points is at ∞). The general case reduces to this one using the triply transitive action of $PGL_2(\mathbb{k})$ on $\mathbb{P}^1$. (Also, if $b = \infty$ use $f = z - a$, and if $a = \infty$ use $f = 1/(z - b)$.)

7.3. **Genus.** So far we have only looked into the cubics/elliptic curves and now we will consider all compact complex curves C. The basic difference is visible on the topological level, and it is captured by the *genus* of the curve. Topologically, *genus* is simply the number of pretzel-type holes in C. Another way to say this is that the abelian group $\mathcal{P}aths$ of *closed paths on C up to homology*, is a free group of rank $2g$.

Holomorphically, *genus* is the number of objects on C that one can integrate over paths in C – the global holomorphic 1-forms on C.

7.3.1. *Topological genus (homology).* Connected compact orientable real manifolds of dimension two ("surfaces") are classified by their *genus*. They are all (extended) pretzels and genus is defined as the number of holes. So $g(S^2) = 0$, $g(S^1 \times S^1) = 1$, and $g = 2$ for the surface of the standard pretzel etc.

Any compact connected complex curve C (a compact connected complex one-dimensional manifold) is in particular a compact orientable real manifold of dimension two.[25] This

[24] $C^{(n)}$ is better since it has some algebraic structure and weaker because it sees only the effective divisors.

[25] Orientation is given by multiplication with i.

gives the notion of the topologically, *genus* $g_T(C)$ of C. This is the simplest invariant of C. For example, $g = 0$ for $\mathbb{P}^1$ and $g = 1$ for cubics.

7.3.2. *The homology* $H_1(C,\mathbb{Z})$. Look at at a picture of C – at the i^{th} hole we can choose a circle b_i which bounds the hole, and a transversal circle a_i that connects the hole with the outer boundary of C. We can choose $a_i, b_i,\ 1 \le i \le g$, so that

$$a_i \cap b_j \;=\; \delta_{ij}\cdot \text{pt} \quad \text{and} \qquad \text{for}\ \ i \neq j:\quad a_i \cap a_j \;=\; \emptyset = \ b_i \cap b_j.$$

7.3.3. *Lemma.* Abelian group $\mathcal{P}aths(C) = H_1(C,\mathbb{Z})$ of *closed paths on* C *up to homology*, is a free group of rank $2g$, so we will choose a $\mathbb{Z}$-basis $a_1, ..., a_g., b_1, ..., b_g$.

Remark. Here paths α, β are said to be homologous if (roughly) $\alpha - \beta$ is the boundary of some open part of C. Notice that in complex analysis homology is described differently: α, β are homologous in an open $U{\subseteq}C$ if they wind up the same number of times around each point of the complement $\mathbb{C} - U$. However it amounts to the same thing for $U{\subseteq}\mathbb{C}$ and the first definition is meaningful on curves.

Once we believe this, $\mathcal{P}aths(C) \;=\; H_1(C,\mathbb{Z})$ is clearly the interesting object since in complex analysis path integrals depend on the path *only up to homology.*

7.4. **Holomorphic differential 1-forms.** The (global) differential 1-forms on C are the (global) holomorphic sections of the cotangent bundle $T^*C \to C$ (a line bundle!).

7.4.1. *Example: a non-vanishing 1-form on curves in* $\mathbb{A}^2$. Let $\mathcal{C}{\subseteq}\mathbb{A}^2$ be the curve given by $F =$ for some polynomial $F \in \mathcal{O}(\mathbb{A}^2) = \Bbbk[x,y]$. On $\mathbb{A}^2 = \mathbb{A}^2_{x,y}$, there are many global 1-forms:

$$\Omega^1(\mathbb{A}^2) = \ \mathcal{O}(\mathbb{A}^2){\cdot}dx \oplus \mathcal{O}(\mathbb{A}^2){\cdot}dy.$$

The inclusion $\mathcal{C}\overset{i}{\hookrightarrow}\mathbb{A}^2$ can be used to pull-back (restrict) these 1-forms to $\mathcal{C}_\lambda$.[26] So, on $\mathcal{C}$ we get 1-forms $(d^*i)dx$ and $(d^*i)dy$, which we call simply dx and dy.

The function F is zero on $\mathcal{C}$ (i.e., restrictions of F and 0 on $\mathcal{C}_\lambda$ are the same), so the restriction i^*dF of the 1-form $dF \;=\; F_x{\cdot}dx + F_y{\cdot}dy$ to $\mathcal{C}$ is zero(i.e., equals i^*0). This means that dx and dy on $\mathcal{C}$ satisfy $F_x{\cdot}dx \;=\; -F_y{\cdot}dy$ (the precise meaning is that $(d^*i)dx$ and $(d^*i)dy$ satisfy this equation). We use this to define a 1-form on $\mathcal{C}$

$$\omega \overset{\text{def}}{=} \frac{dx}{F_x} \;=\; -\frac{dy}{F_y}.$$

[26]Any map of manifolds $f : X \to Y$ has the differential df which can be viewed as a family of linear maps $d_af : T_a(X) \to T_{f(a)}(Y),\ a \in X$. It gives a pull-back operation on 1-forms $d^*f : \Omega^1(Y) \to \Omega^1(X)$, the value of the pull-back $(d^*f)\omega$ at $a \in X$ is

$$[(d^*f)\omega]_a \overset{\text{def}}{=} [T_a(X) \xrightarrow{d_af} T_{f(a)}(Y) \xrightarrow{\omega_a} \mathbb{C}] \;=\; (d_af)^*(\omega_a).$$

Lemma. ω is well defined and does not vanish wherever $\mathcal{C}$ is a submanifold of $\mathbb{A}^2$.

Proof. Since $\frac{dx}{F_x} = -\frac{dy}{F_y}$ the only problem can appear at points $(a,b) \in \mathcal{C}$ such that $F_x(a,b) = F_y(a,b)$, however these are precisely the points where $\mathcal{C}$ fails to be a submanifold.

The values of dx and dy at any point $p = (a,b)$ of $\mathbb{A}^2$ give a basis of $T_p^*\mathbb{A}^2$. If $p \in \mathcal{C}$ and $\mathcal{C}$ is a submanifold at p then $d_p i : T_p\mathcal{C} \to T_p\mathbb{A}^2$ is injective. So, its adjoint $d_p^* i : T_p^*\mathbb{A}^2 \to T_p^*\mathcal{C}$ is surjective. Therefore, at p one of $(d^*i)dx$ and $(d^*i)dy$ is non-zero. This means that by looking at the version of the definition of ω which is appropriate at p we find that $\omega_p \neq 0$.

7.4.2. *Example: 1-form ω_λ on a cubic C_λ.* For $\lambda \neq 0,1$ cubic $\mathcal{C}_\lambda$ is a submanifold of $\mathbb{A}^2$, defined by the function $F(x,y) = x^3 - x^2(1+\lambda) + x\cdot\lambda - y^2$, so the restriction of $dF = (3x^2 - 2(1+\lambda)x + \lambda)dx - 2y\,dy$ to $\mathcal{C}_\lambda$ is zero, hence

$$\textit{1-forms } dx \textit{ and } dy \textit{ on } \mathcal{C}_\lambda \textit{ satisfy} \quad (3x^2 - 2(1+\lambda)x + \lambda)\cdot dx = 2y\cdot dy.$$

We use this to define a 1-form on C_λ

$$\omega_\lambda \stackrel{\text{def}}{=} \frac{2\,dy}{3x^2 - 2(1+\lambda)x + \lambda} = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

Corollary. ω_λ is well defined on C_λ and it has no zeros (nor poles).

Proof. Since $\mathcal{C}_\lambda \subseteq \mathbb{A}^2$ is a submanifold for $\lambda \neq 0,1$, lemma shows that ω_λ is well defined on $\mathcal{C}_\lambda$ and does not vanish on $\mathcal{C}_\lambda$. It remains to check the coordinates at the infinite point of C_λ.

Remark. A non-vanishing section of the cotangent line bundle T^*C_λ over C_λ can be used to trivialize this line bundle – it gives an isomorphism $C_\lambda \times \Bbbk \xrightarrow{\cong} T^*(C_\lambda)$ by $(p,c) \mapsto c\cdot\omega_\lambda(p)$. So for cubics (equivalently for elliptic curves), the cotangent line bundle is trivial.

This is not surprising if we remember that cubics have a group structure and on any group G any *natural* vector bundle V (such as the (co)tangent vector bundles) can be trivialized by $G \times V_1 \xrightarrow{\cong} V$, $(g,v) \mapsto g\cdot v$ (here left multiplication $L_g : G \to G$ lifts to an action on V which I denote $g : V_x \to V_{gx}$).

7.4.3. *Holomorphic genus.* We say that the dimension of the vector space $\Omega^1(C)$ of differential 1-forms on C is the *holomorphic genus* $g_H(C)$ of C.

Theorem. Holomorphic genus and topological genus are the same.

7.4.4. *Strategy of the proof.* The theorem relates some topological data (the genus) and holomorphic data (sections of T^*C). The standard way to do this is:

(1) Express topology through real analysis:

Extend the coefficients to real numbers: $H_1(C,\mathbb{R}) \overset{def}{=} H_1(C,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}$ *and consider the dual vector space* $H^1(C,\mathbb{R}) \overset{def}{=} H^1(C,\mathbb{R})^*$*. Then one has*

- *(De Rham theorem) This space* $H^1(C,\mathbb{R})$ *can be calculated in terms of the smooth differential forms on* C *considered as a 2-dimensional real manifold.*

(2) Relate real analysis and complex analysis – find out which part of the real analysis data is captured by the complex analysis data.

Extend the coefficients to complex numbers: $H^1(C,\mathbb{C}) \overset{def}{=} H^1(C,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$*. Then*

- *(Hodge theorem)* $H^1(C,\mathbb{C})$ *decomposes canonically into two complex vector spaces of the same dimension*
$$H^1(C,\mathbb{C}) = H^{1,0}(C)\oplus H^{0,1}(C)$$
with
$$H^{1,0}(C) \cong \Omega^1(C) \quad \textit{and} \quad H^{0,1}(C) \cong \Omega^1(\bar{C}).$$
So, $H^{1,0}(C)$ *is the contribution of holomorphic analysis to real analysis. By* $\bar{C}$ *I mean the manifold* C *with the opposite complex structure.*

This is the background we need for the theorem: Now everything is in place:
$$\dim_{\mathbb{C}}[H^1(C,\mathbb{C})] = \dim_{\mathbb{R}}[H^1(C,\mathbb{R})] = \dim_{\mathbb{R}}[H_1(C,\mathbb{R})] = \dim_{\mathbb{Z}}[H_1(C,\mathbb{Z})] = 2g.$$
On the other hand, $\dim_{\mathbb{C}}[H^{1,0}(C)] = \dim_{\mathbb{C}}[H^{0,1}(\bar{C})]$ (you can guess this since $\bar{C}$ should behave somewhat as $\mathbb{C}$: one passes from C to $\bar{C}$ by conjugating all complex numbers in sight). Therefore,
$$\dim_{\mathbb{C}}[\Omega^1(C)] = \dim_{\mathbb{C}}[H^{1,0}(C)] = \frac{1}{2}\dim_{\mathbb{C}}[H^1(C,\mathbb{C})] = g.$$

7.4.5. *Examples.* (1) Curves with $g = 0$ will all turn out to be isomorphic to $\mathbb{P}^1$ and we have $\Omega^1(\mathbb{P}^1) = 0$.

(2) T^*C is trivial precisely for $g = 1$. In the case $g = 1$ we already noticed the triviality. If $T^*C \cong C\times\Bbbk$ then $\Omega^1(C) \cong \mathcal{O}(C)$ and on a compact curve $\mathcal{O}(C) =$ constants. So, $g = \dim(\Omega^1(C)) = 1$.

7.4.6. *Integration of holomorphic 1-forms over paths in a curve.* Let us reconsider the integration of algebraic functions in the setting of a complex curve C.

For that we need a path γ in C and a differential form ω on $\mathbb{C}$. Here, $\gamma : [0,1] \to C$ and ω is a global differential 1-form on C, i.e., a global holomorphic sections of the cotangent bundle $T^*C \to C$ (a line bundle!). What this means is that ω assigns to each $c \in C$ a cotangent vector $\omega(c) \in T^*_c(C)$ at c, and the differential of γ gives tangent vectors

$\gamma'(t) = (d_t\gamma)\frac{\partial}{\partial t} \in T_{\gamma(t)}(C)$ at $\gamma(t)$, and finally these two kinds of vectors contract to numbers which we integrate over $[0,1]$

$$\int_\gamma \omega \stackrel{\text{def}}{=} \int_0^1 dt\ \langle \omega(\gamma(t)), \gamma'(t)\rangle.$$

So the point is that on manifolds one can not quite integrate functions but only the differential forms.

Remark. However, you may remember that we have already considered integrals $\int_\gamma y(x)\ dx$ of a *function* $y(x)$ on a curve $\mathcal{C}$. We were able to do this when $\mathcal{C}$ happened to be a branched cover $\mathcal{C} \xrightarrow{\pi} \mathbb{C}$ of $\mathbb{C}$. The point is really that to make sense of this integral we appealed to the possibility of calculating it on the image of γ in $\mathbb{C}$. So we used the relation to $\mathbb{C}$. In terms of the integration of differential 1-forms on C this means that in the background, without mentioning, we really used the differential form π^*dx on $\mathcal{C}$ which was the pull-back of dx on $\mathbb{C}$! So we have really been integrating the 1-form $y(x){\cdot}\pi^*dx$ on C. It turns out that this is a global 1-form on C_λ:

7.5. **The connected Jacobian $J_0(C)$: integration of 1-forms.** Let C be a complete complex curve of genus g.

We will adopt the approach through integrals. (It is close to the topological interpretation!) This will be a generalization of the idea of integrals of algebraic functions (on curves associated to algebraic functions). However, we will consider all compact complex curves C while so far we only looked into the elliptic curves.

7.5.1. *Topological and holomorphic data.* The basic information about a curve is its *genus.* Topologically, this is simply the number of pretzel-type holes in C. Another way to say this is that the abelian group $\mathcal{P}aths(C) = H_1(C,\mathbb{Z})$ of *closed paths on C up to homology,* is a free group of rank $2g$, with a $\mathbb{Z}$-basis give by circles $a_1, ..., a_g., b_1, ..., b_g$.

Holomorphically, *genus* is the number of objects on C that one can integrate over paths in C – the global holomorphic 1-forms on C. So, we choose a basis $\omega_1, ..., \omega_g$ and points $\alpha, \beta \in C$.

7.5.2. *Integrals between two points in C.* For any $\alpha, \beta \in C$, the integrals would like to produce a vector

$$\int_\alpha^\beta (\omega_1, ..., \omega_g) = (\int_\alpha^\beta \omega_1, ..., \int_\alpha^\beta \omega_g) \in \mathbb{C}^g.$$

However, this depends on a choice of a path from α to β, and any two paths differ by a closed path. Integrals depend on a closed path only up to homology, so all ambiguity in $\int_\alpha^\beta$ will be contained in the image of the *period map*

$$P : H_1(C,\mathbb{Z}) \ni [\gamma] \mapsto \int_\gamma (\omega^1, ..., \omega^g) \in \mathbb{C}^g.$$

This is a subgroup

$$\mathcal{P}eriods \stackrel{\text{def}}{=} P(H_1(C,\mathbb{Z})) = \sum_1^g \mathbb{Z}P_{a_i} + \mathbb{Z}P_{b_i} \subseteq \mathbb{C}^g.$$

So, each circle a_i produces ambiguity in the vector $(\int_\alpha^\beta (\omega_1, ..., \omega_g) \in \mathbb{C}^g$, the ambiguity is given by its *period vector*

$$P_{a_i} = \int_{a_i} (\omega_1, ..., \omega_g) \in \mathbb{C}^g.$$

Now $(\int_\alpha^\beta (\omega^1, ..., \omega^g)$ is defined as an element of the quotient group

$$\mathbb{C}^g/\mathcal{P}eriods.$$

7.5.3. *Complex torus J_0.*

Theorem. The subgroup of periods, $\mathcal{P}eriods \subseteq \mathbb{C}^g$ is a lattice:

$$\mathcal{P}eriods = \oplus_i \ \mathbb{Z}{\cdot}P_{a_i} \oplus \mathbb{Z}{\cdot}P_{b_i} \subseteq \mathbb{C}^g;$$

i.e., the periods $P_{a_1}, ..., P_{a_g}, P_{b_1}, ..., P_{b_g}$ are $\mathbb{R}$-independent.

Now we define the connected component of the Jacobian by

$$J_0 \stackrel{\text{def}}{=} \mathbb{C}^g/\mathcal{P}eriods$$

and this is a complex torus of dimension g. This is the universal target of integrals on C (a place where integrals take values).

7.5.4. *Example: cubics.* Now, if we remember that we have studied the integrals on a cubic C_λ with values in an elliptic curve E_τ, then from this general point of view E_τ was the (connected component J_0 of the Jacobian $J = J(C_\lambda)$ of C_λ. The isomorphism $C_\lambda \stackrel{\cong}{\to} E_\tau$ that we found using the $\mathfrak{p}$-function, is a special property of elliptic curves (in general $\dim(C) = 1$ and $\dim(J(C)) = g$)!

7.6. **Comparison of $J_0(C)$ and $Cl(C)$ (periods and divisors).** The integration construction can be restated in terms of divisors of degree 0. Any divisor $D = \sum D - i{\cdot}\alpha_i$ of degree zero, can be organized as $D = \sum p_j - q_j$ for some $p_j, q_j \in C$. To this we can attach an element of the Jacobian $J_0(C) = \mathbb{C}^g/\mathcal{P}eriods$, the sum of integrals

$$Int(D) \stackrel{\text{def}}{=} \sum_j \int_{p_j}^{q_j} (\omega^1, ..., \omega^g) \in J_0(C).$$

7.6.1. *Theorem.* (a) The map $Div_0(C) \xrightarrow{\int} J_0(C) = \mathbb{C}^g/\mathcal{P}eriods$ is well defined, it is a map of groups and it is surjective.

(b) The kernel is the subgroup of principal divisors (divisors of all meromorphic functions).

Proof. We just indicate the easy steps which amount to existence of interesting maps.

In (a), "well defined" means that $\sum_j (\int_{p_j}^{q_j} (\omega^1, ..., \omega^g)$ does not change if we regroup p's and q's. So, we need $\int_A^a + \int_B^b = \int_A^b + \int_B^a$, but this is equivalent to $\int_A^a - \int_B^a = \int_A^b - \int_B^b$, and here both sides are $\int_A^B$. It is clear that *int* is a map of groups:

$$\int(\sum p_i - q_i) + \int(\sum u_j - v_i) = \sum \int_{q_i}^{p_i} + \sum \int_{v_j}^{u_j} = \int(\sum p_i - q_i + \sum u_j - v_i).$$

In (b) we check that divisors of meromorphic functions give zero integrals. This is exciting, why would a specific integral be zero? For a non-zero meromorphic function f we want $\int(div(f)) = 0$. It is certainly true if $f = 1$. It follows that it is true for any f by the following *moving sublemma*:

Let f, g be meromorphic functions which are $\mathbb{C}$-independent. To $(\lambda, \mu) \in \mathbb{C}^2 - 0$ we associate a non-zero meromorphic function $\lambda f + \mu g$, its divisor $div(\lambda f + \mu g)$ and its image in the Jacobian $\phi(\lambda, \mu) \in \int div(\lambda f + \mu g) \in J_0(C) = \mathbb{C}^g/L$ for the period lattice L. Then ϕ factors to a map Φ from $\mathbb{P}^1$ to $\mathbb{C}^g/L$. Moreover, this lifts to a map $\widetilde{\Phi} : \mathbb{P}^1 \to \mathbb{C}^g$ by the lifting sublemma 6.5.2. Now, $\widetilde{\Phi}$ which must be constant, so $\phi(\lambda, g)$ is constant. In particular, $\int(div(f)) = \phi(1,0) = \phi(0,1) = \int(div(g))$.

7.6.2. *Corollary.* As a group, the connected component $J_0(C)$ of the Jacobian is the degree zero part $Cl_0(C)$ of the divisor class group.

7.7. **Picard group $Pic(C)$: line bundles and invertible sheaves.** Let C be a complex curve (i.e., a complex manifold of dimension one). We want to construct the complex Lie group $J(C)$ which is freely generate by C. We start with the same idea on the level of sets. The group $Div(C)$ of divisors on C is the abelian group freely generated by the set C. If $J(C)$ exists, as a group it has to be a quotient of $Div(C)$, but the question of finding a quotient of $Div(C)$ with a manifold structure is a priori mysterious. However there is a natural quotient of $Div(C)$ – the Picard group $Pic(C)$. This is the group of line bundles on C (for tensoring). For flexibility we observe that line bundles can be viewed as certain kinds of sheaves, the *invertible sheaves.* We use this point of view to attach to each divisor a line bundle.

In the end, $Pic(C)$ turns out to be one of incarnations of $J(C)$.

7.7.1. *Vector bundles are the same as locally free sheaves.* Any holomorphic vector bundle V over C gives a sheaf $\mathcal{V}$ on C – the sheaf of sections of L:

$$\mathcal{V}(U) \overset{def}{=} \Gamma(U,V) \overset{def}{=} \textit{ holomorphic section of } V|U, \textit{ the restriction of } V \textit{ to } U \subseteq C.$$

We will see that $\mathcal{V}$ is a locally free sheaf of rank n on C, i.e.,

- $\mathcal{V}$ is a module for the algebra sheaf $\mathcal{O}_C$ of holomorphic functions on C.[27]
- Locally, $\mathcal{O}_C$-module $\mathcal{V}$ is isomorphic to $\mathcal{O}_C^n$.[28]

$\mathcal{L}$ is A locally free sheaf of rank one is called an *invertible sheaf.*

7.7.2. *Proposition.* Construction $V \mapsto \mathcal{V}$ gives a bijection of isomorphism classes of

- vector bundles of rank n on C, and
- locally free sheaves of rank n on C.

In particular we get a bijection of line bundles and invertible sheaves on C.

Proof. **(A)** *$\mathcal{V}$ locally free of rank n.* Let $\mathcal{V}$ be the sheaf of holomorphic sections of a vector bundle V of rank n. Vector bundles are locally trivial, so there is an open cover $\mathcal{U} = (U_i)_{i\in I}$ of C and there are isomorphisms $U_i \times \mathbb{C}^n \xrightarrow{\phi_i} V|U_i$ which preserve fibers and on each fiber ϕ_i's are invertible linear operators. Each ϕ_i identifies the restriction $\mathcal{V}|U_i$ with the sheaf of section of the trivial vector bundle $U_i \times \mathbb{C}^n \to U_i$. But this is the sheaf of functions on U_i with values in $\mathbb{C}^n$, so sections are the same as n-tuples of functions on U_i. So, $\mathcal{V}|U_i \cong \mathcal{O}_{U_i}^n = \mathcal{O}_C^n|U_i$.

We constructed an identification of sheaves of abelian groups. It is clear how one would like to make $\mathcal{V}$ into an $\mathcal{O}_C$-module: $f \in \mathcal{O})C(U)$ should multiply $v \in \mathcal{V}(U)$ pointwise, i.e., $(f \cdot v)(x) = f(x) \cdot v(x) \in V_x$ for $x \in U$. The $f \cdot v$ is clearly a section of V. We need it to be holomorphic and this is checked in local coordinates where it becomes the obvious action of $\mathcal{O}_C$ on $\mathcal{O}_C^n$.

(B) *$V \mapsto \mathcal{V}$ is a bijection.* There is an explicit inverse construction: to a locally free $\mathcal{V}$ one associated vector bundle V which is the spectrum of $S(\mathcal{V}^*)$, the symmetric algebra of the dual sheaf $\mathcal{V}^* \overset{\text{def}}{=} \mathcal{H}\text{om}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{O}_C)$. However we can also argue on a more elementary level, by checking that both objects can be encoded by the same kind of combinatorial data.

(D) *Transition functions for V.* Local triviality of a vector bundle V implies that there is an open cover $\mathcal{U} = (U_i)_{i\in I}$ of C and fiberwise linear isomorphisms $U_i \times \mathbb{C}^n \xrightarrow{\phi_i} V|U_i$. On U_{ij} we get a fiberwise linear automorphism $\phi_{ji} = \phi_j \circ \phi_i{}^{-1}$ of $U_{ij} \times \mathbb{C}^n$, this means that ϕ_{ij} is a holomorphic function $U_{ij} \to GL_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$. Notice that the data ϕ_{ij}, $(i,j) \in I^2$, are sufficient for reconstructing V – one recovers V by gluing trivial vector bundles $U_i \times \mathbb{C}^n$ using identifications ϕ_{ij} over U_{ij}, i.e.,

$$V \cong [\sqcup_i U_i \times \mathbb{C}^n] / \sim$$

[27]This means that each $\mathcal{V}(U)$ is a module for the algebra $\mathcal{O}_C(U)$, and that the actions $\mathcal{O}_C(V) \times \mathcal{V}(U) \to \mathcal{V}(U)$ are compatible with restrictions.

[28]The sum of two sheaves of abelian groups is defined by $(\mathcal{A} \oplus \mathcal{B})(U) \overset{\text{def}}{=} \mathcal{A}(U) \oplus \mathcal{B}(U)$.

for the equivalence relation: $(a,v)\in U_i{\times}\mathbb{C}^n$ and $(b,w)\in U_j{\times}\mathbb{C}^n$ are equivalent if $b=a$ and $w=\phi_{ji}(a){\cdot}v$.

(D) *Transition functions for $\mathcal{V}$.* This works the same. Local triviality of a locally free sheaf $\mathcal{V}$ of rank n implies that there is an open cover $\mathcal{U}=(U_i)_{i\in I}$ of C and isomorphisms of sheaves of $\mathcal{O}_C$-modules $\mathcal{O}_{U_i}^n \xrightarrow{\Phi_i} \mathcal{V}|U_i$. On U_{ij} we get a an automorphism $\Phi_{ji}=\Phi_j{\circ}\Phi_i{}^{-1}$ of the $\mathcal{O}_{U_{ij}}$-module $\mathcal{O}_{U_{ij}}^n$. This (again) means that Φ_{ij} is a holomorphic function $U_{ij}\to GL_n(\mathbb{C}){\subseteq}M_n(\mathbb{C})$.[29] The data ϕ_{ij}, $(i,j)\in I^2$, are (again) sufficient for reconstructing $\mathcal{V}$ by gluing $\mathcal{O}_{U_i}$-modules $\mathcal{O}_{U_i}^n$ using identifications Φ_{ij} over U_{ij}.

(E) *Conclusion.* Both kind of objects are captured by the same kind of transition functions data. It remains to notice that the passage $V{\mapsto}\mathcal{V}$ does not affect the data.

7.7.3. *Corollary.* Line bundles are the same as locally free sheaves of rank one invertible sheaves.

Proof. An invertible sheaf just means a locally free sheaf of rank one.

7.7.4. *Tensoring of vector bundles and of locally free sheaves.*

Lemma. (a) One can tensor (local) sections of vector bundles, i.e., for $W{\subseteq}C$ open, any section $\alpha\in\Gamma(W,U)$ and $\beta\in\Gamma(W,V)$ define a section $\alpha{\otimes}\beta\in\Gamma(W,U{\otimes}V)$ by

$$(\alpha{\otimes}\beta)(c) \overset{\text{def}}{=} \alpha(c){\otimes}\beta(c)\in\ U_c{\otimes}V_c=(U{\otimes}V)_c,\quad c\in W.$$

(b) Under the above correspondence, the tensoring of vector bundles $U{\otimes}V$ corresponds to tensoring of invertible sheaves viewed as $\mathcal{O}_C$-modules: $\mathcal{U}{\otimes}_{\mathcal{O}_C}\mathcal{V}$.

(c) One can invert the non-vanishing sections of line bundles: If L is a line bundle and $s\in\Gamma(W,L)$ does not vanish at any point $a\in W$ then there is a section $s^{-1}\in\Gamma(W,L^*)$ such that $\langle s,s^{-1}\rangle=1$ on W for the pairing of sections of dual vector bundles into functions.

Proof. (a) is clear. It gives for each open $W{\subseteq}C$ a map $\Gamma(W,U){\times}\Gamma(W,V)\to\Gamma(W,U{\otimes}V)$ which is $\mathcal{O}_C(W)$-bilinear, i.e., for $f\in\mathcal{O}(C(W)$, $(f\alpha){\otimes}\beta=\alpha{\otimes}(f\beta)$. So, it gives a map $\mathcal{U}(W){\otimes}_{\mathcal{O}_{C(W)}}\mathcal{V}(W)\to\Gamma(W,U{\otimes}V)$.

Now, the tensor product of sheaves $\mathcal{U}$ and $\mathcal{V}$ over $\mathcal{O}_C$ is essentially obtained by associating to each open W the group $(\mathcal{U}{\otimes}_{\mathcal{O}_C}\mathcal{V})(W)\overset{\text{def}}{=}\mathcal{U}(W){\otimes}_{\mathcal{O}_{C(W)}}\mathcal{V}(W)$, so we have constructed maps $(\mathcal{U}{\otimes}_{\mathcal{O}_C}\mathcal{V})(W)\to\Gamma(W,U{\otimes}V)$. These are clearly compatible with restrictions so we have a map of sheaves $\mathcal{U}{\otimes}_{\mathcal{O}_C}\mathcal{V}\xrightarrow{\mu} U{\otimes}V$. Finally, local trivializations of U and Vgive isomorphisms $\mathcal{U}\cong\oplus\ \mathcal{O}_C{\cdot}e_i$, $\mathcal{V}\cong\oplus\ \mathcal{O}_C{\cdot}f_j$ hence $\mathcal{U}{\otimes}\mathcal{V}\cong\oplus\ \mathcal{O}_C{\cdot}(e_i{\otimes}f_j)$, and then μ is clearly an isomorphism.

[29] Canonical basis e_i of $\mathbb{C}^n$ gives an $\mathcal{O}_{U_{ij}}$-basis E_p of $\mathcal{O}_{U_{ij}}^n$. So, $\Phi_{ij}E_p=\sum_q c_{ij}^{pq}E_q$ for some $c_{ij}^{pq}\in\mathcal{O}(U_{ij})$ which form a matrix function $U_{ij}\to M_n(\mathbb{C})$, and the values are actually in $GL_n(\mathbb{C})$.

(c) The pairing of dual line bundles gives isomorphism $\langle -,-\rangle : L\otimes L^* \to\ Y = C\times\mathbb{C}$, so define $(s^{-1})(p) \in (L^*)_p = (L_p)^*$ so that $\langle s(p), s^{-1}(p)\rangle = 1$.

7.7.5. *Meromorphic sections of line bundles.* A meromorphic section of a line bundle L over an open $U\subseteq C$ means a holomorphic section s on $U-\mathcal{P}$ for some discrete subset $\mathcal{P}$ (the set of *possible poles*), such that at each $a\in\mathcal{P}$, when we use a local trivialization of L near a, the function corresponding to s is meromorphic at a, i.e., it has at most a pole at a. We can be more precise and define the order of the section s at a as

$$ord_a^L(s) \stackrel{\text{def}}{=} ord_a(f)$$

when s corresponds to a function f holomorphic off a, in terms of *some* local trivialization of L near a.

We will denote the vector space of meromorphic sections of L on U by $\mathfrak{M}(U,L)$ and the global meromorphic sections by $\mathfrak{M}(L) \stackrel{\text{def}}{=} \mathfrak{M}(C,L)$. Finally, $\mathfrak{M}(L) \stackrel{\text{def}}{=} \mathfrak{M}(L)-\{0\}$ are the non-zero meromorphic sections.

Corollary. (a) One can tensor (local) meromorphic sections of line bundles, i.e., for $U\subseteq C$ open, any meromorphic sections $\mathfrak{M}(U,L)\otimes_{\mathfrak{M}(U)}\mathfrak{M}(U,M) \to\ \mathfrak{M}(U,L\otimes M)$. The zeros and poles add up as usual: $ord_a^{L\otimes M}(\alpha\otimes\beta) =\ ord_a^L(\alpha) + ord_a^M(\beta)$.

(b) One can invert the non-zero meromorphic sections of line bundles: $\mathfrak{M}^*(U,L)\ni s\mapsto s^{-1}\in\mathfrak{M}^*(U,L^*)$ and $ord_a^{L^*}(s^{-1}) = -ord_a^L(s)$.

Proof. (a) This really means that for the sets $\mathcal{P}_\alpha,\mathcal{P}_\beta\subseteq W$ of poles of $\alpha\in\mathfrak{m}(U,L)$, $\beta\in\mathfrak{m}(U,M)$ we are tensoring holomorphic sections over $U-(\mathcal{P}_\alpha\cup\mathcal{P}_\beta)$. The formula for the order: in terms of local; trivializations of line bundles, this is just the multiplication of meromorphic functions.

7.7.6. *Picard group $Pic(C)$.* Let $\mathfrak{P}ic(C)$ be the set of isomorphism classes of line bundles (invertible sheaves) on C. Let $Pic(C)\subseteq\mathfrak{P}ic(C)$ be the subset of all line bundles with a meromorphic global section.

Lemma. (a) The tensoring of line bundle makes $\mathfrak{P}ic(C)$ into a group. The trivial line bundle $T = C\times\mathbb{C}$ is the neutral element and the inverse of L is the dual line bundle L^*.

(b) $Pic(C)$ is a subgroup.

Proof. (a) Tensor product of line bundles is associative and produces again a line bundle. The trivial line bundle $T = C\times\mathbb{C}$ is clearly a neutral element. Finally, for any line bundle L, the dual vector bundle $L^* = \mathrm{Hom}(L,T)$ is again a line bundle and the canonical pairing $L\otimes L^*\to T$ is clearly an isomorphism.

For (b) we recall that one can tensor and invert meromorphic sections (and T has a meromorphic section 1).

7.8. **GAGA: Geometrie Algebrique and Geometrie Analytic (comparison).** We will indicate that the distinction between $\mathfrak{Pic}$ and Pic does not really appear in algebraic geometry (corollary bellow), and though it does exist in holomorphic geometry there is again no difference for curves (proposition bellow).

So far we have been using both algebraic and holomorphic point of view for algebraic varieties over $\mathbb{C}$. Fortunately, the two pictures are often the same by Serre's comparison of Algebraic Geometry and Analytic Geometry:

7.8.1. *Theorem.* (Serre) If a complex manifold X has a structure of a projective variety, then

- (a) Any holomorphic vector bundle V has a structure of an algebraic vector bundle over X, and
- (b) Global holomorphic sections of V are the same as the global algebraic sections.

7.8.2. *Remarks.* (1) With a little care, the two statements can be combined into a single categorical claim: the operation $U\mapsto U^{an}$ that associates to each algebraic vector bundle U on X "the same" vector bundle but now viewed as a holomorphic vector bundle, is an equivalence of categories.

(2) Claim (b) is quite striking. It is clearly wrong if $X = \mathbb{A}^1$ and $V = X\times\mathbb{C}$ is the trivial line bundle, since there are many more functions in complex geometry (all entire functions on $\mathbb{C}$) than in algebraic geometry (polynomials $\mathbb{C}[x]$). However, if we replace $\mathbb{A}^1$ by a little larger *projective* object $\mathbb{P}^1$ we know that all global holomorphic functions are constant, so they are clearly of algebraic nature. (It helped that we filled in the hole at ∞ of $\mathbb{A}^1$ where holomorphic functions had more freedom then the polynomial ones).

Similarly, on any X, *locally* there are many more functions or sections in complex geometry than in algebraic geometry polynomials). However, if one asks which of these local sections extend to *global* sections, i.e., which ones make sense on all of X, then only the ones of algebraic nature have a chance.

7.8.3. *Corollary.* On a projective X any holomorphic line bundle has a global meromorphic section, i.e.,

Proof. Any algebraic line bundle has a global meromorphic section. Let us check this if X is a curve. Then there is a Zariski open cover U_i, $i\in I$, on which V trivializes: $V|U_i\overset{\cong}{\leftarrow}U_i\times\mathbb{C}^n$. Choose an $i\in I$, then any $0\neq v\in\mathbb{C}^n$ gives a section $s\neq 0$ of V on U_i. Since $X-U_i$ is finite, s has only isolated singularities off U_i. To see how bad these are we need to view s in charts U_j for $j\neq i$. In such chart s is given by $f_{ji}v$ for a transition function f_{ji}. Since we are in algebraic geometry, f_{ji} (locally a restriction of a polynomial) has no essential singularities, hence neither does s.

7.8.4. *Proposition.* Any compact holomorphic curve C has a projective structure, so $Pic(C) = \mathfrak{P}ic(C)$.

Proof. We will postpone it, the idea (*Kodaira embedding* is that if line bundle L over a compact complex manifold X has "sufficiently many global sections" then it gives an explicit embedding of X into a projective space: $X \hookrightarrow \mathbb{P}[\Gamma(X, L)^*]$.

The point is that on a curve one has many effective divisors D and for sufficiently large D we will see that $\mathcal{O}_C(D)$ has sufficiently many sections!

7.8.5. *Remark.* One can check that $\mathfrak{P}ic(C) = Pic(C)$ for curves without using the GAGA theorem – we really used this problem as an excuse to introduce GAGA.

7.9. Comparison of $Pic(C)$ and $Cl(C)$ (line bundles and divisors).

7.9.1. *Divisors give invertible sheaves.* We can use a divisor $D \in Div(C)$ to modify the sheaf $\mathcal{O}_C^{an}$ of holomorphic (=analytic) functions on C. For any open $U \subseteq C$ we define

$$\mathcal{O}_C(D)(U) \stackrel{\text{def}}{=} \{f \in \mathfrak{M}(U);\ ord_a(f) \geq -ord_a(D),\ a \in U\}.$$

Here, if ϕ is a function holomorphic on some open $V \subseteq C$ and $\alpha \in C$ is an isolated singularity of ϕ (in the sense that V contains some punctured neighborhood of α), we define the order of ϕ at α, $ord_\alpha(\phi) \in \mathbb{Z}$, by using a local chart on C near α.

At points $a \in U$ which are not in the support of D the condition is $ord_a(f) \geq 0$, i.e., we ask that f is holomorphic on $U - supp(D)$. If $ord_a(D) < 0$ we impose on f the vanishing of order $|ord_a(D)|$ at a, and if $ord_a(D) > 0$ we allow a pole of order $ord_a(D)$ at a. So, for instance, $\mathcal{O}_C(D) \subseteq \mathcal{O}_C$ iff $-D$ is effective, and $\mathcal{O}_C(D) \supseteq \mathcal{O}_C$ iff D is effective.

Lemma. (a) For any divisor D on a complex curve C, $\mathcal{O}_C(D)$ is an invertible sheaf on C.

(b) $Div(C) \ni D \mapsto \mathcal{O}_C(D) \in \mathfrak{P}ic(C)$ is a map of groups.

(c) A holomorphic section $f \in \mathcal{O}_C(D)(U)$ is by definition a meromorphic function on U. These two points of view give two notions of the divisor of f :

$$ord_a^{\mathcal{O}_C(D)}(f) \ = \ ord_a(f) + ord_a(D),\ a \in U, \quad i.e., \quad div^{\mathcal{O}_C(D)}(f) \ = \ div(f) + D|U.$$

(d) Divisor D gives a trivial line bundle iff D is a divisor of a meromorphic function (i.e., a principal divisor). So,

$$Pic(C) \cong Div(C)/div(\mathfrak{M}^*(C)) = \ Cl(C).$$

Proof. (a) $\mathcal{O}_C(D)$ is a sheaf because the defining conditions are checked locally. Any point $a \in C$ lies in some chart, i.e., a lies in an open $U \subseteq C$ such that on U there is a holomorphic identification $z : U \to \mathbb{C}$. We can choose U small enough so that $U - \{a\}$ does not meet D. Then on U one has $\mathcal{O}_C(D) = \ (z - z(a))^{-ord_a(D)} \cdot \mathcal{O}_C$, hence on U we have a is an isomorphism of $\mathcal{O}_C$-modules, i.e., $\mathcal{O}_C \xrightarrow{(z-z(a))^{-ord_a(D)}} \mathcal{O}_C(D)$, a trivialization of our invertible sheaf $\mathcal{O}_C(D)$.

(b) We want an isomorphism $\mathcal{O}_C(D')\otimes_{\mathcal{O}_C}\mathcal{O}_C(D'') \xrightarrow{\iota} \mathcal{O}_C(D'+D'')$. For any open U the multiplication of meromorphic functions gives a map $\mathcal{O}_C(D')(U)\times\mathcal{O}_C(D'')(U) \xrightarrow{\iota} \mathcal{O}_C(D'+D'')(U)$, since $ord(f'f'') = ord(f') + ord(f'')$.

(c) If we view f as a section of $\mathcal{O}_C(D)$ we calculate $ord_a^{\mathcal{O}_C(D)}(f)$ using the local trivialization near a: $(z-z(a))^{-ord_a(D)} : \mathcal{O}(C)\xrightarrow{\cong}\mathcal{O}_C(D)$ from (a). In terms of this trivialization section f of $\mathcal{O}_C(D)$ corresponds to a function $\frac{f}{(z-z(a))^{-ord_a(D)}}$, hence

$$ord_a^{\mathcal{O}_C(D)}(f) = ord_a\left(\frac{f}{(z-z(a))^{-ord_a(D)}}\right) = ord_a(D) + ord_a(f).$$

(d) $\mathcal{O}_C(D)$ is trivial iff it has a non-vanishing global section f. This means iff there is a meromorphic function $f \in \mathfrak{M}(C)$ such that that at a each point $a \in C$, $ord_a(f) \geq -ord_a(f)$ (so that f is a section of $\mathcal{O}_C(D)$), and $ord_a(f) + ord_a(f) = ord_a^{\mathcal{O}(D)}(f) \leq 0$ (so that f does not vanish at a as a section of $\mathcal{O}_C(D)$). This is equivalent to $D = -ord(f)$ for some $f \in \mathfrak{M}^*(C)$, i.e., to D being principal.

Corollary. On $Pic(C)$ there is a notion of degree: $deg[\mathcal{O}_C(D)] \stackrel{\text{def}}{=} deg(D)$. In particular, there is a subgroup $Pic_0(C)$ of line bundles of degree zero.

Proof. If $\mathcal{O}_C(D')$ and $\mathcal{O}_C(D'')$ are isomorphic then $\mathcal{O}_C('-D'') \cong \mathcal{O}_C(D')\otimes\mathcal{O}_C(D'')^*$ is a trivial line bundle, hence $D' - D'' = div(f)$ for some $0 \neq f \in \mathfrak{M}(C)$. But then $deg[div(f)] = 0$, hence $deg(D') = deg(D'')$.

7.9.2. *Divisors of meromorphic sections of line bundles.*

Lemma. Let L be a line bundle on C.

(a) The non-zero meromorphic sections $\mathfrak{M}^*(L)$ of L form a torsor for the multiplicative group $\mathfrak{M}^*(C)$ of meromorphic functions on C.

(b) The divisors of non-zero meromorphic sections $\mathfrak{M}^*(L)$ of L form a coset in $Cl(C) = Div(C)/div[\mathfrak{M}^*(C)]$.

(c) Meromorphic sections f of $\mathcal{O}_C(D)$ can be canonically identified with meromorphic functions on C, then

$$ord_a^{\mathcal{O}_C(D)}(f) = ord_a(f) + ord_a(D),\ a \in U, \quad i.e., \quad div^{\mathcal{O}_C(D)}(f) = div(f) + D.$$

Proof. (a) First, we know that L has a meromorphic section $\sigma \neq 0$, i.e., $\mathfrak{M}^*(L) \neq \emptyset$. Then, $\mathfrak{M}^*(C)$ acts on $\mathfrak{M}^*(L)$ by multiplication of meromorphic sections by meromorphic functions f. Finally, for any two (non-zero) meromorphic sections s_1, s_2 of L there is a unique $f \in \mathfrak{M}^*(C)$ such that $s_2 = f{\cdot}s_1$. Here $f = s_2/s_1$, or more precisely we have a meromorphic section ${s_2}^{-1}$ of L^* and then a meromorphic section $f = s_1 \otimes {s_2}^{-1}$ of $L \otimes L^* \cong T = C \times \mathbb{C}$. Now, $f \cdot s_2 = s_1$ is clear. Obviously, (a) implies (b) since for any $s \in \mathfrak{M}^*(L)$ we have $\mathfrak{M}^*(L) = \mathfrak{M}^*(C) \cdot s$, hence:

$$div[\mathfrak{M}^*(L)] = \ div[\mathfrak{M}^*(C) \cdot s] = \ div[\mathfrak{M}^*(C)] + div(s) \in Div(C)/div[\mathfrak{M}^*(C)] = \ Cl(C).$$

(c) By definition, the local holomorphic sections of $\mathcal{O}_C(D)$ are meromorphic functions $\mathcal{O}_C(D)(U) \subseteq \mathfrak{M}(U)$. We now extend this to meromorphic sections $s \in \mathfrak{M}(U, \mathcal{O}_C(D))$. First, in a small neighborhood V_a of a point $a \in U$, there is a coordinate function z with $z(a) = 0$ (= *a chart centered at* a). Then $z^{-ord_a^{\mathcal{O}_C(D)}(s)} \cdot s$ is a holomorphic section of $\mathcal{O}_C(D)$ over V_a (we have just killed the pole!), hence it is a meromorphic function f_a on V_a (with a property $div(f_a) + D \geq 0$ on V_a). Now, we multiply back the transition factor and define a *meromorphic function* $z^{ord_a^{\mathcal{O}_C(D)}(s)} \cdot f_a$ on V_a. Now it remains to check that all $z^{ord_a^{\mathcal{O}_C(D)}(s)} \cdot f_a$ glue into one meromorphic function f on U, which we then attach to s.[30] The statement about order at a point now follows from the same statements for holomorphic sections.

Remark. One says that divisors D_1, D_2 are linearly equivalent if $D_1 - D_2$ is a principal divisor (i.e., the images in $Cl(C)$ are the same. We see that this is equivalent to: D_1, D_2 define isomorphic line bundles: $\mathcal{O}_C(D_1) \cong \mathcal{O}_C(D_2)$.

7.9.3. *Recovering a line bundle from meromorphic sections.*

Lemma. Let L be a line bundle on C.

(a) If s is any non-zero meromorphic section of a line bundle L, multiplication with s gives a canonical isomorphism $\mathcal{O}_C(div^L(s)) \xrightarrow{s} \mathcal{L}$, with the sheaf of sections $\mathcal{L}$ of L.

(b) For a divisor D, L is isomorphic to $\mathcal{O}_C(D)$ iff D is the divisor of some meromorphic section of L.

Proof. (a) We need to understand the sections of L to compare them with the known sections of $\mathcal{O}_C(D)$ for $D = div^L(s)$. Let $\phi \in \Gamma(U, L) = \mathcal{L}(U)$ be some holomorphic section of L over an open $U \subseteq C$. Now we have to meromorphic sections of L and from them we can cook up a meromorphic function $f = \phi/s$ on U. We define it as a meromorphic section $f = \phi \otimes s^{-1} \in \mathfrak{M}(U, L \otimes L^*)$, and since $L \otimes L^*$ is canonically isomorphic to the trivial line bundle $T = C \times \mathbb{C}$, f is really a meromorphic function $f \in \cong \mathfrak{M}(U, C \times \mathbb{C}) = \mathfrak{M}(U)$. Now one goes backwards and finds that $\phi = f \cdot s$ in $L \cong T \otimes L$.

In this way we may try to go from any meromorphic functions $f \in \mathfrak{M}(U)$ to a holomorphic section ϕ of L by $f \mapsto \phi \stackrel{\text{def}}{=} f \cdot s$. But, in general ϕ is only going to be another meromorphic section of L on U. For which f's is ϕ holomorphic? We need $ord_a^L(f \cdot s) = ord_a(f) + ord_a^L(s)$ to be ≥ 0 at each point $a \in U$, i.e., $f \in \mathcal{O}_C(div^L(s))$.

[30]Certainly they glue, because we are not really doing anything – we divide and then multiply in the same factor.

So the multiplication by s is a surjective map $\mathcal{O}_C(U) \xrightarrow{-\cdot s} \mathcal{L}(U)$. It is clearly injective because locally (on a dense subset obtained by removing zeros and poles of s), this is a multiplication with an invertible function.

(b) We will see that the constant function 1 on C can be viewed as a meromorphic section σ of $\mathcal{O}_C(D)$ with $div^{\mathcal{O}_C(D)}(s) = D$. According to the lemma 7.9.2.c, since 1 is a meromorphic function on C it can also be viewed as a meromorphic section s of $\mathcal{O}_C(D)$, and moreover $div^{\mathcal{O}_C(D)}(s) = div(1) + D = 0 + D = D$.

So, if $L \cong \mathcal{O}_C(D)$ then it has a meromorphic section with divisor D. The opposite direction is just the part (a).

Corollary. (a) The image of $Div(C) \to \mathfrak{P}ic(C)$ is Pic(C), i.e., a line bundles comes from a divisor iff it has a global meromorphic section.

(b) $Div(C) \to Pic(C)$ factors to $Cl(C) \xrightarrow{\cong} Pic(C)$. The inverse is given by

$$Pic(C) \ni L \mapsto div^L(\mathfrak{M}^*(L)) \in Div(C)/div[\mathfrak{M}^*(C)] = Cl(C).$$

Proof. (a) is clear from the lemma. The first part of (b) then follows since the kernel of $Div(C) \to Pic(C)$ was found to consist precisely of principal divisors. We noticed above that there is a map $Pic(C) \to Cl(C)$ by $L \mapsto div^L(\mathfrak{M}^*(L))$. To see that the two maps are inverse, we check $div(\mathfrak{M}^*[\mathcal{O}_C(D)]) = D + div[\mathfrak{M}^*(C)]$ for $D \in Div(C)$. Recall that D is a divisor of a meromorphic section of $\mathcal{O}_C(D)$ and that $div(\mathfrak{M}^*[\mathcal{O}_C(D)])$. is a coset of principal divisor in $Div(C)$, so $div(\mathfrak{M}^*[\mathcal{O}_C(D)]) = D + div[\mathfrak{M}^*(C)]$.

7.10. **Conclusion: $J_0(C)$ and $Pic(C)$ (periods and line bundles).**

7.10.1. *Comparison.* We have constructed the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & div(\mathfrak{M}^*(C)) & \xrightarrow{\subseteq} & Div_0(C) & \xrightarrow{\int} & J_0(C) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow{\subseteq} & & \downarrow{\iota} & & \\
0 & \longrightarrow & div(\mathfrak{M}^*(C)) & \xrightarrow{\subseteq} & Div(C) & \longrightarrow & Cl(C) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow{=} & & \downarrow{\iota} & & \\
0 & \longrightarrow & div(\mathfrak{M}^*(C)) & \xrightarrow{\subseteq} & Div(C) & \xrightarrow{D \mapsto \mathcal{O}_C(D)} & Pic(C) & \longrightarrow & 0
\end{array}$$

except for the maps i and ι. However, the rows are known to be exact so i and ι are obtained as factorizations of the maps in the middle, and i is injective while ι is a bijection.

All-together, we have proved:

7.10.2. *Theorem.* $J_0(C)$ is the subgroup $Pic_0(C)$ of $Pic(C)$. The canonical isomorphism $Pic_0(C) \to J_0(C)$ associates to a line bundle L the integral $\int(div(s))$ of any meromorphic section s of L.

Remark. Notice that the inclusion i has a lot of content (properties of integrals) while isomorphism ι is largely a matter of permuting the definitions.

7.10.3. $\mathbb{P}^1$, *i.e., curves of genus* 0. Recall that any holomorphic line bundle on $\mathbb{P}^1$ is algebraic. Denote $\mathcal{O}(n) \stackrel{\text{def}}{=} \mathcal{O}_{\mathbb{P}^1}(n{\cdot}\infty),\ n \in \mathbb{Z}$.

7.10.4. *Lemma.* (a) Any line bundle L on $\mathbb{P}^1$ is isomorphic to precisely one of the line bundles $\mathcal{O}(n)$ (and $n = deg(L)$). So, $deg : \text{Pic}(\mathbb{P}^1) \stackrel{\cong}{\to} \mathbb{Z}$ and $Pic_0(\mathbb{P}^1) = 0$.

(b) The global sections can be viewed as the polynomials of degree $\leq n$ or as the degree n polynomials in two variables:

$$\Gamma[\mathbb{P}^1, \mathcal{O}(n)] \;\cong\; \mathbb{C}_{\leq n}[z] \;\cong \mathbb{C}_n[x, y],\ n \in \mathbb{Z}.$$

(c) $T\mathbb{P}^1 \cong \mathcal{O}(2)$ and $\Omega^1 \cong \mathcal{O}(-2)$.

(d) There are no global differential forms on $\mathbb{P}^1$.[31] There are three independent vector fields on $\mathbb{P}^1$: $\partial_z, z\partial_z, z^2\partial_z$.[32]

Proof. (a) Group $Pic_0(\mathbb{P}^1) \cong Cl_0(\mathbb{P}^1)$ is known to be trivial by lemma 7.2.4. So, $Pic(\mathbb{P}^1) \xrightarrow{deg} \mathbb{Z}$ is an isomorphism. Since $deg(n{\cdot}\infty) = n$, $n \mapsto \mathcal{O}(n)$ is the inverse of the degree isomorphism.

(b) $\Gamma[\mathbb{P}^1, \mathcal{O}(n)] = \Gamma[\mathbb{P}^1, \mathcal{O}(n{\cdot}\infty)]$ consists of all functions holomorphic on $\mathbb{A}^1$, i.e., series $\sum_0^{+\infty} f_i z^i$ with the radius of convergence $+\infty$, such that at ∞ where $w = 1/z$ is a parameter, $\sum_0^{+\infty} f_i w^{-i}$ has order $\geq -n$, i.e., $f_i = 0$ for $-i < -n$. So we allow precisely the sums $\sum_0^n f_i z^i$. For $n < 0$ this is nothing and for $n \geq 0$ these are polynomials in z of degree $\leq n$, and for $z = y/x$, the multiplication with x^n identifies them with $\mathbb{C}_n[x, y]$.

(c) $T\mathbb{P}^1$ has a global holomorphic section ∂ with a double zero at ∞ (in terms of $w = 1/z$ we have $\partial = \frac{d}{dz} = -w^2 \frac{d}{dw}$). So, $div(\partial) = 2{\cdot}\infty$ and $TC \cong \mathcal{O}(2{\cdot}\infty) = \mathcal{O}(2)$. Therefore, $T^*\mathbb{P}^1 = [T\mathbb{P}^1]^* \cong \mathcal{O}_{\mathbb{P}^1}(2\infty)^* \cong \mathcal{O}(-2{\cdot}\infty)$. We see that $\partial, z\partial$ and $z^2\partial$ are holomorphic sections with divisors $2{\cdot}\infty, \mathbf{0} + \infty, 2{\cdot}\mathbf{0}$.

(d) $\Gamma(\mathbb{P}^1, \Omega^1_C) \cong \Gamma(\mathbb{P}^1, \mathcal{O}_C(-2)) = 0$ and $\dim[\Gamma(\mathbb{P}^1, T\mathbb{P}^1)] = \dim[\Gamma(\mathbb{P}^1, \mathcal{O}_C(2))] = \dim[\mathbb{C}_2[x, y] = 3$.

7.11. **Group structure on projective cubics.** The cubics[33] play a very special role, these are the only projective curves that admit a group structure. One obvious consequence is that cubics have no special points: at all points they look the same. This is also

[31] Too bad: integrals of differential forms are well defined on $\mathbb{P}^1$. However, at least one can integrate meromorphic differential forms.

[32] Actually, the space of global vector fields always has a structure of a Lie algebra, and in nice cases – like this one – it is the Lie algebra of the automorphism group $\text{Aut}(\mathbb{P}^1) = PGL_2$.

[33] Better to say *projective curves of genus* 1.

true for $\mathbb{P}^1$ (a homogeneous space of the group PGL_2), but *not at all* for curves of genus $g > 1$.

Why do cubics have a group structure? The magic of degree being 3 is that it provides just enough space for the story *mother, father, child*, which is the prototype of our standard algebraic structures (two produce one in groups and rings).

7.11.1. *The origin.* On $C = C_\lambda$ we need to choose a point which will be 0 for the group structure, the simplest choice is the point $\infty = C_\lambda - \mathcal{C}_\lambda$.

7.11.2. *Operation $a\circ b$.* For $a, b \in C \subseteq \mathbb{P}^2$ we define $a\circ b$ as the third point of the intersection of the line $L_{a,b} \subseteq \mathbb{P}^2$ through a, b with C. This requires some explanation.

- Lines in $\mathbb{P}^2 = \mathbb{P}(C^3)$ means the projectivizations $\mathbb{P}(P)$ of two dimensional vector subspaces $P \subseteq \mathbb{C}^3$. This makes sense since $\mathbb{P}(P) \cap \mathbb{A}^2$ is really an affine line in $\mathbb{A}^2$ (except in the one case when the intersection is empty, i.e., $\mathbb{P}(P) = \mathbb{P}^2 - \mathbb{A}^2$).
- There is a unique line $L_{a,b}$ in $\mathbb{P}^2$ that passes through a, b – if $a \neq b$. However, $L_{a,a} \stackrel{\text{def}}{=} \lim_{b\to a}\ L_{a,b}$ is always well-defined – this is (the definition of) the tangent line $\boldsymbol{T}_a(C)$ to C at a.[34]
- The number of points in the intersection of a line L with the cubic C is (by Bezout's theorem) $deg(L)\cdot deg(C) = 1\cdot 3 = 3$ (counted with multiplicities!). So, the intersection of $L_{a,b}$ with C contains a, b but also the third point which we call $a\circ b$. Notice that for instance $a\circ b = b$ if $L_{a,b}$ is tangent to C at b.

Notice the symmetry between $a, b, a\circ b$, i.e., S_3 acts on $\{(a,b,c) \in C^3;\ a\circ b = c\}$. So, this can not be the addition operation, however in any abelian group A there is an S_3-invariant subset of A^3 given by $a + b + a\circ b = 0$. So, we hope that $a\circ b = -(a+b)$. If so, then addition will be given by $(a\circ b)\circ\infty = -[-(a+b)+0] = a+b$.

7.11.3. *Addition $a+b$.* Now define $a+b \stackrel{\text{def}}{=} (a\circ b)\circ\infty$. Then

(1) $\infty + a = a$, i.e., ∞ is the neutral element so we will call it 0. (clearly, $(\infty\circ a)\circ\infty = a$).
(2) $a\circ b$ is commutative, hence so is $a+b$.
(3) $a + b = 0$ means that $(a\circ b)\circ\infty = \infty$, i.e., $a\circ b = \infty\circ\infty$, or $b = a\circ(\infty\circ\infty)$. Here, $\infty\circ\infty$ is the third point at which the tangent line $\boldsymbol{T}_\infty(C)$ at ∞ meets C.

It remains to check associativity. However, we can get around that by checking the first two claims in

[34]Notice that the notion we have defined here has the property $\boldsymbol{T}_a(C) \cong \mathbb{P}^1 \subseteq \mathbb{P}^2$, rather then $\mathbb{A}^1$.

7.11.4. *Theorem.* If we define the group structure on $C = C_\lambda$ using operation $a{\circ}b$, and do that ∞ is the origin in the group, then the map

$\phi \overset{\text{def}}{=} [C \xrightarrow{a\mapsto a-\infty} Div_0(C) \to Cl_0(C)]$, *i.e.*, $\phi(a) = [a{-}\infty]$ (the class of $a - \infty$ in $Cl_0(C)$), satisfies

- (1) $\phi(a+b) = \phi(a) + \phi(b)$, (2) ϕ is injective, (3) ϕ is surjective.

So, all-together, ϕ is an isomorphism of groups $C \to Cl_0(C)$.

Proof. We check (1). We denote the above addition in C by $\oplus$. Then $\phi(a{\oplus}b) = \phi(a){+}\phi(b)$ means that $[(a{\oplus}b) - \infty] = [a-\infty] + [b-\infty]$, i.e., that $(a{\oplus}b) + \infty - a - b$ is a principal divisor.

However, whenever $\alpha, \beta, \gamma \in C \in \mathbb{A}^2$ are colinear, i.e., they lie on some line $L{\subseteq}\mathbb{A}^2$, then $\alpha+\beta+\gamma-3\infty$ is the divisor of the meromorphic function f on C which is the equation of L. Actually, the equation of L is a polynomial function on $\mathbb{A}^2$ which we restrict to $C \cap \mathbb{A}^2$ and extend to a meromorphic function f on C. Then $div(f) = \alpha + \beta + \gamma$ in $C \cap \mathbb{A}^2$, and since the degree has to be zero, the multiplicity of ∞ is -3. So, for any $\alpha, \beta \in C$

$$[\alpha{\circ}\beta] = [\gamma] = [3\infty - \alpha - \beta].$$

Therefore,

$$[(a{\oplus}b)] = [(a{\circ}b){\circ}\infty] = [3\infty{-}((a{\circ}b){+}\infty)] = [2\infty{-}(a{\circ}b)] = [2\infty{-}(3\infty{-}a{-}b)] = [a{+}b{-}\infty].$$

The claims (2-3) we already met (and we proved (3)), because, this is the isomorphism of C with $J_0(C)$ which is in this case an elliptic curve.