ALGEBRAIC GEOMETRY
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Contents

5. **Constructing moduli spaces**

This is one of the basic applications of algebraic geometry. The most popular ones are related to curves:

- moduli of curves,
- moduli of vector bundles on one curve,
- moduli of maps from a given curve C to a variety X.

These spaces are among the most studied ones in algebraic geometry.

The *moduli idea* is that in order to study objects of a certain type it is useful to look at the *space of all such objects* (called the *moduli*). It should be some kind of geometric space that parameterizes all these objects in a useful way. For instance, a path in the moduli space will mean a deformation of such objects. The most familiar example may be the projective space $\mathbb{P}(V)$ (resp. Grassmannian $Gr_r(V)$), which is the moduli of all lines (resp. dimension r subspaces) in a vector space V.

We will see that in practice, the moduli is often constructed as a quotient of a scheme by a group. It turns out that it is not most important to think of the quotient as the set of orbits, but to have some geometric structure on the quotient. This can accomplished in various ways useful from a particular point of view.

The simplest construction of this form is the Invariant theory quotient $X//G$ of an affine variety (or scheme) by a group, the result is again an affine variety (or a scheme). We will construct n^{th} symmetric powers $X^{(n)} \stackrel{\text{def}}{=} X^n//S_n$ of varieties – these are moduli of n unordered points in X with possible repetitions (*moduli of configurations of n identical particles in X*). This moduli turns out quite satisfactory when X is a curve over a field (we look at $X = \mathbb{A}^1$).

When X is a surface (we look at $X = \mathbb{A}^2$), the symmetric powers are *singular*, and we find this unsatisfactory for a moduli – we want to be the objects of a given kind to deform nicely. This we use an excuse to introduce the (punctual) Hilbert schemes $X^{[n]}$ which do give a smooth moduli of n unordered points for surfaces.[1]

The last approach we visit is that of a stack quotient, which one can call the *true quotient* X/G. It has highly desirable properties that

- If X is smooth then X/G is smooth.
- One takes the *total* quotient by G, in the sense that the fibers of $X \to X/G$ are copies of G. For instance the set-theoretic quotient is not it, when we take the set-theoretic quotient, i.e., the set of orbits, we are forgetting to divide by stabilizers of points.

[1]They are *not* quotients by groups!

By this time it is clear that the familiar constructions do not have these properties. For instance the set-theoretic quotient is not it, when we take the set-theoretic quotient, i.e., the set of orbits, we are forgetting to divide by stabilizers of points. Also, invariant theory quotient often introduces singularities when the size of the stabilizer groups varies.

So we need to enlarge the world of varieties (or schemes) to have a hope of finding the "true quotients". The first such enlargement we consider is the category of *spaces* which is the Yoneda completion of the category of varieties. The second such enlargement we is the category of *stacks* which is obtained from the category of varieties using Grothendieck's group theoretic refinement of the Yoneda completion.

Both of these procedures of enlarging a given category $\mathcal{A}$ are simply the categorical analogues of the the idea of extending calculus from functions of distributions.

The end product is that to an action of a group G on a variety X we associate the natural *space quotient* $\underline{X/G}$ (it is quite a natural idea but it does not satisfy the desired properties) and the natural *stack quotient* $\overline{X/G}$ (slightly more refined and it finally does satisfy the desiderata).

5.1. **Moduli $\mathcal{M}(T)$ of objects of type T.** Consider objects of a certain type, say, type T.[2] Experience shows that the set $Isom(T)$ of isomorphism[3] classes of objects of type T often has more structure then just a set. For instance, there may be T-objects of "more general nature", that degenerate to some "special" T-objects, indicating at least some topology on $Isom(T)$. Furthermore, once we find that we are really interested in the space $Isom(T)$ we would like to do calculations on it, so we should make it Therefore, we would like to organize $Isom(T)$ into a geometric space $\mathcal{M} = \mathcal{M}(T)$ which we will call the moduli of of objects of type T.

5.1.1. *The standard strategy.* In practice, the first step is to find some bigger space $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}(T)$ that naturally parameterizes objects of type T, but with possible repetitions. So,

(1) to each $m \in \widetilde{\mathcal{M}}$ one naturally attaches an object V_m of type T,
(2) each object of type T is isomorphic to one of V_m

Actually, one would like one more property (no repetitions!):

$$(*) \quad V_p \cong V_q \textit{ only when } p = q,$$

but this is often not possible. What often happens is that a weaker version of $(*)$ is true

$$(\star) \quad \textit{there is a group } G \textit{ acting on } \widetilde{M}, \textit{ so that } V_p \cong V_q \textit{ iff } q \in G{\cdot}p.$$

[2]For instance, we will look at the examples of quadric curves in $\mathbb{P}^2$ and of cubic curves in $\mathbb{P}^2$.

[3]The word isomorphic means literally "of the same shape". We specify what we mean by it in each situation, but in general it will mean that two objects behave the same in any sense that we are interested in.

Then it seems clear that the solution is the quotient $\mathcal{M} \stackrel{\text{def}}{=} \widetilde{\mathcal{M}}/G$.

Examples: projective and Grassmannian spaces. One can construct a moduli $\mathbb{P}(V)$ of all lines in a vector space V over a field $\Bbbk$ by following the above strategy. We notice that each non-zero vector v produces a line $\Bbbk v$, and all lines appear this way. So $\widetilde{\mathcal{M}} = V - 0$ can be viewed as a "moduli with repetitions" of lines in V. Now, $u, v \in \widetilde{\mathcal{M}} = V - 0$ give the same line iff $v \in \Bbbk^* \cdot u$ for the obvious action of the group $\Bbbk^* = GL_1(\Bbbk)$ on $V - 0$. So, the moduli $\mathcal{M}$ is $\widetilde{M}/GL_1(\Bbbk^*) = (V-0)/\Bbbk^*$.

Similarly one obtains the moduli $Gr_k(V)$ of all $\Bbbk$-dimensional vector subspaces of V. A "moduli with repetitions" can be chosen as $Fr_k(V)$ the set of all $\Bbbk$-tuples $v_\bullet = (v_1, ..., v_k)$ – any such $v_\bullet$ gives $span(v_1, ..., v_k) \in Gr_k(V)$ and we get all $U \in Gr_k(V)$ in this way. The fiber at U is the set $Fr_k(U)$ of all ordered basis of U, so repetitions come from all choices of a basis of U.

To see that one can account for repetitions by an action of a group, we need a group that acts on $Fr_\Bbbk(V)$ and does not affect the span. One way to see this is to view $Fr_k(V)$ as the set $Inj_\Bbbk(\Bbbk^k, V) \subseteq \mathrm{Hom}_\Bbbk(\mathbb{K}^k, V)$ of all injective linear maps from $\Bbbk^k$ to V (here $v_\bullet$ corresponds to a map that sends e_i to v_i). Now, $Fr_k(V) \xrightarrow{span} Gr_k(V)$ is identified with the operation of taking the image of a linear map $Inj_\Bbbk(\Bbbk^k, V) \xrightarrow{image} Gr_\Bbbk(V)$. Now $GL(V) \times GL_k(\Bbbk)$ act on $\mathrm{Hom}_\Bbbk(\mathbb{K}^k, V)$ by $(g, \sigma)A = g \circ A \circ \sigma^{-1}$ and two maps A, B have the same image iff they a re in the same orbit of $GL_k(\Bbbk)$. So, the moduli is

$$Gr_k(V) = Inj_\Bbbk(\Bbbk^k, V)/GL_k(\Bbbk) = Fr_k(V)/GL_k(\Bbbk).$$

(The passage to $Inj_\Bbbk$ was only used to explain the action of GL_k on $Fr_k(V)$.)

5.1.2. *The need for quotients of spaces by groups.* The last step of our strategy requires taking the quotient $\mathcal{M} \stackrel{\text{def}}{=} \widetilde{\mathcal{M}}/G$. So, we see that we have not solved our problem of constructing moduli, we have only reformulated the original problem as:

For a space X *with an action of a group* G *construct an* *adequate* *space* X/G.

Here, *space* could mean: algebraic variety, scheme, or something like that. Another part of the richness of the subject comes from the phrase *adequate* which has different meanings in different situations.

The simplest approach to constructing quotients with a geometric structure is

5.2. **Invariant Theory quotients** $X//G$. We will consider this construction in the example of *symmetric powers* of a variety X, i.e., the moduli of *unordered* n*-tuples of points in* X.

5.2.1. *Unordered pairs of points.* Let $X = \mathbb{A}^1(\Bbbk)$, we are interested in the moduli $\mathcal{M}$ of objects of the following kind:

All unordered pairs $\{\{a,b\}\}$ *of points* $a,b\in X$.
Precisely, what we mean by the symbol $\{\{b,a\}\}$ *is that*
$\{\{b,a\}\}=\{\{a,b\}\}$ *and we allow repetitions* $\{\{a,a\}\}$.

So our moduli $\mathcal{M}$ can be thought of as

All possible positions of two particles of the same kind on a line $X=\mathbb{A}^1$.

Notice that in this case, there is nothing to the idea of isomorphism that is incorporated into the notion of moduli. (Two pairs are considered isomorphic iff they are the same!)

In this situation, $\widetilde{\mathcal{M}}$ can be taken to be the set X^2 of ordered pairs, since an ordered pair (a,b) defines an unordered pair $\{\{a,b\}\}$ by forgetting the order. So $\widetilde{M}=X^2$ is an affine variety with functions $\mathcal{O}(X^2)=\Bbbk[X_1,X_2]$. Group $S_2=\{1,\sigma\}$ acts on X^2 by $\sigma(a,b)=(b,a)$, and we see that the moduli of unordered pairs $\mathcal{M}$ should be

$$\mathcal{M}=\ \widetilde{\mathcal{M}}/S_2=X^2/S_2.$$

The question is how to give the set X^2/S_2 the structure of a geometric space.

5.2.2. *Invariant theory quotients.* If Y is an affine variety with an action of a group G, there is a canonical way to produce an affine variety $Y//G$ that plays the role of the quotient of Y by G. First observe that G acts on the algebra $\mathcal{O}(Y)$ by

$$(g{\cdot}\phi)(y)\overset{\text{def}}{=}\phi(g^{-1}y).$$

To start with, we consider the case when the quotient set Y/G can be given a natural structure of an affine variety. Now, the quotient map $Y\xrightarrow{\pi}Y/G$ gives the pull-back map of algebras of functions $\mathcal{O}(Y/G)\xrightarrow{\pi^*}\mathcal{O}(G)$. It is injective, so it makes $\mathcal{O}(Y/G)$ into a subalgebra of $\mathcal{O}(G)$. Moreover, the pull-backs of functions from the quotient are special among all functions on Y because they are invariant under G:

$$(g{\cdot}\pi^*\phi)(y)\overset{\text{def}}{=}(\pi^*\phi)(g^{-1}y)=\phi(\pi(g^{-1}y))=\phi(\pi(y))=(\pi^*\phi)(y).$$

Actually, we expect that $\mathcal{O}(Y/G)$ is precisely the subalgebra $\mathcal{O}(Y)^G$ of G-invariant functions on Y.

Now, we use the above observation as a definition for any group action on an affine variety:

The invariant theory quotient $Y//G$ *is the space with functions* $\mathcal{O}(Y//G)\overset{def}{=}\mathcal{O}(Y)^G$.

5.2.3. *Symmetric powers* $X^{(n)}$ *(unordered* n*-tuples of points).* For an affine variety X we can now make sense of the moduli of unordered n-tuples of points in X. This moduli is the affine variety

$$X^{(n)}\overset{\text{def}}{=}X^n//S_n,\quad\text{i.e.,}\quad\mathcal{O}(X^{(n)})\overset{\text{def}}{=}\mathcal{O}(X^n)^{S_n}.$$

We call it the n^{th} symmetric power of X since it is the symmetric version of the n^{th} power X^n.

5.2.4. *Symmetric powers of a line.*

Lemma. $(\mathbb{A}^1)^{(n)} \cong \mathbb{A}^n$.

Proof. If $X = \mathbb{A}^1$ then $X^n = \mathbb{A}^n$ with $\mathcal{O}(X^n) = \Bbbk[X_1, ..., X_n]$, and S_n acts on it by permuting variables. The S_n-invariant functions (called the *symmetric polynomials in* n *variables*), are polynomials in the elementary symmetric functions $e_1 = X_1 + \cdots + X_n$, $e_2 = \sum_{i<j} X_i X_j, ..., e_n = X_1 \cdots X_n$, i.e.,

$$\mathcal{O}((\mathbb{A}^1)^{(n)}) \stackrel{def}{=} \mathcal{O}((\mathbb{A}^1)^n)^{S_n} = \Bbbk[X_1, ..., X_n]^{S_n} = \Bbbk[e_1, ..., e_n]$$
$$\textit{for } \ e_p \stackrel{def}{=} \sum_{i_1 < \cdots < i_p} X_{i_1} \cdots X_{i_p}.$$

For instance,

$$\mathcal{O}((\mathbb{A}^1)^{(2)}) = \Bbbk[X_1 + X_2, X_1 X_2].$$

In order to prepare for symmetric powers of surfaces we look at an example of a space with a singularity:

5.2.5. *Matrices and nilpotent cones.* Let M_{mn} be the $m \times n$ matrices over $\Bbbk$. This is an affine variety isomorphic to $\mathbb{A}^{mn}$

$$\mathcal{O}(M_{mn}) = \Bbbk[X_{ij},\ 1 \le i \le m,\ 1 \le j \le n].$$

In the square matrices $M_n = M_{nn}$ we have we have the *nilpotent cone* which consists of all nilpotent matrices

$$\mathcal{N}_n \stackrel{\text{def}}{=} \{x \in \mathbb{M}_n;\ x^p = 0 \text{ for some } p > 0\}$$

For instance

Lemma. (a) For $x \in M_n$ the following is equivalent

(1) $x \in \mathcal{N}_n$, i.e., x is nilpotent,
(2) $x^n = 0$
(3) all eigenvalues are 0,
(4) The characteristic polynomial $\det(\lambda - x)$ equals λ^n.

(b) $\mathcal{N}_n$ is an algebraic variety.

Proof. (a) can be seen using the Jordan form of x. (b) follows from either (2) or (4) in (a).[4]ctually, the equations for $\mathcal{N}_n$ one obtains from (4) are more economical.

[4]A

Corollary. (a) $\mathcal{N}_2 = \{\begin{pmatrix} a & b \\ c & -a \end{pmatrix};\ a^2 + bc = 0\}$.

(b) $\mathbb{A}^2//\{\pm 1\} \cong \mathcal{N}_2$.

Proof. (a) For $x \in M_2$, $\det(\lambda - x) = \lambda^2 - Tr(x)\cdot\lambda + \det(x)$.

(b) We mean the action of $\pm 1 \subseteq \Bbbk^*$ on the vector space $\mathbb{A}^2(\Bbbk)$. So, $-1 \in \{\pm 1\}$ acts on the generators of $\mathcal{O}(\mathbb{A}^2) = \Bbbk[X,Y]$ by $X \mapsto -X,\ Y \mapsto Y$. Therefore, the invariant functions are the polynomials in $\alpha = X^2,\ \beta = Y^2,\ \gamma = XY$. So,

$$\Bbbk[X,Y]^{\{\pm 1\}} \cong \Bbbk[\alpha,\beta,\gamma]/(\alpha\beta = \gamma^2) \cong \mathcal{O}(\mathcal{N}_2).$$

5.2.6. *Singularities.* One of the successes of algebraic geometry is its treatment of singularities. Singularities naturally appear in all kinds of problems. For instance the fibers of a map $\pi : X \to Y$ between two manifolds, are often not manifolds. For instance look at the fiber at zero, of the functions $\mathbb{R}^2 \ni (x,y) \mapsto xy \in \mathbb{R}$ or $\mathbb{R}^3 \ni (x,y,z) \mapsto xy - z^2 \in \mathbb{R}$. This is awkward in the manifolds theory, but it is not a problem in a algebraic geometry (the fibers of a map of varieties are again varieties).

The nilpotent cone $\mathcal{N}_2$ is one of the simplest singular affine varieties. It has singularity at the origin (the zero matrix).[5] To get a feeling for it, one can draw the picture for $\Bbbk = \mathbb{R}$! We find it is a cone over a circle, i.e., it is the union of all lines that pass through one circle C and a fixed point v (not in C).

One basic way we deal with singularities is by finding

5.2.7. *Resolutions of singularities.*

Lemma. Let μ and π be the projection maps from

$$\widetilde{\mathcal{N}_2} \stackrel{\text{def}}{=} \{(x,L) \in \mathcal{N}_2 \times \mathbb{P}(\Bbbk^2);\ xL = 0\} \subseteq \mathcal{N}_2 \times \mathbb{P}(\Bbbk^2)$$

to $\mathcal{N}_2$ and $\mathbb{P}^1$.

(a) $\widetilde{\mathcal{N}_2}$ is a line bundle over $\mathbb{P}^1$.

(b) π is a bijection over $\mathcal{N}_2 - \{0\}$, and $\pi^{-1}(0) \cong \mathbb{P}^1$.

(c) $\mathcal{N}_2$ is a cone, i.e., it is a union of lines (one for each point of $\mathbb{P}^1$), and all these lines meet at one point, the vertex of the cone.

[5] If $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$, having a singularity means that X is not a manifold, i.e., near v the space does not look like an open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. The precise definition in general algebraic geometry will come later.

Remarks. (1) So, one obtains $\mathcal{N}_2$ from a nice space $\widetilde{\mathcal{N}_2}$ by contracting one $\mathbb{P}^1$ to a point. To remind us that something spectacular has happened, the space $\mathcal{N}_2$ is singular at this point.

We say that $\mu : \widetilde{\mathcal{N}_2} \to \mathcal{N}_2$ is a *resolution of the singularity* in $\mathcal{N}_2$. This means that

- $\widetilde{\mathcal{N}_2}$ is smooth,
- the map is generically an isomorphism, and
- the fibers are compact.

(2) Let us look at $\mathcal{C} = \{(x, y, z);\ xy = z^2\}$ over $\mathbb{R}$. First change the coordinates via $x = u + v,\ y = u - v$ to get $z^2 = u^2 - v^2$, hence $u^2 = z^2 + v^2$. Now, for each u we get a circle. All together, one can say that we start with a circle C (say on the height $u = -1$) and a point p (where $u = v = z = 0$). Then $\mathcal{C}(\mathbb{R})$ is the union of all lines through p that meet C. Again, we see the singularity at p.

5.2.8. *Symmetric powers of surfaces.* We will find that if X is a *surface*, i.e., a 2-dimensional algebraic variety then $X^{(2)}$ is singular. So, the moduli of all positions of a pair of identical particles in a plane is singular. Moreover, we will see that the singularity occurs on the diagonal, i.e., when the particles collide!

Lemma. The symmetric square $(\mathbb{A}^2)^{(2)}$ of a plane, is isomorphic to $\mathbb{A}^2 \times \mathcal{N}_2$.

Proof. Let $\mathcal{O}(\mathbb{A}^2) = \Bbbk[X, Y]$, then $\mathcal{O}(\mathbb{A}^2 \times \mathbb{A}^2) = \Bbbk[X_1, Y_1, X_2, Y_2]$ and σ exchanges $X_{\leftrightarrow} X_2,\ Y_1 \leftrightarrow Y_2$. We change the variables to $X = X_1 + X_2,\ Y = Y_1 + Y_2,\ x = X_1 - X_2, y = Y_1 - Y_2$. Then σ fixes X, Y and $x \mapsto -x,\ y \mapsto -y$, hence $\mathcal{O}(\mathbb{A}^2 \times \mathbb{A}^2)^{S_2} = \Bbbk[X, Y, x, y]^{S_2} = \Bbbk[X, Y, x^2, y^2, xy]$.

Remark. The singularity occurs at $x = y = 0$. This means that $X_2 = X_1,\ Y_2 = Y_1$, i.e., the points are the same $(X_1, Y_1) = (X_2, Y_2)$.

5.2.9. *The Moduli Principle or Deformation Principle of Deligne, Drinfeld, Feigin, Kontsevich.* It says that we must have made a mistake because the Moduli Principle (in a vague formulation) says that

Any moduli should be smooth if constructed correctly, i.e.,
if we do not forget any information.

At the moment we know next to nothing about producing and using moduli, so unfortunately we can not justify this principle. It is supposed to make sense only with more examples of moduli. What it claims is that for any reasonable kind of objects, one can organize all examples of such objects in such a way that going from one example to another happens in a smooth way, without any jumps and unnecessary dramatics.

We will use it as an excuse to look for more subtle constructions of moduli.

5.3. **The moduli of unordered points on surfaces: Hilbert schemes.** Among the features of $X^{(n)}$

(1) It is an algebraic variety.
(2) As a set, it really is a set of unordered n-tuples of points in X.
(3) It is smooth when $\dim(X) = 1$.
(4) It is not smooth when $\dim(X) = 2$.

we like the first three but not the fourth. The above principle in suggests that there should be a better notion ? of the moduli of unordered pairs of points then the symmetric powers $X^{(n)} = X^n//S_n$. Moreover, the difference should be that we have forgotten some information when we constructed $X^{(n)}$, so there should be a "forgetting" map $? \xrightarrow{\pi} X^{(n)}$. Finally, in light of (2) above, as a set ? should not be exactly the set of unordered n-tuples of points – to have a *good moduli* of unordered n-tuples we will occasionally need to add more information.

5.3.1. *Geometric view on unordered n-tuples of points.* A priori, the idea of "unordered n-tuples of points of X" is a set theoretic idea which is formalized (made precise) in sets as the set of orbits of S_n in X^n. Its "more geometric version" will involve "more geometric" analogues of unordered n-tuples. So, let us look for more geometric ways to think of an unordered pair $\{\{a, b\}\}$, if we can do this, hopefully a geometric moduli of such objects will just pop-out from this geometric picture.

First, we can think of a two point subset $\{a, b\}\subseteq\mathbb{A}^1$ of a line $X = \mathbb{A}^1$ as a subvariety of $\mathbb{A}^1$. Then it corresponds to an ideal $I_{a,b} = (x-a)(x-b)\Bbbk[x]$ in $\Bbbk[x] = \mathcal{O}(X)$, and a two-dimensional quotient $\mathcal{O}(\{a, b\}) = \mathcal{O}(X)/I_{a,b}$ of $\mathcal{O}(X)$. However, when $a = b$, we fall off the horse because the ring of functions changes drastically:[6] the ring of functions $\mathcal{O}(\{a\})$ on the subvariety $\{a, a\} = \{a\}$ is one dimensional. So "subvariety" is not a perfect idea because we forget the multiplicity (a occurs twice in $\{\{a, a\}\}$).

By now, we know that the thing to do to remember the multiplicity is to think of sub*schemes* rather then sub*varieties.* then $\{\{a, a\}\}$ will be thought of as the *double point* subscheme of $\mathbb{A}^1$. From this point of view the moduli of unordered n-tuples of points in X should be viewed as the set of all

- subschemes $S\subseteq X$ with $\dim(\mathcal{O}(S)) = n$ (we say that S has length n or order n), i.e.,
- all quotients of $\mathcal{O}(X)$ of dimension n, i.e.,
- all ideals of codimension n in $\mathcal{O}(X)$.

This space is denoted $X^{[n]}$ and called the Hilbert scheme of n points in X.

5.3.2. *Subschemes.* Though it is not necessary at this point, let us clarify the meaning of *subscheme.*

[6] We want a smooth moduli – no jumps.

Remember that we defined algebraic subvarieties of $\mathbb{A}^n(\Bbbk)$ as subsets X given by finitely many polynomial equations $X = \{F_1 = \cdots = F_c = 0\}$, and we associated to X the ideal $I_X \subseteq \mathcal{O}(\mathbb{A}^n)$ of functions that vanish on X, and the ring of functions on X which is the quotient $\mathcal{O}(X) = \mathcal{O}(\mathbb{A}^n)/I_X$. Moreover, such X was closed for the Zariski topology on X.

We expect that $\mathbb{A}^n$ also contains some schemes which are not varieties (such as double points). An affine scheme S is determined by its ring of global functions $\mathcal{O}(S)$. Inclusion $S \subseteq \mathbb{A}^n$ will correspond to the restriction map $\mathcal{O}(\mathbb{A}^n) \to \mathcal{O}(S)$, and this should be a surjection, i.e., functions on S should all be restrictions of functions on $\mathbb{A}^n$. This in turn gives an ideal $I_S \stackrel{\text{def}}{=} \text{Ker}[\mathcal{O}(\mathbb{A}^n) \to \mathcal{O}(S)] \subseteq \mathcal{O}(\mathbb{A}^n)$. We now turn these expectations into a definition:

Closed subschemes of an affine scheme X correspond to ideals $I \subseteq \mathcal{O}(X)$. To an ideal I one associates closed subscheme $S \stackrel{def}{=} Spec[\mathcal{O}(X)/I]$.

Now, the ideal $I_S \stackrel{\text{def}}{=} \text{Ker}[\mathcal{O}(X) \to \mathcal{O}(S)]$ is just the ideal I we started with.

5.3.3. *Modules.* Even for the abstract reasons, modules have appear in algebraic geometry. If a commutative ring A can be thought of geometrically as an affine scheme $X = \text{Spec}(A)$, there should be a geometric way to think of A-modules. Some examples:

(1) Any map of affine schemes $\text{Spec}(A) = X \to Y = \text{Spec}(B)$ corresponds to a map of rings $B \to A$, and this map makes A into a B-module!
(2) Closed subschemes S of an affine scheme X correspond to ideals I_S, i.e., to $= \mathcal{O}(X)$-submodules of the $\mathcal{O}(X)$-module The basic example is provided by subschemes $S \subseteq X = \text{Spec}(A)$, the above reasoning reminds us that $\mathcal{O}(S) = \mathcal{O}(X)/I_S$ is a module for $\mathcal{O}(X)$.

5.3.4. *Support of a module.* Let X be an affine scheme and M a module for $\mathcal{O}(X)$. We will say that

(1) M is *supported* in a closed subscheme S if the action of $\mathcal{O}(X)$ factors through an action of $\mathcal{O}(X)/I_S = \mathcal{O}(S)$. This means that I_S acts trivially on M (*kills* M):
$$I_S \cdot M = 0.$$
(2) M is *set-theoretically supported* in a closed subscheme S if for each $m \in M$ there is some n such that $(I_S)^n \cdot m = 0$;, i.e.,
$$f_i \in I_S \Rightarrow f_1 \cdots f_n \cdot m = 0.$$

Examples. (1) Let $X = \mathbb{A}^1 \supseteq S = \{0, 1\}$ so that $\mathcal{O}(X) = \Bbbk[x]$ and $I_S = x(x-1) \cdot \Bbbk[x]$. Now, $\Bbbk[x]/x \cdot \Bbbk[x] = \mathcal{O}(\mathbb{A}^1)/I_{\{0\}} = \mathcal{O}(\{0\})$ is supported on S, but $\Bbbk[x]/x^2 \cdot \Bbbk[x]$ is only set theoretically supported on S.

(2) Let $X = \mathbb{A}^2 \supseteq S = \{(0,0)\}$. Then $I_S = \langle x, y \rangle = x \cdot \Bbbk[x,y] + y \cdot \Bbbk[x,y]$ and $\mathcal{O}(S) \cong \Bbbk$.

Since $\mathcal{O}(X) = \Bbbk[x,y]$, a module for $\mathcal{O}(X)$ is the same as a vector space M with two operators X, Y that commute! Now, M is supported by S if it is killed by I_S, i.e., if it is killed by x and y, i.e., iff the operators X and Y are 0!

On the other hand, M is set-theoretically supported by S if it is killed by I_S, i.e., if each $m \in M$ is – for some n – killed by all n-fold products $f_1 \cdots f_n$ of functions f_i in I_S. However, this is equivalent to to the same condition

$$f_1 \cdots f_n \cdot m = 0$$

whenever all f_i's are in the set $\{x, y\}$ of generators of the ideal I_S. Finally, it suffices that there is some p such that

$$x^p \cdot m \;=\; 0 \;=\; y^p \cdot m$$

(then $n = 2p$ satisfies the preceding condition!). So, the condition is that the operators X, Y act nilpotently on each vector of M!

5.3.5. *Support cycle of a module.* Now we give a more refined notion of the support of a module M called the *support cycle* $\mathbf{supp}(M)$. However, for simplicity we give it only in a special case sufficient for our current purposes.

Let X be an affine variety. For a *finite-dimensional* $\mathcal{O}(X)$-module M which has a filtration by submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ such that the i^{th} graded piece

$$Gr_i(M) \stackrel{\text{def}}{=} M_i / M_{i-1}$$

is isomorphic (as an $\mathcal{O}(X)$-module) to $\mathcal{O}(\{p_i\})$ for a point $p_i \in X$, we say that

$$\mathbf{supp}(M) \stackrel{\text{def}}{=} \sum p_i .$$

So the support cycle is an element of the free abelian group

$$Z^0(X) \stackrel{\text{def}}{=} \oplus_{a \in X} \mathbb{Z} \cdot a$$

with the basis X which we call the "*group of* 0*-cycles in* X". A more geometric way to think of the support cycle $\text{supp}M$ is as an unordered tuple of points, i.e., an element of the symmetric power $X^{(n)}$.

Lemma. $\text{supp}M$ is well defined, i.e.,

(1) for any finite dimensional $\mathcal{O}(X)$-module M a filtration with above properties exists, and
(2) $\mathbf{supp}(M)$ depends only on M, and not on the choice of a filtration.

Proof. This is the Jordan-Hoelder lemma.

5.3.6. *Hilbert-Chow map.* We have just explained on the set theoretical level the following

Theorem. $X^{[n]}$ maps canonically to $X^{(n)}$, this is called the Hilbert-Chow map

$$X^{[n]} \xrightarrow{\pi} X^{(n)}, \quad S \mapsto \mathbf{supp}(\mathcal{O}(S)) \ .$$

5.3.7. *Hilbert schemes of affine spaces.* Here X us an affine space $\mathbb{A}^n$. We will see that our commutative algebra definition of the Hilbert schemes $X^{[p]}$ in this case becomes standard linear algebra.[7] Elements of Hilbert schemes are given by linear operators and the Hilbert-Chow map (i.e., the support cycle map), is given by taking eigenvalues of these operators.

How much does this example tell us about Hilbert schemes of arbitrary varieties X? The answer is that any smooth variety X is in some (not very obvious) sense, locally similar to some $\mathbb{A}^n$, and that therefore $X^{[n]}$ is *locally* similar to $(\mathbb{A}^n)^{(p)}$ (again this is not obvious). So this example tells us about the local structure of Hilbert schemes of smooth varieties.

Lemma. (a) $X^{(p)}$ is the set of isomorphism classes of $(n+2)$-tuples $(M, v, x_1, ..., x_n)$ of

- (0) a p-dimensional vector space M,
- (1) n commuting linear operators x_i on M,
- (2) a vector $v \in M$ which is *cyclic* for the operators – this means that v generators the whole space M under the action of operators $x_1, ..., x_n$, i.e.,

$$\textit{the map } \Bbbk[X_1, ..., X_n] \ni P \overset{\mathcal{E}_v}{\mapsto} P(x_1, ..., x_n) \cdot v \in M \ \textit{ is surjective.}$$

(b) If one thinks of a subscheme $S \in X^{[p]}$ in terms of the data $(M, v, x_1, ..., x_n)$, observe that the commuting operators x_i diagonalize simultaneously, i.e., there is a basis $v_1, ..., v_p$ in which all x_i's simultaneously have triangular matrices:

$$x_j \cdot v_i = \ c^i_j \cdot v_i + \sum_{k<i} \ \gamma^i_j(k) \cdot v_k \qquad \text{for all } i, j..$$

Then

$\mathbf{supp}(M, v, x_1, ..., x_n)$ *is the unordered p-tuple* $\{\{c^1, ..., c^p\}\}$ *of points* $c^i = (c^i_1, ..., c^i_n)$ *of* $X = \mathbb{A}^n$.

(So, the i^{th} point c^i consists of eigenvalues of x_j's on the i^{th} vector v_i).

Proof. (a) First, any closed subscheme $S \subseteq X$ of order n, defines an $\mathcal{O}(X)$-module $M = \mathcal{O}(S) = \mathcal{O}(X)/I_S$ of dimension n and $v = 1$ is a cyclic vector for the operators on M given by the action of generators X_i on M.

In the opposite direction, conditions (0) and (1) say that M is an n-dimensional module for $\mathcal{O}(X) = \Bbbk[X_1, ..., X_n]$. For any vector $v \in M$, the kernel of $\mathcal{E}_v$ is an ideal in $\Bbbk[X_1, ..., X_n]$ (called the *annihilator* of v). Now condition (3) means that there is a vector $v \in M$ such that $\mathcal{E}_v$ gives an isomorphism of $\mathcal{O}(X)$-modules $\mathcal{O}(X)/\mathrm{Ker}(\mathcal{E}_v) \xrightarrow{\cong} M$, in particular the ideal $\mathrm{Ker}(\mathcal{E}_v)$ has codimension p, i.e., it is an element of $X^{[p]}$. So, M obtains the

[7] "Most things work this way."

structure of algebra of functions on the subscheme $S \subseteq X$ of order n, given by the ideal $\mathrm{Ker}(\mathcal{E}_v)$.

Finally, "isomorphic" for $(M', v', x'_1, ..., x'_n)$ and $(M'', v'', x''_1, ..., x''_n)$ means that there is an invertible linear operator $g : M' \to M''$ such that $v'' = gv'$ and $x''_i = g \circ x'_i \circ g^{-1}$. It appears because isomorphic tuples give the same ideal $\mathrm{Ker}(\mathcal{E}_v)$.

(b) is now clear: a basis v_i with a triangular action of x_j's gives a filtration $M_l = span\{v_1, ..., v_l\}$ of M by $\mathcal{O}(X)$-submodules, and on $M_i/M_{i-1} \cong \mathbb{k} \cdot v_i$ each X_j acts by c^i_j, the same as on $\mathcal{O}(\{c^i\})$.

Corollary. (a) $X^{(p)}$ is the set of GL_p-orbits in the set of $(n+1)$-tuples $(v, x_1, ..., x_n)$ of

- (1) n commuting linear operators x_i on $\mathbb{k}^p$,
- (2) a vector $v \in \mathbb{k}^p$ which is *cyclic* for the operators.

Proof. (a) We can assume that $M = \mathbb{k}^p$, so the datum consists of $(n+1)$-tuples $\mathcal{D} = (v, x_1, ..., x_n)$ as above. The isomorphisms of such data $\mathcal{D}'$ and $\mathcal{D}''$ are then given by elements of $g \in GL_p$.

Corollary. (b) (The diagonal fibers of the Hilbert-Chow map.) Denote by $\mathbf{0}$ the zero point in X and by $p \cdot \mathbf{0}$ the corresponding p-fold point in $X^{(p)}$. The fiber $\pi^{-1}(n \cdot \mathbf{0})$ can be described as

- GL_p-orbits in the set of $(n+1)$-tuples $(v, x_1, ..., x_n)$ that satisfy (1), (2) and

  (3) x_i's are nilpotent.
- all ideals I of codimension n in $\mathcal{O}(X)$ that lie between the ideals $I_{\{\mathbf{0}\}} = \sum X_i \mathcal{O}(X)$ and $(I_{\{\mathbf{0}\}})^n$, i.e.,

$$I_{\{\mathbf{0}\}} \supseteq I \supseteq (I_{\{\mathbf{0}\}})^n.$$

(c) (Locality property of the fibers of the Hilbert-Chow map.) Let $\boldsymbol{a} = \sum_{i=1}^k p_i \cdot a_i \in X^{(p)}$ be an unordered p-tuple where different points $a_i \in X$ appear with multiplicity p_i (so $\sum p_i = p$). Then the fiber is the product of contributions at different points:

$$\pi_p{}^{-1}(\sum_{i=1}^k p_i \cdot a_i) \cong \prod_1^k \pi_{p_i}{}^{-1}(n \cdot p_i \cdot a_i).$$

Explicitly,

- An ideal $I \in \pi_p{}^{-1}(\sum_{i=1}^k p_i \cdot a_i)$ is the same as k ideals $I_1, ..., I_k$ with $I_{a_i} \supseteq I_i \supseteq (I_{a_i})^{p_i}$.
- The relation is

$$I = I_1 \cap \cdots \cap I_k.$$

Proof. (a) *The first characterization.* Nilpotency means that all eigenvalues are 0, i.e., all points c^i equal $\mathbf{0} \in X$.

The second characterization. If I is in the fiber then $M=\mathcal{O}(X)/I$ has an $\mathcal{O}(X)$-filtration $0=M_0\subseteq\cdots\subseteq M_n=M$ such that $M_i/M_{i-1}\cong\mathcal{O}(\{0\})$. In other words, there are ideals $I=I_0\subseteq\cdots\subseteq I_n=\mathcal{O}(X)$ such that $I_i/I_{i-1}\cong M_i/M_{i-1}\cong\mathcal{O}(\{0\})$. In particular, $\mathcal{O}(\{\mathbf{0}\})\cong I_n/I_{n-1}=\mathcal{O}(X)/I_{i-1}$, and this implies that $I_{\{\mathbf{0}\}}=I_{n-1}\supseteq I_0=I$.

Moreover, as x_i is zero on $M_i/M_{i-1}\cong\mathcal{O}(\{\mathbf{0}\})$, it sends M_i to M_{i-1}. So, any product of p factors, all from $x_1,...,x_n$, kills M. This means that $(I_{\{\mathbf{0}\}})^n$ acts on $M=\mathcal{O}(X)/I$ by zero, i.e., that $(I_{\{0\}})^n\subseteq I$.

Conversely, let $I_{\{\mathbf{0}\}}\supseteq I\supseteq(I_{\{\mathbf{0}\}})^n$. First, observe that $\mathbf{supp}(I_{\{\mathbf{0}\}}/(I_{\{\mathbf{0}\}})^n)$ is a multiple of $\mathbf{0}$. It dominates $\mathbf{supp}(I_{\{\mathbf{0}\}}/I)$ which is therefore also a multiple of $\mathbf{0}$. Finally, so is

$$\mathbf{supp}(\mathcal{O}(X)/I)\ =\mathbf{supp}(\mathcal{O}(X)/I_{\{\mathbf{0}\}})\ +\mathbf{supp}(I_{\{\mathbf{0}\}}/I)\ =\mathbf{0}+\mathbf{supp}(I_{\{\mathbf{0}\}}/I)\ .$$

(c) Let $a_i=(a_{i1},...,a_{in})\in X=\mathbb{A}^n$ and $I\in{\pi_p}^{-1}(\sum_{i=1}^k\ p_i{\cdot}a_i)$. Then on $M=\mathcal{O}_X/I$ there is an $\mathcal{O}(X)$-filtration $0=M_0\subseteq\cdots\subseteq\mathbb{M}_n=M$ with $Gr_i(M)\cong\mathcal{O}(b_i)$ where points $b_1,...,b_n$ are the same as $a_1,...,a_n$ up to reordering. This implies that $\prod_i\ x_1-a_{i1}$ kills M. Therefore,

$$M=\oplus_{\alpha\in\{a_{11},...,a_{n1}\}}\ M_\alpha^{x_1}\ \ \text{for}\ \ M_\alpha^{x_1}=\{m\in M;\ (x_1-\alpha)^k=0\ \ \text{for}\ \ k>>0\}.$$

Since x_i's commute, each $M_\alpha^{x_1}$ is an $\mathcal{O}(X)$-submodule. So, by inductions we find that

$$M=\oplus_{\alpha\in\mathbb{A}^n}\ M_{\alpha_1,...,\alpha_n}^{x_1,...,x_n},\quad\text{for}\ \ M_{\alpha_1,...,\alpha_n}^{x_1,...,x_n}\overset{\text{def}}{=}\{m\in M;\ \text{for all}\ i,\ \ (x_i-\alpha_i)^k=0\ \ \text{for}\ \ k>>0\}.$$

Clearly, each $M_\alpha\overset{\text{def}}{=}M_{\alpha_1,...,\alpha_n}^{x_1,...,x_n}$ has a filtration with graded pieces isomorphic to $\mathcal{O}(\alpha)$. So, $M_\alpha\neq0$ iff α is one of a_i's, and then $\dim(M_\alpha)=p_i$.

Now, each M_α is a quotient of M, hence also of $\mathcal{O}(X)$. Let $I_\alpha\overset{\text{def}}{=}\text{Ker}[\mathcal{O}(X)\twoheadleftarrow M_\alpha]$, then $M=\oplus_1^k\ M_{a_i}$ gives $I=\cap\ I_{a_i}$. So, $I\in{\pi_n}^{-1}(\sum_1^k\ p_i{\cdot}a_i)$ corresponds to a k-tuple of ideals $I_{a_i}\in{\pi_n}^{-1}(p_i{\cdot}a_i)$

5.3.8. *Examples.* (a) If X is a curve, Hilbert powers are the same as symmetric powers, i.e., the Hilbert-Chow maps $\pi_n:X^{[n]}\to X^{(n)}$ are isomorphisms.

(b) Always $X^{[1]}=X^{(1)}=X$.

(c) If $a_i\in X$ are all different, then ${\pi_n}^{-1}(\{\{a_1,...,a_n\}\})$ is a point.

(d) If $X=\mathbb{A}^n$ then $X^{[2]}\xrightarrow{\pi}X^{(2)}$ is an isomorphism off the diagonal while the fibers over the diagonal are all isomorphic to $\mathbb{P}^{n-1}$. Actually, more canonically

$${\pi_2}^{-1}(2{\cdot}a)=\ \mathbb{P}[T_a(X)]$$

is the set of all lines in the tangent vector space to X at a.

Proof. (a) For $a_i\in X=\mathbb{A}^1$ and $p\in\mathbb{N}$, $\pi^{-1}(\sum\ p_i{\cdot}a_i)\cong\prod\ \pi^{-1}(p_i{\cdot}a_i)$, so, it remains to see that $\pi^{-1}(p{\cdot}a)$ is a point. We can assume that $a=\mathbf{0}$, then the fiber consists of all ideals I of codimension p that lie between $I_\mathbf{0}=x\mathbb{k}[x]$ and $(I_\mathbf{0})^p=x^p\mathbb{k}[x]$. However, the codimension of $(I_\mathbf{0})^p=x^p\mathbb{k}[x]$ is p, the same as for I, hence $(I_\mathbf{0})^p\subseteq I$ is equality !

(b) $\mathcal{O}(X^{(1)}) = \mathcal{O}(X^1)^{S_1} = \mathcal{O}(X)$ hence $X^{(1)} = X$. Now, for $a \in X = X^{(1)}$, ${\pi_1}^{-1}(1{\cdot}a)$ consists of all ideals $I \subseteq \mathcal{O}(X)$ of codimension one such that $I_a \supseteq I \supseteq (I_a)^1$, i.e., the only I is I_a.

(c) By locality, ${\pi_n}^{-1}(\{\{a_1, ..., a_n\}\} \cong$ *prod* ${\pi_1}^{-1}(\{\{a_i\}\}$ is a point by (b).

(d) For $a \in X$, ${\pi_2}^{-1}(2{\cdot}a)$ consists of all codimension 2 ideals $I \subseteq \mathcal{O}(X)$ that lie between $I_a = \sum_i (X_i - a_i)\Bbbk[X_1, ..., X-n]$ and $I_a^2 = \sum_{i,j} (X_i - a_i)(X_j - a_j)\Bbbk[X_1, ..., X-n]$. So it corresponds to all codimension one subspaces (*hyperplanes*), in $I_a/I_a^2 \cong \oplus\Bbbk{\cdot}(X_i - a_i)$, that are submodules for $\mathcal{O}(X)$. However, all X_i's act on I_a/I_a^2 by zero, so all subspaces are submodules. Therefore, the fiber is the set of all of hyperplanes H in $I_a/I_a^2] \cong \mathbb{P}[\oplus\Bbbk{\cdot}(X_i - a_i)]$. This is the same as the set of all lines L in projective space $(I_a/I_a^2)^*$, i.e., the projective space

$$\mathbb{P}[(I_a/I_a^2)^*] \cong \mathbb{P}[(\oplus\Bbbk{\cdot}(X_i - a_i))^*].$$

For the above geometric interpretation it remains to look at definitions

5.3.9. *The (co)tangent spaces to affine varieties.* For a point a in an affine variety X we define the cotangent space at a by

$$T_a^*(X) \stackrel{\text{def}}{=} I_a/I_a^2$$

and the tangent space by

$$T_a(X) \stackrel{\text{def}}{=} [T_a^*(X)]^* = [I_a/I_a^2]^*.$$

To see that this makes sense look at $X = \mathbb{A}^n$. Then

- $I_a = \sum_i (X_i - a_i)\Bbbk[X_1, ..., X-n]$,
- $I_a^2 = \sum_{i,j} (X_i - a_i)(X_j - a_j)\Bbbk[X_1, ..., X-n]$,
- $I_a/I_a^2 \cong \oplus\Bbbk{\cdot}(X_i - a_i)$,
- $[I_a/I_a^2] \cong \oplus\Bbbk{\cdot}\, \partial_{i,a}$.

Here we denote the basis $X_i - a_i$ of T_a^*X by $dx_{i,a}$, and the dual basis by $\partial_{i,a} = \partial_{\partial} x_i|_a$.

5.3.10. *Hilbert scheme as a moduli of configurations of identical particles.* First notice that so far I have in a sense cheated – Hilbert schemes were defined as sets but I never explained the structure of a variety (or a scheme). This is easy but will be postponed. Next, all calculations were done for affine spaces $X = \mathbb{A}^n$. Actually, the results are the same for smooth varieties nd the proofs are the same once one knows the basic facts on smooth varieties.

Theorem. [Fogarty] If X is a smooth surface then $X^{[n]}$ is smooth.

So, Fogarty noticed that the Hilbert schemes $X^{[n]}$ provide a smooth version of the moduli of unordered n-tuples of points for any smooth surface X.

This observation is the foundation of current attempts to extend our fine understanding of curves (i.e., the one dimensional mathematics), to surfaces (Nakajima etc.). This is an important project with applications in mathematics and stringy physics.

5.3.11. *The extra information that makes moduli smooth.* Our first idea for the moduli was $X^{(n)}$ and it turned out to be satisfactory[8] when $\dim(X) = 1$, but singular when $\dim(X)=2$. However, Fogarty says that when $\dim(X) = 2$ then $X^{[n]}$ is a smooth moduli of unordered n-tuples.

Question. What did we forget when we took symmetric square of the plane rather than the Hilbert square? What is the extra information in $X^{[2]}$ that the Hilbert-Chow maps forgets? The fiber of the Hilbert-Chow map at a double point is the space of lines in the tangent space at a (see 5.3.8.d). So the extra information that is missing in the symmetric square is a direction at double points:

At double points the extra information can be thought of as the direction in which one point approached the other.

So, the conclusion is that if we want unordered pairs to change smoothly, at the diagonal when the points collide we should remember how they collided.

5.3.12. *The moduli of unordered points beyond surfaces.* However, for X of dimension > 2 the problem seems to persist, neither of $(\mathbb{A}^3)^{[n]}$ and $(\mathbb{A}^3)^{(n)}$ is smooth.[9]

5.4. **The need for stacks.** The *stacks* are a certain generalization of varieties and schemes. We will not go through the formal definition of stacks, we will only understand them in terms of the *Interaction Principle* bellow. Here, we notice in examples that there are moduli that can not be constructed by IT quotients. Equivalently, there are actions of groups on varieties for which the IT quotient does not do the job well. The main example will be the moduli of quadrics in $\mathbb{P}^n$, for instance the familiar situation of quadric curves in $\mathbb{P}^2$.

5.4.1. *Need more then the invariant theory quotients.* Here are some examples of the failure, i.e., invariant theory quotient does not produce what we expect:

(1) When the multiplicative group G_m acts on $\mathbb{A}^n$, there are many orbits:

$$0 \textit{ and one orbit } L-\{0\} \textit{ for each line } L.$$

However, $\mathbb{A}^n//G_m$ is a point since $\mathcal{O}(\mathbb{A}^n)^{G_m} = \mathbb{k}$.

[8] and very very important!

[9] There is a solution in terms of dg-schemes but it is not clear to me whether this is what one wants, i.e., how useful it is.

(2) When GL_n acts on $\mathbb{A}^n$, over a field $\mathbb{k}$, there are two orbits: $\{0\}$ and the rest. Again, $\mathbb{A}^n//GL_n$ is just a point since $\mathcal{O}(\mathbb{A}^n)^{GL_n} = \mathbb{k}[X_1, ..., X_n]^{GL_n} = \mathbb{k}$.

For instance, for $n = 1$, the multiplicative group $G_m \stackrel{\text{def}}{=} GL_1 = \mathbb{A}^1 - \{0\}$ acts on $\mathbb{A}^1$ with orbits 0 and G_m, but $\mathbb{A}^1//G_m = \text{pt}$.

(3) G_m acts freely on $\mathbb{A}^n - \{0\}$. Here, the set theoretic quotient is $\mathbb{P}^{n-1}$. However $\mathcal{O}(\mathbb{A}^n - 0) = \mathcal{O}(\mathbb{A}^n)$ (for $n \geq 2$), hence $\mathcal{O}(\mathbb{A}^n - 0)^{G_m} = \mathbb{k}$ and $(\mathbb{A}^n - \{0\})//G_m = \text{pt}$.

The problems:

- The first obvious problem arises from different sizes of orbits: say in (2), the non-zero orbit is open and dense (if $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$), so invariant functions are constant on an open set and therefore by continuity everywhere. So the smaller orbit had no say, it was eaten by the larger orbit.

  Another way to say this is that problem arises from different sizes of stabilizers $G_x = \{g \in G;\ gx = x\}$ of points of X. Therefore, the best kind of action will be the *free* action, i.e., the action for which there are no stabilizers.
- The problem in (3) is that $\mathbb{P}^n$ can not be captured by global functions.

Let us also consider an example of a moduli problem where invariant theory quotient does not work, because there is an open orbit as in (2) above:

5.4.2. *Moduli of quadrics.* By a quadric in $\mathbb{P}^n$ we mean any projective subvariety Q given by a degree 2 homogeneous polynomial $G = \sum_{i \leq j} g_{ij} X_i X_j$. In this case we will have an interesting notion of when two quadrics are the same:

we say that two quadrics $P \subseteq \mathbb{P}^n$ and $Q \subseteq \mathbb{P}^n$ are isomorphic if there is an automorphism $g : \mathbb{P}^n \xrightarrow{\cong} \mathbb{P}^n$ of $\mathbb{P}^n$, such that $g(P) = Q$.

Lemma. (a) $\text{Aut}(\mathbb{P}^n) = PGL_{n+1}$.

(b) Isomorphism classes of quadrics in $\mathbb{P}^n$ are given by the orbits of GL_{n+1} in non-zero symmetric matrices: $(\mathcal{S}_{n+1} - \{0\}) \subseteq M_{n+1}$, for the action $g(S) \stackrel{\text{def}}{=} g \cdot S \cdot g^{tr}$.

(c) If $\mathbb{k} = \mathbb{C}$, the isomorphism classes are determined by the rank of the matrix.

Proof. (a) First, GL_{n+1} acts on the vector space $\mathbb{k}^{n+1}$, and then also on the set $\mathbb{P}^n(\mathbb{k})$ of lines in $\mathbb{k}^{n+1}$. Observe that the subgroup D of scalar matrices is isomorphic to $\mathbb{k}^*$ by $\mathbb{k}^* \ni c \mapsto c \cdot I_n \in D$, and that D fixes all lines. So D acts trivially, and therefore we get the action of the quotient group $GL_n/D \stackrel{\text{def}}{=} PGL_n$. The resulting map $PGL_n \to \text{Aut}(\mathbb{P}^n)$ is an isomorphism.[10]

[10] Left for later.

(b) To a symmetric matrix S one can attach a quadratic for Q_S. If we put the variables X_i into a row vector $\begin{pmatrix} X_1 & \cdots & X_n \end{pmatrix}$, then

$$Q_S(X) = X{\cdot}S{\cdot}X^{tr} = \sum_{i,j} s_{ij}X_iX_j.$$

(This is $G = \sum_{i\leq j} g_{ij}X_iX_j$ if $s_{ij} = \begin{cases} g_{ij} & \text{if } i=j \\ \frac{1}{2}g_{ij} & \text{if } i\neq j \end{cases}$.) Now, any $g \in GL_n$ takes X to $X{\cdot}g$, and in this way it affects the quadratic form:

$${}^{g}(Q_S)(X) \stackrel{\text{def}}{=} Q_S(X{\cdot}g) = (X{\cdot}g){\cdot}S{\cdot}(X{\cdot}g)^{tr} = X{\cdot}(g{\cdot}S{\cdot}g^{tr}){\cdot}X^{tr} = \ Q_{gSg^{tr}}(X).$$

(c) We use the fact that for $\Bbbk = \mathbb{C}$, any quadratic form can be diagonalized, i.e., after a linear change of variables it becomes a sum of squares $G_p = \ sum_1^p\ X_i^2$. So, each G-orbit in $\mathcal{S}$ contains a matrix of the form S_p that has p ones on the diagonal and the remaining entries are 0. Clearly, $rank(S_p) = p$. It remains to check that $rank(g{\cdot}S{\cdot}g^{tr}) = rank(S)$.

Conclusion. In $\mathbb{A}^n$ or $\mathbb{P}^n$ there are $n+1$ different quadrics, examples are given by $G_i = X_1^2 + \cdots + X_i^2$ for $1 \leq i \leq n+1$. So the moduli $\mathcal{M}$ of quadrics should have $n+1$ points $q_1, ..., q_{n+1}$. However, the quadrics of higher rank can degenerate to quadrics of lower rank, say the rank of $X^2 + tY^2$ is generically two, but it is one when $t = 0$. This means that q_1 should be approachable from q_2, i.e., that q_1 lies in the closure of q_2. So the moduli is a funny space with points q_i such that $\overline{q_{n+1}} \supseteq \overline{q_n} \supseteq \cdots \supseteq \overline{q_2} \supseteq \ \overline{q_1} = \ q_1$. In particular, point q_{n+1} is dense in $\mathcal{M}$ and the only closed point is q_1.

Certainly, such $\mathcal{M}$ is not an affine variety! Also, the invariant theory quotient not adequate (it doe not produce $\mathcal{M}$), since $\mathcal{S}_{n+1}//GL_{n+1}$ is just a point. The meaning of having only one point in $\mathcal{S}_{n+1}//GL_{n+1}$ (coming from the dense orbit of GL_{n+1} in $\mathcal{S}_{n+1}$), is that invariant theory construction notices the non-degenerate quadrics of type q_{n+1}, but not the degenerate ones.

Question. Can we make $\mathcal{M}$ into *something* like an affine variety?

Let us frame this in terms of group quotients into:

Can one make a quotient $\mathcal{S}_n/GL_n$ which will be a geometric space with $n+1$ $\mathbb{C}$-points?

5.5. **Adding spaces (and stacks) to varieties by the Interaction Principle.**

5.5.1. *Interaction Principle.* Here we push the *Observation Principle* idea, to the following level

To know a space X is the same as to know how it interacts with other spaces.

What this will mean for us is that we know variety X if for each variety Y we know the set $Map(Y, X)$. Taken step further, the principle suggests that

- If we have a "reasonable construction that associates to each variety Y a set F_Y", we can hope that this construction is a description of some space $\mathfrak{X}$ such that $Map(Y,\mathfrak{X}) = F_Y$ for each Y.

5.5.2. *Interaction Principle in categories: Yoneda lemma.* So, it suggests that a natural way to extend a given category of spaces $\mathcal{C}$ is to add to it all *functors* F on $\mathcal{C}$ that are in some sense alike the functors $Map(-,X)$ defined by spaces $X \in \mathcal{C}$.

This is a very general idea. The most familiar use is the introduction of distributions in analysis. In category theory, this idea is called *Yoneda lemma*, we explain in the appendix **??**, in **??**.

5.5.3. *Interaction Principle in sets: Distributions.* This is the most familiar instance of applying the above *Interaction Principle* in mathematics. It is more elementary then the Yoneda lemma in the sense that here an interaction will produce one number (rather then one set).

The idea of delta-functions $\delta_a,\ a \in \mathbb{R}$, is quite useful, say in physics δ_a appears when some particle is imagined to be concentrated at the point a, or δ_t is a unit impulse which is applied in one moment t. However, it has no existence in standard calculus: it should be a function that is zero outside a and still $\int_{-\infty}^{+\infty}\ \delta_a(x) = 1$. One way to make it into a mathematical object is to observe that it interacts with nice functions such as $C^\infty(\mathcal{R})$ using integral:

$$\int_{-\infty}^{+\infty} f(x)\cdot\delta_a(x) \ = \ f(a).$$

This leads to the definition of the the space $\mathcal{D}(\mathbb{R})$ of distributions on $\mathcal{R}$ as "things that interact reasonably with functions". Precisely, a distribution is a (continuous) linear functional on the space $\mathcal{S}$ of "nice functions" (or "test functions"). Then the delta-function at a can be defined as a distribution: this is the functional

$$\delta_a(f) = f(a),\ f \in \mathcal{S}.$$

What makes it useful is that

(1) Functions embed into distributions $C^\infty(\mathbb{R})\hookrightarrow\mathcal{D}(\mathbb{R})$ (function ϕ gives distribution $f\mapsto \int_{-\infty}^{+\infty}\ f(x)\cdot\phi(x))$.
(2) Calculus extends to distributions (there are notions of derivative, integral ...)

A similar pattern appears when we use the interaction idea to extend the range of objects in algebraic geometry. Our basic example will be the

5.6. The $\boxed{\textbf{true quotients } X/G}$ require spaces and stacks.

5.6.1. *The desire for good quotients.* We would like for a group G acting on an algebraic variety X to construct a geometric space X/G which will be a "good quotient" of X by G, in the sense that

(1) If X is smooth then so is X/G.
(2) The fibers of the quotient map $X \to X/G$ are all isomorphic to G.

Let us comment on the these conditions.

(1) is desirable – then we can calculate well on the quotient X/G, however it seems unreasonable. Remember that when $\{\pm 1\}$ acts on $\mathbb{A}^2_{x,y}$ the IT quotient is the cone $\{(u,v,z) \in \mathbb{A}^3;\ uv = z^2\}$. The singularity of the quotient cone at $(u,v,z) = (0,0,0)$ is there for a good reason. $(0,0,0)$ is the image of the origin $0 \in \mathbb{A}^2_{x,y}$, and while orbits $\{\pm p\}$ of $\{\pm 1\}$ in $\mathbb{A}^2$ usually have order two, there is a jump at the origin since $\{\pm 0\}$ has order one. So the discontinuity in the size of the orbit (or stabilizer) may cause singularity in the IT quotient.
(2) is satisfied for a set theoretic quotient when there are no stabilizers. Again, it seems impossible in general since for instance if the fibers of $pt \to pt/G$ should be G then we should find G inside pt.

We see that usually the desired quotient X/G does not exist – *as a variety (or a scheme).* Still, it exists and is important, for instance

- pt/G is called the classifying space of G,
- the G-equivariant cohomology of X (whatever that is) is best understood as the ordinary cohomology of X/G
$$H^*_G(X,\mathbb{Z}) \;=\; H^*(X/G,\mathbb{Z}),$$
- pt/C^* has a standard approximation $\mathbb{P}^\infty$.

5.6.2. *Spaces and stacks.* To find X/G we need to look into a world larger then $\mathbb{k}$ − *varieties* and we will look into two such

$$\mathbb{k}\text{-}Varieties \subseteq \mathbb{k}\text{-}Spaces \subseteq \mathbb{k}\text{-}Stacks.$$

Roughly, these are some distributional versions of $\mathbb{k}$-*Varieties*:

- The enlargement $\mathbb{k}$-*Varieties*⊆$\mathbb{k}$-*Spaces* is obtained by the Yoneda lemma, so
$$\mathbb{k}\text{-}Spaces \;=\; Funct(\mathbb{k}\text{-}Varieties^o, \mathcal{S}ets)$$
are functors from $\mathbb{k}$-varieties to sets, i.e.,
Spaces are variety-like objects that interact with varieties and produce sets.
- The enlargement $\mathbb{k}$-*Varieties*⊆$\mathbb{k}$-*Spaces* involves a generalization of the Yoneda lemma, in which sets are replaced by finer objects: groupoid categories. Roughly:
$$\mathbb{k}\text{-}Spaces \;=\; Funct(\mathbb{k}\text{-}Varieties^o, \mathcal{G}roupoid\mathcal{C}ategories).$$

Therefore,

Stacks are variety-like objects that interact with varieties and produce groupoid categories.

The idea is that when one tries to construct a useful (interesting) space, , i.e., a functor $\mathfrak{X} : \Bbbk\text{-}Varieties^o \to \mathcal{S}ets$, it often happens that the relevant sets $\mathfrak{X}(Y),\ Y \in \Bbbk\text{-}Varieties$; are often sets of isomorphism classes in some groupoid category, $\widetilde{\mathfrak{X}}(Y)$. So, one can ask whether the fundamental construction in this situation is $Y \mapsto \widetilde{\mathfrak{X}}(Y)$ (a stack!), rather then $Y \mapsto \mathfrak{X}(Y)$ (a space) ? The answer is YES: usually $\widetilde{\mathfrak{X}}$ is a better object, say $\mathfrak{X}$ may be singular and $\widetilde{\mathfrak{X}}$ smooth (the singularity is therefore not necessary, it is a produced by forgetting relevant information).

The step from spaces to stacks can be thought of as adding some group theory to the mix ("remembering the automorphisms groups").

5.6.3. $\Bbbk$*-space quotients* $\underline{X/G}$. Following the *Interaction Principle* above, to define X/G as a $\Bbbk$-space $\underline{X/G}$, we will specify for any algebraic variety Y how it interacts with X/G, i.e., we will

describe the set of maps $Map(Y, X/G)$ without invoking the quotient X/G,
i.e., in terms of the G-action on X.

We start with trying to answer the same question in the simplest situation where a "good quotient" X/G exists on the level of varieties, this happens in the case of

5.7. **Free actions (torsors).** We will formulate what we mean by a "free' action, first on the level of sets and then in a way that makes sense in other situations (i.e., categories): topological spaces, manifolds, varieties,...

5.7.1. *Torsors in sets.* Let G be a group acting on a set X. The quotient set X/G is the set of G-orbits in X. Recall that we say that G-action on Y is

- *transitive* if for any $a, b \in Y$ there is a $g \in G$ such that $b = g{\cdot}a$, i.e., G has one orbit in Y. Then for any $y \in Y$ we get a canonical G-identification $G/G_a \xrightarrow{\cong} X,\ gG_a \mapsto g{\cdot}a$.
- *simply transitive* if for any $a, b \in Y$ there is precisely one $g \in G$ such that $b = g{\cdot}a$. In other words, it is transitive and the stabilizers are trivial. Equivalently, for each $a \in Y$, the map $G \ni g \mapsto ga \in Y$ is a bijection.

We say that the action is <u>*free*</u> (in the set theoretic sense) if there are no stabilizers: $G_x = 1,\ x \in X$.[11] An equivalent way to describe this situation is the following notion of *G-torsors* [12], which is standard in mathematical physics.

A <u>*G-torsor*</u> over a set $\mathfrak{Y}$ consists of a map $P \xrightarrow{\pi} \mathfrak{Y}$ and a G-action on P, such that

$$(*) \qquad G \text{ acts simply transitively on each fiber.}$$

Notice that we in particular ask that G-preserves fibers of π, i.e., that π is a G-map for the trivial action on $\mathfrak{Y}$.

Lemma. (a) G-action on X is free iff the quotient map $X \to X/G$ is a G-torsor.

(b) $P \xrightarrow{\pi} \mathfrak{Y}$ is a G-torsor iff the map $G \times P \to P \times \mathfrak{Y} P,\ (g,p) \mapsto (gp,p)$ is a bijection.

Examples.

(1) $GL(V)$ acts simply transitively on the set $Fr(V)$ of bases $v = (v_1, ..., v_n)$ of a vector space, i.e., V is a $GL(V)$-torsor over a point.
The set $Fr(V)$ of bases $v = (v_1, ..., v_n)$ of a vector space V is a torsor for $GL(V)$
The set $Fr(V)$ of bases $v = (v_1, ..., v_n)$ of a vector space V is a torsor for $GL(V)$
(2) $V - 0 \to \mathbb{P}(V)$ is a torsor for G_m over $\mathbb{P}(V)$.
(3) If B is a subgroup of A then $A \to B \backslash A$ is an A-torsor.
(4) Any map $Y \xrightarrow{f} \mathfrak{X}$ can be used to <u>*pull-back*</u> a G-torsor $P \xrightarrow{\pi} \mathfrak{X}$ to a a G-torsor $f^*P \xrightarrow{f^*(\pi)} Y$. Space f^*P and the map $f^*\pi$ can be described fiber by fiber. The fiber $(f^*P)_y$ at $y \in Y$ is just the fiber $P_{f(y)}$ of π at $\pi(y) \in \mathfrak{X}$. A standard way to say this is (for more details see 5.11)

$$f^*P = \{(y,p) \in Y \times P;\ p \in P_{f(y)}\}.$$

Remark. Notice that the pull-back torsor f^*P is related to the original P by the G-map $f^*P \xrightarrow{\widetilde{f}} P,\ (y,p) \mapsto p$; which is characterized by the commutativity of the following diagram:

$$\begin{array}{ccc} f^*P & \xrightarrow{\widetilde{f}} & P \\ {\scriptstyle f^*(\pi)}\downarrow & & {\scriptstyle \pi}\downarrow \\ Y & \xrightarrow{f} & X \end{array} .$$

If we say that by definition of f^*P one has $(f^*X)_y = X_{\pi(y)}$, then the restriction of $\widetilde{f}$ to a fiber $(f^*X)_y \to X_{\pi(y)}$ is just the identity map.

[11] Recall that in some of the above examples, the stabilizers $G_x = \{g \in G;\ gx = x\}$ of points of X caused problems.

[12] Also called *principal G-bundles* or just *G-bundles.*

5.7.2. *Torsors in other categories.* Let us now consider group actions a category $\mathcal{C}$ which is something like topological spaces, manifolds or $\Bbbk$-varieties.

We will say that an action of G on X (in a category $\mathcal{C}$) is *free*, if it can be completed to a G-torsor $X \to \mathfrak{X}$ (in the category $\mathcal{C}$). Then we say that $\mathfrak{X}$ is the *free quotient* of X by G (in $\mathcal{C}$).

A *G-torsor* over $\mathfrak{Y}$ consists of a map $P \xrightarrow{\pi} \mathfrak{Y}$ and a G-action on P, which is locally trivial in the sense that

each $y \in \mathfrak{Y}$ *has a neighborhood* U *such that over* U *one can identify* P *with* $U \times G$.

So we ask that there is a G-isomorphism $\phi : P|U \xrightarrow{\cong} U \times G$, which identifies π with the projection pr_U, i.e.,

$$\begin{array}{ccc} P|_U & \xrightarrow[\cong]{\phi} & U \times G \\ {\scriptstyle \pi|\pi^{-1}U}\downarrow & & \downarrow{\scriptstyle pr_U} \\ U & \xrightarrow{=} & U. \end{array}$$

Examples. (1) Let Σ be a smooth surface in the sense of a 2-dimensional real manifold. At each point $p \in \Sigma$ consider the set $or_{\Sigma,p}$ of orientations of Σ at p. It consists of two opposite orientations, so it has a simply transitive action of $\mathbb{Z}_2 \cong \{\pm 1\}$. Since orientations *locally* extend canonically: (i) $or_\Sigma = \cup_{p\in\Sigma}\ or_{\Sigma,p}$ has a canonical structure of a manifold (a double cover of Σ), (ii) or_Σ is a $\mathbb{Z}_2$-torsor over Σ. Actually, the same holds for any real manifold Σ.

(2) Let X be a $\Bbbk$-variety. Rank n vector bundles V over X are the same as GL_n-torsors over X. For instance, a vector bundle $V \to X$ defines a GL_n-torsor $Fr(V) \to X$ of *frames* of V. Here, $Fr(V)$ is defined so that the fiber of $Fr(V)$ at $x \in X$ is the set $Fr(V_x)$ of all bases $v = (v_1, ..., v_n)$ of the vector space V_x. Since one can identify $Fr(V_x)$ with the set $Isom(\Bbbk^n, V)$ of isomorphisms of vector spaces, we see how GL_n acts on it.[13]

5.7.3. *Local and global.* By definition, any G-torsor $P \xrightarrow{\pi} \mathfrak{Y}$ is *locally* trivial, i.e., locally can be identified with $\mathfrak{Y} \times G$. So, the interesting part is the global behavior. The first question is whether P is globally trivial, i.e., $P \xrightarrow{\cong} \mathfrak{Y} \times G$?

Lemma. Torsor P is trivial iff it has a global section, i.e., a map $\Sigma \xrightarrow{\sigma} P$ such that $\sigma(s) \in \pi^{-1}(s)$, i.e., $\pi \circ \sigma = id_{\mathfrak{Y}}$.

Proof. Actually sections σ of P are the same as trivializations $\iota : \mathfrak{Y} \times G \xrightarrow{\cong} P$. Here, $\mathfrak{Y} \times G$ has a canonical section 1_G, so ι gives a section $\sigma = \iota \circ 1_G$. In the opposite direction, a section σ gives the trivialization $\iota : P \xrightarrow{\cong} \mathfrak{Y} \times G$ by $\iota(y, g) \stackrel{\text{def}}{=} g \cdot \sigma(y)$, $y \in \mathfrak{Y}$, $g \in G$.

[13]This is not totally the same as the first example of its sort: $Fr(V)$ is a torsor for GL_n and for $GL(V)$ (a bitorsor for $(GL_n, GL(V))$!).

Corollary. The $\{\pm 1\}$-torsor $or_S \to S$ is trivial iff S is orientable.

Proof. An orientation on S is a global section of $or_S \to S$.

Remark. This example indicates what kind of global structure can be encoded in a torsor.

5.7.4. *The category $\mathcal{M}_G(X)$ of G-torsors over X.* For G-torsors $P \xrightarrow{p} X$ and $Q \xrightarrow{q} X$, $\mathrm{Hom}_{\mathcal{M}_G(X)}(P,Q)$ consists of all maps $X \xrightarrow{\alpha} Q$ which are G-maps over X, i.e.,

- Diagram $\begin{array}{ccc} P & \xrightarrow{\alpha} & Q \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{=} & X \end{array}$ commutes. This can be stated as: $q(\alpha(a)) = p(a),\ a \in P$, or:

  for each $x \in X$, α maps the the fiber P_x of to the fiber Q_x.
- $\alpha(g{\cdot}a) = g{\cdot}\alpha(g),\ g \in G,\ a \in P.$

One easily checks that this gives a category.

Lemma. (a) Let ${}_GG$ denote G viewed as a set with an action of the group G by the *left multiplication*, then the right multiplication $R_g(x) \overset{\text{def}}{=} xg^{-1}$ $(x, g \in G)$, gives an identification

$$\mathrm{Hom}_{G-SSets}({}_GG, {}_GG) \underset{\cong}{\xleftarrow{R}} G.$$

(b) The category $\mathcal{M}_G(X)$ of G-torsors over X is a groupoid category, i.e., each map is an isomorphism.

Proof. (b) We need to show that any map $\alpha \in \mathrm{Hom}_{\mathcal{M}_G(X)}(P,Q)$ is an isomorphism. So we need to see that each of the maps of fibers $\alpha : P_x \to Q_x$ is an isomorphism. Therefore is remains to see that if G acts simply transitively on $\mathcal{P}$ and $\mathcal{Q}$, any G-map $\mathcal{P} \xrightarrow{f} \mathcal{Q}$ is an isomorphism, but this is clear.

5.7.5. *Categorical characterizations of quotients X/G in the case of free actions.* In general in a category $\mathcal{C}$, we will say that an action of G on X is free, if it can be completed to a G-torsor $X \to \mathfrak{X}$. Then we say that $\mathfrak{X}$ is the *free quotient* of X by G. This makes sense because for any torsor $X \to \mathfrak{X}$, $\mathfrak{X}$ is really the set of G-orbits in X, so $\mathfrak{X}$ is the set of orbits X/G organized into an object of $\mathcal{C}$.

Now, for any $Y \in \mathcal{C}$ we want to describe the functions $Map(Y, X/G)$ purely in terms of the G-action on X. Since X is G-torsor over X/G, a map $f : Y \to X/G$ gives a pull-back of this torsor to a G-torsor $P \overset{\text{def}}{=} f^*X \xrightarrow{f^*(\pi)} Y$ over Y. Moreover, $P = f^*X$ comes with a

G-map $P \xrightarrow{\widetilde{f}} X$ such that the diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\widetilde{f}} & X \\ {\scriptstyle f^*(\pi)}\downarrow & & {\scriptstyle \pi}\downarrow \\ Y & \xrightarrow{f} & X/G \end{array} .$$

This leads to

Lemma. A map from Y to X/G is the same as an isomorphism class of pairs (P, F) of a G-torsor P over Y, and a G-map $P \xrightarrow{F} X$.

Proof. **(A)** From a pair (P, F) we get a map $F : Y\ to X/G$ by taking quotients: a G-map $F : P \to X$ gives a map $f = [Y \cong P/G \to X/G]$, i.e., $f(y) = F(p){\cdot}G$ when p is any element of the fiber P_y.

(B) If $(P, F) = (f^*X, \widetilde{f})$ for some $f : Y \to X$, then the above procedure recovers f from (P, F).

(C) The meaning of the expression "isomorphism classes" is that we say that two pairs (P, F) and (Q, G) are isomorphic if one can identify P and Q in a way compatible with the G-actions and the relation to X and Y, i.e., if there is an isomorphism of G-spaces $\phi : P \to Q$ such that

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ {\scriptstyle F}\uparrow & & {\scriptstyle G}\uparrow \\ P & \xrightarrow{\phi} & Q \\ \downarrow & & \downarrow \\ Y & \xrightarrow{=} & Y. \end{array}$$

So, the isomorphic pairs really contain the same information and we should not distinguish them. More precisely, what one needs to check is

Two pairs induce the same map $Y \to G/X$ iff the pairs are isomorphic!

5.8. **Space quotients $\underline{X/G}$.**

5.8.1. *Hope.* From the above examination of the case when the quotient exists, we *hope* that a "good quotient" X/G, whatever it is, will satisfy:

- For any variety Y, $Map(Y, X/G)$ is the set of
  all isomorphism classes of pairs (P, F) of a G-torsor P over Y, and a G-map $P \to X$.

5.8.2. *Definition.* Notice that we can turn the story around and use this *hope* as a *definition* of a $\Bbbk$-space $\underline{X/G}$, i.e., a functor $\underline{X/G} : \Bbbk\text{-}Varieties \to \mathcal{S}ets$, by

$$\underline{X/G}(Y) \overset{def}{=} \textit{isomorphism classes of a } G\textit{-torsor } P \textit{ over } Y \textit{ and a } G\textit{-map } P \to X.$$

5.8.3. *Relation to cohomology.* In particular, notice that the case of $X = pt$ is already interesting

- $\underline{pt/G}(Y)$ = the set of all isomorphism classes of G-torsors P over Y.

This set is usually called the 1^{st} *cohomology group of* X *with values in the group* G and denoted

$$H^1(X, G) \overset{\text{def}}{=} \underline{pt/G}(X) = \text{Hom}_{\Bbbk-Spaces}(X, \underline{\text{pt}/G}).$$

The last equality is by Yoneda lemma (theorem **??**).

5.8.4. Is $\underline{X/G}$ the "good quotient" we are looking for? A little checking shows that it does not satisfy one of our requirements, which we stated as

The fibers of $X \to X/G$ *are isomorphic to* G.

(With our enriched vocabulary we can restate this as: "$X \to X/G$ is a G-torsor".)

5.9. **Stack quotient** X/G.

5.9.1. *The correction of* $\underline{X/G}$ *to* X/G. The above $\Bbbk - Space\ version$ of the quotient turns out to be close but not perfect. The subtlety is that when the action is not free we want to remember the stabilizers in some way. A fancy way to say this is: $Map(Y, X/G)$ is not just a set, it is a *category*! Then the correct definition is

5.9.2. *Definition.*

- For any variety Y $Map(Y, X/G)$ is the category of all pairs (P, F) of a G-torsor P over Y, and a G-map $P \to X$.

5.9.3. *Category* $Map(Y, X/G)$. Here, one defines the category structure on the pairs so that $\text{Hom}_{Map(Y,X/G)}[(P, F), (Q, G)]$ is the set of all G-maps $\phi : P \to Q$ such that

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ {\scriptstyle F}\uparrow & & {\scriptstyle G}\uparrow \\ P & \xrightarrow{\phi} & Q \\ \downarrow & & \downarrow \\ Y & \xrightarrow{=} & Y. \end{array}$$

The commutativity of the lower square means that $\phi : P \to Q$ is a map of G-torsors over Y, and the upper square says that ϕ intertwines maps F, G of P, Q to X.

Lemma. Category $Map(Y, X/G)$ is a groupoid category, i.e., any map is an isomorphism.

Proof. This follows from the statement for the category $\mathcal{M}_G(Y)$.

5.9.4. *Remarks.* (1) The step we are taking now takes us beyond the world of categories – the totality $\Bbbk - Stacks$ of all $\Bbbk$-stacks, has more structure then a category since $\mathrm{Hom}_{Stacks}(\mathfrak{X}, \mathfrak{Y})$ is not a just a set but rather a category! Actually the totality of all $\Bbbk$-stacks is an example of a notion of $\underline{2 - categories}$, which is generalization of categories in a way similar to how categories. generalize sets.

(2) The difference between the set $Map(Y, \underline{X/G})$ and the category $Map(Y, X/G)$ is sort of small – the morphisms in the category are all isomorphisms, so this is just the information that is needed to form the isomorphism classes of objects. However, $Map(Y, X/G)$ does remember some information that is lost in $Map(Y, \underline{X/G})$ – the set of all automorphisms of each map (P, F) in $Map(Y, \underline{X/G})$.

(3) To be able to think and calculate with stacks (as with say varieties), requires (of course), extending all our algebraic geometry formalism to stacks (just as we extended calculus to distributions). The first step is (as for distributions) to introduce some natural topology on the category of $\Bbbk$-*Varieties* and restrict ourselves to $\Bbbk$-space which are continuous in this topology. We will skip all the details, but the basic idea will be seen to be useful. For instance, when we study the maps from $\mathbb{P}^1$ to $\mathbb{P}^1$ or to any flag variety.

5.10. **The fibers of $X \to X/G$.** Here we sketch, how the difference between the $\Bbbk$-space $\underline{X/G}$ and the $\Bbbk$-stack X/G influence the fibers of the quotient map.

5.10.1. *Fibered products of sets and stacks.* For two maps of sets $A \xrightarrow{\alpha} C \xleftarrow{\beta} B$, the fibered product $A \times_C B$ (see 5.11), is the set of all pairs $(a, b) \in A \times B$ such that $\alpha(a) = \beta(b)$. So, $Map(Y, A \times_C B)$ is the set of all pairs $(p, q) \in Map(Y, A) \times Map(Y, B)$, such that one has equality $\alpha \circ p = \beta \circ q$ in the set $Map(Y, C)$.

However, for two maps of stacks $A \xrightarrow{\alpha} C \xleftarrow{\beta} B$ and a variety Y, for a pair $(p, q) \in Map(Y, A) \times Map(Y, B)$, $\alpha \circ p$, $\beta \circ q$ live in a category (rather then a set) $Map(Y, C)$. Now it is not very interesting whether they are the same, but rather whether they are isomorphic. However, this is still not enough – for consistent thinking we need more then "objects $\mathcal{P}$, $\mathcal{Q}$ are isomorphic", we need to remember which isomorphism we are using to compare these two objects. This *forces* the following definition of the fibered product for stacks

$Map(Y, A \times_C B)$ is the set of all triples *$(p, q, \rho$ where $p \in Map(Y, A)$, $q \in Map(Y, B)$ and ρ is an isomorphism $\rho : \alpha \circ p \xrightarrow{\cong} \beta \circ q$ in the category $Map(Y, C)$.*

5.10.2. *The fibers of $X \to X/G$.* We would like to see that the fibers of $X \xrightarrow{\pi} X/G$ are really isomorphic to G. A point of X/G is a map $pt \xrightarrow{\phi} X/G$ and the fiber of π at the point ϕ is the ϕ-pull-back $pt\times_{X/G}X$ of $X \xrightarrow{\pi} X/G$. What is $pt\times_{X/G}X$? since we are working with stacks this fiber is again a stack, hence a functor

$$pt\times_{X/G}X : \Bbbk\text{-}Varieties \to \mathcal{S}ets,$$

and we know that $(pt\times_{X/G}X)(Y)$ consists of *triples* $(p,q,\rho$ where $p \in Map(Y,pt)$, $q \in Map(Y,X)$ and ρ is an isomorphism $\rho : \phi\circ p\xrightarrow{\cong}\pi\circ q$ in the category $Map(Y,X/G)$. Now, the set one can associate to a $\Bbbk$-space $\mathfrak{Z}$ is the

$$set\ of\ points\ of\ Z \overset{def}{=} Map(pt,Z).$$

So we are interested in triples of $pt \xrightarrow{p} pt$, $pt \xrightarrow{q} X$ and $\rho : \phi\circ p\xrightarrow{\cong}\pi\circ q$ in the category $Map(pt,X/G)$.

5.10.3. *Case $X = pt$.* Let $X = pt$ for simplicity, then the obvious choice of a map $pt \xrightarrow{\phi} X/G$ is $\phi = \pi$. Actually, this also the only one since the category $Map(pt,X/G)$ is the category of G-torsors on pt and any two are isomorphic[14] such a map ϕ is given by a consists of!).

Now, the points of $(pt\times_{X/G}X)(Y)$ consists of *triples* $(1_{\text{pt}},1_{\text{pt}},\rho)$ where ρ is an isomorphism of $\pi\xrightarrow{\cong}\pi$ in the category $Map(pt,pt/G) = \mathcal{M}_G(pt)$. This category has only one object (up to isomorphism), the trivial G-torsor $P = G \to pt$, and this is our π. However, the choices of ρ are given by $\text{Aut}(P) \cong G$.

5.11. **Appendix: Fibered Products, Base Change, Cartesian Squares.** The following very useful construction is the general background for the construction of the pull-back of torsors. I will state it for the sets but it is important in many other settings.

5.11.1. *Fibered products.* The *set X over a set B* means a map $X \xrightarrow{p} B$.

The product of two sets over B, $X \xrightarrow{p} B$ and $Y \xrightarrow{q} B$ (also called the fibered product) is the set

$$X\times_B Y \overset{\text{def}}{=} \{(x,y) \in X\times Y;\ p(x) = q(y)\} \subseteq X\times Y.$$

Notice that it comes with the projection maps $X \xleftarrow{pr_X} X\times_B Y \xrightarrow{pr_Y} Y$, and all maps fit into *commutative* square

$$\begin{array}{ccc} X\times_B Y & \xrightarrow{pr_Y} & Y \\ {\scriptstyle pr_X}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{p} & B \end{array}.$$

[14] So, *uniqueness* really means here uniqueness up to isomorphism.

5.11.2. A commutative square $\begin{array}{ccc} Z & \xrightarrow{\alpha} & Y \\ \downarrow{\scriptstyle\beta} & & {\scriptstyle q}\downarrow \\ X & \xrightarrow{p} & B \end{array}$ is called Cartesian if it is isomorphic to the square $\begin{array}{ccc} X\times_B Y & \xrightarrow{pr_Y} & Y \\ \downarrow{\scriptstyle pr_X} & & {\scriptstyle p}\downarrow \\ X & \xrightarrow{p} & B \end{array}$. This means that one can identify Z with $X\times_B Y$ so that α and β get identified with pr_X and pr_Y.

5.11.3. *Base Change or pull-back.* When we have a set Y over a set B, i.e., a map $Y \xrightarrow{q} B$, we may call B the base and we may think of what it would mean to change the base ? For any map $X \xrightarrow{p} B$ into B we can think of $X\times_B Y \xrightarrow{pr_X} X$ as the "p-pull-back" of $Y \xrightarrow{q} B$ from the base B to the base X, because for any $a \in X$ the fiber of $X\times_B Y \xrightarrow{pr_X} X$ at a is the same as the fiber of $Y \xrightarrow{q} B$ at $p(a)$:

$$pr_X{}^{-1}(A) \;=\; \{(x,y) \in X\times Y;\ p(x) = q(y) \quad \text{and} \quad x = a\} \;=\; \{(a,y) \in X\times Y;\ q(y) = p(a)\} \;\cong\; \{y \in X\times Y$$

So the base has changed but the fibers are the same. When viewed as the p-pull-back, of $Y \to B$, the fibered square can be denoted p^*P.

Notice that any Cartesian square can be viewed as a base change square in two ways.

5.11.4. *Pull-back of torsors.* If $P \to \mathfrak{Y}$ is a G-torsor, it is clear that for any map $f : \mathfrak{X} \to \mathfrak{Y}$, the pull-back $f^*P \stackrel{\text{def}}{=} \mathfrak{X}\times_{\mathfrak{Y}} P$ is a G-torsor over $\mathfrak{Y}$ ("the fibers do not change").

5.11.5. *Examples.* (1) If $X\subseteq B$ then the fibered square is just the restriction of Y to $X\subseteq B$. If $X, Y\subseteq B$ then the fibered square $X\times_B Y$ is just the intersection $X \cap Y$.

(2) For any map $X \xrightarrow{\pi} S$, the fibered square $X\times_B X\subseteq X^2$ is just the equivalence relation "$\pi(a) = \pi(b)$" on X.

5.11.6. *Algebraic geometry.* If $X \xrightarrow{p} B$ and $Y \xrightarrow{q} B$ are maps of $\Bbbk$-varieties (or schemes) then $X\times_B Y$ can be constructed on the same level. For instance iff X, Y, B are affine varieties then so is $X\times_B Y$ when constructed via

$$\mathcal{O}(X\times_B Y) \;=\; \mathcal{O}(X)\otimes_{\mathcal{O}(B)}\mathcal{O}(Y).$$