IDEA OF SPACE IN ALGEBRAIC GEOMETRY

A picture book of algebraic geometry

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Contents

0. **Intro**

0.1. **This text.** This course is an introduction to the *vocabulary* and *methods* of algebraic geometry, geared towards the use of algebraic geometry in various areas of mathematics: number theory, representation theory, combinatorics, mathematical physics. This is the introductory part. In non-vegetarian terms, these are some of the bones of algebraic geometry, but there is not much meat on these bones. After this one would like to start from the beginning and achieve larger precision, clarity and in particular competence. However, these goals are beyond this text.

0.1.1. *Prerequisites.* The text requires some basic familiarity with *algebra* (rings, modules, groups), and in later parts also the *complex analysis* (one variable).

0.1.2. *Reading.* A source with classical and "elementary" flavor is:

Shafarevich Igor R., Basic algebraic geometry (Springer-Verlag).

(Part 1: *Varieties in projective space*, and Part 2: *Schemes and complex manifolds.*) There is a soft-cover as well as the hard-cover edition.

A (more) modern treatment is:

Hartshorne Robin, Algebraic geometry
(Graduate Texts in Mathematics, No. 52. Springer-Verlag).

Both of these are over 20 years old, and there are many more modern treatments.[1]

0.2. **Formation of spaces useful for a given problem.** One of the characteristics of algebraic geometry is that it has gone particularly far in developing more and more abstract notions of space that are *useful*, i.e., new ways of thinking do solve old problems. We will introduce some of these notions of space, and try to indicate why these classes of spaces were introduced and what do they do for us.

The moral here will be that for a given problem you may want to find the notion of a geometric space that will be useful. So, suppose one wants to understand some object X from a *geometric point of view* . For this one encodes some of its properties into saying that X is a **geometric space of the kind $\mathfrak{X}$**. Here, $\mathfrak{X}$ could be something like

- Set,
- Topological space,
- Manifold,
- Algebraic variety,

[1]However, the most comprehensive treatment of modern algebraic geometry is even older:

Grothendieck, Diedonne, Elements de Geometrie Algebrique

It is published in a number of issues of IHES. Hartshorne's book is largely an introduction to this work.

- Scheme,
- Stack or n-stack,
- Differential Graded Scheme
- Non-commutative space a la Connes, etc.

Rather then going through encyclopedia of definitions, let us try to see the *principles* which historically pushed the introduction of certain classes of spaces. Some ideas we will emphasize:

- *Observation Principle: Space is what you observe.* This we interpret as the *functional* view, where one thinks of a space X in terms of the *algebra* of functions on X, and this introduces the use of ALGEBRA in GEOMETRY. An extended form: *X is how X interacts with others*, leads to a *distributional* extension: , we add to spaces objects that that interact with all spaces.
- *Understand solutions of systems of polynomial equations.* This is the origin of algebraic geometry.
- *Stability Principle.* It says that the set of solutions is stable under perturbations of the system. It would be nice if this were true since we could gain information on a given system by studying its deformations. So we *make it true* by adjusting the definitions, i.e., by fine-tuning our focus. This leads to the use of algebraically closed fields, projective spaces, infinitesimals, homological algebra, etc.
- *Theories are example driven.*
- *Formation of moduli.* This is the wish to make the set of isomorphism classes of objects of a certain kind, into a geometric space. Then one could study the *totality* of such objects by geometric means. This is roughly the same question as being able to make geometric quotients X/G of spaces by groups that act on them.
- *Include the number theory.* We will use this wish to force a geometric meaning to *all commutative rings*.
- *Opportunism.* A specific setting for algebro-geometric problems often offers approaches that go beyond the algebro-geometric techniques.
- *Linearization.* In algebraic geometry, one would like to "linearize" various kind of data on a given variety by encoding it as a data on some algebraic group. The standard example is the Jacobian of a curve. [2]
- *Hidden part of constructions.* The technique of homological algebra (passage from abelian groups to complexes of abelian groups) is used to uncovers some less obvious parts of a picture.
- *Relation of local and global.* The sheaf theory is a technique which assembles local information onto global in a very efficient way.

0.3. **Space is what you observe.** We are likely to think of spaces that consist of points, so such space is a *set of points*. We are usually interested in objects with more organization

[2]The most spectacular version of this idea is roughly to put algebraic varieties into an abelian category setting (*category of motives*).

then just a set. The simplest form of additional organization may be *topology*, i.e., a vague prescription of what is close to what. In the next step we often use the *Observation Principle*

we think of X as a space of kind $\mathfrak{X}$ if on X we observe objects of class $\mathfrak{X}$.

This is a part of terminology in physics: we study a system through *observables*, i.e., things that can be observed, i.e., measured. An observable on our object X will be some kind of a function on X so that it can be measured at each point. For instance on the real line $X = \mathbb{R}$ we have studied

- All $\mathbb{R}$-valued functions,
- Continuous functions $C(\mathbb{R})$,
- Smooth (infinitely differentiable) functions $C^\infty(\mathbb{R})$,
- Polynomials $\mathcal{O}(\mathbb{R}) = \mathbb{R}[x]$,
- Analytic functions $\mathcal{O}_{an}(\mathbb{R})$

etc, and then $\mathfrak{X}$ depended on what functions we were interested in – we would think of X respectively as a set, topological space, manifold, algebraic variety, analytic manifold. On a plane $\mathbb{R}^2$ we would also have holomorphic functions $\mathcal{O}_{an}(\mathbb{C})$ and holomorphic polynomials $\mathcal{O}(\mathbb{C}) = \mathbb{C}[z]$, so we could think of it as 2d real manifold or a 1d complex manifold, 1d algebraic variety etc.

0.4. **Algebraic Geometry: combine A and G.** We view X as a space of kind $\mathfrak{X}$ if on X we can observe functions of kind $\mathfrak{X}$. If our observables are functions $\mathcal{O}(X)$ on X with values in a ring $\Bbbk$ (something like $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$), then $\mathcal{O}(X)$ is a ring (one adds and multiplies functions pointwise). This puts us (in a very general sense) in *Algebraic Geometry*, since we can combine the geometric understanding of X with the algebraic analysis of the ring $\mathcal{O}(X)$.

0.5. **Global spaces in algebraic geometry.** One of the fundamental geometric ideas is the *Relation of local and global objects*. For instance the analysis on $\mathbb{R}^n$ or $\mathbb{C}^n$ is the local precursor of the global subject of analysis on manifolds. In fact, even the notion of a manifold (a "global object"), is obtained by gluing together some open pieces of $\mathbb{R}^n$ or $\mathbb{C}^n$ (the "local pieces" of our LEGGO game).

0.5.1. In algebraic geometry one often introduces a class $\mathcal{C}$ of spaces in two stages.

(1) The *affine $\mathcal{C}$-spaces* X are the ones that are completely controlled by the algebra of ("global") functions $\mathcal{O}(X)$ on X.

    From this point of view defining the class of *affine $\mathcal{C}$-spaces* is the same as defining a certain class $\mathcal{A}$ of commutative rings: the rings which appear as rings of functions on affine $\mathcal{C}$-varieties.

(2) Now the class $\mathcal{C}$ is defined as the class of spaces obtained by gluing together the affine $\mathcal{C}$-spaces.

So, general $\mathcal{C}$-spaces are "global" objects obtained by gluing together several affine $\mathcal{C}$-spaces, so we consider from this point of view the affine $\mathcal{C}$-spaces as the "local version" of the notion of the $\mathcal{C}$-space. In algebraic terms, a global object may not be captured by a *single* algebra, but rather by a *system of algebras*.

Examples of this strategy are notions of *Algebraic Varieties* and *Schemes*.

0.5.2. *Affine, Projective, Quasiprojective and Algebraic varieties.* The class of spaces here is the class $\mathcal{A}lg\mathcal{V}ar_{\Bbbk}$ of Algebraic Varieties over an algebraically closed field $\Bbbk$. The summary bellow will only make sense later.

- One starts with the local version, the class $\mathcal{A}ff\mathcal{V}ar_{\Bbbk}$ of Affine Varieties (short for: Affine Algebraic Varieties). It consists of subsets of affine spaces $\mathbb{A}^n(\Bbbk) = \Bbbk^n$ given by systems of polynomial equations.[3]
- Now $\mathcal{A}lg\mathcal{V}ar_{\Bbbk}$ consists of spaces that have a finite open cover by affine varieties: $X = U_1 \cup \cdots \cup U_n$.
- Projective varieties form a subclass $\mathcal{P}roj\mathcal{V}ar_{\Bbbk}$ of $\mathcal{A}lg\mathcal{V}ar$ that one can describe directly as subsets of projective spaces $\mathbb{P}^n(\Bbbk)$ given by systems of *homogeneous* polynomial equations.
- The class $q\mathcal{P}roj\mathcal{V}ar_{\Bbbk}$ of Quasiprojective Varieties (the most useful generality), consists of all open subvarieties of Projective Varieties. So,

$$\mathcal{P}roj\mathcal{V}ar_{\Bbbk} \ \cup \ \mathcal{A}ff\mathcal{V}ar_{\Bbbk} \ \subseteq \ q\mathcal{P}roj\mathcal{V}ar_{\Bbbk} \ \subseteq \ \mathcal{A}lg\mathcal{V}ar_{\Bbbk}.$$

0.5.3. *Schemes.* First, *affine schemes* are the class of geometric spaces that corresponds to *all commutative rings*. Then, *schemes* are spaces that have an open cover by affine schemes. The main point here will be the first step – finding a geometric way to think of all commutative rings. The basic examples will be the schemes corresponding to the rings

(1) the dual numbers $\Bbbk[X]/X^2$,
(2) formal power series $\Bbbk[[X]]$,
(3) integers $\mathbb{Z}$.

0.6. **Transcendental methods in complex geometry.** The study of algebraic varieties over the field $\mathbb{C}$ of complex numbers (which itself has more then just the algebraic structure of a field), benefits from the use of non-algebraic ("transcendental") methods such as complex analysis (the theory of holomorphic functions), the classical topology and differential geometry.[4]

We will use complex analysis of one variable in the study of complex curves. This provides great insights through the relation of analysis to topology and through the study of

[3]As we will see, the corresponding class of commutative rings are the finitely generated $\Bbbk$-algebras without nilpotents.

[4]Similarly, for varieties defined over a *finite field*, one gets a beautiful information by counting.

integration of 1-forms over paths. But we will also use use complex analysis as a shortcut when we could do as well (and better) with algebraic methods – just because we know complex analysis we skip some parts of algebraic theory which work "the same".

0.7. **Curves.** Algebraic curves are the best understood part of algebraic geometry since these are one-dimensional objects.

We start with the very rich example of curves given in a plane by a degree three equation. Then we sketch the extension of these ideas to general curves. This extension is really the study of the notion of a Jacobian of the curve, and we look at the Jacobian from several points of view (algebra, analysis, geometry). The Jacobian of a curve C is *the* commutative group attached to C, and its main role is that one can "linearize" data on the curve (of certain type) by passing to the Jacobian. We mention that this point of view is essential in number theory (the *geometric class field theory*).

The main calculational principle on a curve is the *Riemann-Roch theorem*, and we will deduce it using sheaves.

The last two topics are general tools (not particular to geometry), that are standard in algebraic geometry: *homological algebra* and *sheaves*.

0.8. **Homological algebra.** One tries to apply homological algebra to constructions that morally should contain more information then meets the eye. If it applies, homological algebra produces "derived" versions of the construction which contain the "hidden" information. Technically, the main idea is to embed the interesting setting into a larger world of *complexes*, in which less information gets lost. We will introduce this technique through a geometric example, the notion of *dg-schemes* (*differential graded schemes*). This is a generalization of schemes that improves some basic operations, such as taking fibers of a map, or tanking intersection of two algebraic varieties inside a third.

Here, the role of homological algebra is that it allows construction of more subtle notions of spaces.[5] However, the most standard application of homological algebra in algebraic geometry is the cohomology of

0.9. **Sheaves.** Sheaves are a framework for dealing with the omnipresent problem of relating local and global information on a space. The global information is codified as the functor $\Gamma(X,-)$ of global sections of sheaves on a topological space X. When a sheaf has few global sections, more information may be contained in the derived construction – the cohomology of sheaves.

We will use sheaf cohomology to count the number of global meromorphic functions on a curve that satisfy specific conditions on the positions of poles and zeros. The basic

[5]Another exciting development of this kind is the notion of a *D-brane*. This is a geometric space of a certain kind in string theory (contemporary physics), whose mathematical formulation is a highly sophisticated construct of homological algebra.

tool in such calculations is the *Riemann-Roch theorem*. The sheaf theoretic point of view reformulates the problem as the question of the size of cohomology of line bundles on the curve. The sheaf theory is quite flexible,[6] and the sheaf-theoretic formulation extend from curves to higher dimensions.

0.10. **Appendices.** Appendices cover various mathematical techniques that we use at some point

- *Multilinear Algebra* deals with tensoring of modules over a ring, this is the algebraic operation that geometrically corresponds to taking intersections and fibers of maps.
- The section on *Categories* explains the idea of adding distributional objects to a given setting, by the "Interaction principle": "X is how X interacts with others" (know in category theory as Yoneda lemma).
- In *Manifolds* we summarize a few facts needed to treat algebraic varieties over $\mathbb{C}$ as complex manifolds.
- The section on *Abelian categories* is a detailed treatment of the standard setting for homological algebra.
- In *Abelian category of sheaves of abelian groups*, we check that the category of sheaves of abelian groups on a given space, is an abelian category. This is what is needed in order to apply the homological algebra to sheaves, i.e., to develop the sheaf cohomology.

[6] It is often used as a bridge between different areas of mathematics.

1. **Algebraic Varieties**

Algebraic geometry historically started with polynomial functions on affine spaces $\mathbb{A}^n$.

1.0.1. *Affine spaces* $\mathbb{A}^n$. We start with a commutative ring $\Bbbk$ (something like $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$) and define the n-dimensional affine space $\mathbb{A}^n = \mathbb{A}^n(\Bbbk)$ as the set $\Bbbk^n$ of n-tuples of numbers from $\Bbbk$, with the ring of functions $\mathcal{O}(\mathbb{A}^n) = \Bbbk[X_1, ..., X_n]$ given by the polynomial functions.

1.0.2. *Affine algebraic varieties.* An *affine algebraic variety* X over $\Bbbk$[7] is a subset X of some $\mathbb{A}^n(\Bbbk)$ that can be described by several polynomial equations

$$X = \ \{a = (a_1, ..., a_n) \in \Bbbk^n;\ 0 = F_j(a),\ 1 \leq j \leq m\}$$

for some polynomials $F_j \in\ \mathcal{O}(\mathbb{A}^n)$. The definition offers at least three points of view on affine algebraic varieties

(1) Sets: X is a subset of $\mathbb{A}^n(\Bbbk)$,
(2) Algebra: On X one naturally has a $\Bbbk$-algebra $\mathcal{O}(X)$ of "polynomial functions on X", which one define as all restrictions of polynomials to X:

$$\mathcal{O}(X) \stackrel{\text{def}}{=} \{f|X;\ f \in \mathcal{O}(\mathbb{A}^n)\}.$$

(3) System of polynomial equations: X is described by equations $F_j = 0,\ 1 \leq j \leq m$.

1.0.3. *Varieties and schemes.* In the world of *algebraic varieties*, the first point is basic. We use algebra but when it gives different picture from sets, we adjust it to fit the sets.

In 1950s, Grothendieck discovered that varieties lie in the next world, the larger world of *schemes*. Here one trusts algebra completely and when differences arise, we massage the set theory.

We will spend most time on varieties and just rudiments of schemes, because schemes become useful when one finds difficulties in working with varieties.

1.1. **Relations between algebraic varieties are reflected in algebras of functions.**

For an affine algebraic variety $X \subseteq \mathbb{A}^n$, inclusion $X \subseteq \mathbb{A}^n$ is reflected in the restriction morphism of algebras $\mathcal{O}(\mathbb{A}^n) \xrightarrow{\rho} \mathcal{O}(X),\ \rho(f) = f|X$. Its kernel is the ideal $I_X \subseteq \mathcal{O}(\mathbb{A}^n)$ that consists of all polynomials that vanish on X. For instance I_X contains the defining equations F_j.

[7]Actually this is the standard terminology only if $\Bbbk$ is an algebraically closed field.

Since the restriction map is surjective by the definition of $\mathcal{O}(X)$, we find that the functions on X are described by

$$\mathcal{O}(X) = \ \mathcal{O}(\mathbb{A}^n)/I_X.$$

Example: Circles in $\mathbb{A}^2$*.* For instance, consider the "circle"

$$X = S_{a,b}(r) \stackrel{\text{def}}{=} \{(x,y)\in\Bbbk^2;\ (x-a)^2+(y-b)^2=r^2\}\subseteq\mathbb{A}^2,$$

In this case, the ideal $I_X{\subseteq}\mathcal{O}(\mathbb{A}^2)$ is generated by the above defining function $F=(X-a)^2+(Y-b)^2-r^2$, i.e., $I_X = \ \Bbbk[X,Y]/F{\cdot}\Bbbk[X,Y]$. So, the $\Bbbk$-algebra of functions on X has two generators X,Y related by one relation $(X-a)^2+(Y-b)^2-r^2=0$. So, $\mathcal{O}(X)$ has a basis

$$X^jY^j,\ 0\le i,\ 0\le j\le 1.$$

1.1.1. *Maps of varieties and maps of algebras.* To any map of varieties $f:X\to Y$ there corresponds a morphism of algebras of functions in the opposite direction

$$\mathcal{O}(Y)\xrightarrow{f^*}\mathcal{O}(X),$$

given by the *pull-back of functions*, i.e.,

$$f^*(\phi)=\phi{\circ}f.$$

Actually, this gives an identification

$$Map(X,Y)\ni f\mapsto f^*\in\mathrm{Hom}_{\Bbbk-alg}[\mathcal{O}(Y),\mathcal{O}(X)].$$

Example: maps into affine spaces. A map $f:Y\to\mathbb{A}^n$ consists of n component functions $f=(f_1,...,f_n),\ f_i\in\mathcal{O}(Y)$. The corresponding map $\Bbbk[x_1,...,x_n]=\mathcal{O}(\mathbb{A}^n)\xrightarrow{f^*}\mathcal{O}(X)$, sends generator x_i to $f^*x_i=x_i{\circ}f=f_i$. So,

The dictionary between maps of varieties $f:Y\to\mathbb{A}^n$ and morphism of algebras $\Bbbk[x_1,...,x_n]=\mathcal{O}(\mathbb{A}^n)\xrightarrow{F}\mathcal{O}(X)$, is:

- *F gives $f=(F(x_1),...,F(x_n))$, and*
- *f gives F such that $F(x_i)$ is the i^{th} component function f_i of f.*

1.1.2. *Constructions in geometry and algebra.* Set theoretic operations have algebraic incarnations. For $X,Y{\subseteq}\mathbb{A}^n$, the equations of the *intersection* $X\cap Y$ are obtained by taking the union of equations of X and of Y, for the algebras it will turn out to involve the operation of tensoring

$$\mathcal{O}(X\cap_{\mathbb{A}^n}Y)= \ \mathcal{O}(X){\otimes}_{\mathcal{O}(\mathbb{A}^n)}\mathcal{O}(Y).$$

The equations of the *union* $X\cup Y$ are obtained by multiplying the equations of X and of Y.

2. **Stability of solutions (intersections)**

By the stability of the intersections of algebraic varieties $X, Y \subseteq \mathbb{A}^n$ we mean that a small motion (perturbation) should usually cause no change in the nature of the intersection.

2.0.3. *Intersections of circles.* Our motivation in this section comes from intersecting lines and circles in an affine plane $\mathbb{A}^2(\Bbbk)$.

We start with the most familiar $\Bbbk = \mathbb{R}$. If two circles $X = \{f = 0\}$ and $Y = \{g = 0\}$ in $\mathbb{A}^2(\mathbb{R}) = \mathbb{R}^2$ meet, they are likely to meet in two points. If we move them a little, they still meet in two points. However, if we move them more, we get two more behaviors:

$$X \cap Y = \begin{cases} \text{two points,} \\ \text{one point,} \\ \text{no points;} \end{cases}$$

so our stability seems to break.

2.1. **Passage to algebraically closed fields.** Why is it that from the situations of having nonempty intersection we get to empty intersection; i.e. from having solutions to the system $f = g = 0$, to no solutions?

The first observation is that such things happen in a simpler case, for $c \in \mathbb{R}$

$$Z_c = \{x \in \Bbbk = \mathbb{R};\ x^2 = c\} \subseteq \mathbb{A}^1(\mathbb{R}) = \mathbb{R}.$$

While there are two points in Z_c (i.e., two solutions) for $c > 0$, there are none for $c < 0$. The problem is familiar: $\mathbb{R}$ is not algebraically closed, i.e. there are polynomial equations over $\mathbb{R}$ that have no solutions over $\mathbb{R}$.

This historically led to the introduction of complex numbers, and it turns out that passing from $\mathbb{R}$ to $\mathbb{C}$ increases the stability solves our problem: for generic circles X and Y in $\mathbb{A}^2(\mathbb{C})$, the intersection of X and Y consists of two points.

2.1.1. *Generic point.* Here, *generic* means "not in a very special position"[8]

2.1.2. *Fewer exceptions over* $\mathbb{C}$. Over $\mathbb{C}$ there are still exceptions:[9]

- (i) $X \neq Y$ but the centers are the same,
- (ii) X and Y are tangent at one point,

[8] It is a standard idea in geometry but it is not easy to give it a precise general meaning. Grothendieck did it elegantly.

[9] So many exceptions! So, did we improve the situation by passing from $\mathbb{R}$ to $\mathbb{C}$? Yes, because in some sense over $\mathbb{R}$ there are as many bad positions as good, but over $\mathbb{C}$ the bad ones are a thin subset of all positions! Do you see this? The same behavior happens in a simpler situation:

The set $U(\mathbb{R})$ of $c \in \mathbb{R}$ such that $X^2 = c$ has two solutions over $\mathbb{R}$ is $[0, +\infty)$, and it is of the same size as its complement (the bad c's). However, the set $U(\mathbb{C})$ of $c \in \mathbb{C}$ such that $X^2 = c$ has two solutions over $\mathbb{C}$ has a small complement $\{0\}$.

- (iii) $X = Y$

Later we will come back and resolve even these exceptions.

2.1.3. *The moral.* It is easier to work over an algebraically closed field. Even if you are interested in what happens over $\mathbb{R}$, you may get the basic orientation by first understanding the solution over $\mathbb{C}$, and then you check what part of the solution appears over $\mathbb{R}$.

2.2. **Passage from affine varieties to projective varieties.** Two lines in $\mathbb{A}^2(\mathbb{R})$ are likely to meet in one point, however they may be parallel. In practice, this makes reasoning more complicated since in a situation with a bunch of lines we need to discuss various cases when some of them are parallel.

2.2.1. *Making parallel lines meet.* One can try to solve this by following the railroad track intuition: two parallel lines in a plane should meet, though only at ∞. So we try passing to a larger space then $\mathbb{A}^2(\mathbb{R})$ by adding something at ∞ of $\mathbb{A}^2$. (We hope that our problem is: "$\mathbb{A}^2$ has a hole at ∞".)

What should we add? If we add just one point, $\mathbb{R}^2 \cup \{\infty\}$, then all lines should go through it and the size of $L_1 \cap L_2$ could be 2. Not good, the infinite points of lines that meet in $\mathbb{A}^2$ should be different. So we add one line for each class of parallel lines. Since each such class contains precisely one line through the origin, we can say this in a simpler way: we add one point per each line through the origin.

2.2.2. *Projective spaces* $\mathbb{P}(V)$. For a vector space V over a field $\Bbbk$ we denote by $\mathbb{P}(V)$ the set of lines through the origin, i.e., the 1-dimensional vector subspaces. With this notation, we are passing from $\mathbb{A}^2$ to $\mathbb{A}^2 \sqcup \mathbb{P}(\Bbbk^2)$.

This actually works for any affine space $\mathbb{A}^n$: we can add $\mathbb{P}^(\Bbbk^n)$ and think of this as adding one point per each class of parallel lines in $\mathbb{A}^n$. This turns out to work beautifully – the new object is natural (i.e., it does not have to be explained starting from $\mathbb{A}^n$).

2.2.3. *Lemma.* $V \sqcup \mathbb{P}(V) \ \cong \mathbb{P}(V \oplus \Bbbk)$.

2.2.4. *Projective coordinates.* We first introduce the "*projective coordinates*" on $\mathbb{P}(V)$. A basis e_i of V gives coordinates x_i on V and we denote the line through a vector $x = (x_1, ..., x_n)$ by $k \cdot x = [x_1 : \cdots : x_n]$. Then:

- A line $[x_1 : \cdots : x_n]$ is given when not all x_i are 0.
- Multiplying all projective coordinates by the same scalar $c \in \Bbbk^* \stackrel{\text{def}}{=} \Bbbk \backslash \{0\}$, does not change the line: $[cx_1 : \cdots : cx_n] = \ [x_1 : \cdots : x_n]$.

2.2.5. *Proof.* Coordinates $x_1, ..., x_n$ on V give coordinates $x_0, x_1, ..., x_n$ on $\Bbbk \oplus V$. Now, $\mathbb{P}(k \oplus V)$ breaks into a subset $x_0 = 0$ which is really $\mathbb{P}(V)$, and a subset $x_0 \neq 0$, which is isomorphic to V (all lines here have unique presentation of the form $[1; x_1 : \cdots : x_n]$ with $x_i \in \Bbbk$ arbitrary). **QED**

2.2.6. $\mathbb{P}^n$. By the "n-dimensional projective space" (over $\Bbbk$), we mean

$$\mathbb{P}^n \overset{\text{def}}{=} \mathbb{P}(\Bbbk^{n+1}).$$

By the lemma,the completion of $\mathbb{A}^n$ obtained by making the parallel lines meet at ∞ is just $\mathbb{A}^n \sqcup \mathbb{P}^{n-1} = \mathbb{P}^n$.

2.2.7. *Corollary.* $\mathbb{P}^n \cong \ \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0$.

Here, the embedding of $\mathbb{A}^i$ into $\mathbb{P}^n$, given by a repeated use of the lemma sends a point $(b_1, ..., b_i) \in \mathbb{A}^i$ to a point $[0 : \cdots : 1 : b_1 : \cdots : b_i]$ in $\mathbb{P}^n$.

2.2.8. *Projective algebraic varieties.* A *projective algebraic variety* Y over $\Bbbk$ is a subset Y of some $\mathbb{P}^n(\Bbbk)$ that can be described by several *homogeneous* polynomial equations

$$Y = \ \{(x_0 : \cdots : x_n) \in \mathbb{P}^n(\Bbbk);\ 0 = G_j(x_0, ..., x_n),\ 1 \leq j \leq m\}$$

for some *homogeneous* polynomials $G_j \in \ \Bbbk[X_0, ..., X_n]$.

2.2.9. *Scarcity of functions.* Observe that in $\mathbb{P}^1 \cong \mathbb{A}^1 \sqcup \mathbb{A}^0$ viewed as lines in $\mathbb{A}^2$ (through $(0, 0)$, the first part is given by lines $L_k = \{(x, Y);\ y = kx$ with a slope $k \in \Bbbk = \mathbb{A}^1$ and the second part is a point, the vertical line $x = 0$ of slope ∞.

Functions on $\mathbb{P}^1$ are functions on $\mathbb{A}^1$ that extend over ∞, i.e., polynomials $P(x)$ that have a finite value $\lim_{x \to \infty}\ P(x)$ at ∞, but these are just constants. Actually, in general $\mathcal{O}(\mathbb{P}^n) = \Bbbk$[10] Later we will remove this problem by noticing that there are many *local* functions though the only *global* ones are constants.

Because of this scarcity of functions on projective spaces we did not use functions to define projective subvarieties of $\mathbb{P}^n$ (as in the case of $\mathbb{A}^n$). If $G \in \Bbbk[X_0, ..., X_n]$ is homogeneous of degree d then $G(cx_0, ..., cx_n) = \ c^d{\cdot}G(x_0, ..., x_n)$, so the value of G on the line $[x_0 : \cdots : x_n]$ does not make sense, so it is not a function on $\mathbb{P}^n$.[11] However " $\underline{G = 0 \text{ on } [x_0 : \cdots : x_n]}$" still does make sense, and this is what we used above.

[10]The same holds for all connected projective varieties.

[11]However, such G's will be seen to be *sections of a line bundle on* $\mathbb{P}^n$, and in some sense they will turn out to be generalizations of functions.

2.2.10. *Completion of affine varieties to projective varieties.* Passing from $\mathbb{A}^n$ to $\mathbb{P}^n$ we need to pass somehow from all polynomials to homogeneous polynomials.

The degree of a polynomial $F = \sum_I c_I x^I \in \Bbbk[X_1, ..., X_n]$ is the maximal degree $|I| = I_1 + \cdots + I_n$ of the monomials that appear (i.e., $c_I \neq 0$). If $F \in \Bbbk[X_1, ..., X_n]$ has a degree d we can use it to produce a *homogeneous* polynomial of the same degree but with one more variable

$$\widetilde{F} = \sum_I c_I \cdot X^I \cdot X_0^{d-|I|} \in \Bbbk[X_0, X_1, ..., X_n].$$

Now if an affine variety $X \subseteq \mathbb{A}^n$ is given by equations $F_j = 0$ then the equations $\widetilde{F}_j = 0$ give a projective subvariety of $|P^n$ that we will call $\overline{X}$. Notice that $\bar{X} \cap \mathbb{A}^n$ consists of lines $[x_0; \cdots; x_n]$ with $0 = \widetilde{F}_j(x_0, ..., x_n)$ and $x_0 \neq 0$. After rescaling x_0 we see that these are the lines $[1; y_1; \cdots; y_n]$ with with $0 = \widetilde{F}_j(1; y_1, ..., y_n)$, i.e., $0 = F_j(y_1, , ..., y_n)$ (since $\widetilde{F}|_{x_0=1} = F$!). So,

$$\bar{X} \cap \mathbb{A}^n = X.$$

When thinking of $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1}$ as a completion of $\mathbb{A}^n$, I will call $\mathbb{P}^{n-1}$ the boundary $\partial \mathbb{A}^n$ of $\mathbb{A}^n$ in $\mathbb{P}^n$. Similarly, $\partial X \stackrel{\text{def}}{=} \overline{X} \backslash X = \overline{X} \cap \partial \mathbb{A}^n = \{x_0 = 0 \text{ in } \overline{X}\}$ will be called the boundary ∂X of X in $\overline{X}$.

2.2.11. *Examples.* (a) The boundary of line is a point.

A line L is given by $aX + bY = c$ with $a \neq 0$ or $b \neq 0$. (Here $X = X_1,\ Y = X_2$.) So, $F = aX + bY - c$ and $\widetilde{F} = aX + bY - cX_0$. ∂X consists of lines $[x_0 : x : y]$ such that $X_0 = 0 = \widetilde{F}$, i.e., lines $[0; x : y]$ such that $\widetilde{F}|_{X_0=0}$ vanishes. Here, $\widetilde{F}|_{X_0=0} = aX + bY$, notice that in genera that $\widetilde{F}|_{X_0=0}$ is the top degree homogeneous part F_{top} of F. So, ∂X consists of lines $[x : y]$ that satisfy $ax + by = 0$. This gives precisely one line which one can describe as $[b, -a]$.

(b) The boundary of a circle in $\mathbb{A}^2(\mathbb{C})$ consists of two points.

A circle C is given by $F = (X-a)^2 + (Y-b)^2 - \gamma$, hence $\widetilde{F} = (X - aX_0)^2 + (Y - bX_0)^2 - \gamma X_0^2$, and ∂C is given by lines killed by $\widetilde{F}|_{X_0=0} = F_{top} = X^2 + Y^2$. These are two lines $[1, \pm i]$.

Notice that

(1) over $\mathbb{R}$, $\partial C = \emptyset$, as expected since a circle does not stretch to ∞.
(2) Over $\mathbb{C}$, all circles pass through the same two points at ∞ !

2.3. **Include infinitesimals.** Bravely, we turn from intersecting two lines to intersecting a circle and a line. Over $\mathbb{R}$ we get two points (secant line), one point (tangent line), or no points. Over $\mathbb{C}$ there are only two case: two points or one point (the tangent case).

2.3.1. *Idea of a double point.* One way to describe this is that when a line degenerates from the generic position with respect to the circle to the special ("degenerate") case of a tangent line, the two points in the intersection degenerate to one point. Traditionally, geometers would go around this instability in the number of solutions by saying that the one point intersection in the tangent case should be counted twice as it is a limit of a pair of points, so it is a *double point.*

Nice but hazy! If this makes you unhappy you can try to study the situation in algebra.

2.3.2. *Functions on the intersection of a line and a circle; the algebraic calculation.* To do an algebraic calculation we choose coordinates conveniently. So the circle is the standard circle $C = \{X^2 + Y^2 = 1\}$, and the line is the horizontal line L_c on height c, i.e., $L_c = \{Y = c\}$. The intersection $C \cap L_c$ is obtained by imposing both equations, so $Y = c$ and $X^2 = 1 - c^2$. Therefore, the algebra of functions on the intersection is obtained as the quotient of $\Bbbk[X, Y] = \mathcal{O}(\mathbb{A}^2)$ obtained by imposing both equations:

$$\mathcal{O}(C \cap L_c) = \Bbbk[X, Y]|_{Y=c \text{ and } X^2=1-c^2} = \Bbbk[X]|_{X^2=1-c^2} = \Bbbk[X]/(X^2 - (1 - c^2))\cdot\Bbbk[X, Y].$$

So, the algebra is two dimensional: $\mathcal{O}(C \cap L_c) = \{a + bX;\ a, b \in \Bbbk\}$ and $X^2 = 1 - c^2$.

This sounds roughly right: the intersection usually consists of two points and the functions are therefore two dimensional (can choose value at each point). However, for $c = 1$ the line is tangent and the intersection is one point $(0, 1)$, while the algebra we got:

$$\mathcal{O}(C \cap L_1) = \{a + bX;\ a, b \in \Bbbk\} \quad \text{with} \quad X^2 = 0,$$

is two dimensional.

2.3.3. *A mistake! (If we are really calculating functions on the variety $C \cap L_1$).* C and L_c are affine varieties given by one equation each. The intersection of these two subsets of $\mathbb{A}^2$ is the affine subvariety given by two equation. The definition of functions on the affine variety $C \cap L_c$ is:

$$\mathcal{O}(C \cap L_c) \overset{def}{=} \textit{algebra of all restrictions of polynomials to the } \underline{set}\ C \cap L_c.$$

So, I have actually made a mistake in the algebraic calculation. When $c = \pm 1$, then $X^2 = 1 - c^2$ has one solution $X = 0$ and the algebra of functions on $C \cap L_1$ is the algebra of restrictions $\Bbbk[X]|_{X^2=1-c^2} = \Bbbk[X]|_{X=0}$ of polynomials to the point $X = 0$, so it is one dimensional: $\Bbbk[X]/X\cdot\mathbb{K}[X] \cong \Bbbk$. The mistake was that I was just imposing algebraic conditions rather then checking what happens on the level of sets as I should have if I am working with algebraic varieties (by definition, they are subsets of $\mathbb{A}^n$).

2.3.4. *Why should I believe that the algebraic calculation was correct in some world?, i.e., that the Double Point really exists?* The calculation with a mistake was better in the sense that the result was more stable since the dimension of functions on the intersection of C and L_c was independent of c ! This offers a way out:

- Algebra suggest that there is a world in which the intersection of C and L_c inside $\mathbb{A}^2$, is literally more then a point. This intersection we will denote $C \cap_{\mathbb{A}^2} L_c$ and it will be a space[12], characterized by its ring of functions:

$$\mathcal{O}(C \cap_{\mathbb{A}^2} L_c) \stackrel{def}{=} \textit{take the quotient of } \mathcal{O}(\mathbb{A}^2) \textit{ by imposing the equations of both } C \textit{ and } L_c.$$

This space we will call a *double point*, and the algebra of functions on a double point is isomorphic to $\Bbbk[X]/X^2{\cdot}\Bbbk[X]$.

Now everything fits:

- The functions on a double point are expressions $a + bX,\ a, b \in \Bbbk$, with $X^2 = 0$.
- We certainly expect to have constant functions (even if there is only one point). More precisely, one should think that the double point contains an ordinary point p, because of the quotient map

$$\mathcal{O}(Double\ Point) = \Bbbk[X]/X^2 \xrightarrow{X \to 0} \Bbbk = \mathcal{O}(pt) = \mathcal{O}(p),$$

which can be viewed as restriction of functions to a point p.
- What is X and what is the meaning of $X^2 = 0$? (There are no such elements in a field such as $\mathbb{R}$ or $\mathbb{C}$, except for 0!) The (intuitive) explanation is that X measures the distance from p in the Double Point. Now $X^2 = 0$ shows that a Double Point is just slightly more then a point – we move so little from p that the function X only has infinitesimally small values, they are so small that X^2 is not only "negligible" but actually 0.

So, we can make sense of the double point. We lifted it from the vague idea that some point should be counted twice to a precise mathematical object (an algebra). But there is a small price: we need to find the geometric way of thinking about rings more general then the rings of polynomials ($=\mathcal{O}(\mathbb{A}^n)$), and their quotients $\mathcal{O}(X)$ obtained by restricting polynomials to algebraic subvarieties X of affine spaces $\mathbb{A}^n$. Because

2.3.5. *A double point is not an algebraic variety.* Notice that our double point is *not* an algebraic variety because

The algebra of functions $\mathcal{O}(Y)$ on an algebraic subvariety $Y \subseteq \mathbb{A}^n$ over a field $\Bbbk$ has no nilpotents.

This is so because $f \in \mathcal{O}(Y)$ is the restriction of some polynomial function F from $\mathbb{A}^n$ to $\Bbbk$, to the subset $Y \subseteq \mathbb{A}^n$. So, f is a function from Y to $\Bbbk$. Now $f^e = 0$ implies that for all $y \in Y$ one has in $\Bbbk$: $0 = f^n(y)$, i.e., $0 = f(y)^e$. But since $\Bbbk$ is a field this implies that $f(y) = 0,\ y \in Y$, i.e., $f = 0$.

[12]This use of the word *space* means that we do not yet know what we want.

2.3.6. *Schemes.* So far we are making the case for the existence of a larger world that includes varieties, but also more things, for instance the double point. This will be true in

$$\boxed{\textit{Grothendieck's world of } \mathcal{SCHEMES}.}$$

2.3.7. *Calculus.* At the break of the calculus dawn, the great minds calculated with infinitesimally small quantities. Later we fixed our view on real numbers and decided infinitesimals are nonsense, but we can fix the problem by translating the original formulations into the ε,δ-language. However, algebra allows quantities h which are "infinitesimally small" in the sense that $h^n = 0$ for some n – they just do not live in $\mathbb{R}$ but in some larger ring such as $\mathbb{R}[h]/h^n$.

Trying to do derivatives in this way, we first rewrite $f'(a) \overset{\text{def}}{=} \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, as

$$f'(a)\d{h} = f(a+h) - f(a) + \mathcal{O}(h^2),$$

where $\mathcal{O}(h^2)$ denotes a quantity which goes to 0 faster then h^2. (This reformulation is anyway necessary in higher dimension.) Then one would say

- $f'(a)$ is the coefficient of h in $f(a+h) - f(a) \in \mathbb{R}[h]/h^2$.

Admittedly, one needs f to be in some sense algebraic so that $f(a+h)$ makes sense, i.e., that f extends naturally from $\mathbb{R}$ to $\mathbb{R}[h]/h^2$. (For instance rational functions and power series have this property.)

2.3.8. *Infinitesimal neighborhoods.* If we believe in infinitesimally small objects, we can ask what is the part of the line $\mathbb{A}^1$ which is infinitesimal close to 0? If there really is such space, the *infinitesimal neighborhood* X *of* 0 *in* $\mathbb{A}^1$, then by taking the clue from the Observation Principle, the question

"what is the infinitesimal neighborhood X *of* 0*?",*

can be restated as

"what is the algebra of function $\mathcal{O}(X)$ *on the infinitesimal neighborhood* X *of* 0*?".*

If we are in the algebraic setting, we want something like polynomials, but larger since polynomials make sense on all of $\mathbb{A}^1$, there should be things that make sense only close to 0. This reminds us that the *convergent power series* are series $\sum_0^\infty a_n x^n$ which converge on some small interval $(-\delta,\delta)$ around 0. We may want something even larger, both in order to distinguish the infinitesimal neighborhood X from actual neighborhoods $(-\delta,\delta)$, and to have a purely algebraic notion (i.e., no use of topology in $\mathbb{R}$). This leads to using *all formal power series*

$$\mathcal{O}(X) \overset{\text{def}}{=} \mathbb{R}[[T]],$$

the idea being that they will all converge when we are *infinitesimally* close to 0.

That's it. We've made sense of infinitesimals. The price is as mentioned before, develop a geometric way of thinking about rings like $\Bbbk[h]/h^n$ or $\Bbbk[[h]]$. Actually, it turns out that

Grothendieck's notion of schemes gives a geometric way of thinking about *all commutative rings.*

This goes very far since we (or most of us) prefer geometric way of thinking then the algebraic way.[13]

2.3.9. *Non-commutative geometry.* Next, we would like to have a geometric approach to all rings, including the scary non-commutative rings. This is in progress. There is a number of partial approaches. The most promising one by Connes.

3. **Spot a theorem**

We have looked at examples $\overline{X}\cap\overline{Y}$ of intersections in $\mathbb{P}^2$ of projective completions of affine curves $X = \{f(X,Y) = 0\}\subseteq\mathbb{A}^2$ and $Y = \{g(X,Y) = 0\}\subseteq\mathbb{A}^2$, given by two polynomials f and g.

3.0.10. *Question.* How many points are there in the intersection $X \cap Y$?

We saw that the number $|X \cap Y|$ behaves better if $\Bbbk = \mathbb{C}$ then when $\Bbbk = \mathbb{R}$ (fewer exceptions from the expected behavior). So let us work over $\mathbb{C}$.

3.0.11. *Examples with lines and circles.* Two lines meet at one point (unless $X = Y$). Line and circle meet at two points (unless tangent). Two circles usually meet at four points: two in $\mathbb{A}^2$ and the two common points at ∞.

3.0.12. *Degree of a planar curve.* The number of solutions of a polynomial in one variable is given by its degree. Seeing that this is likely to be important, we will say that a curve $X\subseteq\mathbb{A}^2$ has degree d if it is given by a polynomial f (in two variables), of degree d.

This pushes us to notice that instead of "circles" we should talk of (non-degenerate) *quadrics*, i.e., curves given by a polynomial (in two variables) of degree 2:

3.0.13. *There are only three quadrics over* $\mathbb{C}$. A quadric will mean a curve given by a quadratic equation $F = aX^2+bXY+cY^2+dX+eY+g$ (i.e., degree 2). By a linear change of coordinates (i.e. just a change of point of view on $\mathbb{A}^2$ and $\mathbb{P}^2$), we can rewrite F using completion to a square, first as $\alpha X'^2+\gamma Y'^2+\delta X'+\varepsilon Y'+\phi$ and then as $AX''^2+BY''^2+C$. The degenerate case is when one of A, B, C is zero, this reduces to two lines $X'''Y''' = 0$ or a double line $(X''')^2 = \rho$. So, non-degenerate quadrics are really circles (over $\mathbb{C}$!).

[13]Because we can draw or visualize pictures.

The projective extension $\overline{C}$ of a quadric curve C is given by a homogeneous quadratic polynomial $G = \sum_{0\le i\le j\le 2} c_{ij}x_ix_j$. If we want to count[14] the projective quadrics (it is not difficult to see that this is really the same problem as affine quadrics), the question has more symmetry. The answer is that (over $\mathbb{C}$), after a linear change of coordinates any quadratic form G diagonalizes to one of the form $\sum_{0\le i\le k} Y_i^2$ with $0 \le k \le 2$. So we again get three quadrics.

Over $\mathbb{R}$, after a linear change of coordinates any quadratic form G diagonalizes to one of the form $(\sum_{1\le i\le k_+} Y_i^2) - (\sum_{1\le j\le k_-} Z_j^2)$, with $1 \le k_+ + k_- \le 3$. So there are 4 non-degenerate quadrics and 9 all together.

What are these three quadrics? For completeness we check geometrically that

Lemma. The three quadric curves are quite different

(1) $Q_2 = \{[Y_0 : Y_1 : Y_2];\ Y_0^2 + Y_1^2 + Y_2^2 = 0\} \cong \mathbb{P}^1$.
(2) $Q_1 = \{[Y_0 : Y_1 : Y_2];\ Y_0^2 + Y_1^2 = 0\}$ consist of two projective lines $L_\pm \xrightarrow{\cong} \mathbb{P}^1$ that intersect in a point.
(3) $Q_0 \overset{\text{def}}{=} \{[Y_0 : Y_1 : Y_2];\ Y_0^2 = 0\} \cong \mathbb{P}^1$ should be counted as a *double line*.

Proof. The affine part Q^o of $Q = Q_2 = \{[Y_0 : Y_1 : Y_2];\ Y_0^2 + Y_1^2 + Y_2^2 = 0\}$ is given by $Y_0 = 1$, i.e., $Y_1^2 + Y_2^2 = -1$ and in term of $Y_\pm = Y_1 \pm 1iY_2$ this is $-1 = Y_+ \cdot Y_-$ which is $\Bbbk^*$. The boundary is given by $\mathbb{P}(\{Y_0 = 0\})$, i.e., the lines in $Y_+ \cdot Y_= 0$ and these are two points. All together we see that $Q^o \cong \mathbb{A}^1 - \{0\}$ and $\partial Q^o = \{0, \infty\}$ glues to $Q \cong \mathbb{P}^1$.

For $Q = Q_1 = \{[Y_0 : Y_1 : Y_2];\ Y_0^2 + Y_1^2 = 0\}$ the affine part $Q^o = \{Y_0 = 1\} = \{Y_1^2 = -1\}$ consists of two affine lines $L_\pm^o = \{Y_1 \in \mathbb{A}^o \ \text{ and } \ Y_2 = \pm i\}$. The boundary $\partial Q^o = \mathbb{P}(\{Y_0 = 0\}) = \mathbb{P}\{[Y_1; Y_2];\ Y_1^2 = 0\}$, is one point common to both lines $L_\pm^o$. So, Q consists of two projective lines $L_\pm \xrightarrow{\cong} \mathbb{P}^1$ that intersect in a point.

Finally, $Q = Q_0 \overset{\text{def}}{=} \{[Y_0 : Y_1 : Y_2];\ Y_0^2 = 0\}$ is just one projective line $\mathbb{P}^1$, however it should be counted as a *double line.*

3.0.14. *Bezout's theorem.* So in the examples above the expected number is the product of degrees:

Conjecture. $|X \cap Y| = deg(f) \cdot \deg(g)$.

This turns out to be always true – once one accounts for exceptions by counting intersection points with multiplicities !

Theorem. $\sum_{p\in X\cap Y} mult_p(X, Y) = deg(X) \cdot \deg(Y)$.

[14]The count of quadrics here really means the count of *different quadrics*, i.e., the isomorphism classes of quadrics.

4. **Include the number theory**

4.1. **The spectrum of $\mathbb{Z}$.** Number theory starts with studying the ring $\mathbb{Z}$ of integers. What one studies are polynomial equations, i.e., algebro geometric questions: solve $x^2 + y^2 = z^2$ in $\mathbb{Z}$, or $x^3 + y^3 = z^3$ in $\mathbb{Z}$, etc. To think of this really as algebraic geometry, we need a space. So, we would like the ring of integers $\mathbb{Z}$ to be the ring of functions on some space S: $\mathcal{O}(S) = \mathbb{Z}$. We will call S the spectrum of $\mathbb{Z}$: $S = \text{Spec}(\mathbb{Z})$. Then, doing algebra in $\mathbb{Z}$ will be the same as geometry on S. The geometric way of thinking involves some basic questions

4.1.1. *What are the points of Spec*($\mathbb{Z}$)*?* A point of an affine space $a \in \mathbb{A}^n(\Bbbk)$ over a field $\Bbbk$ is a vantage point from which we can observe the observables i.e., functions $f \in \mathcal{O}(\mathbb{A}^n)$. Of course, by "observing functions at a", I mean evaluating functions at a. What is the evaluation at a in terms of the algebra $\mathcal{O}(\mathbb{A}^n)$? It is a map from functions to the ground field $\Bbbk$ $ev_a : \mathcal{O}(\mathbb{A}^n) \to \Bbbk,\ f \mapsto f(a)$. We can also think of it as a restriction $\mathcal{O}(\mathbb{A}^n) \to \mathcal{O}(\{a\}) \cong \Bbbk$.

So, algebraically, a point of X is a

(1) homomorphism of rings $\mathcal{O}(X) \to A$,
(2) it is surjective, i.e., it is a quotient map: $A \cong \mathcal{O}(X)/I$ for some ideal I (actually $I = I_a$),
(3) the target is a field, i.e. A is a field.

The last requirement intuitively means that the algebra A is "small". For instance for any algebraic subvariety $X \subseteq \mathbb{A}^n$, the restriction map $\mathcal{O}(\mathbb{A}^n) \to \mathcal{O}(X)$, satisfies (1) and (2) but it satisfies (3) only if X is a single point.

So, a point x of S should be homomorphism of algebras from $\mathcal{O}(S) = \mathbb{Z}$ to some quotient field A. All quotients are of the form $A = \mathbb{Z}/n\mathbb{Z},\ n \in \mathbb{Z}_+$; and this is a field iff n is a prime. So

The points of Spec($\mathbb{Z}$) *are primes!*

4.1.2. *Question.* So the quotient fields of $\mathbb{Z}$ should be thought of as points of Spec($\mathbb{Z}$). What is the geometric meaning of the fraction field $\mathbb{Q}$ of $\mathbb{Z}$?

4.1.3. *Psychological maturity.* The first great triumph of this way of thinking in number theory was (as much as I can remember) I guess the proof of the Mordel conjecture by Faltings. I find it remarkable that it took something like 30 years.

4.2. **Affine schemes.** Thanks to Grothendieck, one can think of any commutative ring A as the ring of functions on a space Spec(A). Such spaces that correspond to commutative rings, are called *affine schemes.* Of course, if $A = \Bbbk[X_1, ..., X_n]$ then Spec(A) should be really $\mathbb{A}^n(\Bbbk)$, in the sense that it is an object that contains the same information.

Unfortunately, we have to accept that we may talk about it a bit differently when we think of it as a scheme rather then as a variety.

4.2.1. *Points of schemes and varieties.* One uses differently the word *point* when one talks about schemes then when one talks of varieties. Since varieties are schemes this introduces some *confusion*, and we will make peace with this in a moment!

4.2.2. *The geometric space Spec*(A). Now we will describe the structure of Spec(A) in stages. To avoid abstraction shock, we will now only explain the first two out of three levels of structure on Spec(A) :

(1) Spec(A) is a set:

The points of Spec(A) *are prime ideals* P *of* A.

(2) Spec(A) is a topological space:

The open sets in Spec(A) *are subsets Spec*$(A)_P$, $P \in$ *Spec*(A), *where Spec*$(A)_P$ *is the complement of the subset* $V_P \overset{def}{=} \{Q \;\; Q \supseteq P\} \subseteq$ *Spec*(A).

(3) Spec(A) is a ringed space (i.e., a topological space supplied with a sheaf of rings):

The ring of function on Spec(A) *is* A. *Moreover, there is a ring of functions* $\mathcal{O}(U)$ *for any open* $U \subseteq$*Spec*(A), *and together they form a sheaf of rings* $\mathcal{O}$ *on Spec*(A). *For instance,* $\mathcal{O}(\text{Spec}(A)_P) = A_P$ *is the localization of* A *at* P, *i.e. one inverts all elements in* $A \backslash P$.

Of course, all of this will only make sense much later.

4.3. **Set Spec**(A).

4.3.1. *Maximal ideals as cpoints of Spec*(A). We have decided above that the algebraic way to think of a point of a variety X as a surjective map $\phi : \mathcal{O}(X) \twoheadrightarrow l$ where l is a field. Then $I = \text{Ker}(\phi)$ is an ideal in $\mathcal{O}(X)$ and $l \cong \mathcal{O}(X)/I$. So, all information is contained in an ideal I of $\mathcal{O}(X)$ such that $\mathcal{O}(X)/I$ is a field. However,

Lemma. A/I is a field iff the ideal I in A is maximal.

4.3.2. *Corollary.* The points of an affine variety X are the maximal ideals in $\mathcal{O}(X)$.

So we would like to say that the points of the spectrum Spec(A) (of any ring A), are the maximal ideals of A.

Actually, according to the standard terminology these are not all points of Spec(A) but only the points of a special kind. We will say that

The cpoints[15] *of Spec*(A) *are maximal ideals of* A.

[15]Closed points.

4.3.3. *The points of Spec*(A)*.* We will use another, more general, notion of points of Spec(A) – we will allow more ideals

The points *of Spec*(A) *are all* <u>*prime*</u> *ideals of* A*.*

4.3.4. *Prime ideals.* An ideal $P \subseteq A$ is said to be prime if $a, b \in A$ and $ab \in P$ implies that $a \in P$ or $b \in P$.

Lemma. (a) P is prime iff A/P has no zero divisors.

(b) Maximal ideals are prime.

(c) Zero ideal is prime iff A has no zero divisors ("A is integral").

4.3.5. *Lemma.* (a) For the ring $\mathbb{Z}$:

(1) all ideals are *principal* i.e., of the form $(n) \stackrel{\text{def}}{=} n\mathbb{Z}$ for some $n \in \mathbb{Z}$.
(2) maximal ideals = all $p\mathbb{Z}$ with p a prime.
(3) prime ideals = maximal ideals $\sqcup \{0\}$.

(b) For the ring $\mathcal{O}(\mathbb{A}^1) = \Bbbk[X]$

(1) All ideals in are principal, i.e. of the form $I = (P) = P \cdot \Bbbk[X]$ for some polynomial P. Actually, if we ask that P is monic this will make it unique.
(2) Ideal (P) is prime iff $P = 0$ or P is irreducible.
(3) If $\Bbbk$ is closed the only irreducible polynomials are of the form $X - a, \ a \in \mathbb{A}^1$. The corresponding ideal $(X - a)$ is exactly the ideal I_a of all functions that vanish at a.
(4) So, if $\Bbbk$ is closed, prime ideals are ideals of the form I_Y for one of the following subvarieties Y of $\mathbb{A}^1$:
   - Y is a point (maximal ideals correspond to points!),
   - $Y = \mathbb{A}^1$.

(c) Consider the ring $\mathcal{O}(X)$ for the affine variety $X = \{xy = 0 \ in \ \mathbb{A}^2\}$.

(1) It has a basis $..., y^2, y, 1, x, x^2, ...$, and $xy = 0$.
(2) Maximal ideals correspond to points.
(3) 0 is not prime but there are two more prime ideals $(y) = I_{x-axis}$ and $(y) = I_{y-axis}$.

(c) Prime ideals in $\mathcal{O}(\mathbb{A}^2)$ correspond to

- (0) points,
- (2) $\mathbb{A}^2$ itself and
- (1) irreducible curves in X ("those that consist of one piece").

4.3.6. *Prime ideals and irreducible components.* We say that an affine variety X is *irreducible* if $\mathcal{O}(X)$ has no zero divisors.

Having zero divisors $f, g \neq 0 = fg$ in $\mathcal{O}(X)$ means that there are two affine subvarieties $Y = \{f = 0\}$ and $Z = \{g = 0\}$, such that $Y \cup Z = X$ and neither $Y \subseteq Z$, nor $Z \subseteq Y$. A basic example is $X = \{xy = 0 \ in \ \mathbb{A}^2\}$ which is the union of the x and y axes in the plane. Therefore, *irreducible* means the opposite, i.e., that we can not decompose X into two smaller affine subvarieties.

4.3.7. *Functoriality of the spectrum requires prime ideals.* To a map of varieties $f : X \to Y$ there corresponds the morphism of algebras of functions in the opposite direction

$$\mathcal{O}(Y) \xrightarrow{f^*} \mathcal{O}(X),$$

given by the *pull-back of functions*, i.e., $f^*(\phi) = \phi \circ f$. Actually, this gives an identification

$$Map(X, Y) \ni f \mapsto f^* \in \mathrm{Hom}_{\Bbbk-alg}[\mathcal{O}(Y), \mathcal{O}(X)].$$

Therefore, we expect for general rings A and B to have a bijection

$$Map[\mathrm{Spec}(A), \mathrm{Spec}(B)] \leftrightarrow \mathrm{Hom}_{\mathcal{R}ings}[B, A].$$

So to a map of rings $F : B \to A$, we expect to associate a map of sets $f = \mathrm{Spec}(F) :$ $\mathrm{Spec}(A), \mathrm{Spec}(B)$.

We have agreed that maximal ideals in A should be points of $\mathrm{Spec}(A)$. So we expect to attach to any maximal ideal $I \subseteq A$ some maximal ideal $f(I) \subseteq B$. The map $F : B \to A$ gives just one way of associating to an ideal I in A an ideal J in B – this is the pull-back $J = F^{-1}I = \{a \in A;\ F(a) \in I\}$. So we want

If $I \subseteq A$ is maximal then $F^{-1}I \subseteq B$ is maximal.

However, it is easy to find counterexamples.

How bad is this? If we insist that maximal ideals in A are points of $\mathrm{Spec}(A)$, we are forced to allow more points in $\mathrm{Spec}(B)$ then we expect, not only the maximal ideals in B but also any ideal which is the pull-back of a maximal ideal.

How much more is this? Actually, the pull-back of a maximal ideal is always a prime ideal, and more is true:

Lemma. The pull-back of a prime ideal is always prime!

Proof. Let $F : B \to A$ and let P be a prime ideal in A. If $ab \in F^{-1}P$, i.e., $P \ni F(ab) = F(a) \cdot F(b)$, since P is prime we know that either $P \ni F(a)$ or $P \ni F(b)$, i.e., $F^{-1}P \ni a$ or $F^{-1}P \ni b$).

Conclusion. Adding all prime ideals to maximal idels solves the functoriality (naturality) problem!

4.3.8. *The scheme-theoretic points of a variety.* For an affine variety $X/\Bbbk$, what are the points of the associated scheme $\mathrm{Spec}(\mathcal{O}(X))$? The answer is simple – instead of only looking at the points of X, which are the smallest subvarieties of X, we are force to look at all subvarieties at once. Actually, *all* is slightly more then is needed – we are not interested in the ones that consist of several pieces, if X is a union of two smaller subvarieties Y and Z we omit X from the list.

Lemma. The points of $\mathrm{Spec}(\mathcal{O}(X))$, i.e., the prime ideals in $\mathcal{O}(X)$ are the same as *irreducible* subvarieties Y of X.

Proof. The prime ideals in $\mathcal{O}(X)$ are precisely the ideals I_Y corresponding to *irreducible* subvarieties $Y \subseteq X$.

4.4. **Topological space Spec**(A)**.**

4.4.1. *Zariski topology on affine varieties.* We define the Zariski topology on an affine variety X so that the closed subsets are precisely the affine subvarieties of Y.

So, $X \subseteq \mathbb{A}^n$ is given by finitely many polynomial equations $X = \{0 = F_1 = \cdots = F_c\}$. We say that a subset $Y \subseteq X$ is Zariski closed if it is given by a few more additional polynomial equations. This is natural in the sense that if we *think* of polynomials as continuous functions the the subsets $\{G = 0\}$ should be closed! All-together, if say $\Bbbk = \mathbb{C}$ then the *Zariski closed* subsets of $\mathbb{A}^n = \mathbb{C}^n$ are the closed subsets which can be described using polynomials.

Lemma. A family $\mathcal{C}$ of subsets of a set X is the set of close subsets in some topology $\mathcal{T}$ on X iff

(1) $\mathcal{C} \ni \emptyset, X$, and
(2) $\mathcal{C}$ is closed under finite unions and arbitrary intersections.

Proof. If $\mathcal{T} = \{X - F;\ F \in \mathcal{C}\}$, the conditions on $\mathcal{C}$ translates into conditions on $\mathcal{T}$ which are precisely the definition of a topology.

Proposition. Zariski topology on an affine variety is well defined.

Proof. This is almost a tautology but there is one thing to check.[16] At the moment we will postpone the proof and prove a more general version for schemes (that one really is a tautology!).

[16]What?

4.4.2. *Zariski topology on Spec*(A). Any ideal $I{\subseteq}A$ defines a subset
$$V_I \stackrel{\text{def}}{=} \{P\in \mathrm{Spec}(A);\ P{\supseteq}I\}.$$
We define the Zariski topology on Spec(A) so that the closed subsets are precisely the subsets V_I given by ideals I.

Example. Let us see what this means when $A = \mathcal{O}(X)$ for an affine variety $X/\Bbbk$. The interesting ideals $I{\subseteq}A$ are the ideals $I_Y =\ \{f\in\mathcal{O}(X);\ f|Y=0\}$, corresponding to the affine subvarieties $Y{\subseteq}X$. We will see that V_{I_Y} is an incarnation of Y itself. In particular, the Zariski closed sets in the scheme Spec$[\mathcal{O}(X)]$ correspond to the Zariski closed sets in X, so $I{\mapsto}V_I$ is the correct generalization of the Zariski topology on varieties to schemes.

For this we gauge the subset V_{I_Y} of the set of prime ideals of A by looking at the prime ideals that make sense geometrically, and these are the maximal ideals I_a in $\mathcal{O}(X)$ corresponding to the points a of X. I_a lies in V_{I_Y} if $I_a{\subseteq}I_Y$, i.e., if each function that vanishes on Y also vanishes at a. This happen precisely if $a\in Y$. So,

- To any subvariety Y of X we attach an ideal I_Y and therefore also a closed subset V_{I_Y} of Spec(A).
- One has
$$V_{I_Y}\ \cap\ X\ =\ Y,$$
i.e., the intersection of V_{I_Y} with the points of X (viewed as maximal ideals of $\mathcal{O}(X)$), consists precisely of points of Y.

4.4.3. *Proposition.* Zariski topology is well defined.

Proof. We have to check that $\mathcal{C} = \{V_I,\ I \text{ an ideal in } A\}$ satisfies the conditions from the lemma 4.4.1.... It all follows from the next lemma.

4.4.4. *Lemma.* (a) For ideals $I,J{\subseteq}A$,
$$V_{I\cap J} = V_I\cup V_J.$$
(b) For ideals $I_p{\subseteq}A$,
$$\cap_p\, V_{I_p}\ =\ V_{\sum_p I_p}.$$

4.4.5. *The closed points of a scheme.* In particular we will see that the points of an affine variety X can be described as the closed scheme-theoretic points of X.

Lemma. (a) The closure of a point $P\in$ Spec(A) is $\overline{\{P\}} = V_P$.

(b) The closed points in Spec(A) are precisely the maximal ideals.[17]

(c) For an affine variety $X/\Bbbk$, the closed points in the associated scheme Spec$(\mathcal{O}(X))$ are the same as the points of X.

[17] i.e., what we called the cpoints of Spec(A).

Proof. (a) $\overline{\{P\}} = V_P$ is the smallest closed set V_I that contain P, i.e., the smallest V_I such that $I \subseteq P$. Since $I \subseteq P \Rightarrow V_P \subseteq V_I$, the smallest one is V_P.

Lemma. The prime ideals in $\mathcal{O}(X)$ are precisely the ideals I_Y corresponding to *irreducible* subvarieties $Y \subseteq X$.

5. **Constructing moduli spaces**

This is one of the basic applications of algebraic geometry. The most popular ones are related to curves:

- moduli of curves,
- moduli of vector bundles on one curve,
- moduli of maps from a given curve C to a variety X.

These spaces are among the most studied ones in algebraic geometry.

The *moduli idea* is that in order to study objects of a certain type it is useful to look at the *space of all such objects* (called the *moduli*). It should be some kind of geometric space that parameterizes all these objects in a useful way. For instance, a path in the moduli space will mean a deformation of such objects. The most familiar example may be the projective space $\mathbb{P}(V)$ (resp. Grassmannian $Gr_r(V)$), which is the moduli of all lines (resp. dimension r subspaces) in a vector space V.

We will see that in practice, the moduli is often constructed as a quotient of a scheme by a group. It turns out that it is not most important to think of the quotient as the set of orbits, but to have some geometric structure on the quotient. This can accomplished in various ways useful from a particular point of view.

The simplest construction of this form is the Invariant theory quotient $X//G$ of an affine variety (or scheme) by a group, the result is again an affine variety (or a scheme). We will construct n^{th} symmetric powers $X^{(n)} \stackrel{\text{def}}{=} X^n//S_n$ of varieties – these are moduli of n unordered points in X with possible repetitions (*moduli of configurations of n identical particles in X*). This moduli turns out quite satisfactory when X is a curve over a field (we look at $X = \mathbb{A}^1$).

When X is a surface (we look at $X = \mathbb{A}^2$), the symmetric powers are *singular*, and we find this unsatisfactory for a moduli – we want to be the objects of a given kind to deform nicely. This we use an excuse to introduce the (punctual) Hilbert schemes $X^{[n]}$ which do give a smooth moduli of n unordered points for surfaces.[18]

The last approach we visit is that of a stack quotient, which one can call the *true quotient* X/G. It has highly desirable properties that

- If X is smooth then X/G is smooth.
- One takes the *total* quotient by G, in the sense that the fibers of $X \to X/G$ are copies of G. For instance the set-theoretic quotient is not it, when we take the set-theoretic quotient, i.e., the set of orbits, we are forgetting to divide by stabilizers of points.

[18]They are *not* quotients by groups!

By this time it is clear that the familiar constructions do not have these properties. For instance the set-theoretic quotient is not it, when we take the set-theoretic quotient, i.e., the set of orbits, we are forgetting to divide by stabilizers of points. Also, invariant theory quotient often introduces singularities when the size of the stabilizer groups varies.

So we need to enlarge the world of varieties (or schemes) to have a hope of finding the "true quotients". The first such enlargement we consider is the category of *spaces* which is the Yoneda completion of the category of varieties. The second such enlargement we is the category of *stacks* which is obtained from the category of varieties using Grothendieck's group theoretic refinement of the Yoneda completion.

Both of these procedures of enlarging a given category $\mathcal{A}$ are simply the categorical analogues of the the idea of extending calculus from functions of distributions.

The end product is that to an action of a group G on a variety X we associate the natural *space quotient* $\underline{X/G}$ (it is quite a natural idea but it does not satisfy the desired properties) and the natural *stack quotient* X/G (slightly more refined and it finally does satisfy the desiderata).

5.1. **Moduli $\mathcal{M}(T)$ of objects of type T.** Consider objects of a certain type, say, type T.[19] Experience shows that the set $Isom(T)$ of isomorphism[20] classes of objects of type T often has more structure then just a set. For instance, there may be T-objects of "more general nature", that degenerate to some "special" T-objects, indicating at least some topology on $Isom(T)$. Furthermore, once we find that we are really interested in the space $Isom(T)$ we would like to do calculations on it, so we should make it Therefore, we would like to organize $Isom(T)$ into a geometric space $\mathcal{M} = \mathcal{M}(T)$ which we will call the moduli of of objects of type T.

5.1.1. *The standard strategy.* In practice, the first step is to find some bigger space $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}(T)$ that naturally parameterizes objects of type T, but with possible repetitions. So,

(1) to each $m \in \widetilde{\mathcal{M}}$ one naturally attaches an object V_m of type T,
(2) each object of type T is isomorphic to one of V_m

Actually, one would like one more property (no repetitions!):

$$(*) \quad V_p \cong V_q \textit{ only when } p = q,$$

but this is often not possible. What often happens is that a weaker version of $(*)$ is true

$$(\star) \quad \textit{there is a group } G \textit{ acting on } \widetilde{M}, \textit{ so that } V_p \cong V_q \textit{ iff } q \in G{\cdot}p.$$

[19] For instance, we will look at the examples of quadric curves in $\mathbb{P}^2$ and of cubic curves in $\mathbb{P}^2$.

[20] The word isomorphic means literally "of the same shape". We specify what we mean by it in each situation, but in general it will mean that two objects behave the same in any sense that we are interested in.

Then it seems clear that the solution is the quotient $\mathcal{M} \stackrel{\text{def}}{=} \widetilde{\mathcal{M}}/G$.

Examples: projective and Grassmannian spaces. One can construct a moduli $\mathbb{P}(V)$ of all lines in a vector space V over a field $\Bbbk$ by following the above strategy. We notice that each non-zero vector v produces a line $\Bbbk v$, and all lines appear this way. So $\widetilde{\mathcal{M}} = V - 0$ can be viewed as a "moduli with repetitions" of lines in V. Now, $u, v \in \widetilde{\mathcal{M}} = V - 0$ give the same line iff $v \in \Bbbk^* \cdot u$ for the obvious action of the group $\Bbbk^* = GL_1(\Bbbk)$ on $V - 0$. So, the moduli $\mathcal{M}$ is $\widetilde{M}/GL_1(\Bbbk^*) = (V-0)/\Bbbk^*$.

Similarly one obtains the moduli $Gr_k(V)$ of all $\Bbbk$-dimensional vector subspaces of V. A "moduli with repetitions" can be chosen as $Fr_k(V)$ the set of all $\Bbbk$-tuples $v_\bullet = (v_1, ..., v_k)$ – any such $v_\bullet$ gives $span(v_1, ..., v_k) \in Gr_k(V)$ and we get all $U \in Gr_k(V)$ in this way. The fiber at U is the set $Fr_k(U)$ of all ordered basis of U, so repetitions come from all choices of a basis of U.

To see that one can account for repetitions by an action of a group, we need a group that acts on $Fr_\Bbbk(V)$ and does not affect the span. One way to see this is to view $Fr_k(V)$ as the set $Inj_\Bbbk(\Bbbk^k, V) \subseteq \mathrm{Hom}_\Bbbk(\mathbb{K}^k, V)$ of all injective linear maps from $\Bbbk^k$ to V (here $v_\bullet$ corresponds to a map that sends e_i to v_i). Now, $Fr_k(V) \xrightarrow{span} Gr_k(V)$ is identified with the operation of taking the image of a linear map $Inj_\Bbbk(\Bbbk^k, V) \xrightarrow{image} Gr_\Bbbk(V)$. Now $GL(V) \times GL_k(\Bbbk)$ act on $\mathrm{Hom}_\Bbbk(\mathbb{K}^k, V)$ by $(g, \sigma)A = \ g \circ A \circ \sigma^{-1}$ and two maps A, B have the same image iff they a re in the same orbit of $GL_k(\Bbbk)$. So, the moduli is

$$Gr_k(V) = \ Inj_\Bbbk(\Bbbk^k, V)/GL_k(\Bbbk) = \ Fr_k(V)/GL_k(\Bbbk).$$

(The passage to $Inj_\Bbbk$ was only used to explain the action of GL_k on $Fr_k(V)$.)

5.1.2. *The need for quotients of spaces by groups.* The last step of our strategy requires taking the quotient $\mathcal{M} \stackrel{\text{def}}{=} \widetilde{\mathcal{M}}/G$. So, we see that we have not solved our problem of constructing moduli, we have only reformulated the original problem as:

For a space X *with an action of a group* G *construct an* <u>adequate</u> *space* X/G.

Here, *space* could mean: algebraic variety, scheme, or something like that. Another part of the richness of the subject comes from the phrase *adequate* which has different meanings in different situations.

The simplest approach to constructing quotients with a geometric structure is

5.2. **Invariant Theory quotients** $X//G$**.** We will consider this construction in the example of *symmetric powers* of a variety X, i.e., the moduli of *unordered n-tuples of points in X*.

5.2.1. *Unordered pairs of points.* Let $X = \mathbb{A}^1(\Bbbk)$, we are interested in the moduli $\mathcal{M}$ of objects of the following kind:

All unordered pairs $\{\{a,b\}\}$ of points $a,b \in X$.
Precisely, what we mean by the symbol $\{\{b,a\}\}$ is that
$\{\{b,a\}\} = \{\{a,b\}\}$ and we allow repetitions $\{\{a,a\}\}$.

So our moduli $\mathcal{M}$ can be thought of as

All possible positions of two particles of the same kind on a line $X = \mathbb{A}^1$.

Notice that in this case, there is nothing to the idea of isomorphism that is incorporated into the notion of moduli. (Two pairs are considered isomorphic iff they are the same!)

In this situation, $\widetilde{\mathcal{M}}$ can be taken to be the set X^2 of ordered pairs, since an ordered pair (a,b) defines an unordered pair $\{\{a,b\}\}$ by forgetting the order. So $\widetilde{M} = X^2$ is an affine variety with functions $\mathcal{O}(X^2) = \Bbbk[X_1, X_2]$. Group $S_2 = \{1, \sigma\}$ acts on X^2 by $\sigma(a,b) = (b,a)$, and we see that the moduli of unordered pairs $\mathcal{M}$ should be

$$\mathcal{M} = \widetilde{\mathcal{M}}/S_2 = X^2/S_2.$$

The question is how to give the set X^2/S_2 the structure of a geometric space.

5.2.2. *Invariant theory quotients.* If Y is an affine variety with an action of a group G, there is a canonical way to produce an affine variety $Y//G$ that plays the role of the quotient of Y by G. First observe that G acts on the algebra $\mathcal{O}(Y)$ by

$$(g{\cdot}\phi)(y) \stackrel{\text{def}}{=} \phi(g^{-1}y).$$

To start with, we consider the case when the quotient set Y/G can be given a natural structure of an affine variety. Now, the quotient map $Y \xrightarrow{\pi} Y/G$ gives the pull-back map of algebras of functions $\mathcal{O}(Y/G) \xrightarrow{\pi^*} \mathcal{O}(G)$. It is injective, so it makes $\mathcal{O}(Y/G)$ into a subalgebra of $\mathcal{O}(G)$. Moreover, the pull-backs of functions from the quotient are special among all functions on Y because they are invariant under G:

$$(g{\cdot}\pi^*\phi)(y) \stackrel{\text{def}}{=} (\pi^*\phi)(g^{-1}y) = \phi(\pi(g^{-1}y)) = \phi(\pi(y)) = (\pi^*\phi)(y).$$

Actually, we expect that $\mathcal{O}(Y/G)$ is precisely the subalgebra $\mathcal{O}(Y)^G$ of G-invariant functions on Y.

Now, we use the above observation as a definition for any group action on an affine variety:

The invariant theory quotient $Y//G$ is the space with functions $\mathcal{O}(Y//G) \stackrel{def}{=} \mathcal{O}(Y)^G$.

5.2.3. *Symmetric powers $X^{(n)}$ (unordered n-tuples of points).* For an affine variety X we can now make sense of the moduli of unordered n-tuples of points in X. This moduli is the affine variety

$$X^{(n)} \stackrel{\text{def}}{=} X^n//S_n, \quad \text{i.e.,} \quad \mathcal{O}(X^{(n)}) \stackrel{\text{def}}{=} \mathcal{O}(X^n)^{S_n}.$$

We call it the n^{th} symmetric power of X since it is the symmetric version of the n^{th} power X^n.

5.2.4. *Symmetric powers of a line.*

Lemma. $(\mathbb{A}^1)^{(n)} \cong \mathbb{A}^n$.

Proof. If $X = \mathbb{A}^1$ then $X^n = \mathbb{A}^n$ with $\mathcal{O}(X^n) = \Bbbk[X_1, ..., X_n]$, and S_n acts on it by permuting variables. The S_n-invariant functions (called the *symmetric polynomials in n variables*), are polynomials in the elementary symmetric functions $e_1 = X_1 + \cdots + X_n$, $e_2 = \sum_{i<j} X_i X_j, ..., e_n = X_1 \cdots X_n$, i.e.,

$$\mathcal{O}((\mathbb{A}^1)^{(n)}) \overset{def}{=} \mathcal{O}((\mathbb{A}^1)^n)^{S_n} = \Bbbk[X_1, ..., X_n]^{S_n} = \Bbbk[e_1, ..., e_n]$$
$$\textit{for} \;\; e_p \overset{def}{=} \sum_{i_1 < \cdots < i_p} X_{i_1} \cdots X_{i_p}.$$

For instance,

$$\mathcal{O}((\mathbb{A}^1)^{(2)}) = \Bbbk[X_1 + X_2, X_1 X_2].$$

In order to prepare for symmetric powers of surfaces we look at an example of a space with a singularity:

5.2.5. *Matrices and nilpotent cones.* Let M_{mn} be the $m \times n$ matrices over $\Bbbk$. This is an affine variety isomorphic to $\mathbb{A}^{mn}$

$$\mathcal{O}(M_{mn}) = \Bbbk[X_{ij}, \; 1 \le i \le m, \; 1 \le j \le n].$$

In the square matrices $M_n = M_{nn}$ we have we have the *nilpotent cone* which consists of all nilpotent matrices

$$\mathcal{N}_n \overset{\text{def}}{=} \{x \in \mathbb{M}_n; \; x^p = 0 \text{ for some } p > 0\}$$

For instance

Lemma. (a) For $x \in M_n$ the following is equivalent

(1) $x \in \mathcal{N}_n$, i.e., x is nilpotent,
(2) $x^n = 0$
(3) all eigenvalues are 0,
(4) The characteristic polynomial $\det(\lambda - x)$ equals λ^n.

(b) $\mathcal{N}_n$ is an algebraic variety.

Proof. (a) can be seen using the Jordan form of x. (b) follows from either (2) or (4) in (a).[21]ctually, the equations for $\mathcal{N}_n$ one obtains from (4) are more economical.

[21] A

Corollary. (a) $\mathcal{N}_2 = \{\begin{pmatrix} a & b \\ c & -a \end{pmatrix};\ a^2+bc=0\}$.

(b) $\mathbb{A}^2//\{\pm 1\} \cong \mathcal{N}_2$.

Proof. (a) For $x \in M_2$, $\det(\lambda - x) = \lambda^2 - Tr(x)\cdot\lambda + \det(x)$.

(b) We mean the action of $\pm 1 \subseteq \Bbbk^*$ on the vector space $\mathbb{A}^2(\Bbbk)$. So, $-1 \in \{\pm 1\}$ acts on the generators of $\mathcal{O}(\mathbb{A}^2) = \Bbbk[X,Y]$ by $X \mapsto -X,\ Y \mapsto Y$. Therefore, the invariant functions are the polynomials in $\alpha = X^2,\ \beta = Y^2,\ \gamma = XY$. So,

$$\Bbbk[X,Y]^{\{\pm 1\}} \cong \Bbbk[\alpha,\beta,\gamma]/(\alpha\beta = \gamma^2) \cong \mathcal{O}(\mathcal{N}_2).$$

5.2.6. *Singularities.* One of the successes of algebraic geometry is its treatment of singularities. Singularities naturally appear in all kinds of problems. For instance the fibers of a map $\pi : X \to Y$ between two manifolds, are often not manifolds. For instance look at the fiber at zero, of the functions $\mathbb{R}^2 \ni (x,y) \mapsto xy \in \mathbb{R}$ or $\mathbb{R}^3 \ni (x,y,z) \mapsto xy - z^2 \in \mathbb{R}$. This is awkward in the manifolds theory, but it is not a problem in a algebraic geometry (the fibers of a map of varieties are again varieties).

The nilpotent cone $\mathcal{N}_2$ is one of the simplest singular affine varieties. It has singularity at the origin (the zero matrix).[22] To get a feeling for it, one can draw the picture for $\Bbbk = \mathbb{R}$! We find it is a cone over a circle, i.e., it is the union of all lines that pass through one circle C and a fixed point v (not in C).

One basic way we deal with singularities is by finding

5.2.7. *Resolutions of singularities.*

Lemma. Let μ and π be the projection maps from

$$\widetilde{\mathcal{N}_2} \stackrel{\text{def}}{=} \{(x,L) \in \mathcal{N}_2 \times \mathbb{P}(\Bbbk^2);\ xL = 0\} \subseteq \mathcal{N}_2 \times \mathbb{P}(\Bbbk^2)$$

to $\mathcal{N}_2$ and $\mathbb{P}^1$.

(a) $\widetilde{\mathcal{N}_2}$ is a line bundle over $\mathbb{P}^1$.

(b) π is a bijection over $\mathcal{N}_2 - \{0\}$, and $\pi^{-1}(0) \cong \mathbb{P}^1$.

(c) $\mathcal{N}_2$ is a cone, i.e., it is a union of lines (one for each point of $\mathbb{P}^1$), and all these lines meet at one point, the vertex of the cone.

[22] If $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$, having a singularity means that X is not a manifold, i.e., near v the space does not look like an open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. The precise definition in general algebraic geometry will come later.

Remarks. (1) So, one obtains $\mathcal{N}_2$ from a nice space $\widetilde{\mathcal{N}_2}$ by contracting one $\mathbb{P}^1$ to a point. To remind us that something spectacular has happened, the space $\mathcal{N}_2$ is singular at this point.

We say that $\mu : \widetilde{\mathcal{N}_2} \to \mathcal{N}_2$ is a *resolution of the singularity* in $\mathcal{N}_2$. This means that

- $\widetilde{\mathcal{N}_2}$ is smooth,
- the map is generically an isomorphism, and
- the fibers are compact.

(2) Let us look at $\mathcal{C} = \{(x, y, z);\ xy = z^2\}$ over $\mathbb{R}$. First change the coordinates via $x = u + v,\ y = u - v$ to get $z^2 = u^2 - v^2$, hence $u^2 = z^2 + v^2$. Now, for each u we get a circle. All together, one can say that we start with a circle C (say on the height $u = -1$) and a point p (where $u = v = z = 0$). Then $\mathcal{C}(\mathbb{R})$ is the union of all lines through p that meet C. Again, we see the singularity at p.

5.2.8. *Symmetric powers of surfaces.* We will find that if X is a *surface*, i.e., a 2-dimensional algebraic variety then $X^{(2)}$ is singular. So, the moduli of all positions of a pair of identical particles in a <u>plane</u> is singular. Moreover, we will see that the singularity occurs on the diagonal, i.e., when the particles collide!

Lemma. The symmetric square $(\mathbb{A}^2)^{(2)}$ of a plane, is isomorphic to $\mathbb{A}^2 \times \mathcal{N}_2$.

Proof. Let $\mathcal{O}(\mathbb{A}^2) = \Bbbk[X, Y]$, then $\mathcal{O}(\mathbb{A}^2 \times \mathbb{A}^2) = \Bbbk[X_1, Y_1, X_2, Y_2]$ and σ exchanges $X_\leftrightarrow X_2,\ Y_1 \leftrightarrow Y_2$. We change the variables to $X = X_1 + X_2,\ Y = Y_1 + Y_2,\ x = X_1 - X_2, y = Y_1 - Y_2$. Then σ fixes X, Y and $x \mapsto -x,\ y \mapsto -y$, hence $\mathcal{O}(\mathbb{A}^2 \times \mathbb{A}^2)^{S_2} = \Bbbk[X, Y, x, y]^{S_2} = \Bbbk[X, Y, x^2, y^2, xy]$.

Remark. The singularity occurs at $x = y = 0$. This means that $X_2 = X_1,\ Y_2 = Y_1$, i.e., the points are the same $(X_1, Y_1) = (X_2, Y_2)$.

5.2.9. *The <u>Moduli Principle</u> or <u>Deformation Principle</u> of Deligne, Drinfeld, Feigin, Kontsevich.* It says that we must <u>have made a mistake</u> because the <u>Moduli Principle</u> (in a vague formulation) says that

Any moduli should be smooth if constructed correctly, i.e.,
if we do not forget any information.

At the moment we know next to nothing about producing and using moduli, so unfortunately we can not justify this principle. It is supposed to make sense only with more examples of moduli. What it claims is that for any reasonable kind of objects, one can organize all examples of such objects in such a way that going from one example to another happens in a smooth way, without any jumps and unnecessary dramatics.

We will use it as an excuse to look for more subtle constructions of moduli.

5.3. **The moduli of unordered points on surfaces: Hilbert schemes.** Among the features of $X^{(n)}$

(1) It is an algebraic variety.
(2) As a set, it really is a set of unordered n-tuples of points in X.
(3) It is smooth when $\dim(X) = 1$.
(4) It is not smooth when $\dim(X) = 2$.

we like the first three but not the fourth. The above principle in suggests that there should be a better notion ? of the moduli of unordered pairs of points then the symmetric powers $X^{(n)} = X^n//S_n$. Moreover, the difference should be that we have forgotten some information when we constructed $X^{(n)}$, so there should be a "forgetting" map $? \xrightarrow{\pi} X^{(n)}$. Finally, in light of (2) above, as a set ? should not be exactly the set of unordered n-tuples of points – to have a *good moduli* of unordered n-tuples we will occasionally need to add more information.

5.3.1. *Geometric view on unordered n-tuples of points.* A priori, the idea of "unordered n-tuples of points of X" is a set theoretic idea which is formalized (made precise) in sets as the set of orbits of S_n in X^n. Its "more geometric version" will involve "more geometric" analogues of unordered n-tuples. So, let us look for more geometric ways to think of an unordered pair $\{\{a, b\}\}$, if we can do this, hopefully a geometric moduli of such objects will just pop-out from this geometric picture.

First, we can think of a two point subset $\{a, b\}\subseteq\mathbb{A}^1$ of a line $X = \mathbb{A}^1$ as a subvariety of $\mathbb{A}^1$. Then it corresponds to an ideal $I_{a,b} = (x - a)(x - b)\Bbbk[x]$ in $\Bbbk[x] = \mathcal{O}(X)$, and a two-dimensional quotient $\mathcal{O}(\{a, b\}) = \mathcal{O}(X)/I_{a,b}$ of $\mathcal{O}(X)$. However, when $a = b$, we fall off the horse because the ring of functions changes drastically:[23] the ring of functions $\mathcal{O}(\{a\})$ on the subvariety $\{a, a\} = \{a\}$ is one dimensional. So "subvariety" is not a perfect idea because we forget the multiplicity (a occurs twice in $\{\{a, a\}\}$).

By now, we know that the thing to do to remember the multiplicity is to think of sub*schemes* rather then sub*varieties*. then $\{\{a, a\}\}$ will be thought of as the *double point* subscheme of $\mathbb{A}^1$. From this point of view the moduli of unordered n-tuples of points in X should be viewed as the set of all

- subschemes $S\subseteq X$ with $\dim(\mathcal{O}(S)) = n$ (we say that S has length n or order n), i.e.,
- all quotients of $\mathcal{O}(X)$ of dimension n, i.e.,
- all ideals of codimension n in $\mathcal{O}(X)$.

This space is denoted $X^{[n]}$ and called the Hilbert scheme of n points in X.

5.3.2. *Subschemes.* Though it is not necessary at this point, let us clarify the meaning of *subscheme*.

[23]We want a smooth moduli – no jumps.

Remember that we defined algebraic subvarieties of $\mathbb{A}^n(\Bbbk)$ as subsets X given by finitely many polynomial equations $X = \{F_1 = \cdots = F_c = 0\}$, and we associated to X the ideal $I_X \subseteq \mathcal{O}(\mathbb{A}^n)$ of functions that vanish on X, and the ring of functions on X which is the quotient $\mathcal{O}(X) = \mathcal{O}(\mathbb{A}^n)/I_X$. Moreover, such X was closed for the Zariski topology on X.

We expect that $\mathbb{A}^n$ also contains some schemes which are not varieties (such as double points). An affine scheme S is determined by its ring of global functions $\mathcal{O}(S)$. Inclusion $S \subseteq \mathbb{A}^n$ will correspond to the restriction map $\mathcal{O}(\mathbb{A}^n) \to \mathcal{O}(S)$, and this should be a surjection, i.e., functions on S should all be restrictions of functions on $\mathbb{A}^n$. This in turn gives an ideal $I_S \overset{\text{def}}{=} \mathrm{Ker}[\mathcal{O}(\mathbb{A}^n) \to \mathcal{O}(S)] \subseteq \mathcal{O}(\mathbb{A}^n)$. We now turn these expectations into a definition:

Closed subschemes of an affine scheme X correspond to ideals $I \subseteq \mathcal{O}(X)$. To an ideal I one associates closed subscheme $S \overset{def}{=} Spec[\mathcal{O}(X)/I]$.

Now, the ideal $I_S \overset{\text{def}}{=} \mathrm{Ker}[\mathcal{O}(X) \to \mathcal{O}(S)]$ is just the ideal I we started with.

5.3.3. *Modules.* Even for the abstract reasons, modules have appear in algebraic geometry. If a commutative ring A can be thought of geometrically as an affine scheme $X = \mathrm{Spec}(A)$, there should be a geometric way to think of A-modules. Some examples:

(1) Any map of affine schemes $\mathrm{Spec}(A) = X \to Y = \mathrm{Spec}(B)$ corresponds to a map of rings $B \to A$, and this map makes A into a B-module!
(2) Closed subschemes S of an affine scheme X correspond to ideals I_S, i.e., to $= \mathcal{O}(X)$-submodules of the $\mathcal{O}(X)$-module The basic example is provided by subschemes $S \subseteq X = \mathrm{Spec}(A)$, the above reasoning reminds us that $\mathcal{O}(S) = \mathcal{O}(X)/I_S$ is a module for $\mathcal{O}(X)$.

5.3.4. *Support of a module.* Let X be an affine scheme and M a module for $\mathcal{O}(X)$. We will say that

(1) M is *supported* in a closed subscheme S if the action of $\mathcal{O}(X)$ factors through an action of $\mathcal{O}(X)/I_S = \mathcal{O}(S)$. This means that I_S acts trivially on M (*kills* M):
$$I_S{\cdot}M \;=\; 0.$$
(2) M is *set-theoretically supported* in a closed subscheme S if for each $m \in M$ there is some n such that $(I_S)^n{\cdot}m = 0$;, i.e.,
$$f_i \in I_S \;\Rightarrow\; f_1{\cdots}f_n \cdot m = 0.$$

Examples. (1) Let $X = \mathbb{A}^1 \supseteq S = \{0, 1\}$ so that $\mathcal{O}(X) = \Bbbk[x]$ and $I_S = x(x-1){\cdot}\Bbbk[x]$. Now, $\Bbbk[x]/x{\cdot}\Bbbk[x] = \mathcal{O}(\mathbb{A}^1)/I_{\{0\}} = \mathcal{O}(\{0\})$ is supported on S, but $\Bbbk[x]/x^2{\cdot}\Bbbk[x]$ is only set theoretically supported on S.

(2) Let $X = \mathbb{A}^2 \supseteq S = \{(0,0)\}$. Then $I_S = \langle x, y\rangle = x{\cdot}\Bbbk[x,y] + y{\cdot}\Bbbk[x,y]$ and $\mathcal{O}(S) \cong \Bbbk$.

Since $\mathcal{O}(X) = \Bbbk[x,y]$, a module for $\mathcal{O}(X)$ is the same as a vector space M with two operators X, Y that commute! Now, M is supported by S if it is killed by I_S, i.e., if it is killed by x and y, i.e., iff the operators X and Y are 0!

On the other hand, M is set-theoretically supported by S if it is killed by I_S, i.e., if each $m \in M$ is – for some n – killed by all n-fold products $f_1 \cdots f_n$ of functions f_i in I_S. However, this is equivalent to to the same condition

$$f_1 \cdots f_n \cdot m = 0$$

whenever all f_i's are in the set $\{x, y\}$ of generators of the ideal I_S. Finally, it suffices that there is some p such that

$$x^p \cdot m \;=\; 0 \;=\; y^p \cdot m$$

(then $n = 2p$ satisfies the preceding condition!). So, the condition is that the operators X, Y act nilpotently on each vector of M!

5.3.5. *Support cycle of a module.* Now we give a more refined notion of the support of a module M called the *support cycle* $\mathbf{supp}(M)$. However, for simplicity we give it only in a special case sufficient for our current purposes.

Let X be an affine variety. For a *finite-dimensional* $\mathcal{O}(X)$-module M which has a filtration by submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ such that the i^{th} graded piece

$$Gr_i(M) \stackrel{\text{def}}{=} M_i/M_{i-1}$$

is isomorphic (as an $\mathcal{O}(X)$-module) to $\mathcal{O}(\{p_i\})$ for a point $p_i \in X$, we say that

$$\mathbf{supp}(M) \;\stackrel{\text{def}}{=}\; \sum\; p_i.$$

So the support cycle is an element of the free abelian group

$$Z^0(X) \stackrel{\text{def}}{=} \oplus_{a \in X}\ \mathbb{Z} \cdot a$$

with the basis X which we call the "*group of* 0*-cycles in* X". A more geometric way to think of the support cycle $\text{supp}M$ is as an unordered tuple of points, i.e., an element of the symmetric power $X^{(n)}$.

Lemma. $\text{supp}M$ is well defined, i.e.,

(1) for any finite dimensional $\mathcal{O}(X)$-module M a filtration with above properties exists, and
(2) $\mathbf{supp}(M)$ depends only on M, and not on the choice of a filtration.

Proof. This is the Jordan-Hoelder lemma.

5.3.6. *Hilbert-Chow map.* We have just explained on the set theoretical level the following

Theorem. $X^{[n]}$ maps canonically to $X^{(n)}$, this is called the Hilbert-Chow map

$$X^{[n]} \xrightarrow{\pi} X^{(n)}, \quad S \mapsto \mathbf{supp}(\mathcal{O}(S)) \ .$$

5.3.7. *Hilbert schemes of affine spaces.* Here X us an affine space $\mathbb{A}^n$. We will see that our commutative algebra definition of the Hilbert schemes $X^{[p]}$ in this case becomes standard linear algebra.[24] Elements of Hilbert schemes are given by linear operators and the Hilbert-Chow map (i.e., the support cycle map), is given by taking eigenvalues of these operators.

How much does this example tell us about Hilbert schemes of arbitrary varieties X? The answer is that any smooth variety X is in some (not very obvious) sense, locally similar to some $\mathbb{A}^n$, and that therefore $X^{[n]}$ is *locally* similar to $(\mathbb{A}^n)^{(p)}$ (again this is not obvious). So this example tells us about the local structure of Hilbert schemes of smooth varieties.

Lemma. (a) $X^{(p)}$ is the set of isomorphism classes of $(n+2)$-tuples $(M, v, x_1, ..., x_n)$ of

- (0) a p-dimensional vector space M,
- (1) n commuting linear operators x_i on M,
- (2) a vector $v \in M$ which is *cyclic* for the operators – this means that v generators the whole space M under the action of operators $x_1, ..., x_n$, i.e.,

$$\textit{the map } \Bbbk[X_1, ..., X_n] \ni P \overset{\mathcal{E}_v}{\mapsto} P(x_1, ..., x_n) \cdot v \in M \ \textit{ is surjective.}$$

(b) If one thinks of a subscheme $S \in X^{[p]}$ in terms of the data $(M, v, x_1, ..., x_n)$, observe that the commuting operators x_i diagonalize simultaneously, i.e., there is a basis $v_1, ..., v_p$ in which all x_i's simultaneously have triangular matrices:

$$x_j \cdot v_i = \ c^i_j \cdot v_i + \sum_{k<i} \ \gamma^i_j(k) \cdot v_k \qquad \text{for all } i, j..$$

Then

$$\mathbf{supp}(M, v, x_1, ..., x_n) \ \ \textit{is the unordered p-tuple } \{\{c^1, ..., c^p\}\} \textit{ of points } c^i = (c^i_1, ..., c^i_n) \textit{ of } X = \mathbb{A}^n.$$

(So, the i^{th} point c^i consists of eigenvalues of x_j's on the i^{th} vector v_i).

Proof. (a) First, any closed subscheme $S \subseteq X$ of order n, defines an $\mathcal{O}(X)$-module $M = \mathcal{O}(S) = \mathcal{O}(X)/I_S$ of dimension n and $v = 1$ is a cyclic vector for the operators on M given by the action of generators X_i on M.

In the opposite direction, conditions (0) and (1) say that M is an n-dimensional module for $\mathcal{O}(X) = \Bbbk[X_1, ..., X_n]$. For any vector $v \in M$, the kernel of $\mathcal{E}_v$ is an ideal in $\Bbbk[X_1, ..., X_n]$ (called the *annihilator* of v). Now condition (3) means that there is a vector $v \in M$ such that $\mathcal{E}_v$ gives an isomorphism of $\mathcal{O}(X)$-modules $\mathcal{O}(X)/\text{Ker}(\mathcal{E}_v) \xrightarrow{\cong} M$, in particular the ideal $\text{Ker}(\mathcal{E}_v)$ has codimension p, i.e., it is an element of $X^{[p]}$. So, M obtains the

[24] "Most things work this way."

structure of algebra of functions on the subscheme $S{\subseteq}X$ of order n, given by the ideal $\mathrm{Ker}(\mathcal{E}_v)$.

Finally, "isomorphic" for $(M',v',x'_1,...,x'_n)$ and $(M'',v'',x''_1,...,x''_n)$ means that there is an invertible linear operator $g: M'\to M''$ such that $v''=gv'$ and $x''_i = g{\circ}x'_i{\circ}g^{-1}$. It appears because isomorphic tuples give the same ideal $\mathrm{Ker}(\mathcal{E}_v)$.

(b) is now clear: a basis v_i with a triangular action of x_j's gives a filtration $M_l = span\{v_1,...,v_l\}$ of M by $\mathcal{O}(X)$-submodules, and on $M_i/M_{i-1}\cong \Bbbk{\cdot}v_i$ each X_j acts by c^i_j, the same as on $\mathcal{O}(\{c^i\})$.

Corollary. (a) $X^{(p)}$ is the set of GL_p-orbits in the set of $(n+1)$-tuples $(v,x_1,...,x_n)$ of

- (1) n commuting linear operators x_i on $\Bbbk^p$,
- (2) a vector $v\in\Bbbk^p$ which is *cyclic* for the operators.

Proof. (a) We can assume that $M=\Bbbk^p$, so the datum consists of $(n+1)$-tuples $\mathcal{D}=(v,x_1,...,x_n)$ as above. The isomorphisms of such data $\mathcal{D}'$ and $\mathcal{D}''$ are then given by elements of $g\in GL_p$.

Corollary. (b) (The diagonal fibers of the Hilbert-Chow map.) Denote by $\mathbf{0}$ the zero point in X and by $p{\cdot}\mathbf{0}$ the corresponding p-fold point in $X^{(p)}$. The fiber $\pi^{-1}(n{\cdot}\mathbf{0})$ can be described as

- GL_p-orbits in the set of $(n+1)$-tuples $(v,x_1,...,x_n)$ that satisfy (1), (2) and

  (3) $\quad$ *x_i's are nilpotent.*
- all ideals I of codimension n in $\mathcal{O}(X)$ that lie between the ideals $I_{\{\mathbf{0}\}}=\sum\ X_i\mathcal{O}(X)$ and $(I_{\{\mathbf{0}\}})^n$, i.e.,
$$I_{\{\mathbf{0}\}}\supseteq I\supseteq (I_{\{\mathbf{0}\}})^n.$$

(c) (Locality property of the fibers of the Hilbert-Chow map.) Let $\boldsymbol{a}=\sum_{i=1}^k\ p_i{\cdot}a_i\in X^{(p)}$ be an unordered p-tuple where different points $a_i\in X$ appear with multiplicity p_i (so $\sum\ p_i=p$). Then the fiber is the product of contributions at different points:
$$\pi_p{}^{-1}(\sum_{i=1}^k\ p_i{\cdot}a_i)\ \cong\ \prod_1^k\ \pi_{p_i}{}^{-1}(n{\cdot}p_i{\cdot}a_i).$$

Explicitly,

- An ideal $I\in\pi_p{}^{-1}(\sum_{i=1}^k\ p_i{\cdot}a_i)$ is the same as k ideals $I_1,...,I_k$ with $I_{a_i}\supseteq I_i \supseteq (I_{a_i})^{p_i}$.
- The relation is
$$I=\ I_1\cap\cdots\cap I_k.$$

Proof. (a) *The first characterization.* Nilpotency means that all eigenvalues are 0, i.e., all points c^i equal $\mathbf{0}\in X$.

The second characterization. If I is in the fiber then $M=\mathcal{O}(X)/I$ has an $\mathcal{O}(X)$-filtration $0=M_0\subseteq\cdots\subseteq M_n=M$ such that $M_i/M_{i-1}\cong\mathcal{O}(\{0\})$. In other words, there are ideals $I=I_0\subseteq\cdots\subseteq I_n=\mathcal{O}(X)$ such that $I_i/I_{i-1}\cong M_i/M_{i-1}\cong\mathcal{O}(\{0\})$. In particular, $\mathcal{O}(\{\mathbf{0}\})\cong I_n/I_{n-1}=\mathcal{O}(X)/I_{i-1}$, and this implies that $I_{\{\mathbf{0}\}}=I_{n-1}\supseteq I_0=I$.

Moreover, as x_i is zero on $M_i/M_{i-1}\cong\mathcal{O}(\{\mathbf{0}\})$, it sends M_i to M_{i-1}. So, any product of p factors, all from $x_1,...,x_n$, kills M. This means that $(I_{\{\mathbf{0}\}})^n$ acts on $M=\mathcal{O}(X)/I$ by zero, i.e., that $(I_{\{0\}})^n\subseteq I$.

Conversely, let $I_{\{\mathbf{0}\}}\supseteq I\supseteq(I_{\{\mathbf{0}\}})^n$. First, observe that $\mathbf{supp}(I_{\{\mathbf{0}\}}/(I_{\{\mathbf{0}\}})^n)$ is a multiple of $\mathbf{0}$. It dominates $\mathbf{supp}(I_{\{\mathbf{0}\}}/I)$ which is therefore also a multiple of $\mathbf{0}$. Finally, so is

$$\mathbf{supp}(\mathcal{O}(X)/I)\ =\mathbf{supp}(\mathcal{O}(X)/I_{\{\mathbf{0}\}})\ +\mathbf{supp}(I_{\{\mathbf{0}\}}/I)\ =\mathbf{0}+\mathbf{supp}(I_{\{\mathbf{0}\}}/I)\ .$$

(c) Let $a_i=(a_{i1},...,a_{in})\in X=\mathbb{A}^n$ and $I\in{\pi_p}^{-1}(\sum_{i=1}^k\ p_i{\cdot}a_i)$. Then on $M=\mathcal{O}_X/I$ there is an $\mathcal{O}(X)$-filtration $0=M_0\subseteq\cdots\subseteq\mathbb{M}_n=M$ with $Gr_i(M)\cong\mathcal{O}(b_i)$ where points $b_1,...,b_n$ are the same as $a_1,...,a_n$ up to reordering. This implies that $\prod_i\ x_1-a_{i1}$ kills M. Therefore,

$$M=\oplus_{\alpha\in\{a_{11},...,a_{n1}\}}\ M_\alpha^{x_1}\quad\text{for}\quad M_\alpha^{x_1}=\{m\in M;\ (x_1-\alpha)^k=0\ \text{ for }\ k>>0\}.$$

Since x_i's commute, each $M_\alpha^{x_1}$ is an $\mathcal{O}(X)$-submodule. So, by inductions we find that

$$M=\oplus_{\alpha\in\mathbb{A}^n}\ M_{\alpha_1,...,\alpha_n}^{x_1,...,x_n},\quad\text{for}\quad M_{\alpha_1,...,\alpha_n}^{x_1,...,x_n}\stackrel{\text{def}}{=}\{m\in M;\ \text{for all } i,\ \ (x_i-\alpha_i)^k=0\ \text{ for }\ k>>0\}.$$

Clearly, each $M_\alpha\stackrel{\text{def}}{=}M_{\alpha_1,...,\alpha_n}^{x_1,...,x_n}$ has a filtration with graded pieces isomorphic to $\mathcal{O}(\alpha)$. So, $M_\alpha\neq 0$ iff α is one of a_i's, and then $\dim(M_\alpha)=p_i$.

Now, each M_α is a quotient of M, hence also of $\mathcal{O}(X)$. Let $I_\alpha\stackrel{\text{def}}{=}\mathrm{Ker}[\mathcal{O}(X)\twoheadleftarrow M_\alpha]$, then $M=\oplus_1^k\ M_{a_i}$ gives $I=\cap\ I_{a_i}$. So, $I\in{\pi_n}^{-1}(\sum_1^k\ p_i{\cdot}a_i)$ corresponds to a k-tuple of ideals $I_{a_i}\in{\pi_n}^{-1}(p_i{\cdot}a_i)$

5.3.8. *Examples.* (a) If X is a curve, Hilbert powers are the same as symmetric powers, i.e., the Hilbert-Chow maps $\pi_n:X^{[n]}\to X^{(n)}$ are isomorphisms.

(b) Always $X^{[1]}=X^{(1)}=X$.

(c) If $a_i\in X$ are all different, then ${\pi_n}^{-1}(\{\{a_1,...,a_n\}\})$ is a point.

(d) If $X=\mathbb{A}^n$ then $X^{[2]}\xrightarrow{\pi}X^{(2)}$ is an isomorphism off the diagonal while the fibers over the diagonal are all isomorphic to $\mathbb{P}^{n-1}$. Actually, more canonically

$${\pi_2}^{-1}(2{\cdot}a)=\ \mathbb{P}[T_a(X)]$$

is the set of all lines in the tangent vector space to X at a.

Proof. (a) For $a_i\in X=\mathbb{A}^1$ and $p\in\mathbb{N}$, $\pi^{-1}(\sum\ p_i{\cdot}a_i)\cong\prod\ \pi^{-1}(p_i{\cdot}a_i)$, so, it remains to see that $\pi^{-1}(p{\cdot}a)$ is a point. We can assume that $a=\mathbf{0}$, then the fiber consists of all ideals I of codimension p that lie between $I_{\mathbf{0}}=x\mathbb{k}[x]$ and $(I_{\mathbf{0}})^p=x^p\mathbb{k}[x]$. However, the codimension of $(I_{\mathbf{0}})^p=x^p\mathbb{k}[x]$ is p, the same as for I, hence $(I_{\mathbf{0}})^p\subseteq I$ is equality !

(b) $\mathcal{O}(X^{(1)}) = \mathcal{O}(X^1)^{S_1} = \mathcal{O}(X)$ hence $X^{(1)} = X$. Now, for $a \in X = X^{(1)}$, ${\pi_1}^{-1}(1{\cdot}a)$ consists of all ideals $I{\subseteq}\mathcal{O}(X)$ of codimension one such that $I_a{\supseteq}I{\supseteq}(I_a)^1$, i.e., the only I is I_a.

(c) By locality, ${\pi_n}^{-1}(\{\{a_1,...,a_n\}\} \cong$ *prod* ${\pi_1}^{-1}(\{\{a_i\}\}$ is a point by (b).

(d) For $a \in X$, ${\pi_2}^{-1}(2{\cdot}a)$ consists of all codimension 2 ideals $I{\subseteq}\mathcal{O}(X)$ that lie between $I_a = \sum_i\ (X_i - a_i)\Bbbk[X_1,...,X-n]$ and $I_a^2 = \sum_{i,j}\ (X_i - a_i)(X_j - a_j)\Bbbk[X_1,...,X-n]$. So it corresponds to all codimension one subspaces (*hyperplanes*), in $I_a/I_a^2 \cong\ \oplus\Bbbk{\cdot}(X_i - a_i)$, that are submodules for $\mathcal{O}(X)$. However, all X_i's act on I_a/I_a^2 by zero, so all subspaces are submodules. Therefore, the fiber is the set of all of hyperplanes H in $I_a/I_a^2] \cong \mathbb{P}[\oplus\Bbbk{\cdot}(X_i - a_i)]$. This is the same as the set of all lines L in projective space $(I_a/I_a^2)^*$, i.e., the projective space

$$\mathbb{P}[(I_a/I_a^2)^*] \cong\ \mathbb{P}[(\oplus\Bbbk{\cdot}(X_i - a_i))^*].$$

For the above geometric interpretation it remains to look at definitions

5.3.9. *The (co)tangent spaces to affine varieties.* For a point a in an affine variety X we define the cotangent space at a by

$$T_a^*(X) \stackrel{\text{def}}{=} I_a/I_a^2$$

and the tangent space by

$$T_a(X) \stackrel{\text{def}}{=} [T_a^*(X)]^* =\ [I_a/I_a^2]^*.$$

To see that this makes sense look at $X = \mathbb{A}^n$. Then

- $I_a = \sum_i\ (X_i - a_i)\Bbbk[X_1,...,X-n]$,
- $I_a^2 = \sum_{i,j}\ (X_i - a_i)(X_j - a_j)\Bbbk[X_1,...,X-n]$,
- $I_a/I_a^2 \cong\ \oplus\Bbbk{\cdot}(X_i - a_i)$,
- $[I_a/I_a^2] \cong\ \oplus\Bbbk{\cdot}\ \partial_{i,a}$.

Here we denote the basis $X_i - a_i$ of T_a^*X by $dx_{i,a}$, and the dual basis by $\partial_{i,a} = \partial\partial x_i|_a$.

5.3.10. *Hilbert scheme as a moduli of configurations of identical particles.* First notice that so far I have in a sense cheated − Hilbert schemes were defined as sets but I never explained the structure of a variety (or a scheme). This is easy but will be postponed. Next, all calculations were done for affine spaces $X = \mathbb{A}^n$. Actually, the results are the same for smooth varieties nd the proofs are the same once one knows the basic facts on smooth varieties.

Theorem. [Fogarty] If X is a smooth surface then $X^{[n]}$ is smooth.

So, Fogarty noticed that the Hilbert schemes $X^{[n]}$ provide a smooth version of the moduli of unordered n-tuples of points for any smooth surface X.

This observation is the foundation of current attempts to extend our fine understanding of curves (i.e., the one dimensional mathematics), to surfaces (Nakajima etc.). This is an important project with applications in mathematics and stringy physics.

5.3.11. *The extra information that makes moduli smooth.* Our first idea for the moduli was $X^{(n)}$ and it turned out to be satisfactory[25] when $\dim(X) = 1$, but singular when $\dim(X)$=2. However, Fogarty says that when $\dim(X) = 2$ then $X^{[n]}$ is a smooth moduli of unordered n-tuples.

Question. What did we forget when we took symmetric square of the plane rather than the Hilbert square? What is the extra information in $X^{[2]}$ that the Hilbert-Chow maps forgets? The fiber of the Hilbert-Chow map at a double point is the space of lines in the tangent space at a (see 5.3.8.d). So the extra information that is missing in the symmetric square is a direction at double points:

At double points the extra information can be thought of as the direction in which one point approached the other.

So, the conclusion is that if we want unordered pairs to change smoothly, at the diagonal when the points collide we should remember how they collided.

5.3.12. *The moduli of unordered points beyond surfaces.* However, for X of dimension > 2 the problem seems to persist, neither of $(\mathbb{A}^3)^{[n]}$ and $(\mathbb{A}^3)^{(n)}$ is smooth.[26]

5.4. **The need for stacks.** The *stacks* are a certain generalization of varieties and schemes. We will not go through the formal definition of stacks, we will only understand them in terms of the *Interaction Principle* bellow. Here, we notice in examples that there are moduli that can not be constructed by IT quotients. Equivalently, there are actions of groups on varieties for which the IT quotient does not do the job well. The main example will be the moduli of quadrics in $\mathbb{P}^n$, for instance the familiar situation of quadric curves in $\mathbb{P}^2$.

5.4.1. *Need more then the invariant theory quotients.* Here are some examples of the failure, i.e., invariant theory quotient does not produce what we expect:

(1) When the multiplicative group G_m acts on $\mathbb{A}^n$, there are many orbits:

$$0 \textit{ and one orbit } L - \{0\} \textit{ for each line } L.$$

However, $\mathbb{A}^n//G_m$ is a point since $\mathcal{O}(\mathbb{A}^n)^{G_m} = \mathbb{k}$.

[25] and very very important!

[26] There is a solution in terms of dg-schemes but it is not clear to me whether this is what one wants, i.e., how useful it is.

(2) When GL_n acts on $\mathbb{A}^n$, over a field $\Bbbk$, there are two orbits: $\{0\}$ and the rest. Again, $\mathbb{A}^n//GL_n$ is just a point since $\mathcal{O}(\mathbb{A}^n)^{GL_n} = \Bbbk[X_1, ..., X_n]^{GL_n} = \Bbbk$.

For instance, for $n = 1$, the multiplicative group $G_m \overset{\text{def}}{=} GL_1 = \mathbb{A}^1 - \{0\}$ acts on $\mathbb{A}^1$ with orbits 0 and G_m, but $\mathbb{A}^1//G_m = \text{pt}$.

(3) G_m acts freely on $\mathbb{A}^n - \{0\}$. Here, the set theoretic quotient is $\mathbb{P}^{n-1}$. However $\mathcal{O}(\mathbb{A}^n - 0) = \mathcal{O}(\mathbb{A}^n)$ (for $n \geq 2$), hence $\mathcal{O}(\mathbb{A}^n - 0)^{G_m} = \Bbbk$ and $(\mathbb{A}^n - \{0\})//G_m = \text{pt}$.

The problems:

- The first obvious problem arises from different sizes of orbits: say in (2), the non-zero orbit is open and dense (if $\Bbbk = \mathbb{R}$ or $\mathbb{C}$), so invariant functions are constant on an open set and therefore by continuity everywhere. So the smaller orbit had no say, it was eaten by the larger orbit.

  Another way to say this is that problem arises from different sizes of stabilizers $G_x = \{g \in G;\ gx = x\}$ of points of X. Therefore, the best kind of action will be the *free* action, i.e., the action for which there are no stabilizers.
- The problem in (3) is that $\mathbb{P}^n$ can not be captured by global functions.

Let us also consider an example of a moduli problem where invariant theory quotient does not work, because there is an open orbit as in (2) above:

5.4.2. *Moduli of quadrics.* By a quadric in $\mathbb{P}^n$ we mean any projective subvariety Q given by a degree 2 homogeneous polynomial $G = \sum_{i \leq j} g_{ij} X_i X_j$. In this case we will have an interesting notion of when two quadrics are the same:

we say that two quadrics $P \subseteq \mathbb{P}^n$ and $Q \subseteq \mathbb{P}^n$ are isomorphic if there is an automorphism $g : \mathbb{P}^n \xrightarrow{\cong} \mathbb{P}^n$ of $\mathbb{P}^n$, such that $g(P) = Q$.

Lemma. (a) $\text{Aut}(\mathbb{P}^n) = PGL_{n+1}$.

(b) Isomorphism classes of quadrics in $\mathbb{P}^n$ are given by the orbits of GL_{n+1} in non-zero symmetric matrices: $(\mathcal{S}_{n+1} - \{0\}) \subseteq M_{n+1}$, for the action $g(S) \overset{\text{def}}{=} g \cdot S \cdot g^{tr}$.

(c) If $\Bbbk = \mathbb{C}$, the isomorphism classes are determined by the rank of the matrix.

Proof. (a) First, GL_{n+1} acts on the vector space $\Bbbk^{n+1}$, and then also on the set $\mathbb{P}^n(\Bbbk)$ of lines in $\Bbbk^{n+1}$. Observe that the subgroup D of scalar matrices is isomorphic to $\Bbbk^*$ by $\Bbbk^* \ni c \mapsto c \cdot I_n \in D$, and that D fixes all lines. So D acts trivially, and therefore we get the action of the quotient group $GL_n/D \overset{\text{def}}{=} PGL_n$. The resulting map $PGL_n \to \text{Aut}(\mathbb{P}^n)$ is an isomorphism.[27]

[27]Left for later.

(b) To a symmetric matrix S one can attach a quadratic for Q_S. If we put the variables X_i into a row vector $\begin{pmatrix} X_1 & \cdots & X_n \end{pmatrix}$, then

$$Q_S(X) = X{\cdot}S{\cdot}X^{tr} = \sum_{i,j} s_{ij}X_iX_j.$$

(This is $G = \sum_{i\le j} g_{ij}X_iX_j$ if $s_{ij} = \begin{cases} g_{ij} & \text{if } i=j \\ \frac{1}{2}g_{ij} & \text{if } i\ne j \end{cases}$.) Now, any $g\in GL_n$ takes X to $X{\cdot}g$, and in this way it affects the quadratic form:

$$^g(Q_S)(X) \stackrel{\text{def}}{=} Q_S(X{\cdot}g) = (X{\cdot}g){\cdot}S{\cdot}(X{\cdot}g)^{tr} = X{\cdot}(g{\cdot}S{\cdot}g^{tr}){\cdot}X^{tr} = Q_{gSg^{tr}}(X).$$

(c) We use the fact that for $\Bbbk = \mathbb{C}$, any quadratic form can be diagonalized, i.e., after a linear change of variables it becomes a sum of squares $G_p = sum_1^p X_i^2$. So, each G-orbit in $\mathcal{S}$ contains a matrix of the form S_p that has p ones on the diagonal and the remaining entries are 0. Clearly, $rank(S_p) = p$. It remains to check that $rank(g{\cdot}S{\cdot}g^{tr}) = rank(S)$.

Conclusion. In $\mathbb{A}^n$ or $\mathbb{P}^n$ there are $n+1$ different quadrics, examples are given by $G_i = X_1^2 + \cdots + X_i^2$ for $1 \le i \le n+1$. So the moduli $\mathcal{M}$ of quadrics should have $n+1$ points $q_1, ..., q_{n+1}$. However, the quadrics of higher rank can degenerate to quadrics of lower rank, say the rank of $X^2 + tY^2$ is generically two, but it is one when $t=0$. This means that q_1 should be approachable from q_2, i.e., that q_1 lies in the closure of q_2. So the moduli is a funny space with points q_i such that $\overline{q_{n+1}} \supseteq \overline{q_n} \supseteq \cdots \supseteq \overline{q_2} \supseteq \overline{q_1} = q_1$. In particular, point q_{n+1} is dense in $\mathcal{M}$ and the only closed point is q_1.

Certainly, such $\mathcal{M}$ is not an affine variety! Also, the invariant theory quotient not adequate (it doe not produce $\mathcal{M}$), since $\mathcal{S}_{n+1}//GL_{n+1}$ is just a point. The meaning of having only one point in $\mathcal{S}_{n+1}//GL_{n+1}$ (coming from the dense orbit of GL_{n+1} in $\mathcal{S}_{n+1}$), is that invariant theory construction notices the non-degenerate quadrics of type q_{n+1}, but not the degenerate ones.

Question. Can we make $\mathcal{M}$ into *something* like an affine variety?

Let us frame this in terms of group quotients into:

Can one make a quotient $\mathcal{S}_n/GL_n$ which will be a geometric space with $n+1$ $\mathbb{C}$-points?

5.5. **Adding spaces (and stacks) to varieties by the Interaction Principle.**

5.5.1. *Interaction Principle.* Here we push the *Observation Principle* idea, to the following level

To know a space X is the same as to know how it interacts with other spaces.

What this will mean for us is that we know variety X if for each variety Y we know the set $Map(Y,X)$. Taken step further, the principle suggests that

- If we have a "reasonable construction that associates to each variety Y a set F_Y", we can hope that this construction is a description of some space $\mathfrak{X}$ such that $Map(Y,\mathfrak{X}) = F_Y$ for each Y.

5.5.2. *Interaction Principle in categories: Yoneda lemma.* So, it suggests that a natural way to extend a given category of spaces $\mathcal{C}$ is to add to it all *functors* F on $\mathcal{C}$ that are in some sense alike the functors $Map(-,X)$ defined by spaces $X \in \mathcal{C}$.

This is a very general idea. The most familiar use is the introduction of distributions in analysis. In category theory, this idea is called *Yoneda lemma*, we explain in the appendix A, in A.4.

5.5.3. *Interaction Principle in sets: Distributions.* This is the most familiar instance of applying the above *Interaction Principle* in mathematics. It is more elementary then the Yoneda lemma in the sense that here an interaction will produce one number (rather then one set).

The idea of delta-functions $\delta_a,\ a \in \mathbb{R}$, is quite useful, say in physics δ_a appears when some particle is imagined to be concentrated at the point a, or δ_t is a unit impulse which is applied in one moment t. However, it has no existence in standard calculus: it should be a function that is zero outside a and still $\int_{-\infty}^{+\infty}\ \delta_a(x)\ =\ 1$. One way to make it into a mathematical object is to observe that it interacts with nice functions such as $C^\infty(\mathcal{R})$ using integral:

$$\int_{-\infty}^{+\infty}\ f(x){\cdot}\delta_a(x)\ =\ f(a).$$

This leads to the definition of the the space $\mathcal{D}(\mathbb{R})$ of distributions on $\mathcal{R}$ as "things that interact reasonably with functions". Precisely, a distribution is a (continuous) linear functional on the space $\mathcal{S}$ of "nice functions" (or "test functions"). Then the delta-function at a can be defined as a distribution: this is the functional

$$\delta_a(f) = f(a),\ f \in \mathcal{S}.$$

What makes it useful is that

(1) Functions embed into distributions $C^\infty(\mathbb{R}) \hookrightarrow \mathcal{D}(\mathbb{R})$ (function ϕ gives distribution $f \mapsto \int_{-\infty}^{+\infty}\ f(x){\cdot}\phi(x))$.
(2) Calculus extends to distributions (there are notions of derivative, integral ...)

A similar pattern appears when we use the interaction idea to extend the range of objects in algebraic geometry. Our basic example will be the

5.6. The $\boxed{\textbf{true quotients } X/G}$ require spaces and stacks.

5.6.1. *The desire for good quotients.* We would like for a group G acting on an algebraic variety X to construct a geometric space X/G which will be a "good quotient" of X by G, in the sense that

(1) If X is smooth then so is X/G.
(2) The fibers of the quotient map $X \to X/G$ are all isomorphic to G.

Let us comment on the these conditions.

(1) is desirable – then we can calculate well on the quotient X/G, however it seems unreasonable. Remember that when $\{\pm 1\}$ acts on $\mathbb{A}^2_{x,y}$ the IT quotient is the cone $\{(u,v,z) \in \mathbb{A}^3;\ uv = z^2\}$. The singularity of the quotient cone at $(u,v,z) = (0,0,0)$ is there for a good reason. $(0,0,0)$ is the image of the origin $0 \in \mathbb{A}^2_{x,y}$, and while orbits $\{\pm p\}$ of $\{\pm 1\}$ in $\mathbb{A}^2$ usually have order two, there is a jump at the origin since $\{\pm 0\}$ has order one. So the discontinuity in the size of the orbit (or stabilizer) may cause singularity in the IT quotient.
(2) is satisfied for a set theoretic quotient when there are no stabilizers. Again, it seems impossible in general since for instance if the fibers of $pt \to pt/G$ should be G then we should find G inside pt.

We see that usually the desired quotient X/G does not exist – *as a variety (or a scheme).* Still, it exists and is important, for instance

- pt/G is called the classifying space of G,
- the G-equivariant cohomology of X (whatever that is) is best understood as the ordinary cohomology of X/G
$$H^*_G(X,\mathbb{Z}) \ = \ H^*(X/G,\mathbb{Z}),$$
- pt/C^* has a standard approximation $\mathbb{P}^\infty$.

5.6.2. *Spaces and stacks.* To find X/G we need to look into a world larger then $\Bbbk -$ *varieties* and we will look into two such
$$\Bbbk\text{-}Varieties \subseteq \Bbbk\text{-}\mathit{Spaces} \subseteq \Bbbk\text{-}\mathit{Stacks}.$$
Roughly, these are some distributional versions of $\Bbbk$-$Varieties$:

- The enlargement $\Bbbk$-$Varieties$⊆$\Bbbk$-*Spaces* is obtained by the Yoneda lemma, so
$$\Bbbk\text{-}\mathit{Spaces} \ = \ Funct(\Bbbk\text{-}Varieties^o, \mathcal{S}ets)$$
are functors from $\Bbbk$-varieties to sets, i.e.,
Spaces are variety-like objects that interact with varieties and produce sets.
- The enlargement $\Bbbk$-$Varieties$⊆$\Bbbk$-*Spaces* involves a generalization of the Yoneda lemma, in which sets are replaced by finer objects: groupoid categories. Roughly:
$$\Bbbk\text{-}\mathit{Spaces} \ = \ Funct(\Bbbk\text{-}Varieties^o, \mathcal{G}roupoid\mathcal{C}ategories).$$

Therefore,

Stacks are variety-like objects that interact with varieties and produce groupoid categories.

The idea is that when one tries to construct a useful (interesting) space, , i.e., a functor $\mathfrak{X} : \Bbbk\text{-}Varieties^o \to \mathcal{S}ets$, it often happens that the relevant sets $\mathfrak{X}(Y),\ Y \in \Bbbk\text{-}Varieties$; are often sets of isomorphism classes in some groupoid category, $\widetilde{\mathfrak{X}}(Y)$. So, one can ask whether the fundamental construction in this situation is $Y \mapsto \widetilde{\mathfrak{X}}(Y)$ (a stack!), rather then $Y \mapsto \mathfrak{X}(Y)$ (a space) ? The answer is YES: usually $\widetilde{\mathfrak{X}}$ is a better object, say $\mathfrak{X}$ may be singular and $\widetilde{\mathfrak{X}}$ smooth (the singularity is therefore not necessary, it is a produced by forgetting relevant information).

The step from spaces to stacks can be thought of as adding some group theory to the mix ("remembering the automorphisms groups").

5.6.3. $\Bbbk$-*space quotients* $\underline{X/G}$. Following the *Interaction Principle* above, to define X/G as a $\Bbbk$-space $\underline{X/G}$, we will specify for any algebraic variety Y how it interacts with X/G, i.e., we will

describe the set of maps $Map(Y, X/G)$ *without invoking the quotient* X/G, *i.e., in terms of the* G*-action on* X*.*

We start with trying to answer the same question in the simplest situation where a "good quotient" X/G exists on the level of varieties, this happens in the case of

5.7. **Free actions (torsors).** We will formulate what we mean by a "free' action, first on the level of sets and then in a way that makes sense in other situations (i.e., categories): topological spaces, manifolds, varieties,...

5.7.1. *Torsors in sets.* Let G be a group acting on a set X. The quotient set X/G is the set of G-orbits in X. Recall that we say that G-action on Y is

- *transitive* if for any $a, b \in Y$ there is a $g \in G$ such that $b = g{\cdot}a$, i.e., G has one orbit in Y. Then for any $y \in Y$ we get a canonical G-identification $G/G_a \xrightarrow{\cong} X,\ gG_a \mapsto g{\cdot}a$.
- *simply transitive* if for any $a, b \in Y$ there is precisely one $g \in G$ such that $b = g{\cdot}a$. In other words, it is transitive and the stabilizers are trivial. Equivalently, for each $a \in Y$, the map $G \ni g \mapsto ga \in Y$ is a bijection.

We say that the action is <u>*free*</u> (in the set theoretic sense) if there are no stabilizers: $G_x = 1,\ x \in X$.[28] An equivalent way to describe this situation is the following notion of *G-torsors* [29], which is standard in mathematical physics.

A <u>*G-torsor*</u> over a set $\mathfrak{Y}$ consists of a map $P \xrightarrow{\pi} \mathfrak{Y}$ and a G-action on P, such that

$$(*) \qquad G \textit{ acts simply transitively on each fiber.}$$

Notice that we in particular ask that G-preserves fibers of π, i.e., that π is a G-map for the trivial action on $\mathfrak{Y}$.

Lemma. (a) G-action on X is free iff the quotient map $X \to X/G$ is a G-torsor.

(b) $P \xrightarrow{\pi} \mathfrak{Y}$ is a G-torsor iff the map $G \times P \to P \times \mathfrak{Y} P,\ (g,p) \mapsto (gp, p)$ is a bijection.

Examples.

(1) $GL(V)$ acts simply transitively on the set $Fr(V)$ of bases $v = (v_1, ..., v_n)$ of a vector space, i.e., V is a $GL(V)$-torsor over a point.
   The set $Fr(V)$ of bases $v = (v_1, ..., v_n)$ of a vector space V is a torsor for $GL(V)$
   The set $Fr(V)$ of bases $v = (v_1, ..., v_n)$ of a vector space V is a torsor for $GL(V)$
(2) $V - 0 \to \mathbb{P}(V)$ is a torsor for G_m over $\mathbb{P}(V)$.
(3) If B is a subgroup of A then $A \to B\backslash A$ is an A-torsor.
(4) Any map $Y \xrightarrow{f} \mathfrak{X}$ can be used to <u>*pull-back*</u> a G-torsor $P \xrightarrow{\pi} \mathfrak{X}$ to a a G-torsor $f^*P \xrightarrow{f^*(\pi)} Y$. Space f^*P and the map $f^*\pi$ can be described fiber by fiber. The fiber $(f^*P)_y$ at $y \in Y$ is just the fiber $P_{f(y)}$ of π at $\pi(y) \in \mathfrak{X}$. A standard way to say this is (for more details see 5.11)
$$f^*P = \{(y,p) \in Y \times P;\ p \in P_{f(y)}\}.$$

Remark. Notice that the pull-back torsor f^*P is related to the original P by the G-map $f^*P \xrightarrow{\widetilde{f}} P,\ (y,p) \mapsto p$; which is characterized by the commutativity of the following diagram:

$$\begin{array}{ccc} f^*P & \xrightarrow{\widetilde{f}} & P \\ {\scriptstyle f^*(\pi)}\downarrow & & {\scriptstyle \pi}\downarrow \\ Y & \xrightarrow{f} & X \end{array}.$$

If we say that by definition of f^*P one has $(f^*X)_y = X_{\pi(y)}$, then the restriction of $\widetilde{f}$ to a fiber $(f^*X)_y \to X_{\pi(y)}$ is just the identity map.

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[28] Recall that in some of the above examples, the stabilizers $G_x = \{g \in G;\ gx = x\}$ of points of X caused problems.

[29] Also called *principal G-bundles* or just *G-bundles.*

5.7.2. *Torsors in other categories.* Let us now consider group actions a category $\mathcal{C}$ which is something like topological spaces, manifolds or $\Bbbk$-varieties.

We will say that an action of G on X (in a category $\mathcal{C}$) is *free*, if it can be completed to a G-torsor $X \to \mathfrak{X}$ (in the category $\mathcal{C}$). Then we say that $\mathfrak{X}$ is the *free quotient* of X by G (in $\mathcal{C}$).

A <u>*G-torsor*</u> over $\mathfrak{Y}$ consists of a map $P \xrightarrow{\pi} \mathfrak{Y}$ and a G-action on P, which is locally trivial in the sense that

each $y \in \mathfrak{Y}$ *has a neighborhood* U *such that over* U *one can identify* P *with* $U \times G$.

So we ask that there is a G-isomorphism $\phi : P|U \xrightarrow{\cong} U \times G$, which identifies π with the projection pr_U, i.e.,

$$\begin{array}{ccc}
P|_U & \xrightarrow[\cong]{\phi} & U \times G \\
{\scriptstyle \pi|\pi^{-1}U}\big\downarrow & & {\scriptstyle pr_U}\big\downarrow \\
U & \xrightarrow{=} & U.
\end{array}$$

Examples. (1) Let Σ be a smooth surface in the sense of a 2-dimensional real manifold. At each point $p \in \Sigma$ consider the set $or_{\Sigma,p}$ of orientations of Σ at p. It consists of two opposite orientations, so it has a simply transitive action of $\mathbb{Z}_2 \cong \{\pm 1\}$. Since orientations *locally* extend canonically: (i) $or_\Sigma = \cup_{p\in\Sigma}\ or_{\Sigma,p}$ has a canonical structure of a manifold (a double cover of Σ), (ii) or_Σ is a $\mathbb{Z}_2$-torsor over Σ. Actually, the same holds for any real manifold Σ.

(2) Let X be a $\Bbbk$-variety. Rank n vector bundles V over X are the same as GL_n-torsors over X. For instance, a vector bundle $V \to X$ defines a GL_n-torsor $Fr(V) \to X$ of *frames* of V. Here, $Fr(V)$ is defined so that the fiber of $Fr(V)$ at $x \in X$ is the set $Fr(V_x)$ of all bases $v = (v_1, ..., v_n)$ of the vector space V_x. Since one can identify $Fr(V_x)$ with the set $Isom(\Bbbk^n, V)$ of isomorphisms of vector spaces, we see how GL_n acts on it.[30]

5.7.3. *Local and global.* By definition, any G-torsor $P \xrightarrow{\pi} \mathfrak{Y}$ is *locally* trivial, i.e., locally can be identified with $\mathfrak{Y} \times G$. So, the interesting part is the global behavior. The first question is whether P is globally trivial, i.e., $P \xrightarrow{\cong} \mathfrak{Y} \times G$?

Lemma. Torsor P is trivial iff it has a global section, i.e., a map $\Sigma \xrightarrow{\sigma} P$ such that $\sigma(s) \in \pi^{-1}(s)$, i.e., $\pi \circ \sigma = id_{\mathfrak{Y}}$.

Proof. Actually sections σ of P are the same as trivializations $\iota : \mathfrak{Y} \times G \xrightarrow{\cong} P$. Here, $\mathfrak{Y} \times G$ has a canonical section 1_G, so ι gives a section $\sigma = \iota \circ 1_G$. In the opposite direction, a section σ gives the trivialization $\iota : P \xrightarrow{\cong} \mathfrak{Y} \times G$ by $\iota(y, g) \stackrel{\text{def}}{=} g \cdot \sigma(y),\ y \in \mathfrak{Y},\ g \in G$.

[30]This is not totally the same as the first example of its sort: $Fr(V)$ is a torsor for GL_n and for $GL(V)$ (a bitorsor for $(GL_n, GL(V))$!).

Corollary. The $\{\pm 1\}$-torsor $or_S \to S$ is trivial iff S is orientable.

Proof. An orientation on S is a global section of $or_S \to S$.

Remark. This example indicates what kind of global structure can be encoded in a torsor.

5.7.4. *The category $\mathcal{M}_G(X)$ of G-torsors over X.* For G-torsors $P \xrightarrow{p} X$ and $Q \xrightarrow{q} X$, $\mathrm{Hom}_{\mathcal{M}_G(X)}(P,Q)$ consists of all maps $X \xrightarrow{\alpha} Q$ which are G-maps over X, i.e.,

- Diagram $\begin{array}{ccc} P & \xrightarrow{\alpha} & Q \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle q}\big\downarrow \\ X & \xrightarrow{=} & X \end{array}$ commutes. This can be stated as: $q(\alpha(a)) = p(a),\ a \in P$, or:

  for each $x \in X$, α maps the the fiber P_x of to the fiber Q_x.
- $\alpha(g{\cdot}a) = g{\cdot}\alpha(g),\ g \in G,\ a \in P.$

One easily checks that this gives a category.

Lemma. (a) Let ${}_GG$ denote G viewed as a set with an action of the group G by the *left multiplication*, then the right multiplication $R_g(x) \stackrel{\text{def}}{=} xg^{-1}$ $(x, g \in G)$, gives an identification

$$\mathrm{Hom}_{G-SSets}({}_GG, {}_GG) \ \underset{\cong}{\xleftarrow{R}}\ G.$$

(b) The category $\mathcal{M}_G(X)$ of G-torsors over X is a groupoid category, i.e., each map is an isomorphism.

Proof. (b) We need to show that any map $\alpha \in \mathrm{Hom}_{\mathcal{M}_G(X)}(P,Q)$ is an isomorphism. So we need to see that each of the maps of fibers $\alpha : P_x \to Q_x$ is an isomorphism. Therefore is remains to see that if G acts simply transitively on $\mathcal{P}$ and $\mathcal{Q}$, any G-map $\mathcal{P} \xrightarrow{f} \mathcal{Q}$ is an isomorphism, but this is clear.

5.7.5. *Categorical characterizations of quotients X/G in the case of free actions.* In general in a category $\mathcal{C}$, we will say that an action of G on X is free, if it can be completed to a G-torsor $X \to \mathfrak{X}$. Then we say that $\mathfrak{X}$ is the *free quotient* of X by G. This makes sense because for any torsor $X \to \mathfrak{X}$, $\mathfrak{X}$ is really the set of G-orbits in X, so $\mathfrak{X}$ is the set of orbits X/G organized into an object of $\mathcal{C}$.

Now, for any $Y \in \mathcal{C}$ we want to describe the functions $Map(Y, X/G)$ purely in terms of the G-action on X. Since X is G-torsor over X/G, a map $f : Y \to X/G$ gives a pull-back of this torsor to a G-torsor $P \stackrel{\text{def}}{=} f^*X \xrightarrow{f^*(\pi)} Y$ over Y. Moreover, $P = f^*X$ comes with a

G-map $P \xrightarrow{\widetilde{f}} X$ such that the diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\widetilde{f}} & X \\ {\scriptstyle f^*(\pi)}\downarrow & & {\scriptstyle \pi}\downarrow \\ Y & \xrightarrow{f} & X/G \end{array} .$$

This leads to

Lemma. A map from Y to X/G is the same as an isomorphism class of pairs (P, F) of a G-torsor P over Y, and a G-map $P \xrightarrow{F} X$.

Proof. **(A)** From a pair (P, F) we get a map $F : Y\ toX/G$ by taking quotients: a G-map $F : P \to X$ gives a map $f = [Y \cong P/G \to X/G]$, i.e., $f(y) = F(p){\cdot}G$ when p is any element of the fiber P_y.

(B) If $(P, F) = (f^*X, \widetilde{f})$ for some $f : Y \to X$, then the above procedure recovers f from (P, F).

(C) The meaning of the expression "isomorphism classes" is that we say that two pairs (P, F) and (Q, G) are isomorphic if one can identify P and Q in a way compatible with the G-actions and the relation to X and Y, i.e., if there is an isomorphism of G-spaces $\phi : P \to Q$ such that

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ {\scriptstyle F}\uparrow & & {\scriptstyle G}\uparrow \\ P & \xrightarrow{\phi} & Q \\ \downarrow & & \downarrow \\ Y & \xrightarrow{=} & Y. \end{array}$$

So, the isomorphic pairs really contain the same information and we should not distinguish them. More precisely, what one needs to check is

Two pairs induce the same map $Y \to G/X$ iff the pairs are isomorphic!

5.8. **Space quotients $\underline{X/G}$.**

5.8.1. *Hope.* From the above examination of the case when the quotient exists, we *hope* that a "good quotient" X/G, whatever it is, will satisfy:

- For any variety Y, $Map(Y, X/G)$ is the set of
  all isomorphism classes of pairs (P, F) of a G-torsor P over Y, and a G-map $P \to X$.

5.8.2. *Definition.* Notice that we can turn the story around and use this *hope* as a *definition* of a $\Bbbk$-space $\underline{X/G}$, i.e., a functor $\underline{X/G} : \Bbbk\text{-}Varieties \to \mathcal{S}ets$, by

$$\underline{X/G}(Y) \stackrel{def}{=} \textit{isomorphism classes of a } G\textit{-torsor } P \textit{ over } Y \textit{ and a } G\textit{-map } P \to X.$$

5.8.3. *Relation to cohomology.* In particular, notice that the case of $X = pt$ is already interesting

- $\underline{pt/G}(Y)$ = the set of all isomorphism classes of G-torsors P over Y.

This set is usually called the 1^{st} *cohomology group of X with values in the group G* and denoted

$$H^1(X, G) \stackrel{\text{def}}{=} \underline{pt/G}(X) \; = \; \mathrm{Hom}_{\Bbbk-Spaces}(X, \underline{\mathrm{pt}/G}).$$

The last equality is by Yoneda lemma (theorem A.4.2).

5.8.4. Is $\underline{X/G}$ the "good quotient" we are looking for? A little checking shows that it does not satisfy one of our requirements, which we stated as

The fibers of $X \to X/G$ are isomorphic to G.

(With our enriched vocabulary we can restate this as: "$X \to X/G$ is a G-torsor".)

5.9. **Stack quotient X/G.**

5.9.1. *The correction of $\underline{X/G}$ to X/G.* The above $\Bbbk-$*Space version* of the quotient turns out to be close but not perfect. The subtlety is that when the action is not free we want to remember the stabilizers in some way. A fancy way to say this is: $Map(Y, X/G)$ is not just a set, it is a *category*! Then the correct definition is

5.9.2. *Definition.*

- For any variety Y $Map(Y, X/G)$ is the category of all pairs (P, F) of a G-torsor P over Y, and a G-map $P \to X$.

5.9.3. *Category $Map(Y, X/G)$.* Here, one defines the category structure on the pairs so that $\mathrm{Hom}_{Map(Y,X/G)}[(P, F), (Q, G)]$ is the set of all G-maps $\phi : P \to Q$ such that

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ {\scriptstyle F}\uparrow & & {\scriptstyle G}\uparrow \\ P & \xrightarrow{\phi} & Q \\ \downarrow & & \downarrow \\ Y & \xrightarrow{=} & Y. \end{array}$$

The commutativity of the lower square means that $\phi : P \to Q$ is a map of G-torsors over Y, and the upper square says that ϕ intertwines maps F, G of P, Q to X.

Lemma. Category $Map(Y, X/G)$ is a groupoid category, i.e., any map is an isomorphism.

Proof. This follows from the statement for the category $\mathcal{M}_G(Y)$.

5.9.4. *Remarks.* (1) The step we are taking now takes us beyond the world of categories – the totality $\Bbbk - Stacks$ of all $\Bbbk$-stacks, has more structure then a category since $\mathrm{Hom}_{Stacks}(\mathfrak{X}, \mathfrak{Y})$ is not a just a set but rather a category! Actually the totality of all $\Bbbk$-stacks is an example of a notion of $\underline{2 - categories}$, which is generalization of categories in a way similar to how categories. generalize sets.

(2) The difference between the set $Map(Y, \underline{X/G})$ and the category $Map(Y, X/G)$ is sort of small – the morphisms in the category are all isomorphisms, so this is just the information that is needed to form the isomorphism classes of objects. However, $Map(Y, X/G)$ does remember some information that is lost in $Map(Y, \underline{X/G})$ – the set of all automorphisms of each map (P, F) in $Map(Y, \underline{X/G})$.

(3) To be able to think and calculate with stacks (as with say varieties), requires (of course), extending all our algebraic geometry formalism to stacks (just as we extended calculus to distributions). The first step is (as for distributions) to introduce some natural topology on the category of $\Bbbk$-$Varieties$ and restrict ourselves to $\Bbbk$-space which are continuous in this topology. We will skip all the details, but the basic idea will be seen to be useful. For instance, when we study the maps from $\mathbb{P}^1$ to $\mathbb{P}^1$ or to any flag variety.

5.10. **The fibers of $X \to X/G$.** Here we sketch, how the difference between the $\Bbbk$-space $\underline{X/G}$ and the $\Bbbk$-stack X/G influence the fibers of the quotient map.

5.10.1. *Fibered products of sets and stacks.* For two maps of sets $A \xrightarrow{\alpha} C \xleftarrow{\beta} B$, the fibered product $A \times_C B$ (see 5.11), is the set of all pairs $(a, b) \in A \times B$ such that $\alpha(a) = \beta(b)$. So, $Map(Y, A \times_C B)$ is the set of all pairs $(p, q) \in Map(Y, A) \times Map(Y, B)$, such that one has equality $\alpha \circ p = \beta \circ q$ in the set $Map(Y, C)$.

However, for two maps of stacks $A \xrightarrow{\alpha} C \xleftarrow{\beta} B$ and a variety Y, for a pair $(p, q) \in Map(Y, A) \times Map(Y, B)$, $\alpha \circ p, \beta \circ q$ live in a category (rather then a set) $Map(Y, C)$. Now it is not very interesting whether they are the same, but rather whether they are isomorphic. However, this is still not enough – for consistent thinking we need more then "objects $\mathcal{P}, \mathcal{Q}$ are isomorphic", we need to remember which isomorphism we are using to compare these two objects. This *forces* the following definition of the fibered product for stacks

$Map(Y, A \times_C B)$ *is the set of all* triples $(p, q, \rho$ *where* $p \in Map(Y, A)$, $q \in Map(Y, B)$ *and* ρ *is an isomorphism* $\rho : \alpha \circ p \xrightarrow{\cong} \beta \circ q$ *in the category* $Map(Y, C)$.

5.10.2. *The fibers of $X \to X/G$.* We would like to see that the fibers of $X \xrightarrow{\pi} X/G$ are really isomorphic to G. A point of X/G is a map $pt \xrightarrow{\phi} X/G$ and the fiber of π at the point ϕ is the ϕ-pull-back $pt\times_{X/G}X$ of $X \xrightarrow{\pi} X/G$. What is $pt\times_{X/G}X$? since we are working with stacks this fiber is again a stack, hence a functor

$$pt\times_{X/G}X : \mathbb{k}\text{-}Varieties \longrightarrow \mathcal{S}ets,$$

and we know that $(pt\times_{X/G}X)(Y)$ consists of *triples* $(p, q, \rho$ where $p \in Map(Y, pt)$, $q \in Map(Y, X)$ and ρ is an isomorphism $\rho : \phi\circ p \xrightarrow{\cong} \pi\circ q$ in the category $Map(Y, X/G)$. Now, the set one can associate to a $\mathbb{k}$-space $\mathfrak{Z}$ is the

$$\textit{set of points of } Z \ \stackrel{def}{=}\ Map(pt, Z).$$

So we are interested in triples of $pt \xrightarrow{p} pt$, $pt \xrightarrow{q} X$ and $\rho : \phi\circ p \xrightarrow{\cong} \pi\circ q$ in the category $Map(pt, X/G)$.

5.10.3. *Case* $X = pt$. Let $X = pt$ for simplicity, then the obvious choice of a map $pt \xrightarrow{\phi} X/G$ is $\phi = \pi$. Actually, this also the only one since the category $Map(pt, X/G)$ is the category of G-torsors on pt and any two are isomorphic[31] such a map ϕ is given by a consists of!).

Now, the points of $(pt\times_{X/G}X)(Y)$ consists of *triples* $(1_{\rm pt}, 1_{\rm pt}, \rho)$ where ρ is an isomorphism of $\pi \xrightarrow{\cong} \pi$ in the category $Map(pt, pt/G) = \mathcal{M}_G(pt)$. This category has only one object (up to isomorphism), the trivial G-torsor $P = G \to pt$, and this is our π. However, the choices of ρ are given by $\mathrm{Aut}(P) \cong G$.

5.11. **Appendix: Fibered Products, Base Change, Cartesian Squares.** The following very useful construction is the general background for the construction of the pull-back of torsors. I will state it for the sets but it is important in many other settings.

5.11.1. *Fibered products.* The *set* X *over a set* B means a map $X \xrightarrow{p} B$.

The product of two sets over B, $X \xrightarrow{p} B$ and $Y \xrightarrow{q} B$ (also called the fibered product) is the set

$$X\times_B Y \stackrel{\rm def}{=} \{(x, y) \in X\times Y;\ p(x) = q(y)\} \subseteq X\times Y.$$

Notice that it comes with the projection maps $X \xleftarrow{pr_X} X\times_B Y \xrightarrow{pr_Y} Y$, and all maps fit into *commutative* square

$$\begin{array}{ccc} X\times_B Y & \xrightarrow{pr_Y} & Y \\ {\scriptstyle pr_X}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{p} & B \end{array}.$$

[31] So, *uniqueness* really means here uniqueness up to isomorphism.

5.11.2. A commutative square $\begin{tikzcd} Z \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y \arrow[d, "q"'] \\ X \arrow[r, "p"] & B \end{tikzcd}$ is called Cartesian if it is isomorphic to the square $\begin{tikzcd} X{\times_B}Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"] & Y \arrow[d, "p"'] \\ X \arrow[r, "p"] & B \end{tikzcd}$. This means that one can identify Z with $X{\times_B}Y$ so that α and β get identified with pr_X and pr_Y.

5.11.3. *Base Change or pull-back.* When we have a set Y over a set B, i.e., a map $Y \xrightarrow{q} B$, we may call B the base and we may think of what it would mean to change the base ? For any map $X \xrightarrow{p} B$ into B we can think of $X{\times_B}Y \xrightarrow{pr_X} X$ as the "p-pull-back" of $Y \xrightarrow{q} B$ from the base B to the base X, because for any $a \in X$ the fiber of $X{\times_B}Y \xrightarrow{pr_X} X$ at a is the same as the fiber of $Y \xrightarrow{q} B$ at $p(a)$:

$pr_X{}^{-1}(A) \ = \ \{(x,y) \in X{\times}Y;\ p(x) = q(y) \quad \text{and} \quad x = a\} \ = \ \{(a,y) \in X{\times}Y;\ q(y) = p(a)\} \ \cong \ \{y \in X{\times}Y$

So the base has changed but the fibers are the same. When viewed as the p-pull-back, of $Y \to B$, the fibered square can be denoted p^*P.

Notice that any Cartesian square can be viewed as a base change square in two ways.

5.11.4. *Pull-back of torsors.* If $P \to \mathfrak{Y}$ is a G-torsor, it is clear that for any map $f : \mathfrak{X} \to \mathfrak{Y}$, the pull-back $f^*P \overset{\text{def}}{=} \mathfrak{X}{\times_{\mathfrak{Y}}}P$ is a G-torsor over $\mathfrak{Y}$ ("the fibers do not change").

5.11.5. *Examples.* (1) If $X{\subseteq}B$ then the fibered square is just the restriction of Y to $X{\subseteq}B$. If $X, Y{\subseteq}B$ then the fibered square $X{\times_B}Y$ is just the intersection $X \cap Y$.

(2) For any map $X \xrightarrow{\pi} S$, the fibered square $X{\times_B}X{\subseteq}X^2$ is just the equivalence relation "$\pi(a) = \pi(b)$" on X.

5.11.6. *Algebraic geometry.* If $X \xrightarrow{p} B$ and $Y \xrightarrow{q} B$ are maps of $\Bbbk$-varieties (or schemes) then $X{\times_B}Y$ can be constructed on the same level. For instance iff X, Y, B are affine varieties then so is $X{\times_B}Y$ when constructed via

$$\mathcal{O}(X{\times_B}Y) \ = \ \mathcal{O}(X){\otimes_{\mathcal{O}(B)}}\mathcal{O}(Y).$$

6. Transcendental methods in algebraic geometry – the complex algebraic geometry (Cubics and Elliptic curves)

6.0.7. *Opportunism Principle.* Suppose that we are interested in algebraic varieties over the field $\mathbb{C}$ of complex numbers. Applying the *Opportunism Principle*

Special situations allow special tools,

we recall the complex analysis and notice that we may use non-algebraic ("transcendental") methods to study algebraic varieties over $\mathbb{C}$.

Another important application of this strategy is the use of the counting methods when we work over the (algebraic closure of) a *finite field*.[32]

6.0.8. *This is even useful for general Algebraic Geometry.* Moreover, this is a great idea even if one is interested in algebraic varieties over some other field $\mathbb{k}$. If by using the complex analysis we discover or prove in this way a claim about complex algebraic varieties that does not explicitly use the fact that the ground field is $\mathbb{C}$, we can hope that the same may be true over any closed field, and start looking for an algebraic proof.

As the basic example of the use of transcendental methods we will use complex analysis to study the simplest non-trivial curves, the cubics in $\mathbb{P}^2$. Then we will sketch the extension to more complicated curves.

6.0.9. *Remark.* A standard (transcendental) tool in complex algebraic geometry is the full use of *differential geometry*, which one relates to holomorphic geometry by statements such as:

If a line bundle L has positive the curvature, then L has many sections.

However we will not cover these methods, rather we just use of our standard proficiency in holomorphic functions.

6.1. **Cubics in $\mathbb{P}^2$.** Recall that we have found 3 quadratic curves in $\mathbb{A}^2$ (and two were degenerate versions of the third) in 3.0.13. Now we classify and study the cubic curves.

The most interesting ones will be the affine cubics of a special form

$$\mathcal{C}_\lambda = \ \{(x,y) \in \mathbb{A}^2;\ y^2 = x(x-1)(x-\lambda)\} \subseteq \mathbb{A}^2$$

for some $\lambda \in \mathbb{A}^1 = \mathbb{k}$, and the corresponding projective cubic curves are

$$C_\lambda \stackrel{\text{def}}{=} \overline{\mathcal{C}}_\lambda = \ \{[x:y:z] \in \mathbb{P}^2;\ y^2 z = x(x-z)(x-\lambda z)\} \subseteq \mathbb{P}^2.$$

[32]An example of this appears in homeworks. It illustrates the idea that the counting the number of elements of a variety over a finite field is related to the cohomology of the same variety over $\mathbb{C}$ (the precise relation is given by *Weil conjectures* proved by Deligne). It also suggests the existence of a non-trivial notion of a field with one element.

The reason we look at these is[33]

6.1.1. *Theorem.* Any cubic is isomorphic to one of C_λ.[34]

6.1.2. *Lemma.* The boundary of $\mathcal{C}_\lambda$ is a (triple) point.

Proof. The boundary of $\mathcal{C}_\lambda$ is obtained by requiring that $z = 0$, so we get all $[x : y : 0] \in \mathbb{P}^2$ with $0 = x^3$. So, it is one point $[0 : 1 : 0]$ (but it should really be regarded as a triple point).

6.2. **Drawing cubics over $\mathbb{C}$.** We will view $\mathcal{C}_\lambda$ in terms of the projection to the x-line ${\mathbb{A}^1}_x$[35]

$$\pi : \mathcal{C}_\lambda \to \mathbb{A}^1, \quad \pi(x,y) = x.$$

Notice that we can extend it continuously to $\pi : C_\lambda \to \mathbb{P}^1 = \mathbb{A}^1 \cup \infty$.

The fiber at x consists of two values $\pm\sqrt{x(x-1)(x-\lambda)}$ of the square root, except that we just get one point when $x = 0, 1, \lambda$ and ∞. We will say that C_λ is a *branched double cover of* $\mathbb{P}^1$ with branching at $0, 1, \lambda, \infty$.

Lemma. Projective cubics C_λ, $\lambda \neq 0, 1$; are all car tires, i.e., on the level of a topological space, C_λ is homeomorphic to a *torus.*

Let us write this lemma again in more details:

Lemma. (a) If D is a disc on the x-line ${\mathbb{A}^1}_x$, that does not contain $0, 1, \lambda$, the restriction $\mathcal{C}_\lambda|D \stackrel{\text{def}}{=} \pi^{-1}D$ is the disjoint union of two discs D_i, such that $\pi|D_i \to D$ is a holomorphic isomorphism.

(b) On the x-line $\mathbb{A}^1_x$ we choose curves a_λ from 0 to λ, b_λ from λ to 1 and c_λ from 1 to ∞, so that they do not intersect.[36] Their inverses $\alpha_\lambda, \beta_\lambda, \gamma_\lambda$ are circles in C_λ.

(c) Above $U = \mathbb{P}^1 - (a_\lambda \cup \gamma_\lambda)$, C_λ consists of two disjoint copies U_1, U_2 of U.

(d) The boundary $\partial U_i \stackrel{\text{def}}{=} \overline{U_i} - U_i$ is the union of two circles $\alpha_\lambda \cup \gamma_\lambda$.

(e) Therefore, $\overline{U_i}$ is homeomorphic to the sphere with two holes which are bounded by circles $\alpha_\lambda, \gamma_\lambda$. So, C_λ is obtained by gluing two "spheres with two holes" $\overline{U_1}$ and $\overline{U_2}$, and the gluing is performed by identifying the boundary circles.

(f) The result is a torus.

[33]Proof postponed.

[34]There are exceptions if the characteristic p of $\Bbbk$ is 2 or 3. We will not be interested in this ($p = 0$ is our main interest), and I will often forget to mention when things get more complicated if the characteristic is too small.

[35]This is the affine line with $\mathcal{O}({\mathbb{A}^1}_x) = \mathbb{C}[x]$.

[36]The best if we do it in some simple way. For instance, for most λ we can choose these curves as straight line segments.

Proof. (a) On D the function $x(x-1)(x-\lambda)$ has no zeros so there are two holomorphic functions $y_i(x) = \sqrt{x(x-1)(x-\lambda)}$, related by $y_2 = -y_1$. Therefore $\mathcal{C}_\lambda|D$ is the union of two discs D_i which are the graphs of two functions.

(b) follows.

(c) The argument is of the same kind as for (a), i.e., one *can* define two holomorphic functions $y_i(x)$ on U that are the two versions of $\sqrt{x(x-1)(x-\lambda)}$. The reason is that

- when one goes around 0 in the expression $\sqrt{x(x-1)(x-\lambda)} = \sqrt{x}\sqrt{x-1}\sqrt{x-\lambda}$ the first factor changes by -1 and the the other two factors do not change.
- when one goes around both 0 and 1, two factors change by -1, so the product does not change!

6.3. **Complex manifold structure.** If X is an affine or projective variety over $\Bbbk = \mathbb{C}$, then it is a subset of $\mathbb{C}^n$ (or $\mathbb{P}^n(\mathbb{C})$). We will see later that if X is *smooth*, i.e, if X has no *singularities*, then X has a canonical structure of a complex manifold.

6.3.1. *Lemma.* If $\lambda \neq 0, 1$, then (a) C_λ is a smooth (non-singular) algebraic variety, and (b) it has a natural structure of a one-dimensional complex manifold.

Proof. (a) We do not yet even know what this means.

(b) Let me check this for $\mathcal{C}_\lambda = C_\lambda \cap \mathbb{A}^2$, Similar calculation works near the infinite point of C_λ once you choose the appropriate local coordinates on $\mathbb{P}^2$ near this point.

So, consider a curve $\mathcal{C} \subseteq \mathbb{A}^2$ given by a polynomial equation $F(x, y) = 0$. If $\mathcal{C}$ has a tangent line at a point $p = (a, b)$ of $\mathcal{C}$, then near p one can use the x-projection (if the tangent line is not vertical), or the y-projection (if the tangent line is not horizontal), to identify a piece of $\mathcal{C}$ with with a piece of $\mathbb{A}^1$. The tangent line is defined if the differential d_pF (i.e., the gradient (F_x, F_y)), is not zero at p – then the equation is $F_x(p)(x-a) + F_y(p)(y-b) = 0$.

So we get a manifold structure on $\mathcal{C}$ except at the points p such that $0 = F(p) = F_x(p) = F_y(p)$.

Now consider F of the form $F = y^p - P(x)$. The above system of equations means now that $y^p = P(x)$ and $py^{p-1} = 0$ and $P'(x) = 0$. It implies that $y = 0 = P(x) = P'(x)$. So the only bad points are $(a, 0)$ with a a double root of $P(x)$. In our case, $P(x) = x(x-1)(x-\lambda)$ has no double roots for $\lambda \neq 0, 1$.

6.3.2. *Complex manifold view on cubics?* The first level of this question is to see whether there is some simple construction of C_λ as a complex manifold. The next level is to use complex analysis to study C_λ.

6.4. **Elliptic curves.** First, on a topological level, a torus can be constructed as $\mathbb{R}^2/\mathbb{Z}^2$. This however works on the level of complex manifolds: for any lattice L in $\mathbb{C}$, the quotient group $\mathbb{C}/L$ is a complex manifold in a shape of a torus. So we can hope that these are related to cubics and they will turn out to be the same thing.

6.4.1. *Lattices.* A lattice in a real vector space V is a subgroup L such that there is an $\mathbb{R}$-basis $v_1, ..., v_n$ of V such that $L = \oplus\ \mathbb{Z}{\cdot}v_i$. We consider the quotient group V/L.

Lemma. (a) The family of all subsets $U{\subseteq}V/L$ such that $\pi^{-1}U{\subseteq}V$ is open forms a topology on V/L.[37]

(b) The open box $B = \{\sum_i\ c_i v_i;\ 0 < c_i < 1\}{\subseteq}\ V$ has the property that for each $v \in V$

(1) $\pi|(\overline{B}+v)$ is surjective,
(2) $\pi(B+v){\subseteq}V/L$ is open and dense and
(3) $B+v \to \pi(B+v)$ is a homeomorphism.

(c) Group V/L is a compact topological group, i.e., group operations are continuous.

(d) As a topological space (and a topological group)

$$V/L \cong\ \oplus\ \mathbb{R}{\cdot}v_i/\mathbb{Z}{\cdot}v_i \cong\ \mathbb{R}^n/\mathbb{Z}^n =\ (\mathbb{R}/\mathbb{Z})^n \cong \mathbb{T}^n$$

for the circle group $\mathbb{T} \stackrel{\text{def}}{=} \{z \in \mathbb{C};\ |z| = 1\}{\subseteq}(\mathbb{C}^*, \cdot)$.

Corollary. V/L has a canonical structure of a real manifold.

Proof. We use the open cover of V/L by all $U_v \stackrel{\text{def}}{=} \pi(B+v),\ v \in V$. Each of these comes with a chart $V \overset{open}{B} + v \xrightarrow{\pi_v} U_v$. These charts are compatible since the transition functions are translations in V, so they form an atlas on V/L.

6.4.2. *Complex tori E_L.*

Lemma. (a) If L is a lattice in $\mathbb{C}^n$ (viewed as a $2n$-dimensional real vector space), then $E_L \stackrel{\text{def}}{=} \mathbb{C}^n/L$ has a unique structure of a complex manifold. such that the map $\mathbb{C}^n \xrightarrow{\pi} E_L$ is holomorphic.

(b) E_L is a compact holomorphic Lie group.[38] In particular for each $e \in E_L$ the translation $x{\mapsto}x+e$ is a holomorphic map (actually a an automorphism of the complex manifold E_L).

Proof. The same as above, except that now we observe that the transition functions are holomorphic, not only differentiable.

[37]Called the quotient topology on V/L induced from the topology V by the surjective map $V \xrightarrow{\pi} V/L$. So, $U{\subseteq}V/L$ is open iff $\pi^{-1}U{\subseteq}V$ is open.

[38]holomorphic Lie group means a complex manifold with a group structure such that operations are holomorphic functions.

Remark. (0) We call these the *complex tori.*

(1) From now on we consider <u>only the one-dimensional case</u> $E_L = \mathbb{C}/L$ for a lattice L in $\mathbb{C}$. This is a 1-dimensional complex manifold in a shape of a torus $S^1 \times S^1$.

(2) The standard examples are the lattices $L_\tau \stackrel{\text{def}}{=} \mathbb{Z}\oplus\mathbb{Z}\tau\subseteq\mathbb{C}$ for τ in the upper hyperplane $\mathbb{H} = \{z \in \mathbb{C};\ Im(\tau) > 0\}$. We denote

$$E_\tau \stackrel{\text{def}}{=} \mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z}).$$

6.5. **The moduli of elliptic curves.** We will find that two elliptic curves E_L, E_M are isomorphic precisely when $M = c{\cdot}L$ for some $c \in \mathbb{C}^*$. Therefore,

$$\textit{Moduli of elliptic curves} = \textit{(Moduli of lattices)}/\mathbb{C}^*.$$

However, we do not understand the RHS. So we work in stages: first we see that each elliptic curve E_L is isomorphic to one of the standard ones $E_\tau,\ \tau \in \mathbb{H}$; and then two standard elliptic curves are isomorphic iff the parameters in $\mathbb{H}$ are in the same orbit of $SL_2(\mathbb{Z})$. So,

$$\textit{Moduli of elliptic curves} = \mathbb{H}/SL_2(\mathbb{Z}).$$

This turns out to be understandable and beautiful.

6.5.1. *Isomorphisms of elliptic curves E_L.* We are interested in the classification of elliptic curves E_L up to holomorphic isomorphisms (i.e., up to isomorphisms of complex manifolds).

If two lattices L and M are related by $M = c{\cdot}L$ for some $c \in \mathbb{C}^*$, then the multiplication by c descends from a holomorphic isomorphism $\mathbb{C} \to \mathbb{C}$ to a a holomorphic isomorphism $\mu_c : E_L \to E_M$. So, E_L and E_M are isomorphic. We will see that the converse is also true, if E_L and E_M are isomorphic then M is a multiple of L.

6.5.2. *Sublemma.* (Lifting.) A holomorphic map $\phi : \mathbb{C} \to E_M$, is always of the form $\pi_M \circ f$ for some holomorphic function $f : \mathbb{C} \to \mathbb{C}$. Moreover, all lifts f of ϕ are in bijection with all lifts $a \in \mathbb{C}$ of $\phi(0) \in E_M$, i.e., for any choice of $a \in \mathbb{C}$ with $\pi_M(a) = \phi(0)$ there is a unique lift f of ϕ such that $f(0) = a$.

Proof. First notice that $\phi'(z) : \mathbb{C} \to \mathbb{C}$ is well defined and holomorphic, by using *any* local chart near $\phi(z) \in C/M$. Then

$$f(z) = a + \int_0^z \phi'(u)\ du$$

is well defined because $\mathbb{C}$ is simply connected (so it does not matter which path I take from 0 to z!).

A topological proof. If we consider a continuous $\phi : \mathbb{C} \to E_M$ that for any a we get a continuous lift f through a. It works like this: a chart identifies $B+a$ with a neighborhood

$\pi(B+a)$ of $\pi(a) = \phi(0)$. Since ϕ is continuous, there is a disc D around 0 that ϕ maps to $\pi(B+a)$. Now on D one can define f as a composition of ϕ and the inverse of the chart $B+a \to \pi(B+a)$. Now one replaces $z_0 = 0 \in C$ with $z_1 \in D = D_0$ which lies near the boundary of D, and one extends f from $D = D_0$ to $D_0 \cup D_1$ for a disc D_1 around z_1, etc. Since π is a local homeomorphism we can extend now from D_{i-1} to D_i forever. However, usually there is a problem: in principle when our sequence of connected discs comes back to itself, the newly obtained value of f need not coincide with what we found earlier. The reason such contradictions do not appear (again!) that the source $\mathbb{C}$ of the map ϕ is simply connected.

6.5.3. *Lemma.* $E_L \cong E_M$ iff $M = c{\cdot}L$ for some $c \in G_m(\mathbb{C}) = \mathbb{C}^*$.

Proof. $\boxed{\text{(A) Lifts to } \mathbb{C}.}$ For lattices L, M let $\mathcal{H}(E_L, E_M)$ be the set of all holomorphic maps $\sigma : E_L \to E_M$ such that $\sigma(0) = 0$.

Such σ gives a holomorphic map $\mathbb{C} \xrightarrow{\pi_L} E_L \xrightarrow{\sigma} E_M$ that sends $0 \in \mathbb{C}$ to $0 \in E_M$. Notice that among the lifts of $0 \in E_M$ to $\mathbb{C}$ there is a canonical choice: $0 \in \mathbb{C}$. Therefore, according to the sublemma $\pi{\circ}\sigma$ lifts *uniquely* to an entire function $\widetilde{\sigma} : \mathbb{C} \to \mathbb{C}$ (i.e., $\sigma{\circ}\pi_L = \pi_M{\circ}\widetilde{\sigma}$) such that $\widetilde{\sigma}(0) = 0$. This implies that

(1) If $\sigma \in \mathcal{H}(E_L, E_M)$, $\tau \in \mathcal{H}(E_M, E_N)$ then $\widetilde{\tau{\circ}\sigma} = \widetilde{\tau}{\circ}\widetilde{\sigma}$.
(2) $\widetilde{id_{E_L}} = id_L$.
(3) If $\sigma \in \mathcal{H}(E_L, E_M)$ is a holomorphic isomorphism then so is $\widetilde{\sigma} : \mathbb{C} \to \mathbb{C}$.

$\boxed{\text{(B) } \widetilde{\sigma}(L) \subseteq M.}$ For any $z \in \mathbb{C}$ and $l \in L$,

$$\pi_M(\widetilde{\sigma}(z+l)) = \sigma(\pi_L(z+l)) = \sigma(\pi_L(z)) = \pi_M(\widetilde{\sigma}(z)),$$

hence $\widetilde{\sigma}(z+l) - \widetilde{\sigma}(z) \in M$. As a function of z this is constant (the image of a non-constant holomorphic function is open!). So, M contains $\widetilde{\sigma}(z+l) - \widetilde{\sigma}(z) = \widetilde{\sigma}(0+l) - \widetilde{\sigma}(0) = \widetilde{\sigma}(l)$. Therefore $\widetilde{\sigma}(L) \subseteq M$ and for $l \in L,\ z \in \mathbb{C}$ we have

$$\widetilde{\sigma}(z+l) = \widetilde{\sigma}(z) + \widetilde{\sigma}(l).$$

$\boxed{\text{The end.}}$ If E_L and E_M are isomorphic, choose some holomorphic isomorphism $S : E_L \to E_M$. Then $\sigma : E_L \to E_M,\ \sigma(x) = S(x) - S(0)$ is also an isomorphism and it lies in $\mathcal{H}(E_L, E_M)$. Therefore $\widetilde{\sigma} : \mathbb{C} \to \mathbb{C}$ is a holomorphic automorphism. This means that $\widetilde{\sigma}(z) = az + b$ for some a, b, since we know that all holomorphic automorphism are linear functions! Now $b = \widetilde{\sigma}(0) = 0$ and $\widetilde{\sigma}(z) = az$ for some $a \in \mathbb{C}^*$.

Remember that $\widetilde{\sigma}(L) \subseteq M$ and $\widetilde{\sigma^{-1}}(M) \subseteq L$. Since $\widetilde{\sigma^{-1}} = \widetilde{\sigma}^{-1}$ we find that actually $\widetilde{\sigma}(L) = M$, i.e., $M = a{\cdot}L$.

Corollary. Any holomorphic isomorphism $\sigma : E_L \to E_M$ is a composition $T_e{\circ}\mu_c$ of some homothety $\mu_c,\ c \in \mathbb{C}^*$, and a translation T_e by an element $e \in E_M$.

Proof. The proof of the lemma actually gives precisely this statement.

6.5.4. *Automorphisms of elliptic curves.* E_L embeds into $\mathrm{Aut}(E_L)$ by translations. The stabilizer $\mathcal{A}_L = \{c \in \mathbb{C}^*;\ cL = L\}$ of a lattice L in $\mathbb{C}^*$ also embeds into $\mathrm{Aut}(E_L)$.

always contains $\{\pm 1\}$.

Lemma. $\mathrm{Aut}(E_L) \cong\ E_L \ltimes \mathcal{A}_L$.

Proof. We know that the multiplication $E_L{\times}\mathcal{A}_L \to \mathrm{Aut}(E_L)$ is surjective. It is also injective since E_L and $\mathcal{A}_L$ do not intersect. Finally, $\mathcal{A}_L$ normalizes E_L: $\mu_c{\circ}\tau_x{\circ}{\mu_c}^{-1} = \tau_{\mu_c(x)}$.

Remark. $\mathcal{A}_L$ certainly contains $\{\pm 1\}$. It is easy to see that $\mathcal{A}_L$ is larger then they only for some special lattices. However, these special cases are very interesting. The theory of theses cases is called *complex multiplication.*

6.5.5. *Elliptic curves E_τ.* Now we that $\mathbb{H}$ parameterizes all elliptic curves (but with repetitions!).

Corollary. Any elliptic curve E_L is isomorphic to some E_τ with τ in the upper half-plane $\mathbb{H}$.

Proof. Pick a $\mathbb{Z}$-basis u_1, u_2 of L. Then $\tau = u_1/{u_2}^{-1} \in \mathbb{C} - \mathbb{R}$ and (up to reordering the basis) we can suppose that $\tau \in \mathbb{H}$ (the signs of $Im(\tau)$ and $Im(\tau^{-1})$ are opposite!). Now $L = u_2{\cdot}L_\tau$ since multiplication by u_2 takes $\tau, 1$ to u_1, u_2.

6.5.6. *The action of $SL_2(\mathbb{R})$ on the upper half-plane.* This will be helpful in understanding which repetitions occur when we use $\mathbb{H} \ni \tau{\mapsto}E_\tau$ to parameterize all elliptic curves.

Lemma. (a) $GL_2(\mathbb{C})$ acts naturally on $\mathbb{C}^2$ and therefore also on the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$. In terms of the identification $\mathbb{C} \cup \infty \cong \mathbb{P}^1(\mathbb{C})$ by $\mathbb{C} \ni \tau{\mapsto}[\tau : 1] = \mathbb{C}{\cdot} \begin{pmatrix} \tau \\ 1 \end{pmatrix} \in \mathbb{P}^1$, the action on $\mathbb{C} \cup \infty$ is by fractional linear transforms

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \bullet \tau = \frac{\alpha + \beta\tau}{\gamma + \delta\tau}.$$

(b) Let $GL_2(\mathbb{R})_\pm{\subseteq}GL_2(\mathbb{R})$ consist of matrices g such that $det(g) > 0$ (resp. $det(g) < 0$). Then the subgroup $GL_2(\mathbb{R})_+$ preserves $\mathbb{H}{\subseteq}\mathbb{C}$ while $GL_2(\mathbb{R})_-$ takes $\mathbb{H}$ to the lower half plane $-\mathbb{H}$.

(c) $SL_2(\mathbb{R})$ acts transitively on $\mathbb{H}$ and the stabilizer of $i \in \mathbb{H}$ is the rotation group

$$K = \{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};\ \theta \in \mathbb{R}\}.$$

The subgroup $B_+ = \{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix};\ \alpha > 0,\ \beta \in \mathbb{R}\}{\subseteq}SL_2(\mathbb{R})$ acts simply transitively on $\mathbb{H}$.

Proof. (a) $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\cdot\mathbb{C}\begin{pmatrix}\tau\\ 1\end{pmatrix} = \mathbb{C}\begin{pmatrix}\alpha\tau+\beta\\ \gamma\tau+\delta\end{pmatrix} = \mathbb{C}\begin{pmatrix}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\\ 1\end{pmatrix}$.

(b) For $g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in GL_2(\mathbb{R})$ and $\tau \in \mathbb{C}-\mathbb{R}$,

$$Im(g\bullet\tau) = Im[\frac{\alpha\tau+\beta}{\gamma\tau+\delta}] = Im[\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\cdot\frac{\gamma\overline{\tau}+\beta}{\gamma\overline{\tau}+\delta}]$$

$$= Im\frac{\alpha\gamma|\tau|^2+\beta\delta+[\alpha\delta\tau+\gamma\beta\overline{\tau}]}{|\gamma\tau+\delta|^2} = \frac{Im(\tau)\cdot(\alpha\delta-\gamma\beta)}{|\gamma\tau+\delta|^2} = \det(g)\cdot\frac{Im(\tau)}{|\gamma\tau+\delta|^2}.$$

(c) First, $SL_2(\mathbb{R})\subseteq GL_2(\mathbb{R})_+$ preserves $\mathbb{H}$. Next, if $g \in SL_2(\mathbb{R})$, the above calculation gives

$$g\bullet i = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\bullet i = \frac{\alpha\gamma|i|^2+\beta\delta+i[\alpha\delta-\gamma\beta]}{|\gamma i+\delta|^2} = \frac{\alpha\gamma+\beta\delta+i}{\gamma^2+\delta^2}.$$

Now if $\gamma\bullet i = i$ then $\gamma^2+\delta^2 = 1$ (so $\gamma = -\sin\theta$ and $\delta = \cos\theta$ for some θ), and $\alpha\delta+\beta\gamma = 0$, i.e., the rows are orthogonal, hence $(\alpha,\beta) = c(\cos\theta,\sin\theta)$ for some $c\in\mathbb{R}$. Finally, $c = \det(g) = 1$, hence $g \in K$.

Finally, for $g = \begin{pmatrix}\alpha & \beta\\ 0 & \alpha^{-1}\end{pmatrix} \in B_+$ we see that $g\bullet i = \alpha^2 i+\alpha\beta$. So any $x+iy \in \mathbb{H}$ is $g\bullet i$ for unique $g \in B_+$.

Corollary. (Iwasava decomposition of $SL_2(\mathbb{R})$.) The product map $B_+\times K \xrightarrow{\cdot} SL_2(\mathbb{R})$ is a bijection.

6.5.7. *The moduli of elliptic curves E_τ.* For this moduli problem the "moduli with repetitions" $\widetilde{\mathcal{M}}$ that we start with is chosen as the upper half-plane $\mathbb{H}$, since it gives a complete family E_τ, $\tau\in\mathbb{H}$, of elliptic curves. Then the true moduli will be

$$\mathcal{M} \cong \mathbb{H}/SL_2(\mathbb{Z}).$$

Lemma. For $\tau_i \in \mathbb{H}$, $E_{\tau_1} \cong E_{\tau_2}$ iff $\tau_2 \in SL_2(\mathbb{Z})\cdot\tau_1$.

Proof. We know that $E_{\tau'} \cong E_\tau$ iff there is some $c\in\mathbb{C}^*$ such that $c\cdot L_{\tau'} = L_\tau$. The last condition is equivalent to: $\{c\tau', c\} = c\{\tau',1\}$ is a basis of L_τ, i.e., to: the transition matrix $g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in GL_2(\mathbb{C})$ given by

$$c\tau' = \alpha\cdot\tau+\beta\cdot 1 \quad\text{and}\quad c = \gamma\cdot\tau+\delta\cdot 1,$$

is actually in $GL_2(\mathbb{Z})$.

Therefore, $E_{\tau'} \cong E_\tau$ iff there is some $g \in GL_2(\mathbb{Z})$ such that the following equivalent conditions hold:

- $g\cdot\binom{\tau}{1} \in \mathbb{C}^*\cdot\binom{\tau'}{1}$.

- $g \cdot \mathbb{C}\left(\begin{smallmatrix}\tau\\1\end{smallmatrix}\right) = \mathbb{C}\left(\begin{smallmatrix}\tau'\\1\end{smallmatrix}\right)$ in $\mathbb{P}^1$.
- $\tau' = g \bullet \tau$.

It remains to notice that (since $\tau, \tau' \in \mathbb{H}$), the last condition can be satisfied only when g is in the subgroup $GL_2(\mathbb{Z}) \cap GL_2(\mathbb{R})_+ = SL_2(\mathbb{Z})$!

Theorem. $\mathcal{M} \cong \mathbb{H}/SL_2(\mathbb{Z})$ is a (set theoretic) moduli of elliptic curves.

Proof. This is all known by now. The theorem says that $\mathbb{H}$ parameterizes all elliptic curves and that the repetitions come exactly from the orbits of $SL_2(\mathbb{Z})$ in $\mathcal{H}$.

6.6. **Space** $\mathbb{H}/SL_2(\mathbb{Z})$**.** We are interested in the geometric structure on the moduli $\mathcal{M}$ of elliptic curves which we have so far constructed on the level of sets.

6.6.1. *The double coset formulation and modular forms (automorphic forms).* From the point of view of groups the moduli can be interpreted as

$$\mathcal{M} \cong SL_2(\mathbb{Z})\backslash\mathbb{H} \cong SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/K.$$

This leads to a far reaching group theoretic generalization – one considers double coset spaces

$$\Gamma\backslash G/K$$

where G is a semisimple real Lie group, K is a compact subgroup (often the maximal compact subgroup) and Γ is a discrete subgroup of arithmetic nature.

The functions on such spaces (*automorphic functions*) are key objects of *number theory* and *representation theory*.[39]

However, I'm telling the story upside-down in the sense that (1) the case $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/K$ is interesting enough (without generalizations), and (2) the beautiful theory of *modular forms* that has been developed in the case $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/K$ is based on the above moduli interpretation of the space (and also, in the general theory the best understood and most interesting cases are the ones where $\Gamma\mathbb{G}/K$ has a nice moduli interpretation in complex geometry).

Anyway, this is the subject of Paul's course on Modular Forms so I stop here.

6.6.2. *Fundamental domains.* Let $\Gamma = SL_2(\mathbb{Z})$. We will approximated the *quotient* $\mathbb{H}/\Gamma$ by a subset D of $\mathcal{M}$, such that (i) D is nice, (ii) $D \to \mathbb{H}/\Gamma$ is close to a bijection. Such D will be called a fundamental domain for the action of Γ on $\mathbb{H}$. More precisely, we ask that (i) D be a closed region in $\mathbb{H}$ bounded by finitely many curves, and that (ii) $D \to \mathbb{H}/\Gamma$ is surjective and injectivity only fails on the boundary ∂D.

[39]More generally one studies the (*automorphic forms*) which are sections of line bundles on these spaces.

We start with two elements $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $\Gamma = SL_2(\mathbb{Z})$ with simple geometric meaning: $T(z) = z+1$ is a translation and $S(z) = -1/z$ is the inversion (minus is need to stay in the upper half plane). Because of the translation T, $\mathbb{H} \xrightarrow{\pi} \mathbb{H}/\Gamma$ will be surjective when restricted to any vertical strip of length one, say to $\mathcal{S} = \{Z \in \mathbb{H};\ |Re(z) \leq \frac{1}{2}\}$.

Next, S sends the lines $L_\pm = \{x = \pm\frac{1}{2}\}$ to the semi-circles $C_\mp$ of radius one with centers at ∓ 1 (it sends $\infty \in \mathbb{L}_\pm$ to 0 and $\pm\frac{1}{2} \in L_\pm$ to ∓ 2, ...). So, it identifies the strip $\mathcal{S}$ to the outside of two semi-discs bounded by $C_\pm$. Moreover, S clearly exchanges the inside and the outside of the circle of the $C = \{|z| = 1\}$, and on C it acts as the symmetry with respect to the imaginary axis $e^{i\phi} \mapsto e^{-i\phi+\pi i} = e^{i(\pi-\phi)}$).

We consider a part of $\mathcal{S}$ outside C:[40]

$$\mathcal{D} \stackrel{\text{def}}{=} \{z \in \mathbb{H};\ |Re(z)| \leq \frac{1}{2} \quad \text{and} \quad |z| \geq 1\}.$$

It is bounded by parts of lines $L_\pm$ and the semicircle C. It has two vertices at two sixth roots of unity $L_+ \cap C = \ e^{\pi/3} = \rho$ and $L_- \cap C = \ e^{2\pi/3} = \rho^2$, and one infinite point $\infty \in \mathbb{P}^1$. Its S-image $S(\mathcal{D})$ is then the region (still inside the strip $\mathcal{S}$), bounded by C above and $C_\pm$ bellow. It has two vertices $\rho^2 = S(\rho)$ and $\rho = S(\rho^2)$ and one infinite point $S(\infty) = 0$.

The meaning of $\mathcal{D}$ is explained in

Lemma. (a) For any $\tau \in \mathbb{H}$ the size of the imaginary part has a maximum in the orbit Γz, and this maximum is achieved in $\mathcal{D}$.

(b) The subgroup $\Gamma' \subseteq \Gamma = SL_2(\mathbb{Z})$ generated by S and T satisfies: $\Gamma' \cdot \mathcal{D} = \mathbb{H}$.

Proof. (a) For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the imaginary part of γz is $\frac{Im(z)}{|cz+d|^2}$. For a fixed $z \in \mathbb{H}$, the set of all numbers $|cz+d|$, $\gamma \in \Gamma'$, contains the smallest value (since $c, d \in \mathbb{Z}$), therefore there is a $\gamma \in \Gamma$ such that $Im(\gamma z)$ is the largest possible. Now let $n \in \mathbb{Z}$ be such that $T^n(\gamma z)$ is in the strip $\mathcal{S}$ and the imaginary part is still the largest possible. Then $(T^n\gamma)z \in \mathcal{D}$ – otherwise $|T^n\gamma)z| < 1$, but then its S image would have a larger imaginary part[41]

(b) The proof of (a) used only S and T, so it also applies here. We notice that there is some $\gamma \in \Gamma'$ such that $Im(\gamma z)$ is the largest possible in $\Gamma' \cdot z$, and then, as above, we pass to $T^n(\gamma z) \in \mathcal{S}$ and observe that it really lies in $\mathcal{D}$.

Theorem. (a) $\mathcal{D}$ is a fundamental domain (and so is $S(\mathcal{D})$).

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[40]Draw $\mathcal{D}$ and $S(\mathcal{D})$!

[41]If $\mathbb{H} \in w$ and $|w| < 1$ then $|Im(w) < Im(-1/w)$ since $0, w, -1/w$ are on the same line.

(b) The only pairs of different $\tau, \tau' \in \mathcal{D}$ which are in the same orbit are either: (i) in different boundary lines $L_\pm$ and exchanged by T, or (ii) in the boundary semicircle C and exchanged by S.

(c) Since $\{\pm 1\} \subseteq SL_2(\mathbb{Z})$ acts trivially on $\mathbb{H}$, the action factors to the quotient group $\overline{\Gamma} = PSL_2(\mathbb{Z}) \stackrel{\text{def}}{=} SL_2(\mathbb{Z})/\{\pm 1\}$. The only points z in $\mathcal{D}$ with stabilizers Γ_z larger then $\{\pm 1\}$ (i.e., with nontrivial stabilizers $\overline{\Gamma}_z$ in $\overline{\Gamma}$), are

- (i) $\Gamma_i = \{\pm 1, \pm S\} \cong \mathbb{Z}_4$ (hence $\overline{\Gamma}_i \cong \mathbb{Z}_2$), and
- (ii) $\Gamma_\rho = \{\pm 1\} \cdot \{1, TS, (TS)^2\}$ and $\Gamma_{\rho^2} = \{\pm 1\} \cdot \{1, ST, (ST)^2\}$ (hence $\overline{\Gamma}_z \cong \mathbb{Z}_3$ in both cases).

(d) S, T generate $\Gamma = SL_2(\mathbb{Z})$.

Proof. Claims (b) and (c). The coincidences listed in (b) and (c) really happen. First, $T : L_- \xrightarrow{\cong} L_+$ and S acts on $\mathcal{D} \cap C$ as the reflection in the y-axis. Also, there are some points with obvious stabilizers:

- $z = i$ is fixed by the subgroup $\{\pm 1, \pm S\} \subseteq \Gamma$ generated by S.
- $z = \rho$ is fixed by $\pm TS$ and therefore by the subgroup $\{\pm 1, \pm TS, \pm (TS)^2\} = \{\pm 1\} \cdot \{1, TS, (TS)^2\} \subseteq \Gamma$.[42]
- $z = \rho^2$ is fixed by the subgroup $\{\pm 1, \pm ST, \pm (ST)^2\} = \{\pm 1\} \cdot \{1, ST, (ST)^2\} \subseteq \Gamma$.[43]

Now, consider $z, z' \in \mathcal{D}$ be in the same Γ-orbit and with $Im(z') \geq Im(z)$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ take z to z'. It remains to show that in any such case if $z' \neq z$ then z, z' appear in the list given by (b), and if $z' = z$ in the list given by (c).

Since $Im(z) \leq Im(z') = \frac{Im(z)}{|cz+d|^2}$, we have $|cz + d| \leq 1$. This implies $|c| \leq 1$ since for $z \in \mathbb{H}$, one has $|cz + d| \geq |c| \cdot Im(z) \geq |c| \cdot \frac{\sqrt{3}}{2}$. Now we discuss the possibilities $c = -1, 0, 1$.

(1) <u>If $c = 0$</u> then $d = \pm 1$ and $d \cdot \gamma = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for some $n \in \mathbb{Z}$. Since $z' = (d\gamma)z$ we see that $n \neq 0$ implies that $n = \pm 1$ and $\gamma = \pm T^{\pm 1}$. Then z, z' are have to be in different walls $L_\pm$, and this is the case (b).i with $z' \neq z$.

(2) <u>If $c = 1$</u> then $1 \geq |z + d|$, so in terms of $z = x + iy$, we have $1 \geq (d + x)^2 + y^2$. However, $y^2 \geq (\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$ and if $d \neq 0$ then $|d + x| \geq \frac{1}{2}$. So, either (i) $d = \pm 1$ and $z = \mp \frac{1}{2} + i\frac{\sqrt{3}}{2} \in \{\rho, \rho^2\}$, or (ii) $d = 0$.

- (i) $d = 0$ implies that $\gamma = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right)$, hence $\gamma \cdot S = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) = T^a$, and $\gamma = T^a S^{-1}$. Also, $1 \geq |z + d| = |z| \geq 1$ gives $z \in \mathcal{D} \cap C$. In particular $S^{-1}z$ lies in $\mathcal{D} \cap S$.

  (•) If $a = 0$ then $\gamma = S^{-1} = -S$ and we are in the case (b).ii, and $z = z'$ iff we are in (c).i.

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[42]Notice that $TS = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ has characteristic polynomial $\lambda^2 - \lambda \cdot Tr + Det = \lambda^2 + \lambda + 1 = \frac{\lambda^3 - 1}{\lambda - 1}$. So, the eigenvalues are the two primitive third roots of 1 and $(TS)^3 = 1$.

[43]Notice that $ST = S(TS)S^{-1}$ is a conjugate of TS.

(•) If $a \neq 0$ then the facts $S^{-1}z \in \mathcal{D} \cap S$ and $T^a(S^{-1}z)w = \gamma z = z' \in \mathcal{D}$, imply that $a = \pm 1$ and $S^{-1}z = -\frac{a}{2} + i\frac{\sqrt{3}}{2}$. Therefore, $z = \frac{a}{2} + i\frac{\sqrt{3}}{2} = z'$ and $\gamma = T^a S^{-1}$ is a known stabilizer element from (c).ii.

- (ii) If $d = 1$ then $\gamma = \left(\begin{smallmatrix} a & b \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} b+1 & b \\ 1 & 1 \end{smallmatrix}\right)$. Since $z = \rho^2$, $\mathcal{D}$ contains

$$z' = \frac{(b+1)z + b}{z+1} = b + \frac{z}{z+1} = b + \frac{\rho^2}{\rho} = b + \rho \ ,$$

and therefore $b = 0$ or $b = -1$. When $b = 0$ then $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = STS^{-1}$ takes $z = \rho^2$ to $z' = \rho \neq z$, but this is already done by T (case (b).i). When $b = -1$ then $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = T \cdot STS^{-1}$ fixes $z = \rho^2$.
$\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = T \cdot STS^{-1}$ fixes $z = \rho^2$. However, $ST = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right) = -\gamma$, so we are in (c.ii).
- If $d = -1$ and $z = \rho$ the argument is symmetric.

(3) <u>If $c = -1$</u> we pass to $-\gamma$, hence to $c = 1$.

Claims (a) and (d). In the preceding lemma, part (a) says that $\Gamma \cdot \mathcal{D} = \mathbb{H}$, so (a) follows from (b). For (d) let $\gamma \in \Gamma$, to show that it is in the subgroup Γ' generated by S and T we choose an *interior* point τ of $\mathcal{D}$. Since $\gamma\tau \in \mathbb{H}$ part (b) of the preceding lemma there is some $\gamma' \in \Gamma'$ with $\gamma'(\gamma\tau) \in \mathcal{D}$. So, $\gamma'\gamma \in \Gamma$ sends an interior point of $\mathcal{D}$ to$\mathcal{D}$, but (b) and (c) then imply that $\gamma'\gamma = 1$, hence $\gamma \in \Gamma'$.

Corollary. For any $\tau \in \mathbb{H}$ the intersection of the orbit $\Gamma \cdot \tau$ with $\mathcal{D}$ consists of all $w \in \Gamma \cdot \tau$ such that

- the imaginary part is maximal and
- w is in the strip $\mathcal{S}$, i.e., $|Re(w)| \leq \frac{1}{2}$.

Proof. In view of the lemma, it suffices to see that if $\tau, \tau' \in \mathcal{D}$ are in the same Γ-orbit then $Im(\tau') = Im(\tau)$, but this is clear from the part (b) of the theorem.

6.6.3. *The topological and holomorphic structure of* $\mathbb{H}/\Gamma$. $\mathbb{H}/\Gamma$ is the quotient of $\mathcal{D}$, obtained by making identifications from the theorem 6.6.2.b. When we identify the boundary pieces on the lines $L_\pm$, then $\mathcal{D}$ gives a tube, and the piece of C on the boundary: $e^{\phi i}$, $\pi/3 \leq \phi \leq 2\pi/3$, becomes a boundary circle on the tube since we identify the two ends. The remaining identification happens on this circle: $e^{\phi i} \leftrightarrow e^{(\pi-\phi)i}$. So, the circle gives a segment and this simply closes the bottom of the tube. So, topologically, $\mathcal{M} = \mathbb{H}/\Gamma$ is the plane.

So, if $\mathcal{M} = \mathbb{H}/\Gamma$ has a structure of a complex manifold, by Riemann's uniformization theorem it is isomorphic to either $\mathbb{C}$ or the unit disc.

To find the holomorphic structure on $\mathbb{H}/\Gamma$ and decide its nature, we will study the Γ-invariant holomorphic functions on $\mathbb{H}$. Certainly, such functions factor to functions on $\mathbb{H}/\Gamma$, and we hope that we will be able to put a complex manifold structure on $\mathbb{H}/\Gamma$ such that the holomorphic functions on $\mathbb{H}/\Gamma$ are precisely these factorizations of Γ-invariant holomorphic functions on $\mathbb{H}$.

6.6.4. *Modular functions and modular forms.* We will define *weak modular functions* as Γ-invariant holomorphic functions on $\mathbb{H}$. We are interested in these, but in practice they seem hard so we look into a larger class of *weak modular forms*. We will say that a *weak modular form of weight* $2k$ is a holomorphic functions on $\mathbb{H}$ which transforms under Γ in the following way:

$$f(\gamma{\cdot}z) = \ (cz+d)^{2k}{\cdot}f(z) \ \text{ for } \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

The idea is that if we can construct a few modular forms we can combine them to cancel the additional factor, so we get a modular function.

We get a really important mathematical object when we impose an additional condition. The *modular functions* are defined as the *weak modular functions* which are meromorphic at ∞ (and the same for modular forms). To make sense of this behavior at ∞, we first notice that the manifold $\mathbb{C}/\mathbb{Z}$ is identified with $\mathbb{C}^*$ using $\mathbb{C} \to \mathbb{C}^*$ by $z{\mapsto}e^{2\pi iz}$. This in particular identifies $\mathbb{H}/\mathbb{Z}$ with the punctured unit disc D^*. So, the T-invariant holomorphic functions f on $\mathbb{H}$ (i.e., $f(z+1) = f(z)$), are the same as holomorphic functions ϕ on D^*, via $f(z) = \phi(e^{2\pi iz})$. Notice that the infinity of the strip $\mathcal{S}$ corresponds to $0 \in D$. So, the behavior of f at ∞ is the same as the behavior of $\phi(q)$ ate $q = 0$. If f is weakly modular, the requirement that it be modular is that the expansion $\phi(q) = \sum_{-\infty}^{\infty} \ \phi_n{\cdot}q^n$ has finitely many negative terms.

If ϕ is regular at 0 we can say that f is regular at put $f(\infty) = \phi(0)$. We say that a modular form f is a *cusp form* if it is regular at ∞ and $f(\infty) = 0$. The origin of the terminology *cusp* or *cuspidal* in mathematics (in particular in representation theory), is that the infinity of $\mathbb{H}/\Gamma$ can be viewed as the infinity of the fundamental domain $\mathcal{D}$, but also as the infinite point 0 of $S(\mathcal{D})$, and $S(\mathcal{D})$ has a cuspidal shape at 0.

Remarks. (1) The geometric meaning of modular forms: $f(z)$ is modular of weight $2k$ iff $f(z)(dz)^k$ is Γ-invariant:

$$d(\gamma z) = \ d\frac{az+b}{cz+d} = \ \frac{ad-bc}{(cz+d)^2}dz = \ (cz+d)^{-2}\ dz.$$

In other words, these are differentials[44] on the moduli $\mathcal{M} = \mathbb{H}/\Gamma$ of elliptic curves.

(2) A weight can not be odd[45] since $f(\gamma{\cdot}z) = \ (cz+d)^l{\cdot}f(z)$ implies for $\gamma = -I$ that $f(z) = (-1)^l{\cdot}f(z)$.

(3) Modular forms are essential in number theory and representation theory. recently they have become important in algebraic topology (computations of homotopy groups of

[44]Here, by *differential* we mean a section of a tensor power of the cotangent bundle.
[45]Actually, odd weights appear but the definition is more complicated.

spheres). One of the mathematically most attractive features of string theory (particle physics) is the ease with which it constructs modular forms).

6.6.5. *Eisenstein series and the j-invariant.* A function f on the moduli of elliptic curves, means that to each (isomorphism class of) elliptic curve we attach a number. Since elliptic curves E_L come from lattices $L\subseteq\mathbb{C}$, in particular we want to attach a number to each lattice L. The obvious idea is to take some kind of average of all lattice elements. This gives Eisenstein series

$$G_k(L) \stackrel{\text{def}}{=} \sum_{0\neq l\in L} \frac{1}{l^{2k}}, \quad k=2,3,....$$

We will restrict the Eisenstein series to $\mathbb{H}$ by

$$G_k(\tau) \stackrel{\text{def}}{=} G_k(L_\tau) \stackrel{\text{def}}{=} \sum_{0\neq l\in L_\tau} \frac{1}{l^{2k}}.$$

The negative power $-2k$ is needed for

Lemma. $G_k(L)$ converges absolutely for $k>1$.

Proof. For absolute convergence we consider $\sum_{0\neq l\in L} \frac{1}{|l|^{2k}}$. We find that it is comparable with the integral

$$\int_{\mathbb{C}-?} \frac{1}{(x^2+y^2)^k}\ dx\ dy,$$

where ? is any union of finitely many L-boxes that contains 0. Here, *comparable* means that, the series and the integral converge for the same k. The reason is that in any L-box B that does not contain 0, $\frac{area(B)}{|p|^2} \leq \int_B \frac{1}{(x^2+y^2)^k}\ dx\ dy \frac{area(B)}{|q|^2}$ for the points p and q in B that are closest to 0 (resp. most distant from 0).

So the question is for which k does $\int_{|(x,y)|\geq 1} \frac{1}{(x^2+y^2)^k}\ dx\ dy$ converge. In polar coordinates this is $\int_0^{2\pi} d\theta \int_1^\infty\ r{\cdot}dr\ \frac{1}{r^{2k}} =\ 2\pi \int_1^\infty\ \frac{dr}{r^{2k-1}}$, so we need $2k-1>1$, i.e., $k>1$.

Lemma. $\Gamma_k(\tau)$ is a weak modular form of weight $2k$.

Proof. Clearly $Ga_k(L)$ is homogeneous of degree $-2k$, i.e., $\Gamma_k(c{\cdot}L) = c^{-2k}{\cdot}\Gamma_k(L)$. So, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$,

$$\Gamma_k(\gamma\tau) = \Gamma_k(L_{\gamma\tau}) = \Gamma_k(\mathbb{Z}\oplus\mathbb{Z}{\cdot}\gamma\tau) = \Gamma_k(\mathbb{Z}\oplus\mathbb{Z}{\cdot}\frac{a\tau+b}{c\tau+d}) = (cz{+}d)^{-2k}{\cdot}\Gamma_k(\mathbb{Z}{\cdot}(c\tau{+}d)\oplus\mathbb{Z}{\cdot}(a\tau{+}b))$$

$$\stackrel{(*)}{=} (cz+d)^{-2k}{\cdot}\Gamma_k(L_\tau) = (cz+d)^{-2k}{\cdot}\Gamma_k(\tau).$$

Here, the meaning of $(*)$ is that since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, the lattice with the basis $a\tau{+}b, c\tau{+}d$ is the same as the lattice with a basis $\tau, 1$.

We also need to know that $G_k(\tau)$ is holomorphic in τ. However,

$$G_k(\tau) \;=\; \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{m+n\tau}$$

and (i) each summand is holomorphic in τ, (ii) locally in τ, the sum converges uniformly. Here (ii) is proved by the same argument we used above for the pointwise convergence.

Proposition. (a) G_k is regular at ∞, actually

$$G_k(\infty) \;=\; 2{\cdot}\zeta(2k),$$

for the Riemann zeta function

$$\zeta(s) \stackrel{\text{def}}{=} \sum_1^\infty \frac{1}{n^2} = \prod_{p \text{ a prime}} \frac{1}{1-p^{-s}}.$$

In particular, this is a modular form of weight $2k$.

(b) $\Delta \stackrel{\text{def}}{=} (60G_2)^3 - 27(140G_3)^2$ is a cusp form of weight 12 with an elegant q-expansion

$$(2\pi)^12 \cdot q{\cdot} \prod_1^\infty (1-q^n)^{24}.$$

(c) $j \stackrel{\text{def}}{=} 1728\frac{\Delta}{(60G_2)^3}$ is a modular function.

Theorem. Γ-invariant function $j : \mathbb{H} \to \mathbb{C}$ factors to a bijection $\mathbb{H}/\Gamma \xrightarrow{\cong} \mathbb{C}$.

Corollary. We can use j to make $\mathbb{H}/\Gamma$ into a complex manifold.

Remark. Caution! $\mathbb{H}/\Gamma$ is "set-theoretic" moduli in the sense that it is a complex manifold and as a set it is the set of isomorphism classes of elliptic curves. However, there is a finer version of the moduli which is a stack – we get it if we do not forget the automorphisms of elliptic curves!

6.7. **Integrals of algebraic functions.** We look at the general problem of making sense of integrals of *algebraic functions*, i.e., functions $y(x)$ defined by solving for each x a polynomial equation $a_0(x)y^n(x) + a_1(x)y^{n-1}(x) + \cdots + a_{n-1}y(x) + a_n(x) = 0$. This is clearly a multi-valued function, so in integrals of the form

$$\int_\alpha^\beta y(x)\; dx$$

we need to specify

(1) which path we use from α to β and
(2) which branch of $y(x)$ we use on this path.

The confusion results in a subgroup $\mathcal{P}eriods\subseteq\mathbb{C}$ of *periods* of the integral, such that

$$\int_\alpha^\beta\ y(x)\ dx \text{ is not defined as a number in } \mathbb{C},$$
$$\text{but only as an element of the group } \mathbb{C}/\mathcal{P}eriods.$$

Our main interest is in the algebraic function $y\sqrt{x(x-1)(x-\lambda)}$ related to the cubic C_λ. In this case the integrals $\int_\alpha^\beta\ y\sqrt{x(x-1)(x-\lambda)}\ dx$ lead to a natural isomorphism of the cubic C_λ and a certain elliptic curve $E\tau$.

6.7.1. *Algebraic functions as branched covers of a line.* By an algebraic function on $\mathbb{C}$ I will mean a multivalued function $y(x)$ determined by a polynomial equation

$$a_0(x)y^n(x)+a_1(x)y^{n-1}(x)+\cdots+a_{n-1}y(x)+a_n(x)=0$$

where a_i's are polynomial functions on $\mathbb{C}$.

Examples. $\sqrt[n]{x}$, $\sqrt{x(x-1)(x-\lambda)}$, $\frac{1}{\sqrt{x(x-1)(x-\lambda)}}$ are all algebraic functions.

For a generic x there will be n different roots of the equation, so y will have n possible values. A geometric home for this non-standard mathematical object (a multi-valued function) is the *algebraic curve* $\mathcal{C}\subseteq\mathbb{A}^2$ defined by

$$\mathcal{C} \stackrel{\text{def}}{=} \{(x,y)\in\mathbb{A}^2;\ a_0(x)y^n+a_1(x)y^{n-1}+\cdots+a_{n-1}y+a_n(x)=0\}.$$

The projection $\mathcal{C}\subseteq\mathbb{A}^2_{x,y}\to\mathbb{A}^1_x$ is an n-fold branched cover of $\mathbb{A}^1$, and the branching happens over a finite subset $F\subseteq\mathbb{C}$ consisting of all $x\in\mathbb{C}$ such that the solutions y of the equation of $\mathcal{C}$ acquire multiplicities or $a_0(x)=0$.

Remark. Notice that to a multivalued function $y(x)$ over $U\subseteq\mathbb{C}$ we have associated a complex curve (a one dimensional complex manifold) which is a branched cover of U, by

$$\mathcal{C}=\{(a,b)\in\mathbb{A}^2;\ b \text{ is one of the values of } y(a)\}.$$

This works for all *multivalued holomorphic functions*. For instance $y=\log(x)$ on $\mathbb{C}^*$ gives $\mathcal{C}\to\mathbb{C}^*$ which can be identified with $\mathbb{C}\xrightarrow{e^z}\mathbb{C}^*$ (the *universal cover* of $\mathbb{C}^*$).

We think of a multivalued function $y(x)$ in terms of "branches", i.e., pairs $(V,Y(x)$ where $V\subseteq U$ is open and a holomorphic function Y on V is a version of $y(x)$ In terms of $\mathcal{C}$ these branches of $y(x)$ correspond to open pieces $W\subseteq\mathcal{C}$ such that $W\xrightarrow{pr_x}pr_x(W)$ is a bijection. Such W gives open $V=pr_x(W)\subseteq U$ and a branch Y on V by $Y=\ pr_y\circ(W\xrightarrow{pr_x}V)^{-1}$. In the opposite direction, $W=\{(a,Y(a));\ a\in V\}$.

6.7.2. *Lifting of paths (branches of $y(x)$ along paths).* Over the open set $U=\mathbb{C}-F$, $\mathcal{C}$ is an n-fold covering. So, on each open disc $D\subseteq U$, the multivalued function $y(x)$ breaks into n holomorphic functions $y_i(x)$, and the restriction $\mathcal{C}|D$ of $\mathcal{C}$ over the discs D, is a union of n disjoint discs D_i, the graphs of y_i, such that the map $D_i\to D$ is a holomorphic isomorphism.

A consequence of this is that for any path $\gamma : [0,1] \to U$, and any lift $p \in \mathcal{C}$ of $\gamma(0) \in U$,[46] there is a unique lift $\widetilde{\gamma} : [0,1] \to \mathcal{C}$ of γ, that starts at p.

Such lift can be viewed as a description of a continuous choice $\widetilde{y}(x)$ of the value of y along the path γ, since $\widetilde{\gamma}(t) = (\gamma(t), \widetilde{y}(t)$ for some $\widetilde{y} : [0,1] \to \mathbb{C}$, i.e., a *choice of a branch of* $y(x)$ *along* γ.

6.7.3. *Monodromy of algebraic functions.* This is a side remark on topological aspects of the multivalued nature of the algebraic function $y(x)$.

Let γ be path in $U = \mathbb{C} - F$, and let $\widetilde{\gamma}$ be a lift of γ to a path in $\mathcal{C}$. If γ is closed, i.e., $\gamma(1) = \gamma(0)$, it does not mean that $\widetilde{\gamma}$ is closed. We only know that $\pi(\widetilde{\gamma}(1)) = \pi(\widetilde{\gamma}(0))$. We say that $\widetilde{\gamma}(1) = M_\gamma(p)$ is obtained by applying to p the γ-monodromy.[47]

Example. If $y(x)$ is given by the equation $y^n - x = 0$, i.e., $y = \sqrt[n]{x}$ the branching happens at $F = \{0\}$ and above $\mathbb{C}^*$ curve $\mathcal{C}$ is the n-fold cover. If γ is a circle around 0 (counterclockwise), the monodromy is $e^{2\pi i/n}$.

6.7.4. *Integrals of algebraic functions.* In order to make sense of integrals

$$\int_\alpha^\beta y(x)\ dx$$

of a multi-valued function $y(x)$ we need to specify

(1) which path γ we use from α to β and
(2) which branch of $y(x)$ we use on this path.

However, we saw that a path γ in $\mathbb{C}$, from α to β, and a branch of $y(x)$ along γ, together amount to a choice of a path $\widetilde{\gamma}$ in $\mathcal{C}$ (a lift of γ), which goes from some lift $\widetilde{\alpha}$ of α to some lift $\widetilde{\beta}$ of β.

So our problem is really to calculate integrals of $y(x)\ dx$ over paths $\widetilde{\gamma}$ in $\mathcal{C}$:

$$\int_{\widetilde{\gamma}} y(x)\ dx \overset{\text{def}}{=} \int_0^1 pr_y(\widetilde{\gamma}(t)) \cdot (pr_x \circ \widetilde{\gamma})'(t)\ dt.$$

Moreover, if we allow path γ to pass through ∞ (i.e. paths on the Riemann sphere $\mathbb{P}^1$), then we have to allow the lift $\widetilde{\gamma}$ to pass through infinite points of $\mathcal{C}$, i.e., $\widetilde{\gamma}$ should be a path in the projective closure $C \subseteq \mathbb{P}^2$ of $\mathcal{C} \subseteq \mathbb{A}^2$.

6.8. **Periods of integrals.**

[46]A lift of $\gamma(0)$ is a point $p \in \pi^{-1}(\gamma(0)) \subseteq \mathcal{C}$ that lies above $\gamma(0)$.

[47]So, γ-monodromy acts on the fibers of $\mathcal{C} \to \mathbb{A}^1_x$. Notice the analogy with the Galois theory: monodromy permutes solutions of a polynomial equation.

6.8.1. *Periods of integrals.* We consider the problem of defining for any $\alpha,\beta\in C$ the integral $\int_\alpha^\beta\ y\ dx$ of y from α to β. For this we need a path γ from α to β in C. This can be done in many ways, however for any two choices the integrals differ by an integral over a closed path:

$$\int_{\gamma_1} y(x)\ dx - \int_{\gamma_2} y(x)\ dx = \int_\gamma y(x)\ dx$$

for the closed path $\gamma=\gamma_1-\gamma_2$ from β to β. The integrals

$$P_\gamma \stackrel{\text{def}}{=} \int_\gamma y(x)\ dx$$

over closed paths γ are called the *periods* of the integral. So, we found that

$$\boxed{\int_\alpha^\beta\ y\ dx\ \ \textit{is well defined up to periods.}}$$

This raises the question of finding all periods.

6.8.2. *Periods depend on closed paths up to homotopy.* How much does the period $P_\gamma = \int_\gamma\ y(x)\ dx$ depend on a choice of a close path γ? One of the basic tricks in complex analysis is the observation that the integrals are homotopy invariant, i.e., integral does not change as long as we move the path continuously.[48] So the basic question in this direction is

$(\star)$ *How many closed paths are there in C <u>up to homotopy</u>?*

To have a standard formulation let us pick a point $a\in C$ and let $\pi_1(C,a)$ be the set of *homotopy classes of closed paths* $\gamma:[0,1]\to C$ such that $\gamma(0)=a=\gamma(1)$. All periods come from $\pi_1(C,c)$ since any closed path can be continuously moved to one that passes through a. So, we are interested in a version of $(\star)$: what is $\pi_1(C,a)$?

Lemma. (a) $\pi_1(C,a)$ is a group.

(b) The map $\pi_1(C)\xrightarrow{\int\ y\ dx}\mathbb{C}$, given by $\pi_1(C)\ni\gamma\mapsto\int_\gamma\ y(x)\ dx\in\ \mathbb{C}$, is a morphism of groups.

Proof. (a) The operation is concatenation (composition) of paths: $\gamma_2\circ\gamma_1$ is the path obtained by first following γ_1 and then γ_2. (b) is now clear from definitions: $\int_{\gamma_2\circ\gamma_1} = \int_{\gamma_2}+\int_{\gamma_1}$.

Corollary. The set of periods $\mathcal{P}eriods =\ (\int\ y\ dx)\pi_1(C,c)\subseteq\mathbb{C}$ is a subgroup.

[48]Recall that there is a stronger version which gives sharper versions of statements bellow: integral depends on path only up to homology.

6.9. **Cubics are elliptic curves (periods of elliptic integrals).** The study of integrals $\int \frac{dx}{\sqrt{x(x-1)(x-la)}}$ appeared a classical mathematical question through its relation to the arc length of ellipses. We will use the above ideas on integration of algebraic functions for $y(x) = \frac{1}{\sqrt{x(x-1)(x-la)}}$. The corresponding algebraic curves that capture the multi-valued nature of $y(x)$ (i.e., of $\sqrt{x(x-1)(x-la)}$) are isomorphic to $\mathcal{C}_\lambda$ and C_λ[49]

6.9.1. *Lemma.* In a cubic C_λ ($\lambda \neq 0, 1$), closed paths up to homotopy form a free abelian group with a basis $\alpha_\lambda, \beta_\lambda$, i.e.,

$$\pi_1(C) \ \cong\ \mathbb{Z}\alpha_\lambda \oplus \mathbb{Z}\beta_\lambda.$$

Proof. This is a topological question so let us consider a torus $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/L \overset{\text{def}}{=} E_L$. First any closed path can be deformed continuously so that it passe through $\mathbf{0} = \pi(0)$. Now, any parameterization $\gamma : [0,1] \to E_L$ with $\gamma(0) = \mathbf{0}$, of the path lifts in a unique way to a path $\widetilde{\gamma} : [0,1] \to \mathbb{C}$ with $\widetilde{\gamma}(0) = 0$. Now $l = \widetilde{\gamma}(1) \in L$ and $\widetilde{\gamma}$ deforms to a straight line path from 0 to l. For any basis u, v of L, if $l = p{\cdot}u + q{\cdot}v$ we can further deform this strait line segment into a composition of straight line segments from 0 to $p{\cdot}u$ and from $p{\cdot}u$ to $p{\cdot}u + q{\cdot}v$. With appropriate choices of u, v the image of this deformed path is $p{\cdot}\alpha_\lambda + q{\cdot}\beta_\lambda$.

6.9.2. *Corollary.* $\int_\alpha^\beta \frac{dx}{\sqrt{x(x-1)(x-la)}}$ is well defined with values in the group

$$\mathbb{C}/[\mathbb{Z}{\cdot}P_{\alpha_\lambda} \oplus \mathbb{Z}{\cdot}P_{\beta_\lambda}].$$

Proof. We know that the difference of values of any two versions of $\int_\alpha^\beta y(x)\ dx$ is $\int_\gamma y$ for some closed path γ on C_λ. If γ is homotopic to $p{\cdot}\alpha_\lambda + q{\cdot}\beta_\lambda$, then

$$\int_\gamma y = \ p{\cdot}\int_{\alpha_\lambda} y \ + \ q{\cdot}\int_{\beta_\lambda} y = \ p{\cdot}P_{\alpha_\lambda} + q{\cdot}P_{\beta_\lambda}.$$

6.9.3. *Independence of periods.* In order to be able to claim that the values of the integral are in an elliptic curve, we need

Theorem. The set $\mathcal{P}$ of all periods of the integral $\int y(x)\ dx$ is a lattice with a basis $P_{\alpha_\lambda}, P_{\beta_\lambda}$.

Proof. This will be based on the study of a differential equation that the periods satisfy as functions of λ, the *Picard-Fuchs* equation.

6.9.4. *Cubics are elliptic curves.*

[49]The values of $y(x)$ and $1/y(x)$ are related by a change of coordinates.

Theorem. Choose $\alpha \in C$ and consider the map

$$C \to \mathcal{C}/\mathcal{P}eriods \text{ by } \beta \mapsto \int_\alpha^\beta \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

This is an isomorphism of complex manifolds.

Proof. Map is holomorphic since in local coordinates we know that the derivative of $\int_a^z f(u)\, du$ is $f(z)$. Next, the map is a local isomorphism since the derivative does not vanish. Surjectivity is easy: the image is open (map is local isomorphism!) and closed (source is compact). Injectivity requires additional thinking.

Corollary. Any cubic C_λ is isomorphic to one of the elliptic curves E_τ, $\tau \in \mathbb{H}$, as a *complex manifold*.

6.10. **Theta functions on complex tori.** Any $\tau \in \mathbb{H}$ (i.e., $Im(\tau) > 0$ gives a lattice $L_\tau = \mathbb{Z} \oplus \mathbb{Z} \cdot \tau$ in $\mathbb{C}$, and an elliptic curve $E_\tau = \mathbb{C}/L_\tau$ which comes with the quotient map $\pi : \mathbb{C} \to E_\tau$.

We would like to find some holomorphic functions on E_τ, and this is the same as a holomorphic function f on $\mathbb{C}$ which is periodic in directions of 1 and τ: $f(u+1) = f(u) = f(u+\tau)$. However, there are no such functions, so we ask for the "next best thing": periodic for 1 and *quasiperiodic* for τ in the sense that $f(u+\tau)$ differs from $f(u)$ by a simple factor.

We will construct such functions as *theta series* $\theta_\tau(u)$, given by a formula. Philosophically, being periodic in one direction and *quasiperiodic* in another, means that θ_τ does not really descend to a function on E_τ, but to something close to a function: a section of a line bundle on E_τ.

However, the root of our interest in theta functions is more elementary. We will actually manage to produce a real life function $\mathfrak{p}_\tau$ on E_τ using θ_τ – a combination of θ and θ' in which the quasi-periodicity factor cancels!). It will have a defect: a pole at one point, but this turns out to be obligatory.

6.10.1. *Theta series.* The theta series[50] in $\tau \in \mathbb{H}$ and $u \in \mathbb{C}$ is

$$\theta_\tau(u) \stackrel{\text{def}}{=} \sum_{-\infty}^{+\infty} e^{\pi i (n^2\tau + 2nu)}.$$

Lemma. (a) For any $\tau \in \mathbb{H}$ it defines an entire function of u.

(b) For any $u \in \mathbb{C}$ it defines a holomorphic function on $\mathbb{H}$.

(c) For any $a \in \mathbb{R}$, $b > 0$, the series converges uniformly on the product

$$\{\tau \in \mathbb{H};\ Im(\tau) > b\} \times \{u \in \mathbb{C};\ Im(u) > a\}.$$

[50] The basic facts bellow are proved in homeworks.

(d) The series can be differentiated any number of times (with respect to τ and u), and the derivatives are calculated term by term.

6.10.2. *Transformation properties of theta functions.* We find that θ_τ is periodic in one direction, quasi-periodic in another direction, and even.

Lemma. (a) $\theta_\tau(u+1) = \ \theta_\tau(u)$.

(b) $\theta_\tau(u+\tau) = \ e^{-\pi i(\tau+2u)}\cdot\theta_\tau(u)$.

(c) $\theta_\tau(-u) = \ \theta_\tau(u)$.

6.10.3. *Zeros of theta functions.*

Lemma. (a) θ_τ has a zero at $u_0 \stackrel{\text{def}}{=} \frac{\tau+1}{2}$.

(b) This is the only zero of θ_τ in the closed parallelogram $\overline{\mathcal{P}}_\tau$ generated by vectors $1,\tau$ in the real vector space $\mathbb{C}$:

$$\mathcal{P}_\tau \stackrel{\text{def}}{=} \{a+b\tau;\ 0<a,b<1\}.$$

Proof. (a)

$$\theta_\tau(\frac{\tau+1}{2}) = \ \theta_\tau(\frac{1-\tau}{2}+\tau) = \theta_\tau(\frac{1-\tau}{2})\cdot e^{-\pi i(\tau+2\frac{1-\tau}{2})} = \ -\theta_\tau(\frac{1-\tau}{2})$$

$$= \ -\theta_\tau(\frac{\tau-1}{2}) = \ -\theta_\tau(\frac{1-\tau}{2}+1) = \ -\theta_\tau(\frac{\tau+1}{2}).$$

6.11. **Weierstrass $\mathfrak{p}$-function (elliptic curves are cubics).** By *elliptic functions* we mean meromorphic functions on elliptic curves E_τ. Our goal is to find some such, since one can not do better:

Lemma. Any holomorphic function on a (connected) compact complex curve is constant.

Proof. The image $f(C)\subseteq\mathbb{C}$ is compact. However, if f were not constant its image would have to be open (*Open mapping theorem*).

6.11.1. *Weierstrass $\mathfrak{p}$-function $\mathfrak{p}_\tau$ on E_τ.* Recall that each $\tau\in\mathbb{H}$ defines the function $\theta_\tau(u)$ on $\mathbb{C}$. The Weierstrass $\mathfrak{p}$-function is a meromorphic function on $\mathbb{C}$ which we will define as the second logarithmic derivative of the theta function

$$\mathfrak{p}_\tau(u) \stackrel{\text{def}}{=} (log(\theta_\tau(u))'' = \ (\frac{\theta_\tau'(u)}{\theta_\tau(u)})'.$$

Lemma. (a) $\mathfrak{p}_\tau(u) \stackrel{\text{def}}{=} (log(\theta_\tau(u))''$ is a well defined holomorphic function on

$$\mathbb{C}\setminus\,(\tfrac{1+\tau}{2}+L_\tau),\quad \textit{i.e., off the } L_\tau\textit{-translates of the point } \tfrac{1+\tau}{2}.$$

(b) $\mathfrak{p}_\tau$ is L_τ invariant, i.e., $\mathfrak{p}_\tau(z+1)=\mathfrak{p}_\tau(z)=\mathfrak{p}_\tau(z+\tau)$.

(c) $\mathfrak{p}_\tau$ has a pole of order two at $\frac{1+\tau}{2}$.

(d) $\mathfrak{p}_\tau$ is meromorphic on $\mathbb{C}$.

Corollary. $\mathfrak{p}_\tau$ factors to a meromorphic function $\mathfrak{p}_\tau$ on E_τ. Its only pole is at $\zeta \stackrel{\text{def}}{=} \pi(\frac{\tau+1}{2}) \in E_\tau$, and it is a double pole.

Remark. In group theoretic terms, point ζ is one of three points of order 2 in E_τ.

6.11.2. *Elliptic curves are cubics.*

Theorem. (a) Map $f=(\mathfrak{p}_\tau,\mathfrak{p}'_\tau): E_\tau-\{\zeta\}\to\mathbb{C}^2$ has image in a cubic $\mathcal{C}$ of the form $y^2=4x^3+Ax^2+Bx+C$.

(b) f extends to a holomorphic isomorphism of E_τ and the projective closure $C=\overline{\mathcal{C}}$ of $\mathcal{C}$.

Proof. $\boxed{\text{(a)}}$ Let us dispense with the index τ, so $\theta_\tau=\theta$ etc.

1. Reduction to killing the pole at ζ. f is a holomorphic map on $E_\tau-\{\zeta\}$, and its component functions $\mathfrak{p}=(\frac{\theta'_\tau(u)}{\theta_\tau(u)})'$, $\mathfrak{p}'$ have poles at ζ of orders 2 and 3. The claim is that for some $A,B\in\mathbb{C}$

$$(\mathfrak{p}')^2-4\mathfrak{p}^3-A\mathfrak{p}^2-B\mathfrak{p}\ \textit{is a constant.}$$

However, it suffices that $(\mathfrak{p}')^2+4\mathfrak{p}^3-A\mathfrak{p}^2-B\mathfrak{p}$ be holomorphic at ζ (holomorphic functions on E_τ are constant!).

2. Strategy. We will study the polar parts of Laurent expansions of $(\mathfrak{p}')^2,\mathfrak{p}^3,\mathfrak{p}^2,\mathfrak{p}$ at ζ and we will see that an appropriate combination has no pole. The expansions will be in the variable $v=u-\frac{1+\tau}{2}$.

3. Expansion of $\mathfrak{p}(u)$ is $-v^{-2}+a+bv^2+O(4)$. Let us denote by $\mathcal{O}(k)$ anything of order $\geq k$ at $v=0$, say $O(0)$ means "holomorphic at $\frac{\tau+1}{2}$". At $u=\frac{\tau+1}{2}$ holomorphic function $\mathfrak{p}(u)$ has a first order zero, so $\mathfrak{p}'(\frac{\tau+1}{2})\neq 0$ and $\frac{\theta'(u)}{\theta(u)}$ has a first order pole at $\frac{1+\tau}{2}$,

$$\frac{\theta'(u)}{\theta(u)}\ =\ c_{-1}v^{-1}+\mathcal{O}(v).$$

Moreover,

$$c_{-1}\ =\ Res_{\frac{1+\tau}{2}}(\frac{\theta'_\tau(u)}{\theta_\tau(u)})\ =\ ord_{\frac{1+\tau}{2}}\theta_\tau(u)\ =\ 1.$$

So, $\frac{\theta'(u)}{\theta(u)} = v^{-1} + \mathcal{O}(v)$ and therefore

$$\mathfrak{p}(u) = (\frac{\theta'(u)}{\theta(u)})' = -v^{-2} + \mathcal{O}(v).$$

Notice the absence of v^{-1}. We will see that more is true: the expansion of $\mathfrak{p}(u)$ in v's only has even terms. First, θ is even, hence θ' and θ'/θ are odd, and therefore its derivative $\mathfrak{p}$ is even: $\mathfrak{p}(-u) = \mathfrak{p}(u)$. Then, since $2{\cdot}\mathfrak{p}(\frac{1+\tau}{2} + v)$ is in the lattice,

$$\mathfrak{p}(\frac{1+\tau}{2} + v) = \mathfrak{p}(-\frac{1+\tau}{2} - v) = \mathfrak{p}(\frac{1+\tau}{2} - v).$$

4. Polar parts of $(\mathfrak{p}')^2, \mathfrak{p}^3, \mathfrak{p}^2, \mathfrak{p}$ at ζ. Now we know the first statement in the following series, and then the rest follows:

(1) $\mathfrak{p}(u) \quad = -v^{-2} + a + bv^2 + O(4)$,
(2) $\mathfrak{p}^2(u) = v^{-4} + v^{-2}(-2a) + (a^2 - 2b) + O(2)$,
(3) $\mathfrak{p}^3(u) = -v^{-6} + v^{-4}(3a) + v^{-2}[(2b - a^2) - 2a^2 + b] + O(0)$
$\quad = -v^{-6} + v^{-4}(3a) + v^{-2}[3(b - a^2)] + O(0)$,
(4) $\mathfrak{p}'(u) = 2v^{-3} + 2bv + O(3)$,
(5) $(\mathfrak{p}')^2(u) = 4v^{-6} + 8bv^{-2} + O(0)$.

All together, we have

four functions whose polar parts are combinations of v^{-6}, v^{-4}, v^{-2};
so an appropriate combination of these will be holomorphic!

This combination can be written explicitly:[51]

(1) $(\mathfrak{p}')^2 + 4\mathfrak{p}^3 = 12v^{-4} + (20b - 12a^2)v^{-2} + O(0)$.
(2) $(\mathfrak{p}')^2 + 4\mathfrak{p}^3 - 12\mathfrak{p}^2 = (20b - 12a^2 + 24a)v^{-2} + O(0)$.
(3) $(\mathfrak{p}')^2 + 4\mathfrak{p}^3 - 12\mathfrak{p}^2 + (20b - 12a^2 + 24a)\mathfrak{p} = O(0)$.

$\boxed{\text{(b)}}$ We want to extend $f:\ E_\tau - \zeta \ \to\ \mathcal{C}{\subseteq}\mathbb{A}^2$ holomorphically to $E_\tau \to\ C{\subseteq}\mathbb{P}^2$, i.e., to check that the (possible) isolated singularity of $f:\ E_\tau - \zeta \ \to\ C$ at ζ, is removable. It suffices for instance to calculate $lim_{u\to(\tau+1)/2}\ f(u)$ in $\mathbb{P}^2$ in suitable coordinates near the infinite point of $\mathcal{C}$.

It is easy to see that f is locally an isomorphism – it suffices to check that the differential of f does not vanish. Similarly, surjectivity follows from abstract reasons:

- since f is a local isomorphism, $f(E_\tau){\subseteq}C$ is open, and
- since E_τ is compact the image is also closed. Now,
- since C is connected $f(E_\tau)$ is everything.

Injectivity requires a little thought.

[51]My numbers may be wrong.

6.11.3. *Corollary.* Any elliptic curve is isomorphic to one of the cubics C_λ.

Proof. The theorem identifies E_τ with a cubic of the form $y^2 = -4x^3 + Ax^2 + Bx + C$. However, after an affine change of coordinates it becomes one of C_λ's. First divide by -4 and change y to get it in the form $y^2 = x^3 + Ax^2 + Bx + C = (x-\alpha)(x-\beta)(x-\gamma)$. Now an affine change $x \mapsto ax + b$ takes α, β to $0, 1$ and γ to λ (for this we need $\alpha \neq \beta$, but for $\alpha = \beta$ C_λ is not a manifold so it can not be isomorphic to E_τ).

7. **Linearization: the Jacobian of a curve**

The basic invariant of a compact connected complex curve C is its *genus* $g_C \in \mathbb{N}$. It is important both from the topological and from the holomorphic point of view. There are three very different situations:

- $g = 0$ iff $C \cong \mathbb{P}^1 = 0$.
- $g = 1$ iff C is a cubic (i.e., an elliptic curve).
- $g \geq 2$. These we understand less explicitly.

The basic object we will associate to a curve C will be its Jacobian J_C. The passage from C to J_C can be viewed from a number of points of view and the chapter is organized around explaining the meaning of these approaches and indicating some relation between them.

7.1. **Jacobian of a curve: points of view.** To a smooth complex complete[52] connected curve C we will associate an abelian complex Lie group J called the Jacobian of C. Though C is connected, the Jacobian still comes with connected components $J_n,\ n \in \mathbb{Z}$. Moreover, it comes with a canonical map $C \xrightarrow{\iota} J_1 \subseteq J$.

As all important mathematical ideas, Jacobians can be viewed from a number of points of view

(1) J is the abelian complex Lie group freely generated by C.
(2) J is built from symmetric powers of C, via the Abel-Jacobi maps $C^{(n)} = C^{[n]} \to J_n$.
(3) J is the moduli of complex line bundles on C,
(4) J appears as the universal target of integrals on C.
(5) Topologically, J_0 is the quotient of the holomorphic part $H^{1,0}$ of the first cohomology with complex coefficients $H^1(C,\mathbb{C}) = H^{1,0} \oplus H^{0,1}$, by the image of the integral cohomology $H^1(C,\mathbb{Z}) \subseteq H^1(C,\mathbb{C})$ (when one projects $H^1(C,\mathbb{C})$ to the summand $H^{1,0}$).
(6) As an abelian group, J is the divisor class group $Cl(C)$.

There is a lot of classical mathematics hidden in the identification of these approaches. If one takes (1) as the most natural definition of the Jacobian, i.e., the most intimate relation to curve, then the other statements are all calculation of the abelian group generated by C.

7.1.1. *Formulation (6): divisors.* The group $Div(C)$ of divisors on C is the abelian group freely generated by the <u>set</u> C. It has a $\mathbb{Z}$ basis given by points of C. We can also characterize it categorically:

[52] *Complete* is used in the meaning of *compact*. The idea is that "a curve can fail to be compact only if something is missing".

Any map $C \xrightarrow{f} A$ from C to a commutative group A, canonically factors through $Div(C)$, i.e., there is precisely one map of groups $Div(C) \xrightarrow{\bar{f}} A$ such that

$$(C \xrightarrow{f} A) \;=\; (C \xrightarrow{\iota} J \xrightarrow{\bar{f}} A).$$

7.1.2. *Formulation (1): Jacobian as a "linearization" of a curve.* The disadvantage of $Div(C)$ is that we have forgotten the structure of a complex manifold on C. Jacobian $J(C)$ is the analogue of $Div(C)$ in the category of manifolds (rather then just sets). This is the meaning of the characterization (1) whose precise form is

Any holomorphic map $C \xrightarrow{f} A$ from C to a commutative complex Lie group A, canonically factors through J, i.e., there is precisely one map of complex Lie groups $J \xrightarrow{\bar{f}} A$ such that

$$(C \xrightarrow{f} A) \;=\; (C \xrightarrow{\iota} J \xrightarrow{\bar{f}} A).$$

This is the most natural way to think of the Jacobian, it relates it most intimately to C.

In mathematics we often approach problems by passing to linear algebra settings. Sometimes this is just an approximation (manifold $\mapsto$ tangent space, differential equation $\mapsto$ its linear approximation), and sometimes we pass to a larger linear setting in which we keep all information (manifold $\mapsto$ vectors space of functions, nonlinear KP-differential equation $\mapsto$ KP-hierarchy of linear differential equations). One can think of the Jacobian in this way – to a geometric object, a curve we associate a "more linear" (usually) larger, geometric object.

7.1.3. *Relation of (1) and (6).* The above universal properties of $Div(C)$ and J provide a map of groups $Div(C) \to J$. More is true – J is a quotient of $Div(C)$, so one can imagine that a manifold structure was imposed on $Div(C)$ by pushing points together. Actually this quotient can be explicitly described as

$$Cl(C) \overset{\text{def}}{=} Div(C)/div(\mathfrak{M}^*(C))$$

for the subgroup of *principal divisors* $div(\mathfrak{M}^*(C))$, i.e., divisors of (non-zero) meromorphic functions, where $div(f) = \sum_{a\in C} ord_a(f) \cdot a$. This is called the *divisor class group* $Cl(C)$. Since $Cl(C)$ has less structure then J it is easier to think of, and we use it as a bridge between different approaches to J.

7.1.4. *Formulation (2): symmetric powers as a "semilinearization" of a curve.* The disjoint union $\sqcup_{n\geq 0} C^{(n)}$ of all symmetric powers of C is an *abelian Lie semigroup.* The semigroup structure comes from maps $C^{(p)} \times C^{(q)} \to C^{(p+q)}$ which on the level of sets mean that if one adds p unordered points to q unordered points, one now has $p+q$ unordered points. Lie semigroup refers to a manifold structure on $C^{(p)}$'s and the fact that the operation $C^{(p)} \times C^{(q)} \to C^{(p+q)}$ is a map of manifolds.[53]

This turns out to be the abelian Lie semigroup freely generated by C. Then the free abelian Lie group J generated by C, will be the group associated to the semigroup $\sqcup_n C^{(n)}$.[54] Geometrically, this relation is of the free semigroup and free group is given by the *Abel Jacobi maps* $\mathcal{AJ}_n : C^{(n)} \to J_n$. For $g > 0$, $\mathcal{AJ}_1 : C^{(1)} \to J_1$ is an embedding, and for sufficiently large n maps $\mathcal{AJ}_n : C^{(n)} \to J_n$ are bundles whose fibers are projective spaces.

7.1.5. *Formulations (4-5): integration.* These we can think of as the down to earth approach to Jacobians (less abstract), however it only makes sense over complex numbers (integration requires manifold over $\mathbb{R}$). We will adopt the approach (4) through integrals, and (5) is just its topological interpretation.

(4) will be a generalization of the idea of integrals of algebraic functions. Integrals of algebraic functions were formulated as integrals on curves associated to algebraic functions. These are curves with a special structure: a map to $\mathbb{P}^1$ which was a branched cover, and the interesting results were obtained only for the elliptic curves. Now we consider all compact complex curves C.

Let g be the genus of C. We will choose a basis $a_1, ..., a_g, b_1, ..., b_g$ of $\mathcal{P}aths(C)$ and a basis $\omega_1, ..., \omega_g$ of $\Omega^1(C)$, the global holomorphic 1-forms on C.

The integrals from $\alpha \in C$ to $\beta \in C$ of 1-forms ω_i should produce a vector

$$(\int_\alpha^\beta \omega_1, ..., \int_\alpha^\beta \omega_g) \in \mathbb{C}^g.$$

However, $\int_\alpha^\beta \omega_i$ is well defined only up to periods $\int_{a_j} \omega_i$, $\int_{b_j} \omega_i$, $1 \le j \le g$. The result is that the periods form the *period lattice* $\mathcal{P}eriods$ in $\mathbb{C}^g$, so $(\int_\alpha^\beta \omega_1, ..., \int_\alpha^\beta \omega_g)$ is defined as an element of the complex torus

$$J_0 \stackrel{\text{def}}{=} \mathbb{C}^g/\mathcal{P}eriods,$$

of complex dimension g. So, the connected component J_0 appears as the universal place where integrals have values.

[53] To see that it is a map of algebraic varieties one can describe it as a factorization of the identity map $C^p \times C^q \to C^{p+q}$ to quotients by the group S_{p+q} and its subgroup $S_p \times S_q$:

$$[C^{(p)} \times C^{(q)} \to C^{(p+q)}] = [C^p//S_p \times C^q//S_q \to C^{p+q}//S_{p+q}] = [(C^p \times C^q)//(S_p \times S_q) \to C^{p+q}//S_{p+q}].$$

[54] To any semigroup S one can naturally attach a group G with a map $S \to G$.

7.1.6. *Abel-Jacobi map $C\to J_1$ in terms of integrals.* If we choose a base point $\alpha\in C$ we get the Abel-Jacobi map

$$C \xrightarrow{\mathcal{AJ}} J_0, \quad \beta\mapsto(\int_\alpha^\beta \omega_1, ..., \int_\alpha^\beta \omega_g) + \mathcal{P}eriods.$$

Theorem. (a) Abel-Jacobi map $\mathcal{AJ}: C\to J_1$ is an embedding for $g>0$.

(b) $\mathcal{AJ}(C)\subseteq J$ generates group J.

Example: cubics. Recall now that the integrals on a cubic C_λ had values in an elliptic curve E_τ, and this gave an isomorphism $C_\lambda\to E_\tau$. Now we can restate it as:

- *The connected component J_0 of the Jacobian $J=J(C_\lambda)$ of of a cubic C_λ. is an elliptic curve E_τ.*
- *The Abel-Jacobi map $C_\lambda \xrightarrow{\mathcal{AJ}} J_0(C_\lambda)$ is an isomorphism.*

Since in general $\dim(C)=1$ and $\dim(J(C))=g$, only a part of this generalizes:

7.1.7. *Formulation (3): line bundles.* The identification of (1) and (3) is the geometric part of *Class Field Theory* which is the central part of *Number Theory.* The content is that the group satisfying (1) really exists and it is the group $Pic(C)$ of all line bundles on C. This is *completely geometric* and works over *any field* and in even *larger generality.*

7.2. **Divisor class group $Cl(C)$: divisors on a curve.** The group $Div(C)$ of divisors on C is the free abelian group with a basis given by all points of C. So, any divisor $D\in Div(C)$ can be written as $D=\sum d_i{\cdot}\alpha_i$ for some distinct points $\alpha_1,...,\alpha_p$ of C, and some integers $d_1,...,d_p$. We sometimes denote $D=\sum_{a\in C} ord_a(D){\cdot}D$.

7.2.1. *Principal divisors and degree.* The simplest interesting way to produce a divisor is from a meromorphic function. Let $\mathfrak{M}(C)$ be the field of meromorphic functions on C and $\mathfrak{M}^*(C)$ the multiplicative group of non-zero meromorphic functions. The divisor of $f\in\mathfrak{M}^*(C)$ is

$$div(f) \overset{\text{def}}{=} \sum_{a\in C} ord_a(f){\cdot}a,$$

such divisor s are called *principal divisors.*

The *degree* of a divisor is defined by $deg(\sum d_i{\cdot}\alpha_i) \overset{\text{def}}{=} \sum d_i\in\mathbb{Z}$.

Lemma. $\mathfrak{M}^*(C) \xrightarrow{div} Div(C) \xrightarrow{degree} \mathbb{Z}$ are maps of abelian groups and the composition is 0.

7.2.2. *Divisor class group $Cl(C)$.* It is defined by $Cl(C) \overset{\text{def}}{=} Div(C)/div[\mathfrak{M}^*(C)]$. Since principal divisors live in the subgroup $Div_0(C)$ of degree 0, the degree is well defined on $Cl(C)$ and $0 \to Cl_0(C) \xrightarrow{\subseteq} Cl(C) \xrightarrow{div} \mathbb{Z} \to 0$.

7.2.3. *Effective divisors.* We say that a divisor $D = \sum d_i{\cdot}\alpha_i$ is *effective* if all *multiplicities* d_i are ≥ 0. Notice that the $Div(C)$ contains symmetric powers of C:

Effective divisors of degree n are the same as elements of $C^{(n)}$
i.e., unordered n-tuples of points.[55]

Remark. The group of divisors $Div(C)$ is the *abelian group freely generated by the* **set** C. If we compare this with the formulation (1) of the Jacobian we *expect* that J will be obtained from $Div(C)$ by imposing identifications such that the quotient has a structure of a complex manifold !

7.2.4. *Lemma.* The degree zero part of the divisor class group of $\mathbb{P}^1$ is trivial:

$$Cl_0(\mathbb{P}^1) = 0.$$

Proof. For $a, b \in \mathbb{P}^1$ there is a meromorphic function f such that $\div(f) = a - b$. What works most of the time is $f = \frac{z-a}{z-b}$ (if none of the points is at ∞). The general case reduces to this one using the triply transitive action of $PGL_2(\Bbbk)$ on $\mathbb{P}^1$. (Also, if $b = \infty$ use $f = z - a$, and if $a = \infty$ use $f = 1/(z - b)$.)

7.3. **Genus.** So far we have only looked into the cubics/elliptic curves and now we will consider all compact complex curves C. The basic difference is visible on the topological level, and it is captured by the *genus* of the curve. Topologically, *genus* is simply the number of pretzel-type holes in C. Another way to say this is that the abelian group $\mathcal{P}aths$ of *closed paths on C up to homology*, is a free group of rank $2g$.

Holomorphically, *genus* is the number of objects on C that one can integrate over paths in C – the global holomorphic 1-forms on C.

7.3.1. *Topological genus (homology).* Connected compact orientable real manifolds of dimension two ("surfaces") are classified by their *genus*. They are all (extended) pretzels and genus is defined as the number of holes. So $g(S^2) = 0$, $g(S^1{\times}S^1) = 1$, and $g = 2$ for the surface of the standard pretzel etc.

Any compact connected complex curve C (a compact connected complex one-dimensional manifold) is in particular a compact orientable real manifold of dimension two.[56] This

[55] $C^{(n)}$ is better since it has some algebraic structure and weaker because it sees only the effective divisors.

[56] Orientation is given by multiplication with i.

gives the notion of the topologically, *genus* $g_T(C)$ of C. This is the simplest invariant of C. For example, $g=0$ for $\mathbb{P}^1$ and $g=1$ for cubics.

7.3.2. *The homology* $H_1(C,\mathbb{Z})$. Look at at a picture of C – at the i^{th} hole we can choose a circle b_i which bounds the hole, and a transversal circle a_i that connects the hole with the outer boundary of C. We can choose $a_i, b_i,\ 1\le i\le g$, so that

$$a_i\cap b_j \ = \ \delta_{ij}\cdot\text{pt} \quad \text{and} \qquad \text{for} \ \ i\neq j: \quad a_i\cap a_j \ = \ \emptyset = \ b_i\cap b_j.$$

7.3.3. *Lemma.* Abelian group $\mathcal{P}aths(C)=H_1(C,\mathbb{Z})$ of *closed paths on* C *up to homology*, is a free group of rank $2g$, so we will choose a $\mathbb{Z}$-basis $a_1,...,a_g.,b_1,...,b_g$.

Remark. Here paths α,β are said to be homologous if (roughly) $\alpha-\beta$ is the boundary of some open part of C. Notice that in complex analysis homology is described differently: α,β are homologous in an open $U{\subseteq}C$ if they wind up the same number of times around each point of the complement $\mathbb{C}-U$. However it amounts to the same thing for $U{\subseteq}\mathbb{C}$ and the first definition is meaningful on curves.

Once we believe this, $\mathcal{P}aths(C) \ = \ H_1(C,\mathbb{Z})$ is clearly the interesting object since in complex analysis path integrals depend on the path *only up to homology.*

7.4. **Holomorphic differential 1-forms.** The (global) differential 1-forms on C are the (global) holomorphic sections of the cotangent bundle $T^*C\to C$ (a line bundle!).

7.4.1. *Example: a non-vanishing 1-form on curves in* $\mathbb{A}^2$. Let $\mathcal{C}{\subseteq}\mathbb{A}^2$ be the curve given by $F=$ for some polynomial $F\in\mathcal{O}(\mathbb{A}^2)=\Bbbk[x,y]$. On $\mathbb{A}^2=\mathbb{A}^2_{x,y}$, there are many global 1-forms:

$$\Omega^1(\mathbb{A}^2) = \ \mathcal{O}(\mathbb{A}^2)\cdot dx\oplus\mathcal{O}(\mathbb{A}^2)\cdot dy.$$

The inclusion $\mathcal{C}\overset{i}{\hookrightarrow}\mathbb{A}^2$ can be used to pull-back (restrict) these 1-forms to $\mathcal{C}_\lambda$.[57] So, on $\mathcal{C}$ we get 1-forms $(d^*i)dx$ and $(d^*i)dy$, which we call simply dx and dy.

The function F is zero on $\mathcal{C}$ (i.e., restrictions of F and 0 on $\mathcal{C}_\lambda$ are the same), so the restriction i^*dF of the 1-form $dF=F_x{\cdot}dx+F_y{\cdot}dy$ to $\mathcal{C}$ is zero(i.e., equals i^*0). This means that dx and dy on $\mathcal{C}$ satisfy $F_x{\cdot}dx \ = \ -F_y{\cdot}dy$ (the precise meaning is that $(d^*i)dx$ and $(d^*i)dy$ satisfy this equation). We use this to define a 1-form on $\mathcal{C}$

$$\omega \overset{\text{def}}{=} \frac{dx}{F_x} \ = \ -\frac{dy}{F_y}.$$

[57]Any map of manifolds $f:X\to Y$ has the differential df which can be viewed as a family of linear maps $d_af: T_a(X)\to T_{f(a)}(Y),\ a\in X$. It gives a pull-back operation on 1-forms $d^*f:\Omega^1(Y)\to\Omega^1(X)$, the value of the pull-back $(d^*f)\omega$ at $a\in X$ is

$$[(d^*f)\omega]_a \overset{\text{def}}{=} [T_a(X)\xrightarrow{d_af}T_{f(a)}(Y)\xrightarrow{\omega_a}\mathbb{C}] \ = \ (d_af)^*(\omega_a).$$

Lemma. ω is well defined and does not vanish wherever $\mathcal{C}$ is a submanifold of $\mathbb{A}^2$.

Proof. Since $\frac{dx}{F_x} = -\frac{dy}{F_y}$ the only problem can appear at points $(a,b) \in \mathcal{C}$ such that $F_x(a,b) = F_y(a,b)$, however these are precisely the points where $\mathcal{C}$ fails to be a submanifold.

The values of dx and dy at any point $p = (a,b)$ of $\mathbb{A}^2$ give a basis of $T_p^*\mathbb{A}^2$. If $p \in \mathcal{C}$ and $\mathcal{C}$ is a submanifold at p then $d_p i : T_p\mathcal{C} \to T_p\mathbb{A}^2$ is injective. So, its adjoint $d_p^* i : T_p^*\mathbb{A}^2 \to T_p^*\mathcal{C}$ is surjective. Therefore, at p one of $(d^*i)dx$ and $(d^*i)dy$ is non-zero. This means that by looking at the version of the definition of ω which is appropriate at p we find that $\omega_p \neq 0$.

7.4.2. *Example: 1-form ω_λ on a cubic C_λ.* For $\lambda \neq 0, 1$ cubic $\mathcal{C}_\lambda$ is a submanifold of $\mathbb{A}^2$, defined by the function $F(x,y) = x^3 - x^2(1+\lambda) + x{\cdot}\lambda - y^2$, so the restriction of $dF = (3x^2 - 2(1+\lambda)x + \lambda)dx - 2y\ dy$ to $\mathcal{C}_\lambda$ is zero, hence

$$\text{\textit{1-forms} } dx \text{ \textit{and} } dy \text{ \textit{on} } \mathcal{C}_\lambda \text{ \textit{satisfy}} \quad (3x^2 - 2(1+\lambda)x + \lambda){\cdot}dx \ = \ 2y{\cdot}dy.$$

We use this to define a 1-form on C_λ

$$\omega_\lambda \overset{\text{def}}{=} \frac{2\ dy}{3x^2 - 2(1+\lambda)x + \lambda} = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

Corollary. ω_λ is well defined on C_λ and it has no zeros (nor poles).

Proof. Since $\mathcal{C}_\lambda \subseteq \mathbb{A}^2$ is a submanifold for $\lambda \neq 0, 1$, lemma shows that ω_λ is well defined on $\mathcal{C}_\lambda$ and does not vanish on $\mathcal{C}_\lambda$. It remains to check the coordinates at the infinite point of C_λ.

Remark. A non-vanishing section of the cotangent line bundle T^*C_λ over C_λ can be used to trivialize this line bundle – it gives an isomorphism $C_\lambda \times \Bbbk \xrightarrow{\cong} T^*(C_\lambda)$ by $(p,c) \mapsto c{\cdot}\omega_\lambda(p)$. So for cubics (equivalently for elliptic curves), the cotangent line bundle is trivial.

This is not surprising if we remember that cubics have a group structure and on any group G any *natural* vector bundle V (such as the (co)tangent vector bundles) can be trivialized by $G \times V_1 \xrightarrow{\cong} V$, $(g,v) \mapsto g{\cdot}v$ (here left multiplication $L_g : G \to G$ lifts to an action on V which I denote $g : V_x \to V_{gx}$).

7.4.3. *Holomorphic genus.* We say that the dimension of the vector space $\Omega^1(C)$ of differential 1-forms on C is the *holomorphic genus* $g_H(C)$ of C.

Theorem. Holomorphic genus and topological genus are the same.

7.4.4. *Strategy of the proof.* The theorem relates some topological data (the genus) and holomorphic data (sections of T^*C). The standard way to do this is:

(1) Express topology through real analysis:

Extend the coefficients to real numbers: $H_1(C,\mathbb{R}) \stackrel{def}{=} H_1(C,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}$ *and consider the dual vector space* $H^1(C,\mathbb{R}) \stackrel{def}{=} H^1(C,\mathbb{R})^*$. *Then one has*

- *(De Rham theorem) This space* $H^1(C,\mathbb{R})$ *can be calculated in terms of the smooth differential forms on* C *considered as a 2-dimensional real manifold.*

(2) Relate real analysis and complex analysis – find out which part of the real analysis data is captured by the complex analysis data.

Extend the coefficients to complex numbers: $H^1(C,\mathbb{C}) \stackrel{def}{=} H^1(C,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$. *Then*

- *(Hodge theorem)* $H^1(C,\mathbb{C})$ *decomposes canonically into two complex vector spaces of the same dimension*

$$H^1(C,\mathbb{C}) = H^{1,0}(C)\oplus H^{0,1}(C)$$

with

$$H^{1,0}(C) \cong \Omega^1(C) \quad \textit{and} \quad H^{0,1}(C) \cong \Omega^1(\bar{C}).$$

So, $H^{1,0}(C)$ *is the contribution of holomorphic analysis to real analysis. By* $\bar{C}$ *I mean the manifold* C *with the opposite complex structure.*

This is the background we need for the theorem: Now everything is in place:

$$\dim_{\mathbb{C}}[H^1(C,\mathbb{C})] = \dim_{\mathbb{R}}[H^1(C,\mathbb{R})] = \dim_{\mathbb{R}}[H_1(C,\mathbb{R})] = \dim_{\mathbb{Z}}[H_1(C,\mathbb{Z})] = 2g.$$

On the other hand, $\dim_{\mathbb{C}}[H^{1,0}(C)] = \dim_{\mathbb{C}}[H^{0,1}(\bar{C})]$ (you can guess this since $\bar{C}$ should behave somewhat as $\mathbb{C}$: one passes from C to $\bar{C}$ by conjugating all complex numbers in sight). Therefore,

$$\dim_{\mathbb{C}}[\Omega^1(C)] = \dim_{\mathbb{C}}[H^{1,0}(C)] = \frac{1}{2}\dim_{\mathbb{C}}[H^1(C,\mathbb{C})] = g.$$

7.4.5. *Examples.* (1) Curves with $g = 0$ will all turn out to be isomorphic to $\mathbb{P}^1$ and we have $\Omega^1(\mathbb{P}^1) = 0$.

(2) T^*C is trivial precisely for $g = 1$. In the case $g = 1$ we already noticed the triviality. If $T^*C \cong C\times\Bbbk$ then $\Omega^1(C) \cong \mathcal{O}(C)$ and on a compact curve $\mathcal{O}(C) =$ constants. So, $g = \dim(\Omega^1(C)) = 1$.

7.4.6. *Integration of holomorphic 1-forms over paths in a curve.* Let us reconsider the integration of algebraic functions in the setting of a complex curve C.

For that we need a path γ in C and a differential form ω on $\mathbb{C}$. Here, $\gamma : [0,1] \to C$ and ω is a global differential 1-form on C, i.e., a global holomorphic sections of the cotangent bundle $T^*C \to C$ (a line bundle!). What this means is that ω assigns to each $c \in C$ a cotangent vector $\omega(c) \in T^*_c(C)$ at c, and the differential of γ gives tangent vectors

$\gamma'(t) = (d_t\gamma)\frac{\partial}{\partial t} \in T_{\gamma(t)}(C)$ at $\gamma(t)$, and finally these two kinds of vectors contract to numbers which we integrate over $[0,1]$

$$\int_\gamma \omega \stackrel{\text{def}}{=} \int_0^1 dt\ \langle \omega(\gamma(t)), \gamma'(t)\rangle.$$

So the point is that on manifolds one can not quite integrate functions but only the differential forms.

Remark. However, you may remember that we have already considered integrals $\int_\gamma y(x)\ dx$ of a *function* $y(x)$ on a curve $\mathcal{C}$. We were able to do this when $\mathcal{C}$ happened to be a branched cover $\mathcal{C} \xrightarrow{\pi} \mathbb{C}$ of $\mathbb{C}$. The point is really that to make sense of this integral we appealed to the possibility of calculating it on the image of γ in $\mathbb{C}$. So we used the relation to $\mathbb{C}$. In terms of the integration of differential 1-forms on C this means that in the background, without mentioning, we really used the differential form π^*dx on $\mathcal{C}$ which was the pull-back of dx on $\mathbb{C}$! So we have really been integrating the 1-form $y(x){\cdot}\pi^*dx$ on C. It turns out that this is a global 1-form on C_λ:

7.5. **The connected Jacobian $J_0(C)$: integration of 1-forms.** Let C be a complete complex curve of genus g.

We will adopt the approach through integrals. (It is close to the topological interpretation!) This will be a generalization of the idea of integrals of algebraic functions (on curves associated to algebraic functions). However, we will consider all compact complex curves C while so far we only looked into the elliptic curves.

7.5.1. *Topological and holomorphic data.* The basic information about a curve is its *genus*. Topologically, this is simply the number of pretzel-type holes in C. Another way to say this is that the abelian group $\mathcal{P}aths(C) = H_1(C,\mathbb{Z})$ of *closed paths on C up to homology*, is a free group of rank $2g$, with a $\mathbb{Z}$-basis give by circles $a_1, ..., a_g., b_1, ..., b_g$.

Holomorphically, *genus* is the number of objects on C that one can integrate over paths in C – the global holomorphic 1-forms on C. So, we choose a basis $\omega_1, ..., \omega_g$ and points $\alpha, \beta \in C$.

7.5.2. *Integrals between two points in C.* For any $\alpha, \beta \in C$, the integrals would like to produce a vector

$$\int_\alpha^\beta (\omega_1, ..., \omega_g) = \Big(\int_\alpha^\beta \omega_1, ..., \int_\alpha^\beta \omega_g\Big) \ \in\ \mathbb{C}^g.$$

However, this depends on a choice of a path from α to β, and any two paths differ by a closed path. Integrals depend on a closed path only up to homology, so all ambiguity in $\int_\alpha^\beta$ will be contained in the image of the *period map*

$$P : H_1(C,\mathbb{Z}) \ni [\gamma] \mapsto \int_\gamma (\omega^1, ..., \omega^g) \ \in\ \mathbb{C}^g.$$

This is a subgroup

$$\mathcal{P}eriods \stackrel{\text{def}}{=} P(H_1(C,\mathbb{Z})) = \sum_1^g \mathbb{Z}P_{a_i} + \mathbb{Z}P_{b_i} \subseteq \mathbb{C}^g.$$

So, each circle a_i produces ambiguity in the vector $(\int_\alpha^\beta (\omega_1, ..., \omega_g) \in \mathbb{C}^g$, the ambiguity is given by its *period vector*

$$P_{a_i} = \int_{a_i} (\omega_1, ..., \omega_g) \in \mathbb{C}^g.$$

Now $(\int_\alpha^\beta (\omega^1, ..., \omega^g)$ is defined as an element of the quotient group

$$\mathbb{C}^g/\mathcal{P}eriods.$$

7.5.3. *Complex torus J_0.*

Theorem. The subgroup of periods, $\mathcal{P}eriods \subseteq \mathbb{C}^g$ is a lattice:

$$\mathcal{P}eriods = \oplus_i \ \mathbb{Z}\cdot P_{a_i} \oplus \mathbb{Z}\cdot P_{b_i} \subseteq \mathbb{C}^g;$$

i.e., the periods $P_{a_1}, ..., P_{a_g}, P_{b_1}, ..., P_{b_g}$ are $\mathbb{R}$-independent.

Now we define the connected component of the Jacobian by

$$J_0 \stackrel{\text{def}}{=} \mathbb{C}^g/\mathcal{P}eriods$$

and this is a complex torus of dimension g. This is the universal target of integrals on C (a place where integrals take values).

7.5.4. *Example: cubics.* Now, if we remember that we have studied the integrals on a cubic C_λ with values in an elliptic curve E_τ, then from this general point of view E_τ was the (connected component J_0 of the Jacobian $J = J(C_\lambda)$ of C_λ. The isomorphism $C_\lambda \xrightarrow{\cong} E_\tau$ that we found using the $\mathfrak{p}$-function, is a special property of elliptic curves (in general $\dim(C) = 1$ and $\dim(J(C)) = g$)!

7.6. **Comparison of $J_0(C)$ and $Cl(C)$ (periods and divisors).** The integration construction can be restated in terms of divisors of degree 0. Any divisor $D = \sum D - i\cdot\alpha_i$ of degree zero, can be organized as $D = \sum p_j - q_j$ for some $p_j, q_j \in C$. To this we can attach an element of the Jacobian $J_0(C) = \mathbb{C}^g/\mathcal{P}eriods$, the sum of integrals

$$Int(D) \stackrel{\text{def}}{=} \sum_j \int_{p_j}^{q_j} (\omega^1, ..., \omega^g) \in J_0(C).$$

7.6.1. *Theorem.* (a) The map $Div_0(C) \xrightarrow{\int} J_0(C) = \mathbb{C}^g/\mathcal{P}eriods$ is well defined, it is a map of groups and it is surjective.

(b) The kernel is the subgroup of principal divisors (divisors of all meromorphic functions).

Proof. We just indicate the easy steps which amount to existence of interesting maps.

In (a), "well defined" means that $\sum_j (\int_{p_j}^{q_j} (\omega^1, ..., \omega^g)$ does not change if we regroup p's and q's. So, we need $\int_A^a + \int_B^b = \int_A^b + \int_B^a$, but this is equivalent to $\int_A^a - \int_B^a = \int_A^b - \int_B^b$, and here both sides are $\int_A^B$. It is clear that int is a map of groups:

$$\int(\sum p_i - q_i) + \int(\sum u_j - v_i) = \sum \int_{q_i}^{p_i} + \sum \int_{v_j}^{u_j} = \int(\sum p_i - q_i + \sum u_j - v_i).$$

In (b) we check that divisors of meromorphic functions give zero integrals. This is exciting, why would a specific integral be zero? For a non-zero meromorphic function f we want $\int(div(f)) = 0$. It is certainly true if $f = 1$. It follows that it is true for any f by the following *moving sublemma*:

Let f, g be meromorphic functions which are $\mathbb{C}$-independent. To $(\lambda, \mu) \in \mathbb{C}^2 - 0$ we associate a non-zero meromorphic function $\lambda f + \mu g$, its divisor $div(\lambda f + \mu g)$ and its image in the Jacobian $\phi(\lambda, \mu) \in \int div(\lambda f + \mu g) \in J_0(C) = \mathbb{C}^g/L$ for the period lattice L. Then ϕ factors to a map Φ from $\mathbb{P}^1$ to $\mathbb{C}^g/L$. Moreover, this lifts to a map $\widetilde{\Phi} : \mathbb{P}^1 \to \mathbb{C}^g$ by the lifting sublemma 6.5.2. Now, $\widetilde{\Phi}$ which must be constant, so $\phi(\lambda, g)$ is constant. In particular, $\int(div(f)) = \phi(1, 0) = \phi(0, 1) = \int(div(g))$.

7.6.2. *Corollary.* As a group, the connected component $J_0(C)$ of the Jacobian is the degree zero part $Cl_0(C)$ of the divisor class group.

7.7. **Picard group $Pic(C)$: line bundles and invertible sheaves.** Let C be a complex curve (i.e., a complex manifold of dimension one). We want to construct the complex Lie group $J(C)$ which is freely generate by C. We start with the same idea on the level of sets. The group $Div(C)$ of divisors on C is the abelian group freely generated by the set C. If $J(C)$ exists, as a group it has to be a quotient of $Div(C)$, but the question of finding a quotient of $Div(C)$ with a manifold structure is a priori mysterious. However there is a natural quotient of $Div(C)$ – the Picard group $Pic(C)$. This is the group of line bundles on C (for tensoring). For flexibility we observe that line bundles can be viewed as certain kinds of sheaves, the *invertible sheaves*. We use this point of view to attach to each divisor a line bundle.

In the end, $Pic(C)$ turns out to be one of incarnations of $J(C)$.

7.7.1. *Vector bundles are the same as locally free sheaves.* Any holomorphic vector bundle V over C gives a sheaf $\mathcal{V}$ on C – the sheaf of sections of L:

$$\mathcal{V}(U) \stackrel{def}{=} \Gamma(U,V) \stackrel{def}{=} \textit{ holomorphic section of } V|U, \textit{ the restriction of } V \textit{ to } U{\subseteq}C.$$

We will see that $\mathcal{V}$ is a locally free sheaf of rank n on C, i.e.,

- $\mathcal{V}$ is a module for the algebra sheaf $\mathcal{O}_C$ of holomorphic functions on C.[58]
- Locally, $\mathcal{O}_C$-module $\mathcal{V}$ is isomorphic to $\mathcal{O}_C^n$.[59]

$\mathcal{L}$ is A locally free sheaf of rank one is called an *invertible sheaf.*

7.7.2. *Proposition.* Construction $V{\mapsto}\mathcal{V}$ gives a bijection of isomorphism classes of

- vector bundles of rank n on C, and
- locally free sheaves of rank n on C.

In particular we get a bijection of line bundles and invertible sheaves on C.

Proof. **(A)** *$\mathcal{V}$ locally free of rank n.* Let $\mathcal{V}$ be the sheaf of holomorphic sections of a vector bundle V of rank n. Vector bundles are locally trivial, so there is an open cover $\mathcal{U} = (U_i)_{i\in I}$ of C and there are isomorphisms $U_i{\times}\mathbb{C}^n \xrightarrow{\phi_i} V|U_i$ which preserve fibers and on each fiber ϕ_i's are invertible linear operators. Each ϕ_i identifies the restriction $\mathcal{V}|U_i$ with the sheaf of section of the trivial vector bundle $U_i{\times}\mathbb{C}^n \to U_i$. But this is the sheaf of functions on U_i with values in $\mathbb{C}^n$, so sections are the same as n-tuples of functions on U_i. So, $\mathcal{V}|U_i \cong \mathcal{O}_{U_i}^n = \mathcal{O}_C^n|U_i$.

We constructed an identification of sheaves of abelian groups. It is clear how one would like to make $\mathcal{V}$ into an $\mathcal{O}_C$-module: $f \in \mathcal{O})C(U)$ should multiply $v \in \mathcal{V}(U)$ pointwise, i.e., $(f{\cdot}v)(x) = f(x){\cdot}v(x) \in V_x$ for $x \in U$. The $f{\cdot}v$ is clearly a section of V. We need it to be holomorphic and this is checked in local coordinates where it becomes the obvious action of $\mathcal{O}_C$ on $\mathcal{O}_C^n$.

(B) *$V{\mapsto}\mathcal{V}$ is a bijection.* There is an explicit inverse construction: to a locally free $\mathcal{V}$ one associated vector bundle V which is the spectrum of $S(\mathcal{V}^*)$, the symmetric algebra of the dual sheaf $\mathcal{V}^* \stackrel{\text{def}}{=} \mathcal{H}\text{om}_{\mathcal{O}_C}(\mathcal{V},\mathcal{O}_C)$. However we can also argue on a more elementary level, by checking that both objects can be encoded by the same kind of combinatorial data.

(D) *Transition functions for V.* Local triviality of a vector bundle V implies that there is an open cover $\mathcal{U} = (U_i)_{i\in I}$ of C and fiberwise linear isomorphisms $U_i{\times}\mathbb{C}^n \xrightarrow{\phi_i} V|U_i$. On U_{ij} we get a fiberwise linear automorphism $\phi_{ji} = \phi_j{\circ}{\phi_i}^{-1}$ of $U_{ij}{\times}\mathbb{C}^n$, this means that ϕ_{ij} is a holomorphic function $U_{ij} \to GL_n(\mathbb{C}){\subseteq}M_n(\mathbb{C})$. Notice that the data ϕ_{ij}, $(i,j) \in I^2$, are sufficient for reconstructing V – one recovers V by gluing trivial vector bundles $U_i{\times}\mathbb{C}^n$ using identifications ϕ_{ij} over U_{ij}, i.e.,

$$V \;\cong\; [\sqcup_i \, U_i{\times}\mathbb{C}^n]/\sim$$

[58]This means that each $\mathcal{V}(U)$ is a module for the algebra $\mathcal{O}_C(U)$, and that the actions $\mathcal{O}_C(V){\times}\mathcal{V}(U) \to \mathcal{V}(U)$ are compatible with restrictions.

[59]The sum of two sheaves of abelian groups is defined by $(\mathcal{A}{\oplus}\mathcal{B})(U) \stackrel{\text{def}}{=} \mathcal{A}(U){\oplus}\mathcal{B}(U)$.

for the equivalence relation: $(a,v) \in U_i{\times}\mathbb{C}^n$ and $(b,w) \in U_j{\times}\mathbb{C}^n$ are equivalent if $b = a$ and $w = \phi_{ji}(a){\cdot}v$.

(D) *Transition functions for* $\mathcal{V}$. This works the same. Local triviality of a locally free sheaf $\mathcal{V}$ of rank n implies that there is an open cover $\mathcal{U} = (U_i)_{i\in I}$ of C and isomorphisms of sheaves of $\mathcal{O}_C$-modules $\mathcal{O}_{U_i}^n \xrightarrow{\Phi_i} \mathcal{V}|U_i$. On U_{ij} we get a an automorphism $\Phi_{ji} = \Phi_j{\circ}\Phi_i{}^{-1}$ of the $\mathcal{O}_{U_{ij}}$-module $\mathcal{O}_{U_{ij}}^n$. This (again) means that Φ_{ij} is a holomorphic function $U_{ij} \to GL_n(\mathbb{C}){\subseteq}M_n(\mathbb{C})$.[60] The data ϕ_{ij}, $(i,j) \in I^2$, are (again) sufficient for reconstructing $\mathcal{V}$ by gluing $\mathcal{O}_{U_i}$-modules $\mathcal{O}_{U_i}^n$ using identifications Φ_{ij} over U_{ij}.

(E) *Conclusion.* Both kind of objects are captured by the same kind of transition functions data. It remains to notice that the passage $V{\mapsto}\mathcal{V}$ does not affect the data.

7.7.3. *Corollary.* Line bundles are the same as locally free sheaves of rank one invertible sheaves.

Proof. An invertible sheaf just means a locally free sheaf of rank one.

7.7.4. *Tensoring of vector bundles and of locally free sheaves.*

Lemma. (a) One can tensor (local) sections of vector bundles, i.e., for $W{\subseteq}C$ open, any section $\alpha \in \Gamma(W,U)$ and $\beta \in \Gamma(W,V)$ define a section $\alpha{\otimes}\beta \in \Gamma(W, U{\otimes}V)$ by

$$(\alpha{\otimes}\beta)(c) \overset{\text{def}}{=} \alpha(c){\otimes}\beta(c) \in\ U_c{\otimes}V_c = (U{\otimes}V)_c, \quad c \in W.$$

(b) Under the above correspondence, the tensoring of vector bundles $U{\otimes}V$ corresponds to tensoring of invertible sheaves viewed as $\mathcal{O}_C$-modules: $\mathcal{U}{\otimes}_{\mathcal{O}_C}\mathcal{V}$.

(c) One can invert the non-vanishing sections of line bundles: If L is a line bundle and $s \in \Gamma(W,L)$ does not vanish at any point $a \in W$ then there is a section $s^{-1} \in \Gamma(W,L^*)$ such that $\langle s, s^{-1}\rangle = 1$ on W for the pairing of sections of dual vector bundles into functions.

Proof. (a) is clear. It gives for each open $W{\subseteq}C$ a map $\Gamma(W,U){\times}\Gamma(W,V) \to \Gamma(W,U{\otimes}V)$ which is $\mathcal{O}_C(W)$-bilinear, i.e., for $f \in \mathcal{O}(C(W)$, $(f\alpha){\otimes}\beta = \alpha{\otimes}(f\beta)$. So, it gives a map $\mathcal{U}(W){\otimes}_{\mathcal{O}_{C(W)}}\mathcal{V}(W) \to \Gamma(W,U{\otimes}V)$.

Now, the tensor product of sheaves $\mathcal{U}$ and $\mathcal{V}$ over $\mathcal{O}_C$ is essentially obtained by associating to each open W the group $(\mathcal{U}{\otimes}_{\mathcal{O}_C}\mathcal{V})(W) \overset{\text{def}}{=} \mathcal{U}(W){\otimes}_{\mathcal{O}_{C(W)}}\mathcal{V}(W)$, so we have constructed maps $(\mathcal{U}{\otimes}_{\mathcal{O}_C}\mathcal{V})(W) \to \Gamma(W,U{\otimes}V)$. These are clearly compatible with restrictions so we have a map of sheaves $\mathcal{U}{\otimes}_{\mathcal{O}_C}\mathcal{V} \xrightarrow{\mu} U{\otimes}V$. Finally, local trivializations of U and V give isomorphisms $\mathcal{U} \cong \oplus\ \mathcal{O}_C{\cdot}e_i$, $\mathcal{V} \cong \oplus\ \mathcal{O}_C{\cdot}f_j$ hence $\mathcal{U}{\otimes}\mathcal{V} \cong \oplus\ \mathcal{O}_C{\cdot}(e_i{\otimes}f_j)$, and then μ is clearly an isomorphism.

[60] Canonical basis e_i of $\mathbb{C}^n$ gives an $\mathcal{O}_{U_{ij}}$-basis E_p of $\mathcal{O}_{U_{ij}}^n$. So, $\Phi_{ij}E_p = \sum_q c_{ij}^{pq}E_q$ for some $c_{ij}^{pq} \in \mathcal{O}(U_{ij})$ which form a matrix function $U_{ij} \to M_n(\mathbb{C})$, and the values are actually in $GL_n(\mathbb{C})$.

(c) The pairing of dual line bundles gives isomorphism $\langle -,-\rangle : L\otimes L^* \rightarrow\ Y = C\times\mathbb{C}$, so define $(s^{-1})(p) \in (L^*)_p = (L_p)^*$ so that $\langle s(p), s^{-1}(p)\rangle = 1$.

7.7.5. *Meromorphic sections of line bundles.* A meromorphic section of a line bundle L over an open $U\subseteq C$ means a holomorphic section s on $U-\mathcal{P}$ for some discrete subset $\mathcal{P}$ (the set of *possible poles*), such that at each $a \in \mathcal{P}$, when we use a local trivialization of L near a, the function corresponding to s is meromorphic at a, i.e., it has at most a pole at a. We can be more precise and define the order of the section s at a as

$$ord_a^L(s) \stackrel{\text{def}}{=} ord_a(f)$$

when s corresponds to a function f holomorphic off a, in terms of *some* local trivialization of L near a.

We will denote the vector space of meromorphic sections of L on U by $\mathfrak{M}(U,L)$ and the global meromorphic sections by $\mathfrak{M}(L) \stackrel{\text{def}}{=} \mathfrak{M}(C,L)$. Finally, $\mathfrak{M}(L) \stackrel{\text{def}}{=} \mathfrak{M}(L)-\{0\}$ are the non-zero meromorphic sections.

Corollary. (a) One can tensor (local) meromorphic sections of line bundles, i.e., for $U\subseteq C$ open, any meromorphic sections $\mathfrak{M}(U,L)\otimes_{\mathfrak{M}(U)}\mathfrak{M}(U,M) \rightarrow\ \mathfrak{M}(U,L\otimes M)$. The zeros and poles add up as usual: $ord_a^{L\otimes M}(\alpha\otimes\beta) =\ ord_a^L(\alpha) + ord_a^M(\beta)$.

(b) One can invert the non-zero meromorphic sections of line bundles: $\mathfrak{M}^*(U,L) \ni s\mapsto s^{-1} \in \mathfrak{M}^*(U,L^*)$ and $ord_a^{L^*}(s^{-1}) = -ord_a^L(s)$.

Proof. (a) This really means that for the sets $\mathcal{P}_\alpha, \mathcal{P}_\beta\subseteq W$ of poles of $\alpha \in \mathfrak{m}(U,L)$, $\beta \in \mathfrak{m}(U,M)$ we are tensoring holomorphic sections over $U-(\mathcal{P}_\alpha\cup\mathcal{P}_\beta)$. The formula for the order: in terms of local; trivializations of line bundles, this is just the multiplication of meromorphic functions.

7.7.6. *Picard group $Pic(C)$.* Let $\mathfrak{P}ic(C)$ be the set of isomorphism classes of line bundles (invertible sheaves) on C. Let $Pic(C)\subseteq\mathfrak{P}ic(C)$ be the subset of all line bundles with a meromorphic global section.

Lemma. (a) The tensoring of line bundle makes $\mathfrak{P}ic(C)$ into a group. The trivial line bundle $T = C\times\mathbb{C}$ is the neutral element and the inverse of L is the dual line bundle L^*.

(b) $Pic(C)$ is a subgroup.

Proof. (a) Tensor product of line bundles is associative and produces again a line bundle. The trivial line bundle $T = C\times\mathbb{C}$ is clearly a neutral element. Finally, for any line bundle L, the dual vector bundle $L^* = \mathrm{Hom}(L,T)$ is again a line bundle and the canonical pairing $L\otimes L^* \rightarrow T$ is clearly an isomorphism.

For (b) we recall that one can tensor and invert meromorphic sections (and T has a meromorphic section 1).

7.8. **GAGA: Geometrie Algebrique and Geometrie Analytic (comparison).** We will indicate that the distinction between $\mathfrak{P}ic$ and Pic does not really appear in algebraic geometry (corollary bellow), and though it does exist in holomorphic geometry there is again no difference for curves (proposition bellow).

So far we have been using both algebraic and holomorphic point of view for algebraic varieties over $\mathbb{C}$. Fortunately, the two pictures are often the same by Serre's comparison of Algebraic Geometry and Analytic Geometry:

7.8.1. *Theorem.* (Serre) If a complex manifold X has a structure of a projective variety, then

- (a) Any holomorphic vector bundle V has a structure of an algebraic vector bundle over X, and
- (b) Global holomorphic sections of V are the same as the global algebraic sections.

7.8.2. *Remarks.* (1) With a little care, the two statements can be combined into a single categorical claim: the operation $U\mapsto U^{an}$ that associates to each algebraic vector bundle U on X "the same" vector bundle but now viewed as a holomorphic vector bundle, is an equivalence of categories.

(2) Claim (b) is quite striking. It is clearly wrong if $X = \mathbb{A}^1$ and $V = X\times\mathbb{C}$ is the trivial line bundle, since there are many more functions in complex geometry (all entire functions on $\mathbb{C}$) than in algebraic geometry (polynomials $\mathbb{C}[x]$). However, if we replace $\mathbb{A}^1$ by a little larger *projective* object $\mathbb{P}^1$ we know that all global holomorphic functions are constant, so they are clearly of algebraic nature. (It helped that we filled in the hole at ∞ of $\mathbb{A}^1$ where holomorphic functions had more freedom then the polynomial ones).

Similarly, on any X, *locally* there are many more functions or sections in complex geometry than in algebraic geometry polynomials). However, if one asks which of these local sections extend to *global* sections, i.e., which ones make sense on all of X, then only the ones of algebraic nature have a chance.

7.8.3. *Corollary.* On a projective X any holomorphic line bundle has a global meromorphic section, i.e.,

Proof. Any algebraic line bundle has a global meromorphic section. Let us check this if X is a curve. Then there is a Zariski open cover U_i, $i \in I$, on which V trivializes: $V|U_i \xleftarrow{\cong} U_i\times\mathbb{C}^n$. Choose an $i \in I$, then any $0 \neq v \in \mathbb{C}^n$ gives a section $s \neq 0$ of V on U_i. Since $X - U_i$ is finite, s has only isolated singularities off U_i. To see how bad these are we need to view s in charts U_j for $j \neq i$. In such chart s is given by $f_{ji}v$ for a transition function f_{ji}. Since we are in algebraic geometry, f_{ji} (locally a restriction of a polynomial) has no essential singularities, hence neither does s.

7.8.4. *Proposition.* Any compact holomorphic curve C has a projective structure, so $Pic(C) = \mathfrak{P}ic(C)$.

Proof. We will postpone it, the idea (*Kodaira embedding* is that if line bundle L over a compact complex manifold X has "sufficiently many global sections" then it gives an explicit embedding of X into a projective space: $X \hookrightarrow \mathbb{P}[\Gamma(X,L)^*]$.

The point is that on a curve one has many effective divisors D and for sufficiently large D we will see that $\mathcal{O}_C(D)$ has sufficiently many sections!

7.8.5. *Remark.* One can check that $\mathfrak{P}ic(C) = Pic(C)$ for curves without using the GAGA theorem – we really used this problem as an excuse to introduce GAGA.

7.9. Comparison of $Pic(C)$ and $Cl(C)$ (line bundles and divisors).

7.9.1. *Divisors give invertible sheaves.* We can use a divisor $D \in Div(C)$ to modify the sheaf $\mathcal{O}_C^{an}$ of holomorphic (=analytic) functions on C. For any open $U \subseteq C$ we define

$$\mathcal{O}_C(D)(U) \stackrel{\text{def}}{=} \{f \in \mathfrak{M}(U);\ ord_a(f) \geq -ord_a(D),\ a \in U\}.$$

Here, if ϕ is a function holomorphic on some open $V \subseteq C$ and $\alpha \in C$ is an isolated singularity of ϕ (in the sense that V contains some punctured neighborhood of α), we define the order of ϕ at α, $ord_\alpha(\phi) \in \mathbb{Z}$, by using a local chart on C near α.

At points $a \in U$ which are not in the support of D the condition is $ord_a(f) \geq 0$, i.e., we ask that f is holomorphic on $U - supp(D)$. If $ord_a(D) < 0$ we impose on f the vanishing of order $|ord_a(D)|$ at a, and if $ord_a(D) > 0$ we allow a pole of order $ord_a(D)$ at a. So, for instance, $\mathcal{O}_C(D) \subseteq \mathcal{O}_C$ iff $-D$ is effective, and $\mathcal{O}_C(D) \supseteq \mathcal{O}_C$ iff D is effective.

Lemma. (a) For any divisor D on a complex curve C, $\mathcal{O}_C(D)$ is an invertible sheaf on C.

(b) $Div(C) \ni D \mapsto \mathcal{O}_C(D) \in \mathfrak{P}ic(C)$ is a map of groups.

(c) A holomorphic section $f \in \mathcal{O}_C(D)(U)$ is by definition a meromorphic function on U. These two points of view give two notions of the divisor of f :

$$ord_a^{\mathcal{O}_C(D)}(f) \ = \ ord_a(f) + ord_a(D),\ a \in U, \quad i.e., \quad div^{\mathcal{O}_C(D)}(f) \ = \ div(f) + D|U.$$

(d) Divisor D gives a trivial line bundle iff D is a divisor of a meromorphic function (i.e., a principal divisor). So,

$$Pic(C) \cong Div(C)/div(\mathfrak{M}^*(C)) = \ Cl(C).$$

Proof. (a) $\mathcal{O}_C(D)$ is a sheaf because the defining conditions are checked locally. Any point $a \in C$ lies in some chart, i.e., a lies in an open $U \subseteq C$ such that on U there is a holomorphic identification $z : U \to \mathbb{C}$. We can choose U small enough so that $U - \{a\}$ does not meet D. Then on U one has $\mathcal{O}_C(D) = \ (z - z(a))^{-ord_a(D)} \cdot \mathcal{O}_C$, hence on U we have a is an isomorphism of $\mathcal{O}_C$-modules, i.e., $\mathcal{O}_C \xrightarrow{(z-z(a))^{-ord_a(D)}} \mathcal{O}_C(D)$, a trivialization of our invertible sheaf $\mathcal{O}_C(D)$.

(b) We want an isomorphism $\mathcal{O}_C(D')\otimes_{\mathcal{O}_C}\mathcal{O}_C(D'') \xrightarrow{\iota} \mathcal{O}_C(D'+D'')$. For any open U the multiplication of meromorphic functions gives a map $\mathcal{O}_C(D')(U)\times\mathcal{O}_C(D'')(U) \xrightarrow{\iota} \mathcal{O}_C(D'+D'')(U)$, since $ord(f'f'') = ord(f') + ord(f'')$.

(c) If we view f as a section of $\mathcal{O}_C(D)$ we calculate $ord_a^{\mathcal{O}_C(D)}(f)$ using the local trivialization near a: $(z-z(a))^{-ord_a(D)} : \mathcal{O}(C)\xrightarrow{\cong}\mathcal{O}_C(D)$ from (a). In terms of this trivialization section f of $\mathcal{O}_C(D)$ corresponds to a function $\frac{f}{(z-z(a))^{-ord_a(D)}}$, hence

$$ord_a^{\mathcal{O}_C(D)}(f) = \ ord_a\left(\frac{f}{(z-z(a))^{-ord_a(D)}}\right) = \ ord_a(D) + ord_a(f).$$

(d) $\mathcal{O}_C(D)$ is trivial iff it has a non-vanishing global section f. This means iff there is a meromorphic function $f \in \mathfrak{M}(C)$ such that that at a each point $a \in C$, $ord_a(f) \geq -ord_a(f)$ (so that f is a section of $\mathcal{O}_C(D)$), and $ord_a(f) + ord_a(f) = ord_a^{\mathcal{O}(D)}(f) \leq 0$ (so that f does not vanish at a as a section of $\mathcal{O}_C(D)$). This is equivalent to $D = -ord(f)$ for some $f \in \mathfrak{M}^*(C)$, i.e., to D being principal.

Corollary. On $Pic(C)$ there is a notion of degree: $deg[\mathcal{O}_C(D)] \stackrel{\text{def}}{=} deg(D)$. In particular, there is a subgroup $Pic_0(C)$ of line bundles of degree zero.

Proof. If $\mathcal{O}_C(D')$ and $\mathcal{O}_C(D'')$ are isomorphic then $\mathcal{O}_C('-D'') \cong \mathcal{O}_C(D')\otimes\mathcal{O}_C(D'')^*$ is a trivial line bundle, hence $D'-D'' = div(f)$ for some $0 \neq f \in \mathfrak{M}(C)$. But then $deg[div(f)] = 0$, hence $deg(D') = deg(D'')$.

7.9.2. *Divisors of meromorphic sections of line bundles.*

Lemma. Let L be a line bundle on C.

(a) The non-zero meromorphic sections $\mathfrak{M}^*(L)$ of L form a torsor for the multiplicative group $\mathfrak{M}^*(C)$ of meromorphic functions on C.

(b) The divisors of non-zero meromorphic sections $\mathfrak{M}^*(L)$ of L form a coset in $Cl(C) = Div(C)/div[\mathfrak{M}^*(C)]$.

(c) Meromorphic sections f of $\mathcal{O}_C(D)$ can be canonically identified with meromorphic functions on C, then

$$ord_a^{\mathcal{O}_C(D)}(f) \ = \ ord_a(f) + ord_a(D), \ a \in U, \quad i.e., \quad div^{\mathcal{O}_C(D)}(f) \ = \ div(f) + D.$$

Proof. (a) First, we know that L has a meromorphic section $\sigma \neq 0$, i.e., $\mathfrak{M}^*(L) \neq \emptyset$. Then, $\mathfrak{M}^*(C)$ acts on $\mathfrak{M}^*(L)$ by multiplication of meromorphic sections by meromorphic functions f. Finally, for any two (non-zero) meromorphic sections s_1, s_2 of L there is a unique $f \in \mathfrak{M}^*(C)$ such that $s_2 = f{\cdot}s_1$. Here $f = s_2/s_1$, or more precisely we have a meromorphic section ${s_2}^{-1}$ of L^* and then a meromorphic section $f = s_1 \otimes {s_2}^{-1}$ of $L \otimes L^* \cong T = C \times \mathbb{C}$. Now, $f \cdot s_2 = s_1$ is clear. Obviously, (a) implies (b) since for any $s \in \mathfrak{M}^*(L)$ we have $\mathfrak{M}^*(L) = \mathfrak{M}^*(C) \cdot s$, hence:

$$div[\mathfrak{M}^*(L)] = \ div[\mathfrak{M}^*(C) \cdot s] = \ div[\mathfrak{M}^*(C)] + div(s) \in Div(C)/div[\mathfrak{M}^*(C)] = \ Cl(C).$$

(c) By definition, the local holomorphic sections of $\mathcal{O}_C(D)$ are meromorphic functions $\mathcal{O}_C(D)(U) \subseteq \mathfrak{M}(U)$. We now extend this to meromorphic sections $s \in \mathfrak{M}(U, \mathcal{O}_C(D))$. First, in a small neighborhood V_a of a point $a \in U$, there is a coordinate function z with $z(a) = 0$ (= *a chart centered at* a). Then $z^{-ord_a^{\mathcal{O}_C(D)}(s)} \cdot s$ is a holomorphic section of $\mathcal{O}_C(D)$ over V_a (we have just killed the pole!), hence it is a meromorphic function f_a on V_a (with a property $div(f_a) + D \geq 0$ on V_a). Now, we multiply back the transition factor and define a *meromorphic function* $z^{ord_a^{\mathcal{O}_C(D)}(s)} \cdot f_a$ on V_a. Now it remains to check that all $z^{ord_a^{\mathcal{O}_C(D)}(s)} \cdot f_a$ glue into one meromorphic function f on U, which we then attach to s.[61] The statement about order at a point now follows from the same statements for holomorphic sections.

Remark. One says that divisors D_1, D_2 are linearly equivalent if $D_1 - D_2$ is a principal divisor (i.e., the images in $Cl(C)$ are the same. We see that this is equivalent to: D_1, D_2 define isomorphic line bundles: $\mathcal{O}_C(D_1) \cong \mathcal{O}_C(D_2)$.

7.9.3. *Recovering a line bundle from meromorphic sections.*

Lemma. Let L be a line bundle on C.

(a) If s is any non-zero meromorphic section of a line bundle L, multiplication with s gives a canonical isomorphism $\mathcal{O}_C(div^L(s)) \xrightarrow{s} \mathcal{L}$, with the sheaf of sections $\mathcal{L}$ of L.

(b) For a divisor D, L is isomorphic to $\mathcal{O}_C(D)$ iff D is the divisor of some meromorphic section of L.

Proof. (a) We need to understand the sections of L to compare them with the known sections of $\mathcal{O}_C(D)$ for $D = div^L(s)$. Let $\phi \in \Gamma(U, L) = \mathcal{L}(U)$ be some holomorphic section of L over an open $U \subseteq C$. Now we have to meromorphic sections of L and from them we can cook up a meromorphic function $f = \phi/s$ on U. We define it as a meromorphic section $f = \phi \otimes s^{-1} \in \mathfrak{M}(U, L \otimes L^*)$, and since $L \otimes L^*$ is canonically isomorphic to the trivial line bundle $T = C \times \mathbb{C}$, f is really a meromorphic function $f \in \cong \mathfrak{M}(U, C \times \mathbb{C}) = \mathfrak{M}(U)$. Now one goes backwards and finds that $\phi = f \cdot s$ in $L \cong T \otimes L$.

In this way we may try to go from any meromorphic functions $f \in \mathfrak{M}(U)$ to a holomorphic section ϕ of L by $f \mapsto \phi \stackrel{\text{def}}{=} f \cdot s$. But, in general ϕ is only going to be another meromorphic section of L on U. For which f's is ϕ holomorphic? We need $ord_a^L(f \cdot s) = ord_a(f) + ord_a^L(s)$ to be ≥ 0 at each point $a \in U$, i.e., $f \in \mathcal{O}_C(div^L(s))$.

[61]Certainly they glue, because we are not really doing anything – we divide and then multiply in the same factor.

So the multiplication by s is a surjective map $\mathcal{O}_C(U) \xrightarrow{-\cdot s} \mathcal{L}(U)$. It is clearly injective because locally (on a dense subset obtained by removing zeros and poles of s), this is a multiplication with an invertible function.

(b) We will see that the constant function 1 on C can be viewed as a meromorphic section σ of $\mathcal{O}_C(D)$ with $div^{\mathcal{O}_C(D)}(s) = D$. According to the lemma 7.9.2.c, since 1 is a meromorphic function on C it can also be viewed as a meromorphic section s of $\mathcal{O}_C(D)$, and moreover $div^{\mathcal{O}_C(D)}(s) = \ div(1) + D = 0 + D = D$.

So, if $L \cong \mathcal{O}_C(D)$ then it has a meromorphic section with divisor D. The opposite direction is just the part (a).

Corollary. (a) The image of $Div(C) \to \mathfrak{P}ic(C)$ is $\mathrm{Pic}(C)$, i.e., a line bundles comes from a divisor iff it has a global meromorphic section.

(b) $Div(C) \to Pic(C)$ factors to $Cl(C) \xrightarrow{\cong} Pic(C)$. The inverse is given by

$$Pic(C) \ni \ L \mapsto div^L(\mathfrak{M}^*(L)) \in \ Div(C)/div[\mathfrak{M}^*(C)] = Cl(C).$$

Proof. (a) is clear from the lemma. The first part of (b) then follows since the kernel of $Div(C) \to Pic(C)$ was found to consist precisely of principal divisors. We noticed above that there is a map $Pic(C) \to Cl(C)$ by $L \mapsto div^L(\mathfrak{M}^*(L))$. To see that the two maps are inverse, we check $div(\mathfrak{M}^*[\mathcal{O}_C(D)]) = \ D + div[\mathfrak{M}^*(C)]$ for $D \in Div(C)$. Recall that D is a divisor of a meromorphic section of $\mathcal{O}_C(D)$ and that $div(\mathfrak{M}^*[\mathcal{O}_C(D)])$. is a coset of principal divisor in $Div(C)$, so $div(\mathfrak{M}^*[\mathcal{O}_C(D)]) = \ D + div[\mathfrak{M}^*(C)]$.

7.10. **Conclusion:** $J_0(C)$ **and** $Pic(C)$ **(periods and line bundles).**

7.10.1. *Comparison.* We have constructed the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & div(\mathfrak{M}^*(C)) & \xrightarrow{\subseteq} & Div_0(C) & \xrightarrow{\int} & J_0(C) & \longrightarrow & 0 \\
 & & {\scriptstyle =}\big\downarrow & & {\scriptstyle \subseteq}\big\downarrow & & {\scriptstyle \iota}\big\downarrow & & \\
0 & \longrightarrow & div(\mathfrak{M}^*(C)) & \xrightarrow{\subseteq} & Div(C) & \longrightarrow & Cl(C) & \longrightarrow & 0 \\
 & & {\scriptstyle =}\big\downarrow & & {\scriptstyle =}\big\downarrow & & {\scriptstyle \iota}\big\downarrow & & \\
0 & \longrightarrow & div(\mathfrak{M}^*(C)) & \xrightarrow{\subseteq} & Div(C) & \xrightarrow{D \mapsto \mathcal{O}_C(D)} & Pic(C) & \longrightarrow & 0
\end{array}$$

except for the maps i and ι. However, the rows are known to be exact so i and ι are obtained as factorizations of the maps in the middle, and i is injective while ι is a bijection.

All-together, we have proved:

7.10.2. *Theorem.* $J_0(C)$ is the subgroup $Pic_0(C)$ of $Pic(C)$. The canonical isomorphism $Pic_0(C) \to J_0(C)$ associates to a line bundle L the integral $\int(div(s))$ of any meromorphic section s of L.

Remark. Notice that the inclusion i has a lot of content (properties of integrals) while isomorphism ι is largely a matter of permuting the definitions.

7.10.3. *Line bundles on* $\mathbb{P}^1$. Recall that any holomorphic line bundle on $\mathbb{P}^1$ is algebraic. Denote $\mathcal{O}(n) \stackrel{\text{def}}{=} \mathcal{O}_{\mathbb{P}^1}(n{\cdot}\infty),\ n \in \mathbb{Z}$.

7.10.4. *Lemma.* (a) Any line bundle L on $\mathbb{P}^1$ is isomorphic to precisely one of the line bundles $\mathcal{O}(n)$ (and $n = deg(L)$). So, $deg : \mathrm{Pic}(\mathbb{P}^1) \xrightarrow{\cong} \mathbb{Z}$ and $Pic_0(\mathbb{P}^1) = 0$.

(b) The global sections can be viewed as the polynomials of degree $\le n$ or as the degree n polynomials in two variables:

$$\Gamma[\mathbb{P}^1, \mathcal{O}(n)] \ \cong\ \mathbb{C}_{\le n}[z] \ \cong \mathbb{C}_n[x, y],\ n \in \mathbb{Z}.$$

(c) $T\mathbb{P}^1 \cong \mathcal{O}(2)$ and $\Omega^1 \cong \mathcal{O}(-2)$.

(d) There are no global differential forms on $\mathbb{P}^1$.[62] There are three independent vector fields on $\mathbb{P}^1$: $\partial_z, z\partial_z, z^2\partial_z$.[63]

Proof. (a) Group $Pic_0(\mathbb{P}^1) \cong Cl_0(\mathbb{P}^1)$ is known to be trivial by lemma 7.2.4. So, $Pic(\mathbb{P}^1) \xrightarrow{deg} \mathbb{Z}$ is an isomorphism. Since $deg(n{\cdot}\infty) = n$, $n{\mapsto}\mathcal{O}(n)$ is the inverse of the degree isomorphism.

(b) $\Gamma[\mathbb{P}^1, \mathcal{O}(n)] = \Gamma[\mathbb{P}^1, \mathcal{O}(n{\cdot}\infty)]$ consists of all functions holomorphic on $\mathbb{A}^1$, i.e., series $\sum_0^{+\infty} f_i z^i$ with the radius of convergence $+\infty$, such that at ∞ where $w = 1/z$ is a parameter, $\sum_0^{+\infty} f_i w^{-i}$ has order $\ge -n$, i.e., $f_i = 0$ for $-i < -n$. So we allow precisely the sums $\sum_0^n f_i z^i$. For $n < 0$ this is nothing and for $n \ge 0$ these are polynomials in z of degree $\le n$, and for $z = y/x$, the multiplication with x^n identifies them with $\mathbb{C}_n[x, y]$.

(c) $T\mathbb{P}^1$ has a global holomorphic section ∂ with a double zero at ∞ (in terms of $w = 1/z$ we have $\partial = \frac{d}{dz} = -w^2 \frac{d}{dw}$). So, $div(\partial) = 2{\cdot}\infty$ and $TC \cong \mathcal{O}(2{\cdot}\infty) = \mathcal{O}(2)$. Therefore, $T^*\mathbb{P}^1 = [T\mathbb{P}^1]^* \cong \mathcal{O}_{\mathbb{P}^1}(2\infty)^* \cong \mathcal{O}(-2{\cdot}\infty)$. We see that $\partial, z\partial$ and $z^2\partial$ are holomorphic sections with divisors $2{\cdot}\infty, \mathbf{0} + \infty, 2{\cdot}\mathbf{0}$.

(d) $\Gamma(\mathbb{P}^1, \Omega^1_C) \cong \Gamma(\mathbb{P}^1, \mathcal{O}_C(-2)) = 0$ and $\dim[\Gamma(\mathbb{P}^1, T\mathbb{P}^1)] = \dim[\Gamma(\mathbb{P}^1, \mathcal{O}_C(2))] = \dim[\mathbb{C}_2[x, y] = 3$.

7.11. **Group structure on projective cubics.** The cubics[64] play a very special role, these are the only projective curves that admit a group structure. One obvious consequence is that cubics have no special points: at all points they look the same. This is also

[62]Too bad: integrals of differential forms are well defined on $\mathbb{P}^1$. However, at least one can integrate meromorphic differential forms.

[63]Actually, the space of global vector fields always has a structure of a Lie algebra, and in nice cases – like this one – it is the Lie algebra of the automorphism group $\mathrm{Aut}(\mathbb{P}^1) = PGL_2$.

[64]Better to say *projective curves of genus* 1.

true for $\mathbb{P}^1$ (a homogeneous space of the group PGL_2), but *not at all* for curves of genus $g > 1$.

Why do cubics have a group structure? The magic of degree being 3 is that it provides just enough space for the story "*mother, father,child*", which is the prototype of our standard algebraic structures (two produce the third in groups and rings).

7.11.1. *The origin.* On $C = C_\lambda$ we need to choose a point which will be 0 for the group structure, the simplest choice is the point $\infty = C_\lambda - \mathcal{C}_\lambda$.

7.11.2. *Operation $a \circ b$.* For $a, b \in C \subseteq \mathbb{P}^2$ we define $a \circ b$ as the third point of the intersection of the line $L_{a,b} \subseteq \mathbb{P}^2$ through a, b with C. This requires some explanation.

- Lines in $\mathbb{P}^2 = \mathbb{P}(C^3)$ means the projectivizations $\mathbb{P}(P)$ of two dimensional vector subspaces $P \subseteq \mathbb{C}^3$. This makes sense since $\mathbb{P}(P) \cap \mathbb{A}^2$ is really an affine line in $\mathbb{A}^2$ (except in the one case when the intersection is empty, i.e., $\mathbb{P}(P) = \mathbb{P}^2 - \mathbb{A}^2$).
- There is a unique line $L_{a,b}$ in $\mathbb{P}^2$ that passes through a, b – if $a \neq b$. However, $L_{a,a} \stackrel{\text{def}}{=} \lim_{b \to a} \ L_{a,b}$ is always well-defined – this is (the definition of) the tangent line $\boldsymbol{T}_a(C)$ to C at a.[65]
- The number of points in the intersection of a line L with the cubic C is (by Bezout's theorem) $deg(L) \cdot deg(C) = 1 \cdot 3 = 3$ (counted with multiplicities!). So, the intersection of $L_{a,b}$ with C contains a, b but also the third point which we call $a \circ b$. Notice that for instance $a \circ b = b$ if $L_{a,b}$ is tangent to C at b.

Notice the symmetry between $a, b, a \circ b$, i.e., S_3 acts on $\{(a, b, c) \in C^3;\ a \circ b = c\}$. So, this can not be the addition operation, however in any abelian group A there is an S_3-invariant subset of A^3 given by $a + b + a \circ b = 0$. So, we hope that $a \circ b = -(a + b)$. If so, then addition will be given by $(a \circ b) \circ \infty = -[-(a + b) + 0] = a + b$.

7.11.3. *Addition $a + b$.* Now define $a + b \stackrel{\text{def}}{=} (a \circ b) \circ \infty$. Then

(1) $\infty + a = a$, i.e., ∞ is the neutral element so we will call it 0. (clearly, $(\infty \circ a) \circ \infty = a$).
(2) $a \circ b$ is commutative, hence so is $a + b$.
(3) $a + b = 0$ means that $(a \circ b) \circ \infty = \infty$, i.e., $a \circ b = \infty \circ \infty$, or
$b = a \circ (\infty \circ \infty)$. Here, $\infty \circ \infty$ is the third point at which the tangent line $\boldsymbol{T}_\infty(C)$ at ∞ meets C.

It remains to check associativity. However, we can get around that by checking the first two claims in

[65] Notice that the notion we have defined here has the property $\boldsymbol{T}_a(C) \cong \mathbb{P}^1 \subseteq \mathbb{P}^2$, rather then $\mathbb{A}^1$.

7.11.4. *Theorem.* If we define the group structure on $C = C_\lambda$ using operation $a{\circ}b$, and do that ∞ is the origin in the group, then the map

$\phi \stackrel{\text{def}}{=} [C \xrightarrow{a \mapsto a-\infty} Div_0(C) \to Cl_0(C)]$, *i.e.*, $\phi(a) = [a{-}\infty]$ (the class of $a - \infty$ in $Cl_0(C)$), satisfies

- (1) $\phi(a + b) = \phi(a) + \phi(b)$, (2) ϕ is injective, (3) ϕ is surjective.

So, all-together, ϕ is an isomorphism of groups $C \to Cl_0(C)$.

Proof. We check (1). We denote the above addition in C by $\oplus$. Then $\phi(a{\oplus}b) = \phi(a){+}\phi(b)$ means that $[(a{\oplus}b) - \infty] = [a - \infty] + [b - \infty]$, i.e., that $(a{\oplus}b) + \infty - a - b$ is a principal divisor.

However, whenever $\alpha, \beta, \gamma \in C \in \mathbb{A}^2$ are collinear, i.e., they lie on some line $L{\subseteq}\mathbb{A}^2$, then $\alpha + \beta + \gamma - 3\infty$ is the divisor of the meromorphic function f on C which is the equation of L. Actually, the equation of L is a polynomial function on $\mathbb{A}^2$ which we restrict to $C \cap \mathbb{A}^2$ and extend to a meromorphic function f on C. Then $div(f) = \alpha + \beta + \gamma$ in $C \cap \mathbb{A}^2$, and since the degree has to be zero, the multiplicity of ∞ is -3. So, for any $\alpha, \beta \in C$

$$[\alpha{\circ}\beta] = [\gamma] = [3\infty - \alpha - \beta].$$

Therefore,

$$[(a{\oplus}b)] = [(a{\circ}b){\circ}\infty] = [3\infty{-}((a{\circ}b){+}\infty)] = [2\infty{-}(a{\circ}b)] = [2\infty{-}(3\infty{-}a{-}b)] = [a{+}b{-}\infty].$$

The claims (2-3) we already met (and we proved (3)), because, this is the isomorphism of C with $J_0(C)$ which is in this case an elliptic curve.

The remaining two topics are **homological algebra** and **sheaves**. These are two general tools (not particular to geometry), that are useful for many kinds of mathematics and are standard in algebraic geometry.

We introduce homological algebra on the example of improved versions of intersections and fibers of maps. However the geometric content is mostly for fun, the real point is the homological algebra idea of uncovering a hidden part of constructions. This is then used to produce the cohomology of sheaves – a hidden part of the construction of taking global sections of sheaves.

The sheaves we are interested in are the (sheaves of sections of) line bundles on curves. So we will be calculating the cohomology of line bundles $\mathcal{O}_C(D)$ on a curve C, corresponding to various divisors D. Let $h^i(D)$ be the dimension of the i^{th} cohomology group $H^i[C, \mathcal{O}_C(D)]$. The number $h^0(D)$, has geometric content, this is the dimension of the vector space $\Gamma(X, \mathcal{O}_C(D)) = \mathcal{O}_C(D)\ (C)$, i.e., the number of global meromorphic functions on C that satisfy some restrictions on the positions of poles and zeros (which we specify by the choice of the divisor D). The reason we treat this geometric question in terms of sheaves is that it makes situation quite flexible (there are more sheaves then just the line bundles) and we can effectively do many calculations. Here, the higher cohomologies will be mostly a tool for calculating the zero$^{\text{th}}$ cohomology, i.e., the numbers $H^0(D)$.

We will first go through this basic application of homological algebra to algebraic geometry (cohomology of line bundles on curves), and then we will fill in some gaps by checking that the category of sheaves really has a structure of an abelian category, hence provides a setting for the use of homological algebra.

8. The hidden part of constructions: homological algebra (differential graded schemes)

Homological algebra is a general tool, one can describe it as being in the business of observing the hidden part of the iceberg that is beneath the water level. That makes it useful in various areas of mathematics, and so in particular in algebraic geometry.

8.0.5. *What does the Homological algebra do?* Homological algebra is a general tool useful in various areas of mathematics. One tries to apply it to constructions that morally should contain more information then meets the eye. The homological algebra, if it applies, produces "derived" versions of the construction ("the higher cohomology"), which contain the "hidden" information. We will visit some examples of the use of homological algebra:

(1) **Cohomology of sheaves.** This is a very standard tool in algebraic geometry and we will try to understand how it works.

    Sheaves are a framework for dealing with an omnipresent problem of relating local and global information on a space. The global information is codified as the functor $\Gamma(X,-)$ of global sections of sheaves on a topological space X. When a sheaf has few global sections, more information may be contained in the derived construction – the cohomology of sheaves.

(2) **Subtle spaces.** This is a more advanced topic, so we will get just a glimpse.
   - The notion of *dg-schemes* (*differential graded schemes*) is a generalization of the notion of a scheme.[66] Formally, the difference is that the functions on a dg-scheme form a *commutative dg-algebra*, i.e., as we expect, we get a commutative algebra but it has some extra structure from homological algebra – the structure of a complex. The most obvious application is that such refined objects contain some more subtle information. However, we will only see how they appear in order to get *stable versions*[67] of calculations with ordinary algebraic varieties: the derived intersection, derived fiber etc.
   - A *D-brane* is a geometric space of a certain kind in string theory (contemporary physics). Mathematical formulation of a D-brane turns out to be a highly sophisticated constructs of homological algebra.

8.0.6. *Contents.* We will introduce the ideas of homological algebra on the example of *dg-schemes*. Actually, we only do one extremely simple illustrative computation and find one (sorry looking) example of a dg-scheme. The motivating idea is that the honored *Stability Principle* suggests that the fibers of maps should not jump in size. The reality does not comply until we change the notion of a fiber to a *dg-fiber* or *derived fiber*.

[66] A few years old.
[67] As in *Stability Principle*.

After that we go back to school and learn quickly the formalism of homological algebra in abelian categories. This will be needed for the more extensive use of homological algebra when we study sheaves in the next section.

8.1. **"Continuity" of fibers.** Consider a map $f : X \to Y$ of algebraic varieties. The fibers $f^{-1}(b) \subseteq X$ may jump in size as one varies b in Y. The class of maps for which the fiber does not jump are the so called *flat maps*, however many important maps are not flat. For instance we know that the blow up $\widetilde{V} \to V$ is not flat !

However, $f^{-1}(b) = \{a \in X;\ f(a) = b\}$ is the set of solutions of an algebraic equation (or system of such if we use coordinates). Now recall our beloved *Stability principle*: if a system of equations changes, the solutions should change continuously (i.e. no jumps in size). Clearly it does not apply to this situation. That's it.

Or maybe not. Remember that for $X = \mathbb{A}^2$, the simplest version $X^{(2)} = X^2/S_2$ of the moduli of unordered pairs of points turned out to be singular, we managed to modify it to a smooth version $X^{[2]}$ (the second Hilbert scheme of X), by remembering more data. $X^{(2)}$ remembers all unordered pairs of points, but $X^{[2]}$ in addition also remembers when two points collide the direction in which they approached each other.

So, if we just think of the fiber as a set the fibers most certainly may jump. In order to make this set theoretic jump less central we may try to add more to the notion of the fiber – it should be influenced by nearby fibers. So we should have an enhanced notion of a fiber which would remember something about the map f near b. Such notion appears naturally in algebra. To see this remember the construction of the

8.1.1. *Fibered product of varieties and tensor product of algebras.* Let X and Y are *varieties over* Z, in the sense that we remember certain maps $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ to a variety Z. The fibered product of X and Y over Z is defined as a set by

$$X \times_Z Y \stackrel{\text{def}}{=} \{(x, y) \in X \times Y;\ f(x) = g(y)\}.$$

It is actually an algebraic subvariety of $X \times Y$ since it is defined by an algebraic equation $f(x) = g(y)$.

Maps f and g give morphisms of algebras $\mathcal{O}(X) \xleftarrow{f^*} \mathcal{O}(Z) \xrightarrow{g^*} \mathcal{O}(Y)$, so one can form a tensor product of algebras

$$\mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y).$$

Now let X, Y, Z be affine (for simplicity), so that they are captured by their algebras of functions. Then there is the same information in the diagrams

$$X \xrightarrow{f} Z \xleftarrow{g} Y \quad \text{and} \quad \mathcal{O}(X) \xleftarrow{f^*} \mathcal{O}(Z) \xrightarrow{g^*} \mathcal{O}(Y),$$

so we can *hope* that the resulting constructions $X \times_Z Y$ and $\mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y)$ contain the same amount of information, i.e., that

Lemma. $\mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y) = \mathcal{O}(X\times_Z Y)$.

Proof. $X\times_Z Y$ is a subset of the affine variety $X\times Y$ given by algebraic equation $f(x) = g(y)$, so it is closed affine subvariety of $X\times Y$.

We will check that for any affine variety W, there is a canonical identification

$$Map(W, X\times_Z Y) \cong \mathrm{Hom}[\mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y), \mathcal{O}(W)].$$

But, an algebra map $F \in \mathrm{Hom}[\mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y), \mathcal{O}(W)]$ is the same as as a pair of algebra maps $\alpha \in \mathrm{Hom}[\mathcal{O}(X), \mathcal{O}(W)]$ and $\beta \in \mathrm{Hom}[\mathcal{O}(Y), \mathcal{O}(W)]$, such that the compositions are the same:

$$[\mathcal{O}(Z) \xrightarrow{f^*} \mathcal{O}(X) \xrightarrow{\alpha} \mathcal{O}(W)] \quad = \quad [\mathcal{O}(Z) \xrightarrow{g^*} \mathcal{O}(Y) \xrightarrow{\beta} \mathcal{O}(W)].$$

(In one direction α, β give $F(u\otimes v) = \alpha(u)\cdot\beta(v)$, in the opposite F gives $\alpha(u) = F(u\otimes 1)$.) But α and beta are the same as maps $W \xrightarrow{a} X$ and $W \xrightarrow{b} Y$, and the condition $\alpha\circ f^* = \beta\circ g^*$ is then the same as $f\circ A = g\circ B$. So the data contained in F amount to a map $W \xrightarrow{(A,B)} X\times Y$, such that the corresponding maps to Z, $f\circ A$ and $g\circ B$, are the same. But this precisely means that (A, B), maps W, to $X\times Y$.

Corollary. (a) If $X\subseteq Z\supseteq Y$ then $X\times_Z Y = X\cap Y$, so

$$\mathcal{O}(X\cap Y) = \mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y).$$

(b) If $X \xrightarrow{f} Z$ and $y\in Z$ we denote by $X_y = f^{-1}(y)$ the fiber of f at y. Then $X\times_Z Y = f^{-1}(y)$, so

$$\mathcal{O}(f^{-1}y) = \mathcal{O}(X_y) = \mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\Bbbk,$$

here the tensor product uses a homomorphism $\mathcal{O}(Z) \to \Bbbk = \mathcal{O}(\{y\})$ given by $\{y\} \to Z$.

8.1.2. *Higher fibered products.* The algebraic construction of tensor product $M\otimes_A N$ of two modules over a commutative algebra A, has a refinement which produces a sequence of A-modules called

$$M\otimes_A N = Tor_0^A(M, n),\ Tor_1^A(M, n),\ Tor_2^A(M, n),\ \dots$$

Together they form and object called the derived tensor product $M\overset{L}{\otimes}_A N$. This object is a *complex* of A-modules. Complexes are standard objects in *Homological algebras.*

When applied to the situation of a fibered product of affine varieties we get $\mathcal{O}(Z)$-modules

$$\mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y) = Tor_0^{\mathcal{O}(Z)}(\mathcal{O}(X), \mathcal{O}(Y)),\ Tor_0^{\mathcal{O}(Z)}(\mathcal{O}(X), \mathcal{O}(Y)),\ Tor_0^{\mathcal{O}(Z)}(\mathcal{O}(X), \mathcal{O}(Y)),\ \dots$$

and they glue together into the derived tensor product $\mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y)$ This object is a *complex* of $\mathcal{O}(Z)$-modules, but since $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are more then just $\mathcal{O}(Z)$-modules – they are $\mathcal{O}(Z)$-algebras – the derived tensor product $\mathcal{O}(X)\otimes_{\mathcal{O}(Z)}\mathcal{O}(Y)$ is more then just a complex of $\mathcal{O}(Z)$-modules, it is an algebra type object in the world of complexes, and this is called a *differential graded* $\mathcal{O}(Z)$*-algebra*, or just a *dg-algebra over* $\mathcal{O}(Z)$.

Now this is going to produce a more refined version of the fibered product:

- *the derived fibered product of affine varieties X and Y over Z, is the space $X\overset{L}{\times}_Z Y$, such that the algebra of functions on $X\overset{L}{\times}_Z Y$ is*
$$\mathcal{O}(X)\overset{L}{\otimes}_{\mathcal{O}(Z)}\mathcal{O}(Y).$$

Now, since $\mathcal{O}(X)\overset{L}{\otimes}_{\mathcal{O}(Z)}\mathcal{O}(Y)$ is not just an algebra, $X\overset{L}{\times}_Z Y$ is not just a variety or a scheme. It is what is called a *differential graded scheme* or just a dg-scheme.

So, we get a refined version of a fibered product $X\times_Z Y$ of two varieties, and this refined version is a dg-scheme. (In particular we get refined versions of intersections and of fibers of maps: the derived intersection and the derived fiber.) So the entrance fee we pay for this game is that we have to extend the algebraic geometry to the category of dg-schemes, i.e., geometric objects such that their algebras of functions are not necessarily just commutative algebras but *commutative dg-algebras.*

8.2. **Homological algebra.** We will explain it on the example of categories $\mathfrak{m}(\Bbbk)$ of modules over a ring $\Bbbk$ (need not be commutative). The prototype is the example of the category of abelian groups $\mathcal{A}b=\mathfrak{m}(\mathbb{Z})$. However, the formalism works in other cases such as the all important category $\mathcal{S}h\mathcal{A}b(X)$ of sheaves of abelian groups on a topological space X. What is required of the category $\mathcal{A}$ in order to use homological algebra is that in $\mathcal{A}$ we have the basic notions that we use when we calculate with abelian groups: such as subobject, quotient, image of a map, addition of maps. Such categories are called *abelian categories.*

8.2.1. *Notion of a complex.* A complex of cochains is a sequence of $\Bbbk$-modules and maps
$$\cdots\xrightarrow{\partial^{-2}}C^{-1}\xrightarrow{\partial^{-1}}C^0\xrightarrow{\partial^0}C^1\xrightarrow{\partial^1}C^2\xrightarrow{\partial^2}\cdots,$$
such that the compositions of coboundary operators ∂^i are zero: $\partial^{i+1}\partial^i=0,\ i\in\mathbb{Z}$. We often omit the index on the coboundary operator, so we can write the preceding requirement as $\partial\circ\partial=0$.

From a complex of cochains we get three sequences of $\Bbbk$-modules

- i-cocycles $Z^i\overset{\text{def}}{=}\mathrm{Ker}(\partial^i)\subseteq C^i$,
- i-coboundaries $B^i\overset{\text{def}}{=}Im(\partial^{i-1})=Im(\partial^{i-1})\subseteq C^i$,
- i-cohomologies $H^i\overset{\text{def}}{=}Z^i/B^i$,

Here we used $B^i\subseteq Z^i$ which follows from $\partial\partial=0$.

8.2.2. *Resolutions.* The basic example of complexes are resolutions. We will consider *resolutions of modules by free modules.*

Our motivational example will be a point b on a line $\mathbb{A}^1$. Then $\mathcal{O}(\{b\})=\Bbbk_b$ is a module for $\mathcal{O}(\mathbb{A}^1)=\Bbbk[x]$ (via the restriction map $\mathcal{O}(\mathbb{A}^1)\to\mathcal{O}(\{b\})$). In order to remember that we are looking at a point of $\mathbb{A}^1$ (or that "our point b is allowed to roam through $\mathbb{A}^1$), we consider the relation of $\Bbbk_b = \mathcal{O}(\{b\})$ to the algebra $\mathcal{O}(\mathbb{A}^1) = \Bbbk[x]$ functions on the *entire* $\mathbb{A}^1$. We capture this relation in the sense of a *resolution* of $\Bbbk_b$ – a way of encoding $\Bbbk_b$ in terms of several copies of $\Bbbk[x]$.

We first observe that $\Bbbk_b$ is a quotient of $\Bbbk[x]$. Inclusion $\{b\}\overset{j}{\hookrightarrow}\mathbb{A}^1$ gives the restriction map

$$\Bbbk[x] \xrightarrow{j^*} \Bbbk_b$$

which is surjective. However, since the map is not an isomorphism, by itself it does not quite capture $\Bbbk_b$. The error is in the kernel $\operatorname{Ker}(j^*) = (x-b)\Bbbk[x]$. So we try to record the kernel in terms of $\Bbbk[x]$. However, this is quite simple since we have an isomorphism $\Bbbk[x] \xrightarrow{x-b} (x-b)\Bbbk[x]$. This is a complete success, we expressed $\Bbbk_b$ via $\Bbbk[x]$, the summary of our thinking is just a way of interpreting the short exact sequence

$$0 \longrightarrow (x-b)\Bbbk[x] \xrightarrow{\subseteq} \Bbbk[x] \xrightarrow{j^*} \Bbbk_b \longrightarrow 0,$$

as an isomorphic short exact sequence the

$$0 \longrightarrow \Bbbk[x] \xrightarrow{x-b} \Bbbk[x] \xrightarrow{j^*} \Bbbk_b \longrightarrow 0.$$

This is called a resolution of $\Bbbk_b$ in terms of $\Bbbk[x]$.

In more complicated situations the exact sequence may be longer and one may need several copies of $\Bbbk[x]$ in each position (the reason is that the kernel of j^* need not be a free module of rank one, and indeed need not be free at all. Say for a point $\{(0,0)\}\overset{i}{\mathbb{A}}{}^2$ in a plane, one has a Koszul resolution of $\mathcal{O}(\{(0,0)\}) = \Bbbk_{(0,0)}$ by *free modules* for $\mathcal{O}(\mathbb{A}^2) = \Bbbk[x,y]$:

$$0 \to \Bbbk[x,y] \xrightarrow{h\to(h,-h)} \Bbbk[x,y]\oplus\Bbbk[x,y] \xrightarrow{(f,g)\mapsto xf+yg} \Bbbk[x,y] \xrightarrow{i^*} \Bbbk_{0,0} \to 0.$$

Precise definitions. A *left resolution* of a module M by free modules is an exact complex

$$\cdots \longrightarrow P^{-2} \xrightarrow{\partial^{-2}} P^{-1} \xrightarrow{\partial^{-1}} P^0 \xrightarrow{q} M \longrightarrow 0 \longrightarrow \cdots,$$

in which all P^i's are free modules. However one also uses the term *resolution of* M for the complex (together with a map q)

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow 0 \longrightarrow \cdots.$$

Notice one particular property of resolutions of M: these are complexes with $H^i = 0$ for $i \neq 0$ and $H^0 = M$. This again says that a resolution is a way of encoding M – the total information we can extract from the resolution is just M itself.

In this terminology $\Bbbk[x]$-module $\Bbbk_b$ has a resolution (we remember the map j^*)

$$\cdots \to \underset{\boxed{-2}}{0} \to \underset{\boxed{-1}}{\Bbbk[x]} \xrightarrow{x-b} \underset{\boxed{0}}{\Bbbk[x]} \to \underset{\boxed{1}}{0} \to \cdots.$$

The boxed numbers bellow indicate that we consider the guy that was next to $\Bbbk_b$ as being in position 0, the next one in position -1, etc.

8.2.3. *The derived versions of constructions.* Roughly speaking, the left derived version LF of a construction F (i.e., a functor F) obtained by replacing a module M by its free module resolution $P^\bullet$:

$$LF(M) \stackrel{\text{def}}{=} F(P^\bullet).$$

Since $P^\bullet$ is a complex, $F(P^\bullet)$ will again be a complex. Its cohomologies will be called the derived functors of F

$$L^iF(M) \stackrel{\text{def}}{=} H^i[LF(M)] = H^i[F(P^\bullet)].$$

8.2.4. *Exactness properties.* We say that functor F is right exact if it preserves exactness for sequences of type $M' \to M \to M'' \to 0$, i.e., if $M' \to M \to M'' \to 0$ is exact then $F(M') \to F(M) \to F(M'') \to 0$ is exact.

takes any

Lemma. (a) If F is right exact then $L^0F = F$.

(b) Functors $N \otimes_{\Bbbk} -$ of tensor product with N are right exact.

Remark. So we recover F on the level zero of the derived construction – the rest, i.e., the higher derived functors L^iF are a *bonus*.

8.2.5. *The size of a complex.* The main thing about the complex $C^\bullet$ are its cohomology groups. Actually, the complex is eventually thought of as a way of gluing all $H^i(C^\bullet)$ in one object. So the notion of the size will have to unchanged when one passes to cohomology. This is satisfied by the idea of *Euler characteristic.* If in a complex $C^\bullet$ all terms are finite dimensional vector spaces and only finitely many terms are non-zero, the Euler characteristic is defined by

$$\chi(C^\bullet) = \sum_i (-1)^i \dim(C^i).$$

Lemma. $\chi[H^*(C^\bullet)] = \chi[C^\bullet]$.

8.2.6. *Differential graded algebras (algebra objects in complexes).* A *differential graded algebra*[68] $(A^\bullet, d, cd)$, is a complex $(A^\bullet, d)$ with a structure of an algebra $A \times A \xrightarrow{cd} A$ on $A = \oplus_{\mathbb{Z}} A^n$, and the two should be compatible:

- $A_p \cdot A_q \subseteq A_{p+q}$, and
- for $a \in A_p,\ b \in A_q$

$$d(a \cdot b) = d(a) \cdot b + (-1)^p a \cdot d(b).$$

[68]Usually we just say *dg-algebra*.

The last property is a version of the product rule $\partial(fg)d(a{\cdot}b) \;=\; \partial(f){\cdot}g + f{\cdot}\partial(g)$, except that it has been *enriched* by signs. The appearance of signs in homological algebra is particularly striking in the definition

dg-algebra A is commutative if for any $a \in A_p,\ b \in A_q$ one has

$$b{\cdot}a = \;(-1)^{pq} a{\cdot}b.$$

This property is also called *graded commutativity* and in physics *super commutativity* (this is the mathematical basis of super-symmetry).

A very basic example of commutative dg-algebras is the exterior algebra $A = \wedge^{\bullet} V$ of a vector space V, here $A_p = \wedge^p V$ and the differential is zero. Any basis $x_1, ..., x_p$ of V generates A and since $deg(x_i) = 1$, these generators "anti-commute", i.e.,

$$x_i x_j \;=\; -x_j x_i.$$

In particular $x_i^2 = 0$ (at least if the characteristic of the field is not 2). A_p has a basis of monomials $x_{i_1}{\cdots}x_{i_p}$ with $1 \leq i_1 < \cdots < i_p \leq 1$. This dg-algebra is also called the *Grassmannian algebra* on generators $x_1, ..., x_n$.[69]

8.3. **Example: intersection of points on a line.** We will calculate the simplest example of derived fiber product. It is indeed so simple that it does not impress at all. However, already here we will have to use the above machinery.

8.3.1. *The set theoretic level of the problem.* The problem is two take the derived intersection of two points $a, b \in \mathbb{A}^1$ on a line. Set theoretically, $\{a\} \cap \{b\}$ is empty if $a \neq b$ and it is a point $\{a\}$ if $a = b$. So the intersection jumps when we move b.

We can also view this as calculating the fiber at $b \in \mathbb{A}^1$ of the map $\{a\} \overset{i}{\hookrightarrow} \mathbb{A}^1$: $i^{-1}(b) \cong \{a\} \cap \{b\}$. So we are working on our original problem: set theoretically the fiber jumps when b has a special value $b = a$.

Let us remember the wish to create a refined notion of a fiber $i^{-1}b$ which will take into account the nearby fibers $i^{-1}c$. If $b \neq a$ the same is true for nearby c's, so $i^{-1}b = \emptyset = i^{-1}c$ and there is clearly nothing to refine. So we want a refined version of $i^{-1}a$ which not be $\emptyset$ but will take into account that nearby fibers are $\emptyset$.

In terms of the intersection picture $\{a\} \cap \{b\}$ we want a derived version $\{a\} \overset{L}{\cap} \{b\}$ which is in some sense continuous in b: so $\{a\} \overset{L}{\cap} \{a\}$ should take into account that $\{a\} \overset{L}{\cap} \{b\} = \{a\} \cap \{b\} = \emptyset$ for nearby b's. Another way to say this is that $\{a\} \overset{L}{\cap} \{a\}$ should take into account that the intersection is happening inside $\mathbb{A}^1$ (so it should be as continuous as possible in the variable $b \in \mathbb{A}^1$).

[69] In super mathematics one thinks of it as the algebra of functions on the super point $\mathbb{A}^{0,n}$.

8.3.2. *Algebraic recalculation of the set theoretic intersection.* Let $\mathcal{O}(\mathbb{A}^1) = \Bbbk[x]$ and denote by $\Bbbk_a = \mathcal{O}(\{a\})$ the functions on the point a, so algebra $\Bbbk_a$ is just the field $\Bbbk$ but we remember that this copy of $\Bbbk$ is related to the point $a \in \mathbb{A}^1 = \Bbbk$. Actually, $\{a\} \overset{i}{\hookrightarrow} \mathbb{A}^1$ corresponds to a map $i^* : \Bbbk[x] \ = \mathcal{O}(\mathbb{A}^1) \to \mathcal{O}(\{a\}) = \Bbbk_a \cong \Bbbk$ and i^* is the evaluation at a: $i^*(P(x)) = P(a)$ for a polynomial $P(x) \in \Bbbk[x]$. Similarly, $\Bbbk_b = \mathcal{O}(\{b\}) \cong \Bbbk$ and $\{b\} \overset{j}{\hookrightarrow} \mathbb{A}^1$ corresponds to $j^* : \Bbbk[x] \to \Bbbk_b \cong \Bbbk$ and $j^*(P(x)) = P(b)$. Now,

$$\mathcal{O}(\{a\} \cap \{b\}) = \ \mathcal{O}(\{a\} \times_{\mathbb{A}^1} \{b\}) = \ \mathcal{O}(\{a\}) \otimes_{\mathcal{O}(\mathbb{A}^1)} \mathcal{O}(\{b\}) = \ \Bbbk_a \otimes_{\Bbbk[x]} \Bbbk_b.$$

Observe that $\Bbbk[x] \xrightarrow{i^*} \Bbbk_a$ is surjective and the kernel is the ideal $I_b = \langle x - b \rangle = (x-b)\Bbbk[x]$ of all functions that vanish at b. So, algebra $\Bbbk_b \cong \Bbbk[x]/\langle x - b\rangle$ is a quotient of $\Bbbk[x]$. This allows us to use tensor product identities:

$$\mathcal{O}(\{a\} \cap \{b\}) = \ \Bbbk_a \otimes_{\Bbbk[x]} \Bbbk[x]/\langle x - b\rangle \cong \ \Bbbk_a/\langle x - b\rangle \cdot \Bbbk_a = \ \Bbbk_a/(x-b) \cdot \Bbbk_a.$$

Here $x - b \in \mathcal{O}(\mathbb{A}^1)$ acts on $\Bbbk_a$ via i^*, the evaluation at a. So, $(x-b)\cdot\Bbbk_a = \ (a-b)\cdot\Bbbk_a$ and this is $\Bbbk_a$ if $b \neq a$ and 0 if $b = a$. So, we have calculated

$$\mathcal{O}(\{a\} \cap \{b\}) = \ \Bbbk_a/(x-b)\cdot\Bbbk_a = \begin{cases} \Bbbk_a & \text{if } b \neq a, \\ 0 & \text{if } b = a. \end{cases} = \begin{cases} \mathcal{O}(\{a\}) & \text{if } b \neq a, \\ \mathcal{O}(\emptyset) & \text{if } b = a. \end{cases},$$

i.e., we have recalculated the obvious claim $\{a\} \cap \{b\} = \begin{cases} \{a\} & \text{if } b \neq a, \\ \emptyset & \text{if } b = a. \end{cases}$, in algebraic language.

8.3.3. *Derived tensor product.* Remember that

The derived tensor product $\Bbbk_a \overset{L}{\otimes}_{\Bbbk[x]} \Bbbk_b$ is obtained by replacing in $\Bbbk_a \otimes_{\Bbbk[x]} \Bbbk_b$, the $\Bbbk[x]$-module $\Bbbk_b$ by its resolution.

We know such resolution

$$\Big[\cdots \to \underset{\boxed{-2}}{0} \to \underset{\boxed{-1}}{\Bbbk[x]} \xrightarrow{x-b} \underset{\boxed{0}}{\Bbbk[x]} \to \underset{\boxed{1}}{0} \to \cdot\Big],$$

so we get a complex

$$\Bbbk_a \overset{L}{\otimes}_{\Bbbk[x]} \Bbbk_b \overset{\text{def}}{=} \Bbbk_a \otimes_{\Bbbk[x]} \ \ \Big[\cdots \to \underset{\boxed{-2}}{0} \to \underset{\boxed{-1}}{\Bbbk[x]} \xrightarrow{x-b} \underset{\boxed{0}}{\Bbbk[x]} \to \underset{\boxed{1}}{0} \to \cdots\Big]$$

$$= \ \Big[\cdots \to \underset{\boxed{-2}}{\Bbbk_a \otimes_{\Bbbk[x]} 0} \ \to \ \underset{\boxed{-1}}{\Bbbk_a \otimes_{\Bbbk[x]} \Bbbk[x]} \xrightarrow{id \otimes (x-b)} \underset{\boxed{0}}{\Bbbk_a \otimes_{\Bbbk[x]} \Bbbk[x]} \to \underset{\boxed{1}}{\Bbbk_a \otimes_{\Bbbk[x]} 0} \ \to \ \cdots\Big]$$

$$\cong \ \Big[\cdots \to \underset{\boxed{-2}}{0} \ \to \ \underset{\boxed{-1}}{\Bbbk_a} \xrightarrow{x-b} \underset{\boxed{0}}{\Bbbk_a} \to \underset{\boxed{1}}{0} \to \cdots\Big]$$

$$\cong \ \Big[\cdots \to \underset{\boxed{-2}}{0} \ \to \ \underset{\boxed{-1}}{\Bbbk_a} \xrightarrow{a-b} \underset{\boxed{0}}{\Bbbk_a} \to \underset{\boxed{1}}{0} \to \cdots\Big].$$

Now,

- if $a \neq b$ this is isomorphic to

$$\big[\cdots \to \underset{\boxed{\text{-2}}}{0} \to \underset{\boxed{\text{-1}}}{\Bbbk_a} \xrightarrow{id} \underset{\boxed{0}}{\Bbbk_a} \to \underset{\boxed{1}}{0} \to \cdots\big].$$

This we can think of as a resolution of 0 by free $\Bbbk[x]$-modules, so it is just a way to encode 0. Therefore, as desired

$$\Bbbk_a \overset{L}{\times}_{\Bbbk[x]} \Bbbk_b = \ 0 \ \text{ if } \ a \neq b.$$

- if $a = b$ this is isomorphic to

$$\big[\cdots \to \underset{\boxed{\text{-2}}}{0} \to \underset{\boxed{\text{-1}}}{\Bbbk_a} \xrightarrow{0} \underset{\boxed{0}}{\Bbbk_a} \to \underset{\boxed{1}}{0} \to \cdots\big].$$

Notice that this is *not* a resolution of anything. In fact as all maps are zero, the objects in various degrees are not related ("there is no cancellation"), so one should regard this object as a sum of its contributions in degrees 0 and -1, we call these constituents of $\Bbbk_a \overset{L}{\otimes}_{\Bbbk[x]} \Bbbk_b$, the 0^{th} and -1^{st} Tor functor:

$$Tor_{-1}^{\Bbbk[x]}(\Bbbk_a, \Bbbk_a) \ = \ \Bbbk_a \ = \ Tor_0^{\Bbbk[x]}(\Bbbk_a, \Bbbk_a).$$

In particular in degree 0 we get the correct value, i.e., just what the ordinary tensor product produces:

$$Tor_0^{\Bbbk[x]}(\Bbbk_a, \Bbbk_a) \ = \ \Bbbk_a \ = \ \Bbbk_a \otimes_{\Bbbk[x]} \Bbbk_a.$$

However, the derived computation gives a completely new ingredient in degree -1 :

$$Tor_{-1}^{\Bbbk[x]}(\Bbbk_a, \Bbbk_a) \ = \ \Bbbk_a.$$

This is a remainder that we were allowed to move b through $\mathbb{A}^1$, i.e., that we are taking an *ambiental intersection* of the two points– intersections which remembers the ambient $\mathbb{A}^1$.

8.3.4. *The size of* $\mathcal{O}(a \overset{L}{\cap}_{\mathbb{A}^1} b) \ = \ \Bbbk_a \overset{L}{\otimes}_{\Bbbk[x]} \Bbbk_b$ *is continuous in* b. We get different results depending on whether $a = b$, however the *size* does not change:

$$\chi(\Bbbk_a \overset{L}{\otimes}_{\Bbbk[x]} \Bbbk_b) \ = \ \dim[H^0(\Bbbk_a \overset{L}{\otimes}_{\Bbbk[x]} \Bbbk_b)] - \dim[H^0(\Bbbk_a \overset{L}{\otimes}_{\Bbbk[x]} \Bbbk_b)]$$

$$= \ \dim[Tor^0_{\Bbbk[x]}(\Bbbk_a, \Bbbk_b)] - \dim[Tor^{-1}_{\Bbbk[x]}(\Bbbk_a, \Bbbk_b)] = \ \begin{cases} 0 - 0 & \text{if } b \neq a, \\ 1 - 1 & \text{if } b = a. \end{cases} = \ 0.$$

8.4. **Differential graded schemes.** As in the ordinary (rather then homological) algebra, we will define *affine differential graded schemes* as geometric spaces $\mathfrak{X}$ that correspond to commutative dg-algebras $\mathcal{A}$ by $\mathcal{O}(\mathfrak{X})=\mathcal{A}$, then $\mathfrak{X}$ is called $\mathrm{Spec}(\mathcal{A})$.

As we have seen this idea provides a refined notion of fibered product of $X \xrightarrow{f} Z \xleftarrow{g} Y$, the derived fibered product

$$X\overset{L}{\times}_Z Y \overset{\text{def}}{=} \mathrm{Spec}(\mathcal{O}(X)\overset{L}{\otimes}_{\mathcal{O}(Z)}\mathcal{O}(Y)].$$

Roughly, the refined version remembers that one could vary the maps f and g, and the this makes the size of the fibered product stable under such motions of maps. In particular, we get the notion of *derived intersection* $X\overset{L}{\cap}_Z Y$ of two subspace X,Y of Z, which is an *ambiental* intersection since it remembers that we may move X and Y inside Z.

8.4.1. *Example: self intersection of a point.* Let us see what did we get as the self intersection of a point on a line.

Let $a,b\in\mathbb{A}^1$. If $a\neq b$ we got $\mathcal{O}(a\overset{L}{\cap}_{\mathbb{A}^1} b)=0$, hence $a\overset{L}{\cap}_{\mathbb{A}^1} b=\emptyset$ (an empty scheme), as we should. If $a=b$ we got $A=\mathcal{O}(a\overset{L}{\cap}_{\mathbb{A}^1} a)$ to be a complex with $\Bbbk$ in degrees -1 and 0 and zero differential. What is the algebra structure? $A_0=\Bbbk$ acts on A as multiplication by scalars. The remaining multiplication is zero since $A_1{\times}A_1 \xrightarrow{\cdot} A_2=0$. So, we got a Grassmannian algebra on one generator, since $A\cong\Bbbk{\oplus}\Bbbk x=\ \wedge^\bullet\ \Bbbk x$ for a one dimensional space with a basis x.[70]

One can similarly calculate the self intersection of a point in $\mathbb{A}^n$:

Lemma. For a point $a\in\mathbb{A}^n$, $\mathcal{O}(a\overset{L}{\cap}_{\mathbb{A}^n}a)$ is the Grassmannian algebra $\wedge^\bullet\ \Bbbk^n$. So, $a\overset{L}{\cap}_{\mathbb{A}^n}$ is the spectrum of the Grassmannian algebra $\wedge^\bullet\ \Bbbk^n$.[71]

Proof. This is easily seen by using the *Koszul resolution.* We will do the calculation based on *faith* that dg-geometry exists and works reasonably, so that

$$(X_1\overset{L}{\cap}_{Z_1}Y_1)\times(X_2\overset{L}{\cap}_{Z_2}Y_2)\ \cong\ (X_1{\times}X_2)\overset{L}{\times}_{Z_1\times Z_2}(Y_1{\times}Y_2).$$

Then for $a,b\in\mathbb{A}^n$

$$a\overset{L}{\times}_{\mathbb{A}^n}b\ \cong\ (a_1{\times}\cdots{\times}a_n)\overset{L}{\times}_{\mathbb{A}^1\times\cdots\times\mathbb{A}^1}(b_1{\times}\cdots{\times}b_n)\ \cong\ (a_1\overset{L}{\cap}_{\mathbb{A}^1}b_1){\times}\cdots{\times}(a_n\overset{L}{\cap}_{\mathbb{A}^1}b_n),$$

hence

$$\mathcal{O}(a\overset{L}{\times}_{\mathbb{A}^n}b)\ \cong\ \mathcal{O}(a_1\overset{L}{\cap}_{\mathbb{A}^1}b_1){\otimes}\cdots{\otimes}\mathcal{O}(a_n\overset{L}{\cap}_{\mathbb{A}^1}b_n)$$
$$\cong\ (\Bbbk{\oplus}\Bbbk x_1){\otimes}\cdots{\otimes}(\Bbbk{\oplus}\Bbbk x_n)\ \cong\ (\wedge^\bullet\ \Bbbk x_1){\otimes}\cdots{\otimes}(\wedge^\bullet\ \Bbbk x_n)\cong\ \wedge^\bullet\ (\Bbbk x_1{\oplus}\cdots{\oplus}\Bbbk x_n).$$

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[70]In terms of super mathematics we would say that $A=\mathcal{O}(a\overset{L}{\cap}_{\mathbb{A}^1} a)$ is a super point $\mathbb{A}^{0,1}$.

[71]In terms of super mathematics again, $A=\mathcal{O}(a\overset{L}{\cap}_{\mathbb{A}^n}a)$ is a super point of type $\mathbb{A}^{0,n}$.

8.4.2. *Koszul duality.* To indicate that Grassmannian algebra is not non-sense, let me mention the following fact without giving details[72]

8.4.3. *Theorem.* (Priddy) Let x_i and y_i be dual bases of two dual vector spaces. Then the categories of modules over dg-algebras[73] $\wedge^\bullet$ $(\Bbbk x_1 \oplus \cdots \oplus \Bbbk x_n)$ and $\Bbbk[y_1, ..., y_n]$, are canonically equivalent.

A fancy version of this gives an unexpected relations between familiar objects:

8.4.4. *Theorem.* Let X, Y be vector subspaces of a vector space Z. Then the dual vector space Z^* contains vector subspaces $X^\perp, Y^\perp$. The categories of modules over dg-algebras of functions on dg-schemes $X \overset{L}{\cap}_Z Y$ and $X^\perp \overset{L}{\cap}_{Z^*} Y^\perp$ are canonically equivalent. [74]

8.5. **Abelian categories.** An abelian category is a category $\mathcal{A}$ which has the formal properties of the category $\mathcal{A}b$, i.e., we can do in $\mathcal{A}$ all computations that one can do in $\mathcal{A}b$. The basic example: categories $\mathfrak{m}(\Bbbk)$ of modules over a ring $\Bbbk$. Here is a (long) list of properties that make a category $\mathcal{A}$ abelian

(1) Category $\mathcal{A}$ is additive if
  - (A0) For any $a, b \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(a, b)$ has a structure of abelian group such that then compositions are bilinear.
  - (A1) $\mathcal{A}$ has a zero object,
  - (A2) $\mathcal{A}$ has sums of two objects,
  - (A3) $\mathcal{A}$ has products of two objects,[75]

(2) Category $\mathcal{A}$ is abelian if it is additive and
  - (A4) It has kernels and cokernels (hence in particular it has images and coimages!).
  - (A5) The canonical maps $Coim(\phi) \to Im(\phi)$ are isomorphisms.

Let us recall what this means

8.5.1. *(Co)kernels and (co)images.* In module categories a map has kernel, cokernel and image. To incorporate these notions into our project of defining abelian categories we will find their abstract formulations. Two of these notions are primary and dual to each other:

(1) Kernels
  - *Intuition.* For a map of $\Bbbk$-modules $M \xrightarrow{\alpha} N$

[72] A module over a dg-algebra is of course a complex with extra structure – you should be able to cook up the definition.

[73] Since we are dealing with homological algebra, the appropriate categories are the so called *derived categories* of modules, which we will not define.

[74] The same holds if X, Y are vector subbundles of a vector bundle Z over S.

[75] In an additive category $a \oplus b$ is canonically the same as $a \times b$.

– the kernel $\mathrm{Ker}(\alpha)$ is a subobject of M,
– the restriction of α to it is zero,
– and this is the largest subobject with this property

- *Definition in an additive category $\mathcal{A}$.* k is a kernel of a map $a \xrightarrow{\alpha} b$ if
  – we have a map $k \xrightarrow{\sigma} a$ from k to a,
  – if we follow this map by α the composition is zero,
  – map $k \xrightarrow{\sigma} a$ is universal among all such maps.[76]

(2) *Cokernels*

- In $\mathfrak{m}(\Bbbk)$ the cokernel of $M \xrightarrow{\alpha} N$ is $N/\alpha(M)$. So there is a map $N \longrightarrow Coker(\alpha)$, the composition with α kills it, and the cokernel is universal among all such objects.
- In additive $\mathcal{A}$, the cokernel of $a \xrightarrow{\sigma} b$ is an object c supplied with a map $b \longrightarrow c$ which is universal among maps from b that kill α.

Now, the two secondary notions (they use kernels and cokernels).

(1) *Images.*

- In $\mathfrak{m}(\Bbbk)$, $Im(\alpha)$ is a subobject of N which is the kernel of $N \longrightarrow \alpha(M)$.
- In additive $\mathcal{A}$, if $a \xrightarrow{\sigma} b$ has cokernel $b \longrightarrow \mathrm{Coker}(\alpha)$, then the image of σ is $Im(\sigma) \overset{\text{def}}{=} \mathrm{Ker}[b \longrightarrow \mathrm{Coker}(\sigma)]$ (if it exists).

(2) *Coimages.*

- In $\mathfrak{m}(\Bbbk)$ the coimage of α is $M/\mathrm{Ker}(\alpha)$.
- In additive $\mathcal{A}$, if $a \xrightarrow{\sigma} b$ has kernel $\mathrm{Ker}(\sigma) \longrightarrow a$, then the coimage of σ is $Coim(\sigma) \overset{\text{def}}{=} Coker[\mathrm{Ker}(\sigma) \longrightarrow a]$. (if it exists).

In $\mathfrak{m}(\Bbbk)$, the canonical map $Coim(\alpha) = M/\mathrm{Ker}(\alpha) \longrightarrow Im(\alpha)$ is an isomorphism. This observation is the final ingredient (A5) in the definition of abelian categories.[77][78]

8.5.2. *Extending the rest of the vocabulary from modules to abelian categories.* Once we have the notion of kernel and cokernel everything follows:

- a map $i : a \longrightarrow b$ makes a into a subobject of b if $\mathrm{Ker}(i) = 0$ (we denote it $a \hookrightarrow b$ or even informally by $a \subseteq b$, one also says that i is a monomorphism or informally that it is an inclusion),

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[76] The "universality" means that all maps into a, $x \xrightarrow{\tau} a$, which are killed by α, factor uniquely through k (i.e., through $k \xrightarrow{\sigma} a$). So, all such maps τ are obtained from σ (by composing it with some map $x \to k$).

[77] This is also a reason why you never hear of coimages.

[78] For (A5) we also need:

Lemma. In additive $\mathcal{A}$, if $\sigma : a \to b$ has image and coimage then there is a canonical map $Coim(\sigma) \to Im(\sigma)$. It appears in a canonical factorization of σ into a composition

$$a \to Coim(\sigma) \to Im(\sigma) \to b.$$

- a map $q : b \to c$ makes c into a quotient of b if $\mathrm{Coker}(q) = 0$ (we denote it $b \twoheadrightarrow c$ and say that q is an epimorphism or informally that q is surjective),
- the quotient of b by a subobject $a \xrightarrow{i} b$ is $b/a \stackrel{\text{def}}{=} \mathrm{Coker}(i)$,
- a complex in $\mathcal{A}$ is a sequence of maps $\cdots A^n \xrightarrow{d^n} A^{n+1} \to \cdots$ such that $d^{n+1}\circ d^n = 0$, its cocycles, coboundaries and cohomologies are defined by $B^n = Im(d^n)$ is a subobject of $Z^n = \mathrm{Ker}(d^n)$ and $H^n = Z^n/B^n$;
- sequence of maps $a \xrightarrow{\mu} b \xrightarrow{\nu} c$ is exact (at b) if $\nu\circ\mu = 0$ and the canonical map $Im(\mu) \to \mathrm{Ker}(\nu)$ is an isomorphism.

Now with all these definitions we are in a familiar world, i.e., they work as we expect. For instance, sequence $0 \to a' \xrightarrow{\alpha} a \xrightarrow{\beta} a'' \to 0$ is exact iff a' is a subobject of a and a'' is the quotient of a by a', and if this is true then

$$\mathrm{Ker}(\alpha) = 0, \mathrm{Ker}(\beta) = a', \mathrm{Coker}(\alpha) = a'', \mathrm{Coker}(\beta) = 0, \mathrm{Im}(\alpha) = a', \mathrm{Im}(\beta) = a''.$$

8.5.3. *The difference between general abelian categories and module categories.* In a module category $\mathfrak{m}(\Bbbk)$ our arguments often use the fact that $\Bbbk$-modules are after all abelian groups and *sets* – so we can think in terms of their elements. A reasoning valid in any abelian category has to be done more formally: via *composing maps* and *factoring maps through intermediate objects*. However, this is mostly appearances – if we try to use set theoretic arguments we will not go wrong:

8.5.4. *Theorem.* [Mitchell] Any abelian category is equivalent to a full subcategory of some category of modules $\mathfrak{m}(\Bbbk)$.

8.6. **Category $C(\mathcal{A})$ of complexes with values in an abelian category $\mathcal{A}$.** Let $\mathcal{A}$ be the category $\mathfrak{m}(\Bbbk)$ of modules over a ring $\Bbbk$.[79] A map of complexes $f : A^\bullet \to B^\bullet$ is a system of maps f^n of the corresponding terms in complexes, which "preserves" the differential in the sense that in the diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A^{-2} & \longrightarrow & A^{-1} & \longrightarrow & A^{0} & \longrightarrow & A^{1} & \longrightarrow & \cdots \\
\Big\downarrow & & {\scriptstyle f^{-2}}\Big\downarrow & & {\scriptstyle f^{-1}}\Big\downarrow & & {\scriptstyle f^{0}}\Big\downarrow & & {\scriptstyle f^{1}}\Big\downarrow & & \Big\downarrow \\
\cdots & \longrightarrow & B^{-2} & \longrightarrow & B^{-1} & \longrightarrow & B^{0} & \longrightarrow & B^{1} & \longrightarrow & \cdots
\end{array}$$

all squares commute.[80] This clearly defines a category of complexes $\mathcal{C}^\bullet[\mathcal{A}]$, objects are complexes and morphisms are maps of complexes.

We observe some of the properties of the category $C(\mathcal{A})$.

8.6.1. *Properties.* The next two lemmas give basic properties of the above structures on the category $C(\mathcal{A})$.

[79] However everything works the same in any abelian category $\mathcal{A}$.

[80] Meaning that two possible ways of following arrows give the same result: $f^n\circ d_A^{n-1} = d_B^{n-1}\circ f^{n-1}$, for all n; i.e., $f\circ d = d\circ f$.

8.6.2. *Lemma.* $C(\mathcal{A})$ is an abelian category and a sequence of complexes is exact iff it is exact on each level!

Proof. For a map of complexes $A \xrightarrow{\alpha} B$ we can define $K^n = \mathrm{Ker}(A^n \xrightarrow{\alpha^n} B^n)$ and $C^n = A^n/\alpha^n(B^n)$. This gives complexes since d_A induces a differential d_K on K and d_B a differential d_C on C. Moreover, it is easy to check that in category $C(\mathcal{A})$ one has $K = \mathrm{Ker}(\alpha)$ and $C = \mathrm{Coker}(\alpha)$. Now one finds that $Im(\alpha)^n = Im(\alpha^n) = \alpha^n(A^n)$ and $Coim(\alpha)^n = Coim(\alpha^n) = A^n/\mathrm{Ker}(\alpha^n)$, so the canonical map $Coim \to Im$ is given by isomorphisms $A^n/\mathrm{Ker}(\alpha^n) \xrightarrow{\cong} \alpha^n(A^n)$. Exactness claim follows.

8.6.3. *Lemma.* A short exact sequence of complexes $0 \to A \to B \to C \to 0$ gives a long exact sequence of cohomologies.

$$\cdots \xrightarrow{\partial^{n-1}} \mathrm{H}^n(A) \xrightarrow{\mathrm{H}^n(\alpha)} \mathrm{H}^n(B) \xrightarrow{\mathrm{H}^n(\beta)} \mathrm{H}^n(C) \xrightarrow{\partial^n} \mathrm{H}^{n+1}(A) \xrightarrow{\mathrm{H}^{n+1}(\alpha)} \mathrm{H}^{n+1}(B) \xrightarrow{\mathrm{H}^{n+1}(\beta)} \cdots$$

8.7. **Exactness of functors and the derived functors.** Remember that derived versions are suppose to improve some constructions, i.e., *functors.* How this is exactly done depends on *exactness properties* of the functor in question. We will consider *additive functors* $F : \mathcal{A} \to \mathcal{B}$ between additive categories,[81] this means that the maps $\mathrm{Hom}_{\mathcal{A}}(a', a'') \to \mathrm{Hom}_{\mathcal{B}}(Fa', Fa'')$ are required to be morphisms of abelian groups.

Lemma. If F is additive then $F(0) = 0$ and $F(a \oplus b) \cong F(a) \oplus F(b)$.

8.7.1. *Exactness properties.*

(1) *Exact functors.* We will say that F is exact if it preserves short exact sequences, i.e., for any SES $0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$ in $\mathcal{A}$, its F-image in $\mathcal{B}$ is exact, i.e., the sequence $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ is a SES in $\mathcal{B}$.
(2) *Left exact functors.* [82] We say, that F is left exact if for any SES its F-image $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ is exact except possibly in the A''-term, i.e., $F(\beta)$ *need not be surjective.*
(3) *Right exact functors.* F is right exact if it the F-image $F(0) \to F(A') \xrightarrow{F(\alpha)} F(A) \xrightarrow{F(\beta)} F(A'') \to F(0)$ of a SES is exact except possibly in the A'-term, i.e., $F(\alpha)$ may fail to be injective.

Proposition. Let $\mathcal{A}$ be an abelian category, for any $a \in \mathcal{A}$,

- (a) $\mathrm{Hom}_{\mathcal{A}}(a, -) : \mathcal{A} \to \mathcal{A}b$ is left exact!,
- (b) $\mathrm{Hom}_{\mathcal{A}}(-, a) : \mathcal{A}^o \to \mathcal{A}b$ is right exact!

[81] One can think of the case where $\mathcal{A} = \mathfrak{m}(\Bbbk)$ and $\mathcal{B} = \mathfrak{m}(l)$ since the general case works the same.
[82] Few interesting functors are exact so we have to relax the notion of exactness.

Proof. (a) For any exact sequence $0 \to b' \xrightarrow{\alpha} b \xrightarrow{\beta} b'' \to 0$ we consider the corresponding sequence $\mathrm{Hom}_{\mathcal{A}}(a, b') \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathcal{A}}(a, b) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathcal{A}}(a, b'')$.

(1) α_* is injective. if $a \xrightarrow{\mu} b'$ and $0 = \alpha_*(\mu) \stackrel{\text{def}}{=} \alpha \circ \mu$, then μ factors through the kernel $\mathrm{Ker}(\alpha)$ (by the definition of the kernel). However, $\mathrm{Ker}(\alpha) = 0$ (by definition of a short exact sequence), hence $\mu = 0$.

(2) $\mathrm{Ker}(\beta^*) = Im(\alpha_*)$. First, $\beta_* \circ \alpha_* = (\beta \circ \alpha)_* = 0_* = 0$, hence $Im(\alpha_*) \subseteq \mathrm{Ker}(\beta^*)$. If $a \xrightarrow{\nu} b$ and $0 = \beta_*(\nu)$, i.e., $0 = \beta \circ \nu$, then ν factors through the kernel $\mathrm{Ker}(\beta)$. But $\mathrm{Ker}(\beta) = a'$ and the factorization now means that ν is in $Im(\alpha_*)$.

Remark. $\mathrm{Hom}(a, -)$ *is not always exact.* Let $\mathcal{A} = \mathcal{A}b$ and apply $\mathrm{Hom}(a, -)$ for $a = \mathbb{Z}/2\mathbb{Z}$ to $0 \to 2\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \to 0$. Then $id_{\mathbb{Z}/2\mathbb{Z}}$ does not lift to a map from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$. So β_* need not be surjective.

Lemma. Tensoring is right exact in each argument, i.e., for any left $\Bbbk$-module M the functor $M \underset{\Bbbk}{\otimes} - : \mathfrak{m}^r(\Bbbk) \to \mathcal{A}b$ is right exact, and so is $- \underset{\Bbbk}{\otimes} N : \mathfrak{m}(\Bbbk) \to \mathcal{A}b$ for any right $\Bbbk$-module N.

8.7.2. *Projectives and the existence of projective resolutions.* Let $\mathcal{A}$ be an abelian category. We say that $p \in \mathcal{A}$ is a projective object if the functor $\mathrm{Hom}_{\mathcal{A}}(p, -) : \mathcal{A} \to \mathcal{A}b$ is exact.

Since $\mathrm{Hom}_{\mathcal{A}}(p, -)$ is known to be always left exact, what we need is that for any short exact sequence $0 \to a \xrightarrow{\alpha} b \xrightarrow{\beta} c \to 0$ map $\mathrm{Hom}(p, b) \to \mathrm{Hom}(p, c)$ is surjective. In other words, if c is a quotient of b then any map from p to the quotient $p \xrightarrow{\gamma} c$ lifts to a map to b, i.e., there is a map $p \xrightarrow[\tilde{\gamma}]{} b$ such that $\gamma = \beta \circ \tilde{\gamma}$ for the quotient map $b \xrightarrow{\beta} c$.

Lemma. (a) In $\mathfrak{m}(\Bbbk)$, free modules are projective. More precisely, P is projective iff P is a summand of a free module.

(b) $\oplus_{i \in I}\, p_i$ is projective iff all summands p_i are projective.

We say that abelian category $\mathcal{A}$ has enough projectives if any object is a quotient of a projective object.

Corollary. Module categories have enough projectives.

The importance of "enough projectives" comes from

Proposition. For an abelian category $\mathcal{A}$ the following is equivalent

(1) Any object of $\mathcal{A}$ has a projective resolution (i.e., a left resolution consisting of projective objects).
(2) $\mathcal{A}$ has enough projectives.

8.7.3. *Injectives and the existence of injective resolutions.* Dually, we say that $i \in \mathcal{A}$ is an injective object if the functor $\mathrm{Hom}_{\mathcal{A}}(-,i):\ \mathcal{A} \to \mathcal{A}b^o$ is exact.

Again, since $\mathrm{Hom}_{\mathcal{A}}(-,i)$ is always right exact, we need for any short exact sequence $0 \to a \xrightarrow{\alpha} b \xrightarrow{\beta} c \to 0$ that the map $\mathrm{Hom}(b,p) \xrightarrow{\alpha^*} \mathrm{Hom}(a,p),\ \alpha^*(\phi) = \phi\circ\alpha$; be surjective. This means that if a is a subobject of b then any map $a \xrightarrow{\gamma} i$ from a subobject a to i extends to a map from b to i, i.e., there is a map $b \xrightarrow{\tilde{\gamma}} i$ such that $\gamma =\ \tilde{\gamma}\circ\alpha$. So, an object i is injective if each map from a subobject $a'\hookrightarrow a$ to i, extends to the whole object a.

Proposition. (a) A $\mathbb{Z}$-module I is injective iff I is divisible, i.e., for any $a \in I$ and $n \in \{1,2,3,...\}$ there is some $\tilde{a} \in I$ such that $a = n{\cdot}\tilde{a}$. (So, we ask that the multiplications $n: I \to I$ with $n \in \{1,2,3,...\}$, are all surjective.)[83]

(b) For any abelian group M denote $\widehat{M} = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$. Then the canonical map $M \xrightarrow{\rho} \widehat{\widehat{M}}$ is injective.

Proof. (a) For $m \in M,\ \chi \in \widehat{M},\ \rho(m)\ (\chi) \stackrel{\text{def}}{=} \chi(m)$. So, $\rho(m) = 0$ means that m is killed by each $\chi\widehat{M}$ ("each character of M"). If $m \neq 0$ then $\mathbb{Z}{\cdot}m$ is isomorphic to $\mathbb{Z}$ or to one of $\mathbb{Z}/n\mathbb{Z}$, in each case we can find a $\mathbb{Z}{\cdot}m \xrightarrow{\chi_0} \mathbb{Q}/\mathbb{Z}$ which is $\neq 0$ on the generator m. Since $\mathbb{Q}/\mathbb{Z}$ is injective we can extend χ_0 to M.

Proof. [84] For any $a \in I$ and $n > 0$ we can consider $\frac{1}{n}\mathbb{Z}\supseteq\mathbb{Z} \xrightarrow{\alpha} I$ with $\alpha(1) = a$. If I is injective then α extends to $\widetilde{\alpha}: \frac{1}{n}\mathbb{Z} \to I$ and $a = n\widetilde{\alpha}(\frac{1}{n})$.

Conversely, assume that I is divisible and let $A\supseteq B \xrightarrow{\beta} I$. Consider the set $\mathcal{E}$ of all pairs (C,γ) with $B\subseteq C\subseteq A$ and $\gamma: C \to I$ an extension of β. It is partially ordered with $(C,\gamma) \leq (C',\gamma')$ if $C\subseteq C'$ and γ' extends γ. From Zorn lemma and the following observations it follows that $\mathcal{E}$ has an element (C,γ) with $C = A$:

(1) For any totally ordered subset $\mathcal{E}'\subseteq\mathcal{E}$ there is an element $(C,\gamma) \in \mathcal{E}$ which dominates all elements of $\mathcal{E}'$ (this is clear: take $C = \cup_{(C',\gamma')\in\mathcal{E}'} C'$ and γ is then obvious).
(2) If $(C,\gamma) \in \mathcal{E}$ and $C \neq A$ then (C,γ) is not maximal:
  - choose $a \in A$ which is not in C and let $\widetilde{C} = C + \mathbb{Z}{\cdot}a$ and $C \cap \mathbb{Z}{\cdot}a =\ \mathbb{Z}{\cdot}na$ with $n \geq 0$. If $n = 0$ then $\widetilde{C} = C\oplus\mathbb{Z}{\cdot}a$ and one can extend γ to C by zero on $\mathbb{Z}{\cdot}a$. If $n > 0$ then $\gamma(na) \in I$ is n-divisible, i.e., $\gamma(na) = nx$ for some $x \in I$.

[83] $\mathbb{Z}$ is projective in $\mathcal{A}b$ but it is not injective in $\mathcal{A}b$: $\mathbb{Z}\subseteq\frac{1}{n}\mathbb{Z}$ and the map $1_{\mathbb{Z}}: \mathbb{Z} \to \mathbb{Z}$ does not extend to $\frac{1}{n}\mathbb{Z} \to \mathbb{Z}$.

[84] The proof will use the Zorn lemma which is an essential part of any strict definition of set theory:

- Let $(I,\leq)$ be a (non-empty) partially ordered set such that any chain J in I (i.e., any totally ordered subset) is dominated by some element of I (i.e., there is some $i \in I$ such that $i \geq j,\ j \in J$). Then I has a maximal element.

Then one can extend γ to $\widetilde{C}$ by $\widetilde{\gamma}(a)=x$ (first define a map on $C\oplus\mathbb{Z}{\cdot}a$, and then descend it to the quotient $\widetilde{C}$).

Lemma. (a) For any abelian category $\mathcal{A}$ the following is equivalent

(1) Any object of $\mathcal{A}$ has an injective resolution (i.e., a right resolution consisting of injective objects).
(2) $\mathcal{A}$ has enough injectives. (We say that abelian category $\mathcal{A}$ has enough injectives if any object is a subobject of an injective object.)

(b) Product $\prod_{i\in I}\ J_i$ is injective iff all factors J_i are injective.

Corollary. Category of abelian groups has enough injectives.

Proof. To M we associate a huge injective abelian group $I_M = \prod_{x\in\widehat{M}}\ \mathbb{Q}/\mathbb{Z}{\cdot}x =\ (\mathbb{Q}/\mathbb{Z})^{\widehat{M}}$, its elements are $\widehat{M}$-families $c=(c_\chi)_{\chi\in\widehat{M}}$ of elements of $\mathbb{Q}/\mathbb{Z}$ (we denote such family also as a (possibly infinite) formal sum $\sum_{\chi\in\widehat{M}}\ c_\chi{\cdot}\chi$). By part (a), canonical map *io* is injective

$$M \xrightarrow{\iota} I_M, \qquad \iota(m)=\ (\chi(m))_{\chi\in\widehat{M}} = \sum_{\chi\in\widehat{M}} \chi(m){\cdot}\chi, \quad m\in M.$$

Theorem. Module categories $\mathfrak{m}(\Bbbk)$ have enough injectives.

Proof. The problem can be reduced to the known case $\Bbbk=\mathbb{Z}$ via the canonical map of rings $\mathbb{Z}\xrightarrow{\phi}\Bbbk$.

Remarks. (1) An injective resolution of the $\mathbb{Z}$-module $\mathbb{Z}$ is $0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/Z\to 0$.

(2) Injective resolutions are often big, hence more difficult to use in specific calculations then say, the free resolutions. However, they are necessary for the functor $\Gamma(X,-)$ of global sections of sheaves.

8.7.4. *Left derived functor RF of a right exact functor F.* We observe that if F is right exact then the correct way to extend it to a functor on the derived level is the construction $LF(M)\overset{\text{def}}{=}\ F(P^\bullet)$, i.e., replacement of the object by a projective resolution. "Correct" means here that LF is really more then F – it contains the information of F in its zero$^{\text{th}}$ cohomology, i.e., $L^0F\cong\ F$ for $L^iF(M)\overset{\text{def}}{=}H^i[LF(M)]$. Letter L reminds us that we use a left resolution.

Lemma. If the functor $F:\mathcal{A}\to\mathcal{B}$ is right exact, there is a canonical isomorphism of functors $H^0(LF)\cong\ F$.

Proof. Let $\cdots \to P^{-2} \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{q} M \to 0$ be a projective resolution of M. Then $LF(M) = F[\cdots \to P^{-2} \to P^{-1} \xrightarrow{d^{-1}} P^0 \to 0 \to \cdots]$ equals

$$[\cdots \to F(P^{-2}) \to F(P^{-1}) \xrightarrow{F(d^{-1})} F(P^0) \to 0 \to \cdots],$$

so $H^0[LF(M)] = F(P^0)/F(d^{-1})F(P^{-1})$.

If we apply F to the exact sequence $P^{-1} \xrightarrow{d^{-1}} P^0 \to M \xrightarrow{q} 0$, the right exactness gives an exact sequence $F(P^{-1}) \xrightarrow{F(d^{-1})} F(P^0) \xrightarrow{F(q)} F(M) \to 0$. Therefore, $F(q)$ factors to a canonical map $F(P^0)/F(d^{-1})F(P^{-1}) \to F(M)$ which is an isomorphism.

8.7.5. *Right derived functor RF of a left exact functor F.* Obviously, we want to define for any left exact functor $F : \mathcal{A} \to \mathcal{B}$ a right derived functor RF by replacing an object M by its injective resolution $I^\bullet$

$$(RF)M \stackrel{\text{def}}{=} F(I^\bullet) \quad \text{and} \quad (R^iF)M \stackrel{\text{def}}{=} H^i[(RF)M] \;=\; H^i[F(I^\bullet)].$$

As above, $(R^0F)(M) \stackrel{\text{def}}{=} H^0[(RF)M] \cong F(M)$, i.e., $R^0F = F$.

8.8. **Appendix: The ideal setup for homological algebra.** We have a prescription that corrects a functor which is only half-exact. Say, if F is a left exact functor we had to correct it on the right side, so we replaced a module by its injective resolution $I^\bullet$ which is a right resolution (i.e., it is in degrees ≥ 0). This gives $R^iF(M) \stackrel{\text{def}}{=} H^i[F(I\bullet)]$. However, notice some

8.8.1. *Foundational and calculational problems.*

8.8.2. *Questions.*

(1) There may be more then one injective resolution of M, which one do I use?
(2) Does a map $\alpha : M \to N$ give something relating $R^iF(M)$ and $R^iF(N)$?
(3) If M is obtained by gluing simpler modules M', M'', i.e., if there is a short exact sequence $0 \to M' \to M \to M'' \to 0$, can I analyze $R^iF(M)$ in terms of $R^iF(M')$ and $R^iF(M'')$?

The answer is $\boxed{\textbf{Yes}}$:

8.8.3. *Lemma.*

(1) Use any injective resolution:
$R^iF(M)$ does not depend on the choice, for two injective resolutions $I^\bullet, J^\bullet$ of M, there are canonical isomorphisms $H^i[F(I^\bullet)] \cong H^i[F(J^\bullet)]$.
(2) Each R^iF is a functor:
for any map $M \xrightarrow{\alpha} N$ the following is true:

- *(i) for any injective resolutions $I^\bullet, M^\bullet$ of M and N, there is a lift $\widetilde{a}l : I^\bullet \to J^\bullet$ of α*
- *(ii) the corresponding map $H^i(\widetilde{\alpha}) :\ H^i[F(I^\bullet)] \to H^i[F(J^\bullet)]$? does not depend on the choices of $I^\bullet, J^\bullet, \widetilde{\alpha}$, so*
- *(iii) it is a well defined map $R^iF(M) \xrightarrow{R^i(\alpha)} R^iF(N)$.*

(3) A short exact sequence gives a long exact sequence of derived functors:

$$\text{Any short exact sequence } 0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0,$$

- *(i) lifts to a short exact sequence of injective resolutions $0 \to I'^\bullet \xrightarrow{\widetilde{\alpha}} I^\bullet \xrightarrow{\widetilde{\beta}} I''^\bullet \to 0$,*
- *(ii) A short exact sequence of injective resolutions always splits, i.e., there is a subcomplex $J^\bullet \subseteq I^\bullet$ complementary to $I'^\bullet$. Therefore,*
- *(iii) the sequence of complexes $0 \to F(I'^\bullet) \xrightarrow{F(\widetilde{\alpha})} F(I^\bullet) \xrightarrow{F(\widetilde{\beta})} F(I''^\bullet) \to 0$, is again exact. So,*
- *applying cohomology to it we get a long exact sequence*

$$0 \to F(M') \to F(M) \to F(M'') \to R^1F(M') \to R^1F(M) \to R^1F(M'') \to R^2F(M') \to R^2F(M) \to R^2F(M'') \cdots$$

The direct proofs of these facts are routine and only take finite amount of time. However, there is a (calculationaly superior) conceptual approach which involves finding appropriate categories:

8.8.4. *Homotopy category of complexes.* The origin of messiness is having to *choose* a resolution for each object. Though resolutions $I^\bullet, J^\bullet$ of one object M may be very different complexes, claim (1) above suggests that – in some sense – they are the same. This is achieved by replacing the category of complexes $C^\bullet(\mathcal{A})$ with the *homotopic category of complexes* $K^\bullet(\mathcal{A})$ – the objects are again the complexes but there are more isomorphisms, and any two resolutions of M are *canonically isomorphic* in $K^\bullet(\mathcal{A})$. So, taking injective resolutions becomes a *functor* $\mathcal{I} : \mathcal{A} \to K(\mathcal{A})$. On the other hand, there are some obvious functors on homotopy categories: just by applying F to complexes we get a functor $K(F) : K(\mathcal{A}) \to K(\mathcal{B})$, and also the cohomology of complexes gives functors $H^n : K(\mathcal{B}) \to \mathcal{B}$. Therefore we get *functorial constructions of derived functors* as compositions of known functors:

$$RF \stackrel{\text{def}}{=} K(F)\circ\mathcal{I}\ :\ \mathcal{A} \to K(\mathcal{B}) \quad \text{and} \quad R^iF \stackrel{\text{def}}{=} H^i\circ K(F)\circ\mathcal{I}\ :\ \mathcal{A} \to \mathcal{B}.$$

This is pretty neat, but it turns out that there is an even cleaner point of view on homological algebra:

8.8.5. *Derived category of complexes.* One can do better then make resolutions functorial, one can get all resolutions of M to be *canonically isomorphic to M*, so that when we are using a resolution no complications are introduced. For that one passes to the *derived category* $D(\mathcal{A})$ by adding more isomorphisms to $K(\mathcal{A})$:

- Objects are (again) complexes.
- Any map of complexes $A^\bullet \xrightarrow{\alpha} B^\bullet$ which gives isomorphism of cohomology groups (i.e., all maps $H^i(A^\bullet) \xrightarrow{H^i(\alpha)} H^i(B^)\bullet$ are isomorphisms), acquires an inverse in $D(\mathcal{A})$, i.e., α becomes an isomorphism in $D(\mathcal{A})$.

This derived category is the standard set up for homological algebra. One problem it resolves is how to derive functor FG which is neither left nor right exact, but is a composition of say a left exact functor F and a right exact functor G.[85]

[85]This is not doable without derived categories, the reason is essentially that while taking cohomology of complexes forgets a lot of information, the derived categories take complexes seriously.

9. **Local and global information: sheaves (cohomology of line bundles on curves)**

Sheaves are a machinery which addresses an essential problem – the relation between local and global information – so they appear throughout mathematics, but sheaves are particularly useful and highly developed in algebraic geometry.

The passage from local to global is formalized here as the procedure $\Gamma(X,-)$ of taking global sections of sheaves on a space X. However, this becomes really useful only when combined with homological algebra. The derived functors of $\Gamma(X,-)$ are the sheaf cohomology functors $H^i(X,-),\ i=0,1,....$

We start on sheaf cohomology with an approximate version – the Čech cohomology $\check{H}^i_{\mathcal{U}}(X,\mathcal{A})$ of a sheaf $\mathcal{A}$ with respect to an open cover $\mathcal{U}$ of X. This is a great calculational tool because in many situations (i.e., under some conditions on the relation between the cover and the sheaf) it computes the true cohomology $H^i(X,\mathcal{A})$. However, Čech cohomology is much more down to Earth[86] then the correct version. After this introduction we define the general cohomology of sheaves.

In algebraic geometry the basic example of sheaves are the so called coherent sheaves, for instance the sheaves of sections of vector bundles. We will be most interested in line bundles on curves and the main tool will be the Riemann-Roch theorem.

9.1. **Sheaves.**

9.1.1. *Example of a sheaf: smooth functions on* $\mathbb{R}$. Let X be $\mathbb{R}$ or any smooth manifold. The notion of smooth functions on X gives the following data:

- for each open $U\subseteq X$ an algebra $C^\infty(U)$ (the smooth functions on U),
- for each inclusion of open subsets $V\subseteq U\subseteq X$ a map of algebras $C^\infty(U)\xrightarrow{\rho^U_V}C^\infty(V)$ (the restriction map);

and these data have the following properties

(1) (*transitivity of restriction*) $\rho^U_V\circ\rho^U_V=\ \rho^U_W$ for $W\subseteq V\subseteq U$,
(2) (*gluing*) if the functions $f_i\in C^\infty(U_i)$ on open subsets $U_i\subseteq X,\ i\in I$, are compatible in the sense that $f_i=f_j$ on the intersections $U_{ij}\stackrel{\text{def}}{=}U_i\cap U_j$, then they glue into a unique smooth function f on $U=\cup_{i\in I}\ U_i$.

The context of dealing with objects which can be restricted and glued compatible pieces is formalized in the notion of sheaves. The definition is formal (precise) way of saying that a given class $\mathcal{C}$ of objects forms a sheaf if it is *defined by local conditions*, i.e., conditions which can be checked in a neighborhood of each point:

[86] Gea.

9.1.2. *Definition of sheaves on a topological space.* A sheaf of sets $\mathcal{S}$ on a topological space $(X,\mathcal{T})$ consists of the following data:

- for each open $U\subseteq X$ a set $\mathcal{S}(U)$,
- for each inclusion of open subsets $V\subseteq U\subseteq X$ a map $\mathcal{S}(U) \xrightarrow{\rho_V^U} \mathcal{S}(V)$ (called the restriction map);

and these data are required to satisfy

(1) (*identity*) $\rho_U^U = \ id_{\mathcal{S}(U)}$.
(2) (*transitivity of restriction*) $\rho_W^V\circ\rho_V^U = \ \rho_W^U$ for $W\subseteq V\subseteq U$,
(3) (*gluing*) Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of an open $U\subseteq X$. For a family of elements $f_i \in \mathcal{S}(U_i),\ i \in I$, compatible in the sense that $\rho^{U_i}_{U_{ij}} f_i = \rho^{U_i}_{U_{ij}} f_j$ in $\mathcal{S}(U_{ij})$ for $i,j \in I$; there is a unique $f \in \mathcal{S}(U)$ such that on the intersections $\rho^{U_i}_{U_{ij}} f = f_i$ in $\mathcal{S}(U_i),\ i \in I$.
(4) $\mathcal{S}(\emptyset) = \emptyset$.

9.1.3. *Sheaves with values in a category $\mathcal{A}$.* We can equally define sheaves of abelian groups, rings, modules, etc – only the last, and least interesting requirement has to be modified, say in abelian groups we would ask that $\mathcal{S}(\emptyset)$ is the trivial group $\{0\}$.

9.1.4. *Examples.* (1) *Structure sheaves.* On a topological space X one has a sheaf of continuous functions C_X. If X is a smooth manifold there is a sheaf C_X^∞ of smooth functions, etc., holomorphic functions $\mathcal{H}_X$ on a complex manifold, "polynomial" functions $\mathcal{O}_X$ on an algebraic variety. In each case the topology on X and the sheaf contain all information on the structure of X.

(2) The constant sheaf S_X on X associated to a set S: $S_X(U)$ is the set of *locally* constant functions from U to X.

(3) Constant functions do not form a sheaf, neither do the functions with compact support. A given class $\mathcal{C}$ of objects forms a sheaf if it is defined by local conditions. For instance, being a (i) function with values in S, (ii) non-vanishing (i.e., invertible) function, (iii) solution of a given system $(*)$ of differential equations; are all local conditions: they can be checked in a neighborhood of each point.

9.2. **Global sections functor** $\Gamma : \mathcal{S}heaves(X) \to\mathcal{S}ets$**.** Elements of $\mathcal{S}(U)$ are called the *sections* of a sheaf $\mathcal{S}$ on $U\subseteq X$ (this terminology is from classical geometry). By $\Gamma(X,\mathcal{S})$ we denote the set $\mathcal{S}(X)$ of *global sections.*

The construction $\mathcal{S}\mapsto\Gamma(X,\mathcal{S})$ means that we are looking at global objects of a given class $\mathcal{S}$ of objects, which is defined by local conditions. We will see that the procedure $\mathcal{S}\mapsto\Gamma(X,\mathcal{S})$ has a hidden part, the *cohomology* $\mathcal{S}\mapsto H^\bullet(X,\mathcal{S})$ of the sheaf $\mathcal{S}$ on X.

9.2.1. *Smooth manifolds.* On a smooth manifold X, $\Gamma(X,C^\infty)=C^\infty(X)$ is huge. The holomorphic setting will be more subtle.

9.2.2. *Example: sheaves corresponding to multivalued function.* Let $\mathcal{S}$ be the sheaf of solutions of $zy'=\lambda y$ in holomorphic functions on $X=\mathbb{C}^*$. On any disc $c\in D\subseteq X$, evaluation at the center gives $\mathcal{S}(D)\xrightarrow{\cong}\mathbb{C}$ (the solutions are multiples of functions $z^\lambda = e^{\lambda\log(z)}$ defined using a branch of logarithm on D). However, $\Gamma(X,\mathcal{S})=0$ if $\lambda\notin\mathbb{Z}$. So locally there is a lot, but nothing globally. This is a restatement of: multi-valued function z^λ is useful but has no single-valued meaning on $\mathbb{C}^*$.

9.2.3. *Example: global holomorphic functions on* $\mathbb{P}^1$*.* $\mathbb{P}^1=\mathbb{C}\cup\infty$ can be covered by $U_1=U=\mathbb{C}$ and $U_2=V=\mathbb{P}^1-\{0\}$. We think of $X=\mathbb{P}^1$ as a complex manifold by identifying U and V with $\mathbb{C}$ using coordinates u,v such that on $U\cap V$ one has $uv=1$.

Lemma. $\Gamma(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1})=\mathbb{C}$.

Proof. (1) *Proof using a cover.* A holomorphic function f on X restricts to $f|U=\sum_{n\geq 0}\alpha_n u^n$ and to $f|V=\sum_{n\geq 0}\beta_n v^n$. On $U\cap V=\mathbb{C}^*$, $\sum_{n\geq 0}\alpha_n u^n=\sum_{n\geq 0}\beta_n u^{-n}$, and therefore $\alpha_n=\beta_n=0$ for $n\neq 0$.

(2) *Proof using maximum modulus principle.* The restriction of a holomorphic function f on X to $U=\mathbb{C}$ is a bounded holomorphic function (since X is compact), hence a constant.

9.3. **Čech cohomology of sheaves.** Cohomology of sheaves is a machinery which deals with the subtle ("hidden") part of the the relation between local and global information. The Čech cohomology is its simplest calculational tool.

9.3.1. *Cohomology of sheaves.* There is a general *cohomology theory for sheaves* which associates to any sheaf of abelian groups $\mathcal{A}$ a sequence of groups $H^i(X,\mathcal{A})$. The Čech cohomology $\check{H}^i_{\mathcal{U}}(X,\mathcal{A})$ can be viewed as an approximation of the *true cohomology* $H^i(X,\mathcal{A})$, which is calculated using an open cover $\mathcal{U}$ of X. We start with the Čech cohomology which is conceptually much simpler, however it is very useful since in practice, for a specific class of sheaves $\mathcal{A}$ one can find the corresponding class of open covers $\mathcal{U}$ such that $\check{H}^i_{\mathcal{U}}(X,\mathcal{A})=H^i(X,\mathcal{A})$.[87]

9.3.2. *Calculation of global section via an open cover.* The first idea is to find all global sections of a sheaf by examining how one can glue local sections into global sections. Let $\mathcal{U}=(U_i)_{i\in I}$ be an open cover of a topological space X, we will choose a complete ordering on I[88] We will use finite intersections $U_{i_0,\dots,i_p}\overset{\text{def}}{=}U_{i_0}\cap\cdots\cap U_{i_p}$ with $i_0<\cdots<i_p$.

[87]For instance in algebraic geometry one usually considers the *quasicoherent sheaves* and then it suffices if all U_i are affine.

[88]It is not really necessary but it simplifies practical calculations.

To a sheaf of abelian groups $\mathcal{A}$ on X we associate a map of abelian groups (e,

- $C^0(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \prod_{i\in I} \mathcal{A}(U_i)$, its elements are systems $f=(f_i)_{i\in I}$, with one section $f_i\in\mathcal{A}(U_i)$ for each open set U_i,
- $C^1(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \prod_{\{(i,j)\in I^2;\ i<j\}} \mathcal{A}(U_{ij})$, its elements are systems $g=(g_{ij})_{(i,j)\in I^2}$ of sections $g_{ij}\in\mathcal{A}(U_{ij})$ on all intersections U_{ij}.
- map sends $f=(f_i)_{i\in I}\in C^0$ to $df\in C^1$ with

$$(df)_{ij} \stackrel{\text{def}}{=} \rho^{U_j}_{U_{ij}} f_j - \rho^{U_i}_{U_{ij}} f_i.$$

Less formally, we usually state it as $(df)_{ij} = f_j|U_{ij} - f_i|U_{ij}$.

Lemma. For any sheaf of abelian groups $\mathcal{A}$ on X

$$\Gamma(\mathcal{A}) \xrightarrow{\cong} \operatorname{Ker}[C^0(\mathcal{U},\mathcal{A}) \xrightarrow{d} C^1(\mathcal{U},\mathcal{A})].$$

9.3.3. *Čech complex* $C^\bullet(\mathcal{U},\mathcal{A})$. Emboldened, we try more of the same. We want to capture more of the relation between local sections by extending the construction into a sequence of maps of abelian groups

$$C^0(\mathcal{U},\mathcal{A}) \xrightarrow{d^0} C^1(\mathcal{U},\mathcal{A}) \xrightarrow{d^1} C^2(\mathcal{U},\mathcal{A}) \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} C^n(\mathcal{U},\mathcal{A}) \xrightarrow{d^n} C^{n+1}(\mathcal{U},\mathcal{A}) \xrightarrow{d^{n+1}} \cdots.$$

Here,

$$C^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \prod_{i_0<\cdots<i_n} \mathcal{A}(U_{i_0,\dots,i_n})$$

consists of all systems of sections on $(n+1)$-tuple intersections. The map d^n (we call it the n^{th} *differential*), creates from $f=(f_{i_0,\dots,i_n})_{I^n}\in C^n$ an element $d^n(f)\in C^{n+1}$, with

$$d^n(f)_{i_0,\dots,i_{n+1}} = \sum_{s=0}^{n+1} (-1)^s f_{i_0,\dots,i_{s-1},i_{s+1},\dots,i_{n+1}}.$$

From this we construct groups of *n-cocycles* and *n-coboundaries*

$$Z^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \operatorname{Ker}(C^n \xrightarrow{d^n} C^{n+1}) \subseteq C^n \quad \text{and} \quad B^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} Im(C^{n-1} \xrightarrow{d^{n-1}} C^n) \subseteq C^n.$$

Lemma. (Čech complex $C^\bullet(\mathcal{U},\mathcal{A})$.)

(a) Show that d^0 is the same as before.

(b) Show that $(C^\bullet(\mathcal{U},\mathcal{A}),d^\bullet)$ is a complex, i.e., $d^n \circ d^{n-1}=0$.

(c) Show that $B^n(\mathcal{U},\mathcal{A}) \subseteq Z^n(\mathcal{U},\mathcal{A})$.

9.3.4. *Čech cohomology* $\check{H}^\bullet_{\mathcal{U}}(X,\mathcal{A})$. It is defined as the cohomology of the Čech complex $C^\bullet(\mathcal{U},\mathcal{A})$, i.e.,

$$\check{H}^n_{\mathcal{U}}(X,\mathcal{A}) \stackrel{\text{def}}{=} Z^n(\mathcal{U};\mathcal{A})/B^n(\mathcal{U};\mathcal{A}), \quad n = 0,1,2,....$$

This construction is a generalization of the global sections of a sheaf since

$$\check{H}^0_{\mathcal{U}}(X,\mathcal{A}) = Z^0(\mathcal{U},\mathcal{A})B^0(\mathcal{U},\mathcal{A}) = Z^0(\mathcal{U},\mathcal{A}) = \Gamma(\mathcal{A}).$$

Lemma. If the open cover $\mathcal{U}$ consists of two open sets U and V, show that

(1) $\check{H}^0_{\mathcal{U}}(X,\mathcal{A}) = \{(a,b) \in \mathcal{A}(U)\oplus\mathcal{A}(V);\ a = b \text{ on } U\cap V\} \cong \Gamma(X,\mathcal{A})$.
(2) $\check{H}^1_{\mathcal{U}}(X,\mathcal{A}) = \frac{\mathcal{A}(U\cap V)}{\rho^U_{U\cap V}\mathcal{A}(U)+\rho^V_{U\cap V}\mathcal{A}(V)}$.
(3) $\check{H}^i_{\mathcal{U}}(X,\mathcal{A}) = 0$ for $i > 1$.

9.4. **Quasi-coherent sheaves on algebraic varieties.**

9.5. **True cohomology of sheaves.**

9.5.1. *Functors* $H^i(X,\mathcal{F})$. The cohomology of sheaves starts with the functor of global sections considered as a functor

$$\Gamma(X,-):\ \mathcal{S}h\mathcal{A}b(X) \to \mathcal{A}b$$

from the category of sheaves of abelian groups to the category of abelian groups. It turns out that

Lemma. (a) $\mathcal{S}h\mathcal{A}b(X)$ is an *abelian category* with enough injectives.

(b) Functor $\Gamma(X,-)$ is left exact.

Therefore there are right derived functors of $\Gamma(X,-)$ which one denotes

$$H^i(X,-) \stackrel{\text{def}}{=} R^i\Gamma(X,-):\ \mathcal{S}h\mathcal{A}b(X) \to \mathcal{A}b.$$

9.5.2. *Computation.* The unpleasant part here is that by definition this involves injective resolutions in the category of sheaves and injective objects in sheaves tend to be very large. So we try to minimize the use of definitions and use general properties such as

Theorem.

(1) $H^0(X,\mathcal{A}) = \Gamma(X,\mathcal{A})$ and it equals $\check{H}^0_{\mathcal{U}}(X,\mathcal{A})$ so it can be calculated using any open cover.
(2) A short exact sequence of sheaves gives a long exact sequence of cohomologies.

9.5.3. *Computation in algebraic geometry.* In any given setting (topology, analysis, complex manifolds, logic) one develops the understanding of the classes of sheaves relevant for that setting. In algebraic geometry the most relevant sheaves are the *quasi-coherent sheaves*, and here are some basic facts

Theorem. Let $\mathcal{F}$ be a quasi-coherent sheaf on an algebraic variety X.

(1) If X is affine, the higher cohomologies vanish: $H^i(X,\mathcal{F}) = 0$ for $i > 0$.
(2) If $\mathcal{U}$ is an open cover such that all finite intersections $U_{i_1,\dots,i_n}$ are affine, the cohomology is the same as the Čech cohomology:
$$H^i(X,\mathcal{F}) \ = \ \check{H}^i_{\mathcal{U}}(X,\mathcal{F}).$$
(3) Cohomologies vanish beyond dimension of X: $H^i(X,\mathcal{F}) = 0$ for $i > \dim(X)$.

Corollary. On curves we only have to worry about $H^0(C,\mathcal{F})$ and $H^1(C,\mathcal{F})$.

9.6. **Geometric representation theory: cohomology of line bundles on $\mathbb{P}^1$.**

9.6.1. *Cohomology of vector bundles.* Recall that to a vector bundle V on X we associate the sheaf $\mathcal{V}$ of sections of the vector bundle V. If V is obtained by gluing trivial vector bundles $V_i = U_i{\times}\mathbb{C}^n$ by transition functions ϕ_{ij}, then $\mathcal{V}(U)$ consists of all systems of $f_i \in \mathcal{H}(U_i\cap U,\mathbb{C}^n)$ such that on all intersections $U_{ij}\cap U$ one has $f_i = \phi_{ij} f_j$.

By *cohomology $H^*(X,V)$ of the vector bundle V* we mean the cohomology of the associated sheaf $\mathcal{V}$.

9.6.2. *Line bundles L_n on $\mathbb{P}^1$.* On $\mathbb{P}^1$ let L_n be the vector bundle obtained by gluing trivial vector bundles $U{\times}\mathbb{C}$, $V{\times}\mathbb{C}$ over $U\cap V$ by identifying $(u,\xi)\in U{\times}\mathbb{C}$ and $(v,\zeta)\in V{\times}\mathbb{C}$ if $uv = 1$ and $\zeta = u^n{\cdot}\xi$. So for $U_1 = U$ and $U_s = V$ one has $\phi_{12}(u) = u^n$, $U \in U\cap V{\subseteq}U$. Let $\mathcal{L}_n$ be the sheaf of holomorphic sections of L_n.

9.6.3. *Lemma.* (a) $\Gamma(\mathbb{P}^1,\mathcal{L}_n) \cong \ \mathbb{C}_n[x,y] \overset{\text{def}}{=}$ homogeneous polynomials of degree n. So, it is zero if $n < 0$ and for $n \geq 0$ the dimension is $n+1$.

(b) $H^1_{\mathcal{U}}(\mathbb{P}^1,\mathcal{L}_n) \cong ?$.

9.6.4. *Representations of $G = SL_2(\mathbb{C})$.* $SL_2(\mathbb{C})$ acts on $\mathbb{C}_2$ and therefore also on the algebra of functions $\mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x,y]$, and its homogeneous summands $\mathbb{C}_n[x,y]$. Also, action on $\mathbb{C}^2$ factors to an action on $\mathbb{P}^1$ = lines in $\mathbb{C}^2$, and this extends to an action on line bundles $\mathcal{L}_n$ over $\mathbb{P}^1$. In particular, $SL_2(\mathbb{C})$ acts on the vector space $H^i(\mathbb{P}^1,\mathcal{L}_n)$ that are naturally produced from L_n. In fact,

9.6.5. *Lemma.* $\mathbb{C}_n[x,y] = \Gamma(\mathbb{P}^1,\mathcal{L}_n), n = 0,1,2,..$ is the list of *all irreducible finite dimensional holomorphic representations of* $SL_2(\mathbb{C})$.

9.6.6. *Borel-Weil-Bott theorem.* For each semisimple (reductive) complex group G there is a space $\mathcal{B}$ (the flag variety of G) such that all irreducible finite dimensional holomorphic representations of G are obtained as global sections of line bundles on $\mathcal{B}$.

9.7. **Riemann-Roch theorem.** This is the most useful tool for calculations on curves. Classically, the Riemann-Roch theorem is a a deep geometric statement that relates the counts in two related situations, of meromorphic functions and 1-forms satisfying certain conditions on zeros and poles that are allowed. We will present it in terms of cohomology of line bundles and break it into several standard ideas that all generalize individually to higher dimensional geometry. The proof is now almost trivial because of the added flexibility in the sheaf theoretic setting.

9.7.1. *Other approaches.* A purely geometric proof of Riemann-Roch may take a semester. So sheaves are useful, but one pays a price incorporating sheaves into a standard part of our thinking. There is another, faster, approach to Riemann-Roch theorem for curves through the ring of *adels*. This is simpler then learning sheaves,[89] but this approach has a disadvantage that it has not been developed as well in higher dimensions, so it is not a standard tool in mathematics (except in one theory that specializes in the one-dimensional objects: the *Number Theory*).

9.7.2. *Riemann-Roch spaces.* To a divisor D on a curve C one associates the *Riemann-Roch* vector space

$$H(D) \stackrel{\text{def}}{=} \{f \in \mathfrak{M}(C);\ div(f) + D \geq 0\}$$

and the Riemann-Roch number $h(D) \stackrel{\text{def}}{=} \dim[H(D)]$, the number of (linearly independent) global meromorphic functions on C that satisfy some restrictions on the positions of poles and zeros (which we specify by the choice of the divisor D).

9.7.3. *Canonical divisors.* A divisor K on C is called a *canonical divisor* if $\mathcal{O}_C(K) \cong \Omega^1_C$.

9.7.4. *Theorems.*

Theorem. [A. Riemann-Roch theorem.] Let g be the genus of C and let K be any canonical divisor on C. Then, for any divisor D on C

$$h(D) - h(K - D) = deg(D) + 1 - g.$$

[89] It is not obvious, but adels really amount to learning sheaves precisely in the amount needed for line bundles on curves.

Theorem. [B. Riemann-Roch companion.]

(a) $deg(K) = 2(g-1)$.

(b) $deg(D) < 0 \;\Rightarrow\; h(D) = 0$.

The vanishing claim above leads to a special case of the Riemann-Roch theorem which is particularly satisfying:

Corollary. If $deg(D) > deg(K) = 2(g-1)$ then

$$h(D) = \; deg(D) + 1 - g.$$

9.7.5. *$h(D)$ as a function of $deg(D)$.* Notice that the dependence of $h(D)$ on $deg(D)$ is given by the line $y = x + (1-g)$ when $deg(D) > 2(g-1)$ and by the line $y = 0$ when $deg(D) < 0$. In between there is a subtlety interval $0 \le deg(D) \le 2(g-1)$ where $h(D)$ depends on D in subtle ways (in particular, it is not determined by $deg(D)$), and this is a source of a lot of nice mathematics.

9.7.6. *Examples.* There are three different kinds of behavior:

(1) $\boxed{g = 0.}$ Everything is known:

$$h(D) = max[deg(D) + 1, 0].$$

This can be used to show that any curve of genus zero is isomorphic to $\mathbb{P}^1$.

(2) $\boxed{\;g = 1.\;}$ When $deg(D) \neq 0$ then

$$h(D) = max[deg(D), 0].$$

In the only undetermined case $deg(D) = 0$ one has different possibilities since $h(0) = 1$ and $h(a-b) = 0$ when $a \neq b$.

(3) $\boxed{\;g > 1.\;}$ Here the unknown grows.

9.8. Abel-Jacobi maps $C^{(n)} \xrightarrow{\mathcal{AJ}_n} Pic_n(C)$.

Recall the Abel-Jacobi maps

$$C^{(n)} \xrightarrow{\mathcal{AJ}_n} Pic_n(C), \qquad D \mapsto [\mathcal{O}_C(D)] \qquad (n \ge 0).$$

9.8.1. *Lemma.* The fiber ${\mathcal{AJ}_n}^{-1}(L)$ of the n^{th} Abel-Jacobi map, at a line bundle $L \in Pic_n(C)$ is the projective spaces $\mathbb{P}[\Gamma(C,L)]$.

Proof. Recall that $C^{(n)}$ consists of all effective divisors of degree n. So, the fiber ${\mathcal{AJ}_n}^{-1}(L)$ consists of all effective divisors of degree n such that $\mathbb{L} \cong \mathcal{O}_C(D)$. But, $\mathbb{L} \cong \mathcal{O}_C(D)$ means that D is a divisor of a meromorphic section $s \neq 0$ of L. Moreover, since D is effective, such section has to be holomorphic. So, the fiber is

$${\mathcal{AJ}_n}^{-1}(L) \;=\; \{div(s);\; s \in \Gamma(C,L) - 0\}.$$

Therefore, we consider the map

$$\Gamma(C,L)-0 \xrightarrow{div} Div(C).$$

If s_1, s_2 have the same image: $div(s_1)=div(s_2)$ then $s_1{s_2}^{-1}$ is a meromorphic section of $L\otimes L^* \cong T = C\times\mathbb{C}$ and $div(s_1{s_2}^{-1}) = div^L(s_1) - div^L(s_2) = 0$. So this is a holomorphic functions which does not vanish, i.e., a non-zero constant! Therefore,

$$div(s_1)=div(s_2) \;\Leftrightarrow\; s_1\in\mathbb{C}^*s_2 \;\Leftrightarrow\; [s_1]=[s_2]\in\ \mathbb{P}[\Gamma(C,L)].$$

Therefore, we have found that

$$\mathcal{AJ}_n{}^{-1}(L) \;\cong\; div[\Gamma(C,L)-0] \;\cong\; \mathbb{P}[\Gamma(C,L)-0] \;\cong\; \mathbb{P}^{\dim[\Gamma(C,L)]-1}.$$

9.8.2. *Theorem.* When $n>2(g-1)$ the Abel-Jacobi map

$$C^n \xrightarrow{\mathcal{AJ}_n} Pic_n(C), \qquad D\mapsto [\mathcal{O}_C(D)],$$

is a bundle of projective spaces $\mathbb{P}^{n-g}$.

Proof. If $deg(L)=n$ is $>2(g-1)$, we know that

$$\dim[\Gamma(C,L)]-1 = \ [\deg(L)+1-g]-1 = \ n-g.$$

In particular, the fiber is the same at all points of $Pic_n(C)$, and this implies that $\mathcal{AJ}_n$ is a bundle.

9.8.3. *Remarks.* (0) When $\Gamma(C,L)=0$ the fiber is $\mathbb{P}(\{0\})=\emptyset$.

(1) When n is not large enough, each fiber is either empty (if $\Gamma(C,L)=0$) or a projective space $\mathbb{P}[\Gamma(C,L)]$. However, the fibers now may vary with L. For instance

(1) For $g>0$ map $C \xrightarrow{\mathcal{AJ}_1} J_1(C)$ is an embedding (so most fibers are empty).
(2) For $1\le d\le g$ the image of $C^{(d)} \xrightarrow{\mathcal{AJ}_d} J_d(C)$ has dimension d. So $\mathcal{AJ}_g$ is the first surjective map while
    The image of $C^{(g-1)}$ is a codimension one hypersurface called the theta divisor. $\Theta\subseteq J_{g-1}(C)$.

For instance in (1) it is clear that $C \xrightarrow{\mathcal{AJ}_1} J_1(C)$ is injective since the non-empty fibers are projective spaces and if $g>0$ then C does not contain any $\mathbb{P}^n$ with $n>0$. So the non-empty fibers are points. (However, for $g=0$ one has $C\cong\mathbb{P}^1$ and this is indeed a fiber of $\mathcal{AJ}_1$.)

9.9. **Class Field Theory.** The *Class Field Theory* is a central part of Number Theory. It has the *arithmetic part* (study of Spec($\mathbb{Z}$)), and the *geometric part* (study of curves over finite fields). The two areas provide completely parallel theories but the arithmetic part is much deeper and the geometric part is often used as a source of ideas.

The above theorem

When $n > 2(g-1)$ the Abel-Jacobi map

$$C^n \xrightarrow{\mathcal{AJ}_n} Pic_n(C), \qquad D \mapsto [\mathcal{O}_C(D)],$$

is a bundle of projective spaces $\mathbb{P}^{n-g}$.

is essentially the *unramified case* of the geometric Class Field Theory. The transition to the *ramified part* means that we allow curves which are not complete – a few points may be missing. Roughly, the story is the same, except that it takes longer to tell because Jacobians get larger.

What is the use of the above theorem? It gives *linearization* for certain kind of data on C. Imagine some object $\mathcal{L}$ spread over C. The first step in its linearization is the integration over finite unordered subsets of C,[90] it gives an object $\mathcal{L}^{(n)}$ spread over $C^{(n)}$.

$$\mathcal{L}^{(n)}(D) \stackrel{\text{def}}{=} \int_D \mathcal{L}.$$

Next, for sufficiently large n (i.e., $n > 2(g-1)$, one uses the theorem to descend $\mathcal{L}^{(n)}$ to an object $\mathcal{L}_n$ spread over $J_n(C)$, in the sense that

$$\mathcal{L}^{(n)} \;\cong\; (\mathcal{AJ}_n)^* \mathcal{L}_n.$$

Finally, these $\mathcal{L}_n$'s for $n > 2(g-1)$ constitute a version of $\mathcal{L}$ that lives on an abelian group $J(C)$, and $\mathcal{L}_n$'s are in some sense compatible with the group structure on $J(C)$. This compatibility allows one to to extend the construction of objects $\mathcal{L}_n$ on $J_n(C)$, from $n > 2(g-1)$ to *all* integers $n \in \mathbb{Z}$. In the light of this compatibility of the family $\mathcal{L}_\bullet$ spread over $J(C)$ with the group structure, one can view $\mathcal{L}_\bullet$ as a linearization of $\mathcal{L}$.

9.9.1. *Langlands program.* The objects $\mathcal{L}$ that one can linearize in this way (for instance one dimensional local systems, or equivalently the abelian Galois representations), are necessarily simple enough to be understood in terms of an abelian group. In 68 Langlands proposed a program to linearize more complicated objects in terms of non-abelian groups. Ever since, it has been one of the central undertakings in pure math.

9.9.2. *Completely integrable systems.* This is another example of objects (specially nice and interesting partial differential equations), that linearize on Jacobians of curves. So one can ask what is the relation to Class Field Theory?

9.10. **Cohomology of line bundles on curves.**

[90] Remember that because of $C^{(n)} \cong C^{[n]}$ we can view *finite unordered subsets with multiplicities* as *finite subschemes* of C.

9.10.1. *Line bundles and sheaf cohomology.* By definitions, the Riemann-Roch space $H(D)$ is the space of global sections

$$H(D) \ = \ \Gamma(C, \mathcal{O}_C(D)) = \mathcal{O}_C(D)\ (C)$$

of the line bundle $\mathcal{O}_C(D)$ associated to C. So, it is possible that the entire cohomology of line bundles $\mathcal{O}_C(D)$ is relevant, so let us denote by $h^i(D)$ be the dimension of the i^{th} cohomology group $H^i[C, \mathcal{O}_C(D)]$. Then $h^0(D) = h(D)$ and it will turn out that the second ingredient of the Riemann-Roch theorem is

$$h^1(D) = h(K - D).$$

So, the Riemann-Roch theorem really is about cohomology of line bundles. Moreover, the treatment of a geometric question $h(D) =?$, in terms of sheaves makes situation quite flexible since there are more sheaves then just the line bundles.

In terms of sheaf cohomology the Riemann-Roch theorem is explained as a combination of several facts, each of which has important generalizations to higher dimensional objects.

(1) $\boxed{\text{Euler characteristic.}}$ The following statement has important generalizations to higher dimensional varieties (Riemann-Roch-Hirzebruch theorem), and to schemes and maps of schemes (Grothendieck-Riemann-Roch theorem), etc. The Euler characteristic of cohomology of a line bundle is

$$\chi[H^*(X;L)] = \ \dim[H^0(X;L)] - \dim[H^1(X;L)]$$

because we know that on a curve $H^i(X;L) = 0$ for $i > \dim(C) = 1$.

Theorem. For any line bundle L on C,

$$\chi[H^*(X;L)] = \ deg(L) + 1 - g.$$

(2) $\boxed{\text{Serre duality.}}$

Theorem. For any line bundle L on C

$$H^i(C, L)^* \ \cong \ H^{1-i}(C, L^* \otimes \Omega^1_C).$$

(3) $\boxed{\text{Kodaira vanishing}}$

Theorem. (a) If $deg(L) > 2g - 2$ then $H^1(C, L) = 0$.
(b) If $deg(L) < 0$ then $H^0(C, L) = 0$.

(4) $\boxed{\text{Kodaira embedding.}}$

Theorem. If $deg(L) > 2g$ then L gives a projective embedding of C, i.e., C can viewed as a submanifold of a projective space

$$C \hookrightarrow \ \mathbb{P}[\Gamma(C, L)^*].$$

(5) $\boxed{\text{Riemann-Hurwitz.}}$ $deg(\Omega^1_C) = 2(g - 1)$.

Remarks. (1) The two claims of Kodaira vanishing are equivalent by Serre duality.

(2) Serre duality for the line bundle $L = \mathcal{O}_C(D)$ says that

$$h^1(D) = \ h^0(K - D)$$

where K is any canonical divisor (a divisor such that $\mathcal{O}_C(K) \cong \Omega^1_C$). Now, the Euler characteristic statement for $L = \mathcal{O}_C(D)$ gives the Riemann-Roch theorem, while the Kodaira vanishing and Riemann-Hurwitz give the companion theorem.

In the remainder we indicate the proofs of these claims.

9.10.2. *The Euler characteristic.* In general, the Euler characteristic of a line bundle behaves much better then the individual cohomology groups

Proposition. The value of $\chi[H^*(X;L)] - deg(L)$ is the same for all line bundles L on C.

Lemma. For a divisor D any point $a \in D$ gives short exact sequence of sheaves

$$0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D+a) \to \delta_a \to 0,$$

where δ_a is the sheaf analogue of the δ-distribution at a point:

$$\delta_a(V) = \begin{cases} 0 & \text{if } a \not\in V, \text{ and} \\ \mathbb{C} & \text{if } V \ni a. \end{cases}.$$

Proof. Point $a \in C$ gives an inclusion of sheaves $\mathcal{O}_C(D) \subseteq \mathcal{O}_C(D+a)$. The quotient sheaf is the sheafification of the presheaf $V \mapsto Q(V) = \mathcal{O}_C(D+a)(V)/\mathcal{O}_C(D)(V)$. On $U = C - a \subseteq C$, inclusion $\mathcal{O}_C(D) \subseteq \mathcal{O}_C(D+a)$ is equality, so for $V \subseteq C - a$ one has $Q(V) = 0$. On the other hand, if V is a small neighborhood of a then on V $\mathcal{O}_C(D) = \ (z - z(a))^{-ord_a(D)} \mathcal{O}_C$ and so,

$$Q(V) = \ (z-z(a))^{-ord_a(D)-1}\mathcal{O}_C(V)/(z-z(a))^{-ord_a(D)}\mathcal{O}_C(V) \ \cong \ \mathcal{O}_C(V)/(z-z(a)){\cdot}\mathcal{O}_C(V)$$

$$\xrightarrow[\cong]{f \mapsto f(a)} \ \mathbb{C}.$$

The sheafification of Q is then the sheaf δ_a.

Now, any line bundle L is isomorphic to one of $\mathcal{O}_C(D), \ D \in Div(C)$, and we check in the same way that a point $a \in C$ gives a short exact sequence of sheaves

$$0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D+a) \to \delta_a \to 0.$$

Proof of the proposition. The above short exact sequence of sheaves leads to a long exact sequence of cohomology groups

$$0 \to H^0[C,\mathcal{O}_C(D)] \to H^0[C,\mathcal{O}_C(D{+}a)] \to H^0[C,\delta_a] \to H^1[C,\mathcal{O}_C(D)] \to H^1[C,\mathcal{O}_C(D{+}a)] \to H^1[C,\delta_a] \to 0.$$

The zero at the right end appears because we know that on a curve there is only H^0 and H^1. Moreover, we know that $H^0[C,\delta_a] = \mathbb{C}$, and we believe that $H^1[C,\delta_a] = 0$ because δ_a really lives on the point a and on a point there is only H^0. This gives the following relation of Euler characteristics:

$$\chi(H^0[C,\mathcal{O}_C(D+a)]) = \ \chi[H^*[C,\mathcal{O}_C(D)]] + \chi(H^*[C,\delta_a]) = \ \chi[H^*[C,\mathcal{O}_C(D)]] + 1).$$

However, one also has

$$deg[\mathcal{O}_C(D+a)] = \ deg(D+a) = deg(D) + 1 = deg[\mathcal{O}_C(D)] + 1.$$

So, the validity of the theorem for $\mathcal{O}_C(D)$ and for $\mathcal{O}_C(D+a)$ is equivalent (and one connect any two line bundles by a sequence of such changes).

9.10.3. *Cohomology of differential forms.* Now, it suffices to check the Euler characteristic theorem for one line bundle, and we use Ω^1_C. Recall that for a complex curve C, there are two ways to think of the genus g of C (and therefor one needed an argument to check that these are the same).

Theorem. (a) $\dim[H^0(X,\Omega^1_C)] = \ \dim[\Omega^1_C(C)] = \ g.$

(b) There is a canonical isomorphism (called the *trace map*)

$$H^1(C,\Omega^1_C) \xrightarrow[\cong]{tr} \ \mathbb{C}.$$

(c) $deg(\Omega^1_C) = 2g-2$.

Proof. (a) is the holomorphic definition of the genus $g \overset{\text{def}}{=} \ \dim[\Omega^1_C(C)]$.

(c) Any meromorphic function f on C can be viewed as a *holomorphic* map $f : C \to \mathbb{P}^1$. If f is not constant then it is surjective and then this map can be used to reduce the claim from C to $\mathbb{P}^1$. However, the $\mathbb{P}^1$ case is easy since here we know the cohomology of all line bundles.

(b) We will construct the trace map using *residues*. Recall that if U is a neighborhood of a point $a \in C$ and ω is a meromorphic one form on U which is holomorphic off a that we can define the residue $Res_a(\omega)$ either[91]

- analytically as the integral $\int_\gamma \ \omega$ for a curve γ that goes once around a, or
- algebraically as the $(-1)^{\text{st}}$ coefficient of the expansion $\omega = \sum_i \ a_i{\cdot}z^i{\cdot}dz$ of ω in a local coordinate z.

[91]The second definition requires checking that it is independent of the choice of a local coordinate. This can be done either directly, or by the comparison with the first definition.

To apply this observe that Ω^1_C has a trivialization on $C-F$ for a finite subset F (if D is the divisor of any meromorphic section of Ω^1_C then $\Omega^1_C \cong \Omega_C(D)$ so it has a trivialization off the support $F = supp(D)$ of the divisor D). Also it has trivialization on small neighborhood V_a of each $a \in F$, moreover we can choose V_a to be identified with a disc with center a by a local chart, and also small enough so that all V_a's are disjoint. So, we get an open cover $\mathcal{U}$ of C by $U = C - F$ and $V = \cup_{a\in F}$. For some general reasons, one can calculate $H^*(C, \mathbb{P}1^1_C)$ as the Čech cohomology $H^*_{\mathcal{U}}(C, \mathbb{P}1^1_C)$ for this cover. But then

$$H^1(C, \Omega^1_C) \;\cong\; H^1_{\mathcal{U}}(C, \Omega^1_C) \;=\; \frac{\Omega^1_C(U\cap V)}{\rho^U_{U\cap V}\Omega^1_C(U) + \rho^V_{U\cap V}\Omega^1_C(V)}.$$

Now, $U \cap V$ is the disjoint union of punctured neighborhoods $V^*_a = V_a - \{a\}$ of points $a \in F$, so we have a map

$$\Omega^1_C(U\cap V) \xrightarrow{\sum_{a\in F}\ Res_a} \mathbb{C}.$$

This map kills the restrictions $\rho^V_{U\cap V}\Omega^1_C(V)$ of forms holomorphic on V (they are holomorphic at each $a \in F$!), and also the restrictions $\rho^U_{U\cap V}\omega$ of all forms $\omega \in \Omega^1_C(U)$, since for such ω $\sum_{a\in F}\ Res_a\omega = \ \sum_{a\in C}\ Res_a\omega$, however, for any meromorphic form η on C

$$\sum_{a\in C} Res_a\eta \;=\; 0.$$

We proved this before but only in the case when $\eta = df$ is the differential of some meromorphic function f on C (then the result was that $\sum\ ord_a f = \ sum\ Res_a(df/f) = 0$), the proof however works in general. Therefore, the map $\sum_{a\in C}\ Res_a :\ \Omega^1_C(U\cap V) \to\ \mathbb{C}$ factors to $H^1_{\mathcal{U}}(C, \Omega^1_C) \to \mathbb{C}$. This is the trace map.

9.10.4. *Euler characteristic theorem (the end of the proof).* Now we know that for all line bundles L on C, the number $\chi[H^*(X; L)] - deg(L)$ is the same, so we only need to calculate it for $L = \Omega^1_C$. Here

$$\dim[H^0(C, \Omega^1_C)] - \dim[H^1(C, \Omega^1_C)] - deg(\Omega^1_C)] = \ g - 1 - 2(g-1) = \ 1 - g.$$

9.10.5. *Kodaira vanishing.* The claim that $deg(L) < 0$ implies $\Gamma(C, L) = 0$ is quite elementary.

Choose a divisor D such that $L \cong \mathcal{O}_C(D)$. Recall that a holomorphic section $f \in \mathcal{O}_C(D)(C)$ is by definition a meromorphic function on C, and the two points of view are related by :

$$ord^{\mathcal{O}_C(D)}_a(f) \;=\; ord_a(f) + ord_a(D),\ a \in C, \quad i.e., \quad div^{\mathcal{O}_C(D)}(f) \;=\; div(f) + D.$$

This gives a contradiction:

$$deg[div^{\mathcal{O}_C(D)}(f)] = \ deg[div(f)] + deg[D] = deg(D) < 0,$$

while $div^{\mathcal{O}_C(D)}(f)$ has to be effective for holomorphic sections.

9.10.6. *Serre duality.* We need the following cohomological idea. The cohomology of line bundles L and M contributes to the cohomology of their tensor product $L{\otimes}M$ by the canonical maps

$$H^i(C,L){\otimes}H^j(C.M) \xrightarrow{m_{ij}} H^{i+j}(C.L{\otimes}M).$$

This gives a pairing

$$H^i(C,L){\otimes}H^{1-i}(C,L^*{\otimes}\Omega^1_C) \xrightarrow{m_{ij}} H^1(C,L{\otimes}L^*{\otimes}\Omega^1_C) \xrightarrow{\cong} H^1(C,\Omega^1_C) \xrightarrow{tr} \mathbb{C},$$

which one checks is non-degenerate so it induces $H^{1-i}(C,L^*{\otimes}\Omega^1_C)\xrightarrow{\cong} H^i(C,L)^*$.

9.10.7. *Kodaira embedding.* Let us see the meaning of the embedding claim. We will examine how a line bundle L on a compact manifold C gives a natural embedding of C into a projective space, *provided* that L has *sufficiently many sections.*

Any point $c\in C$ gives the evaluation map $e_c:\Gamma(C,L)\to L_c$.

(1) If there is a section s which does not vanish at c then e_c is surjective and $\mathrm{Ker}[e_c]$ is a hyperplane in $H_c\subseteq\Gamma(C,L)$, so it corresponds to a line $\iota(c)\overset{\text{def}}{=} H_c^{\perp}\ \in\mathbb{P}[\Gamma(C,L)^*]$.
(2) If for each point $c\in C$ there is a section that does not vanish at c, then ι is a map $\iota:C\to\mathbb{P}[\Gamma(C,L)^*]$.
(3) If for any pair of different points $(a,b)\in C^2$ there is a section s such that $s(a)\neq 0=s(b)$, then ι is injective.
(4) If for each point $c\in C$ and any tangent vector $v\in T_c(C)$ there is a section s such that $s(c)=0$ and $d_cs\in T_{\iota(c)}(C)$ is not zero, then ι is an embedding of manifolds, i.e., it makes C into a submanifold of the projective space $\mathbb{P}[\Gamma(C,L)^*]$.

Remark. One can state (3) as

sections of L distinguish the points of C.

One can view (4) in the same way :

sections of L distinguish infinitesimally close points of C.

since (in the scheme theoretic language) one can view the tangent vector v as giving the second point $a+v$ which is infinitesimally closed point to a. Now we can summarize the discussion into

Lemma. If L has enough sections (in the sense that they distinguish points and infinitesimally close points of C), then C embeds as a submanifold of $\mathbb{P}[\Gamma(C,L)^*]$.

Remarks. (1) So, the embedding theorem claims that if the degree of L is sufficiently large (precisely, $deg(L) > deg(\Omega^1_C) = 2(g-1)$), then L has enough sections to give a projective embedding of C.

(2) Actually, more is true. Once we embed a compact complex manifold X into some projective space $\mathbb{P}^n$, one can prove that inside $\mathbb{P}^n$, X is described by homogeneous polynomial equations, so the embedding gives a structure of a projective algebraic variety, Therefore, any compact complex manifold that has a line bundle with enough sections has a structure of a projective variety!

Proof. Let L be a line bundle on C of degree $n > 2g$.

(1) <u>If M is a line bundle of degree $\geq 2g$, all evaluation maps are $\neq 0$.</u> At each point $a \in C$ there is the evaluation map $\Gamma(C, M) \xrightarrow{e_a} M_a$ with values in the fiber M_a, $e_a(s) \stackrel{\text{def}}{=} s(a) \in M_a$. Choose a presentation of M as $\mathcal{O}_C(D)$ The short exact sequence of sheaves

$$0 \to \mathcal{O}_C(D-a) \to \mathcal{O}_C(D) \to \delta_a \to 0,$$

gives a long exact sequence of cohomology groups

$$0 \to H^0[C, \mathcal{O}_C(D-a)] \to H^0[C, \mathcal{O}_C(D)] \to H^0[C, \delta_a] \to H^1[C, \mathcal{O}_C(D-a)] \to \cdots.$$

If $deg[\mathcal{O}_C(D-a)] > 2g-2$, i.e., $deg[\mathcal{O}_C(D)] \geq 2g$, then $H^1[C, \mathcal{O}_C(D-a)] = 0$. So, the map $H^0[C, \mathcal{O}_C(D)] \to H^0[C, \delta_a] = \mathbb{C}$ is surjective. But this is exactly the evaluation map.

Also notice (for later) that $H^0[C, \mathcal{O}_C(D-a)] \to H^0[C, \mathcal{O}_C(D)]$ is the inclusion of all sections of $\mathcal{O}_C)D)$ that vanish at a.

(2) <u>Sections distinguish points.</u> Let a, b be two different points of C. If $L \cong \mathcal{O}_C(D)$ then the degree of $\mathcal{O}_C(D-b)$ is $deg(L) - 1 \geq 2g$, so the evaluation e_a is nonzero on sections of $\mathcal{O}_C(D-b)$. However, these are precisely the sections of $\mathcal{O}_C(D)$ that vanish at b. So, $\cong \mathcal{O}_C(D)$ has a holomorphic section which vanishes at b but not at a.

(3) <u>Sections distinguish infinitesimally close points.</u> Let us think of the evaluation $e_a(s) \in L_a$, as the restriction of a section s to a point, i.e., the restriction of a section s of a line bundle L on C to a section $s(a)$ of a line bundle L_a on the point a. This is the 0^{th} order information on s at a. To get the first order information we consider the double point subscheme $a_2 = \text{Spec}(\mathcal{O}_C/\mathcal{I}_a^2)$, the restriction $L|a_2$ of L to a_2, and the restriction map $\Gamma(C, L) \xrightarrow{\rho} \Gamma(a_2, L|a_2)$. The claim that *sections of L distinguish infinitesimally close points*, means literally that the map ρ is surjective.

To check this, we again put it into a sheaf framework. This is given by a slight generalization of the above sublemma:

Sublemma. For a divisor D any point $a \in D$ gives short exact sequence of sheaves

$$0 \to \mathcal{O}_C(D-2a) \to \mathcal{O}_C(D) \to \delta_{a_2} \to 0,$$

where δ_{a_2} is the sheaf[92]

$$\delta_{a_2}(V) = \begin{cases} 0 & \text{if } a \notin V \text{, and} \\ \mathcal{O}(a_2) & \text{if } V \ni a. \end{cases}.$$

Proof. Point $a \in C$ gives an inclusion of sheaves $\mathcal{O}_C(D-2a) \subseteq \mathcal{O}_C(D)$ and the quotient sheaf is the sheafification of the presheaf $V \mapsto Q(V) = \mathcal{O}_C(D)(V)/\mathcal{O}_C(D-2a)(V)$. On $C-a \subseteq C$, inclusion $\mathcal{O}_C(D-2a) \subseteq \mathcal{O}_C(D)$ is equality, so for $V \subseteq C-a$ one has $Q(V) = 0$. On the other hand, if V is a small neighborhood of a then on V $\mathcal{O}_C(D) = (z-z(a))^{-ord_a(D)}\mathcal{O}_C$ and so,

$$Q(V) = (z-z(a))^{-ord_a(D)}\mathcal{O}_C(V)/(z-z(a))^{2-ord_a(D)}\mathcal{O}_C(V) \cong \mathcal{O}_C(V)/(z-z(a))^2 \cdot \mathcal{O}_C(V).$$

The sheafification of Q is then the sheaf δ_{a_2}.

End of the proof of the embedding theorem. The sublemma puts the restriction map ρ into an exact sequence

$$0 \to H^0[C, \mathcal{O}_C(D-2a)] \to H^0[C, \mathcal{O}_C(D)] \xrightarrow{\rho} H^0[C, \delta_{a_2}] \to H^1[C, \mathcal{O}_C(D-2a)] \to \cdots.$$

If $deg[\mathcal{O}_C(D)] > 2g$, then $deg[\mathcal{O}_C(D-2a)] > 2g-2$, hence $H^1[C, \mathcal{O}_C(D-2a)] = 0$. So, ρ is surjective.

10. **Homeworks**

The grade will be based mostly on homeworks, possibly some extra project.

Do as much as you can. At this level one does not require perfection but an honest effort that should help you *learn*. The presentation should be readable, but the level of detail should be sufficient to explain the situation to yourself.

It is acceptable to state that you understand a problem on the level that it is a waste of time for you to write it down, but be sure you do not cheat yourself.

Difficulties: Office hours, we can also have homework help sessions when desired.

It is a very good idea to discuss the problems that resist reasonable effort with other students, however it is *critical* that you write solutions by yourself.

Looking into books I have or have not mentioned may help.

If a problem require expertise in something you did not learn yet (say, manifolds), it is *acceptable* to state so, and not do the problem. It is *preferable* to do the problem anyway by consulting other students or me.

[92]The functions on a_2 are described in terms of a coordinate function z on a small neighborhood W of a, by $\mathcal{O}(a_2) \cong \mathcal{O}(W)/(z-z-z(a))^2\mathcal{O}(W)$.

Homework 1

$\heartsuit$[93]

1. Explain how the following look like (i.e., how do the obvious pieces fit together into a geometric shape):

(1) $\mathbb{P}^1_{\mathbb{R}}$,
(2) $\mathbb{P}^2_{\mathbb{R}}$,
(3) $\mathbb{P}^1(\mathbb{C})$.

$\heartsuit$

The blow up. The blow up of the vector space V is the set

$$\widetilde{V} \stackrel{\text{def}}{=} \{(L,v) \in \mathbb{P}(V){\times}V;\ v \in L\} \subseteq \mathbb{P}(V){\times}V.$$

Since it is a subset of $\mathbb{P}(V){\times}V$ it comes with the projection maps

$$\mathbb{P}(V) \xleftarrow{\pi} \widetilde{V} \xrightarrow{\mu} V.$$

2. Explain the relation of $\widetilde{V}$ to $\mathbb{P}(V)$ and V, i.e.,

(1) Show that the fibers $\pi^{-1}(L),\ L \in \mathbb{P}(V)$ of $\pi : \widetilde{V} \longrightarrow \mathbb{P}(V)$ are naturally vector spaces. (We say that $\widetilde{V}$ is a *vector bundle* over $\mathbb{P}(V)$.
(2) Describe the fibers $\mu^{-1}(v),\ v \in V$, of $\widetilde{V} \xrightarrow{\mu} V$.

3. Explain how does the blow up $\widetilde{\mathbb{R}^2}$ look like. Can you relate it to the Moebius strip?

$\heartsuit$

Finite Fields. For each prime number, ring $\mathbb{F}_p \stackrel{\text{def}}{=} \mathbb{Z}/p\mathbb{Z}$ is a field with p elements.[94] The advantage of a finite field $\Bbbk$ is that we can do polynomials, affine and projective spaces, affine and projective varieties, but now there is something to count since everything is finite. So we have a new way of measuring the size of algebraic varieties. For instance if we can ask ourselves how many elements there are in the circle $C(\Bbbk) =\ \{(x,y) \in \Bbbk^2;\ x^2+y^2=1\}$ over a finite field $\Bbbk$ with q elements ?

If we go a little further, and consider not one finite field $\mathbb{F}$ but *all of them*, we arrive at the question

How does the number of elements $X(\mathbb{F})$ depend on $q = |F|$?

It turns out that for many nice projective varieties the number $|X(\Bbbk)|$ is a polynomial in q. In general, we encode these numbers into the Zeta function[95] of X,

$$Z_X(T) \stackrel{\text{def}}{=} e^{\sum_1^\infty\ |X(\mathbb{F}_{q^n})|\cdot\frac{T^n}{n}}.$$

[93] Due Thursday Feb 12

[94] Actually, if $q = p^e$ is a power of a prime, then there exists precisely one field with q elements, we denote it $\mathbb{F}_q$. Moreover, these are all finite fields. However, this is not going to be important.

[95] This is a John Cullinan correction of a previous version.

This function turns out to be a a rational function of the variable T which contains deep information about X.[96] Because of this, the passage from $\mathbb{C}$ to a finite field is often the strongest known method for studying algebraic varieties over $\mathbb{C}$!

Grassmannians. For a vector space V over a field $\Bbbk$ we denote by $Gr_p(V)$ the set of all p-dimensional vector subspaces of V. For instance, $Gr_1(V)$ is the projective space $\mathbb{P}(V)$. Since $Gr_p(V)$ really depends only on the dimension of V, we often look at the standard examples

$$Gr_p(n) \overset{\text{def}}{=} Gr_p(\Bbbk^n).$$

4. Consider a finite field $\mathbb{F}$ with q elements.

(1) Find the number of elements in the affine space $\mathbb{A}^n(\mathbb{F})$.
(2) Find the number of elements $N_1(q)$ in $\mathbb{P}^{n-1}(\mathbb{F}) = Gr_1(\mathbb{F}^n)$, i.e., the number of one-dimensional subspaces in $\mathbb{F}^n$.
(3) Find the number of elements $N_2(q)$ in $Gr_2(\mathbb{F}^n)$, i.e., the number of two-dimensional subspaces in $\mathbb{F}^n$.
(4) Find the limits $s_i = \lim_{q\to 1} \ N_i(q)$ for $i = 1, 2$. What does s_i count?

♡

Hints.

1.1. Since we want to understand these on the topological level, we consider the building blocks of these spaces and how do they glue.

1.3. From 1.2 we have two points of view on $\widetilde{\mathbb{R}^2}$: (1) a circle with a bunch of lines, each passing through one point of the circle, (2) replace a point in $\mathbb{R}^2$ by a circle.

1.4. Let $Fr_p(V)$ be the set of all p-tuples $(v_1, ..., v_p)$ of independent vectors in V. Then

$$|Gr_p(\mathbb{F}^n)| = \frac{|Fr_p(\mathbb{F}^n)|}{|Fr_p(\mathbb{F}^p)|}.$$

Numbers $|Fr_p(\mathbb{F}^n)|$ are easy to calculate for $p = 1, 2$.

[96]As conjectured by Weil and proved by Weil, Dwork and (the deepest part) by Deligne.

Homework 2

$\heartsuit$[97]

Ideals. 1. (a) Give an example of a map of rings $\phi : B \to A$ and a maximal ideal $P \subseteq A$ such that $\phi^{-1}P$ is not maximal.

(b) Show that if ϕ is surjective then ϕ^{-1} of a maximal ideal is maximal.

2. (a) In the ring A find two different ideals I and J such that $V_I = V_J$, when

(1) $A = \mathbb{k}[x]$ for $\mathbb{k}$ a field,
(2) $A = \mathbb{Z}$.

(b) The radical of an ideal $I \subseteq A$ is the set

$$\sqrt{I} = \{a \in A;\ a^n \in I \text{ for some } n > 0\}.$$

Show that $\sqrt{I}$ is an ideal and $V_I = V_{\sqrt{I}}$.

$\heartsuit$

Complex curves. 3. Let $X \subseteq \mathbb{A}^2(\mathbb{C})$ be the affine curve

$$y^2 = (x - a_1) \cdots (x - a_{2n})$$

with the numbers a_i distinct. Let $\overline{X} \subseteq \mathbb{P}^2$ be the corresponding projective curve.

(a) What is the boundary $\partial X = \overline{X} - X$ of X in $\overline{X}$?

(b) Draw X with all the necessary explanations.

$\heartsuit$

Zariski topology. Consider a projective space $\mathbb{P}^n$ over a closed field $\mathbb{k}$. Zariski topology on $\mathbb{P}^n$ is defined so that the closed subsets are just the projective subvarieties. We will construct an open cover of $\mathbb{P}^n$ by $n+1$ affine spaces:

4. Let $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n;\ x_i \neq 0$, and let

$$\phi_i :\ \mathbb{A}^n \to U_i,\ \phi_i(a_1, ..., a_n) = [a_1 : \cdots : a_i : 1 : a_{i+1}; \cdots : a_n].$$

Show that

(1) $U_0, ..., U_n$ is an open cover of $\mathbb{P}^n$.
(2) ϕ_i is a bijection.

[97] Due Thursday Feb 26

Homework 3

♡[98]

Projective spaces as manifolds. 1. Prove that

(a) $\mathbb{P}^n(\mathbb{R})$ has a canonical structure of a manifold of dimension n.[99]

(b) $\mathbb{P}^n(\mathbb{C})$ has a canonical structure of a complex manifold of dimension n

(c) Manifolds $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$ are compact.

♡

Theta functions. Let $\tau \in \mathbb{H}$ (i.e., $Im(\tau) > 0$. It gives a lattice $L_\tau = \mathbb{Z}\oplus\mathbb{Z}\cdot\tau$ in $\mathbb{C}$, and an elliptic curve $E_\tau = \mathbb{C}/L_\tau$. It comes with the quotient map $\pi : \mathbb{C} \to E_\tau$.

We would like to find some holomorphic functions on E_τ, and this is the same as a holomorphic function f on $\mathbb{C}$ which is periodic in directions of 1 and τ: $f(z+1) = f(z) = f(z+\tau)$. However, there are no such functions, so we ask for the next best thing: periodic for 1 and *quasiperiodic.*

2. The theta series in $\tau \in \mathbb{H}$ and $u \in \mathbb{C}$ is

$$\theta_\tau(u) \stackrel{\text{def}}{=} \sum_{-\infty}^{+\infty} e^{\pi i(n^2\tau+2nu)}.$$

(a) Show that it defines for any $\tau \in \mathbb{H}$ an entire function of u.

(b) Show that for any $u \in \mathbb{C}$ it defines a holomorphic function on $\mathbb{H}$.

(c) Show that for any $a \in \mathbb{R},\ b > 0$, the series converges uniformly on the product

$$\{\tau \in \mathbb{H};\ Im(\tau) > b\} \times \{u \in \mathbb{C};\ Im(u) > a\}.$$

(d) Show that the series can be differentiated any number of times (with respect to τ and u), and the derivatives are calculated term by term.

[98] Due Thursday March 4

[99] One has to (1) define topology on it, (1) define charts, (3) check that the charts are compatible, i.e., the transition functions are smooth (i.e., infinitely differentiable).

♡

Zariski topology on affine varieties. 3. Consider the Zariski topology on the affine line $\mathbb{A}^1$ over a closed field $\Bbbk$.

(a) What are the open sets of $\mathbb{A}^1$?

(b) Show that any two non-empty open sets in $\mathbb{A}^1$ meet.[100]

(c) Show that $\mathbb{A}^1$ is *quasi-compact*, i.e., any open cover reduces to a finite subcover.[101]

♡

Counting. Let X be a finite set.

- Let $Gr_k(X)$ be the set of all k-element subsets of X.
- For $J = \{j_1 < j_2 \cdots j_k\}$, let $Gr_J(X)$ be the set of partial flags of type J in X, i.e., the set of k-tuples of increasing subsets of correct size

$$Gr_J(X) \stackrel{\text{def}}{=} \{(A_1, ..., A_k) \in \mathcal{P}_{j_1} \times \cdots \mathcal{P}_{j_k},\ A_1 \subseteq \cdots \subseteq A_k\}.$$

Let $n = |X|$ be the number of elements of X. When $J = \{1, ..., n\}$ is the largest possible, we call $Gr_J(X)$ the set of flags $\mathcal{F}(X)$ in X.

4. (a) Find $|Gr_k(X)|$.

(b) For $K \subseteq J$ there is a canonical projection

$$\pi = \pi_{K \subseteq J} : Gr_J \to Gr_K, \qquad (A_j)_{j \in J} \mapsto (A_j)_{j \in K}.$$

Show that it is surjective.

(c) If $J = \{a = j_1 < b = j_2 < \cdots < j_k\} \subseteq \{1, ..., n-1\}$, and $K = J - \{a\}$, calculate the number of elements in any fiber of π.

(d) Find $|Gr_J(X)|$ for $J = \{j_1 < j_2 < \cdots < j_k\}$.

(d) Consider the case when $J = \{1, ..., n\}$ is the largest possible.

(1) Find a canonical identification of $\mathcal{F}(X)$ with the set of all total orders on X.
(2) Show that $\mathcal{F}(X)$ is a torsor for the group S_X of permutations of X.[102]
(3) Use this to (re)calculate $|S_X|$.

---

[100] Such topological spaces are called *irreducible*.
[101] In this terminology "compact" means quasi-compact and Hausdorff.
[102] This means that S_X acts simply transitively on $\mathcal{F}(X)$ for the group $|S_X|$.

Homework 4

♡[103]

Symmetric powers. The n^{th} symmetric power of an affine scheme X over a closed field $\Bbbk$,[104] is defined as the affine variety $X^{(n)}$ such that

$$\mathcal{O}(X^{(n)}) \stackrel{\text{def}}{=} \mathcal{O}(X^n)^{S_n}.$$

This definition is a bit abstract so we want to understand $X^{(n)}$ as a set, and in particular we would like to compare the set $X^{(n)}$ with the set $X^n/S_n \stackrel{\text{def}}{=}$ all S_n-orbits in X^n. Here, we will see that these are the same when $X = \mathbb{A}^1$.[105]

1. Let $X = \mathbb{A}^1$. Show that

(1) To any S_n-orbit α in X^n one can associate a polynomial

$$\chi_\alpha(\lambda) \stackrel{def}{=} \prod_i (\lambda - a_i), \textit{ where } a = (a_1, ..., a_n) \textit{ is } \text{any } \textit{element of } \alpha.$$

(2) Denote the coefficients of this polynomial by

$$\chi_\alpha(\lambda) \stackrel{\text{def}}{=} \lambda^n - e_1(\alpha)\lambda^{n-1} + e_2(\alpha)\lambda^{n-2} - \cdots + (-1)^n e_n(\alpha)\lambda^0.$$

So, e_i's are functions on the set of orbits X^n/S_n, and therefore in particular on X^n. Show that e_i's are polynomials on $X^n = \mathbb{A}^n$.

(3) Show that $e = (e_1, ..., e_n) : X^n/S_n \to \mathbb{A}^n$ is a bijection.

(4) Prove that there is an isomorphism of algebraic varieties $X^{(n)} \cong \mathbb{A}^n$ and that as a set, $X^{(n)}$ consists of S_n-orbits in X^n.

♡

Theta functions. 2. (a) $\theta_\tau(u+1) = \theta_\tau(u)$.

(b) $\theta_\tau(u+\tau) = e^{-\pi i(\tau+2u)} \cdot \theta_\tau(u)$.

(c) $\theta_\tau(-u) = \theta_\tau(u)$.

3. (a) θ_τ has precisely one zero in in the closed parallelogram $\overline{\mathcal{P}}_\tau$ generated by vectors $1, \tau$ in the real vector space $\mathbb{C}$:

$$\mathcal{P}_\tau \stackrel{\text{def}}{=} \{a + b\tau;\ 0 < a, b < 1\}.$$

(b) This zero is at $u_0 \stackrel{\text{def}}{=} \frac{\tau+1}{2}$.

♡

[103] Due Thursday March 11

[104] For simplicity we will assume that the characteristic n of $\Bbbk$ is zero.

[105] The trouble is that we know examples of Invariant Theory quotients do not work well in the sense that, as a set, $Y//G$ is not the set of orbits Y/G. For instance, $G_m = \Bbbk^*$ has two orbits on $\mathbb{A}^1$ but $\mathbb{A}^1//G_m$ is just a point.

Linear counting ("Quantum computing"). We will see that counting subspaces gives an interesting deformation of the standard combinatorics which counts subsets.

Let $\mathbb{F}$ be a finite field with q elements. For the vector space $V = \mathbb{F}^n$, we will consider the sets $Fr_k(V)$ of k-tuples of independent vectors $v = (v_1, ..., v_k)$ in V (the set of "k-frames" in V), and the related set $Gr_k(V)$ of all k-dimensional subspaces of V (this is the k^{th} *Grassmannian variety of* V).

The n^{th} q-integer is the polynomial

$$[n] \stackrel{\text{def}}{=} \frac{\boldsymbol{q}^n - 1}{\boldsymbol{q} - 1} \;=\; 1 + \boldsymbol{q} + \cdots + \boldsymbol{q}^{n-1} \;\in\; \mathbb{Z}[\boldsymbol{q}], \qquad n \in \mathbb{Z}.$$

Similarly one defines quantum factorials and quantum binomial coefficients

$$[n]! \stackrel{\text{def}}{=} [1] \cdots [n] \quad \text{and} \quad \begin{bmatrix} n \\ m \end{bmatrix} \stackrel{\text{def}}{=} \frac{[n]!}{[m]! \cdot [n-m]!}.$$

When we evaluate these polynomials at $\boldsymbol{q} = q$ we put index q (but we often forget to write it). For instance we know that

$$[n]_q \stackrel{\text{def}}{=} [n]|_{\boldsymbol{q}=q}$$

is the number of elements in $\mathbb{P}(\mathbb{F}^n)$.[106]

4. (a) Find $|Fr_k(\mathbb{F}^n)|$.

(b) Show that $|GL_n(\mathbb{F})| = |Fr_n(\mathbb{F}^n)| = \; q^{\binom{n}{2}} \cdot (q-1)^n \cdot [n]!$.

(c) Find $|Gr_k(\mathbb{F}^n)|$, i.e., the number of k-dimensional subspaces in $\mathbb{F}^n$.

(d) Find the limit

$$\lim_{q \to 1} \; |Gr_k(n, \mathbb{F})|$$

(e) What does this limit count? Complete the following intuitive claim, i.e., which *notion* do you think is the limit of the *notion* of a k-dimensional subspace of $\mathbb{F}^n$ as the number of elements of the field $\mathbb{F}$ approaches 1 :

$$\lim_{q \to 1} \; k\text{-dimensional subspaces of } \mathbb{F}^n \;\; = \;\; \ldots\ldots..$$

[106] and this is our motivation for interest in q-integers.

Homework 5

♡[107]

Elliptic functions. Recall that each $\tau \in \mathbb{H}$ defines the function $\theta_\tau(u)$ on $\mathbb{C}$. The Weierstrass $\mathfrak{p}$-function is a meromorphic function on $\mathbb{C}$ which we will define as the second logarithmic derivative of the theta function

$$\mathfrak{p}_\tau(u) \stackrel{\text{def}}{=} (log(\theta_\tau(u))''.$$

Subgroup $L_\tau = \mathbb{Z}\oplus\tau{\cdot}\mathbb{Z}\subseteq\mathbb{C}$ is a lattice in $\mathbb{C}$.

1. (a) Explain why $\mathfrak{p}_\tau(u) \stackrel{\text{def}}{=} (log(\theta_\tau(u))''$ is a well defined holomorphic function on

$$\mathbb{C}\setminus\, (\tfrac{1+\tau}{2} + L_\tau),\ \ \textit{i.e., off the } L_\tau\textit{-translates of the point } \tfrac{1+\tau}{2}.$$

(b) Show that $\mathfrak{p}_\tau$ is L_τ invariant, i.e., $\mathfrak{p}_\tau(z+1) = \mathfrak{p}_\tau(z) = \mathfrak{p}_\tau(z+\tau)$.

(c) Show that $\mathfrak{p}_\tau$ has a pole of order two at $\frac{1+\tau}{2}$.

(d) Show that $\mathfrak{p}_\tau$ is meromorphic on $\mathbb{C}$.

♡

Tensoring of commutative algebras and fibered products of varieties. We consider commutative algebras over a closed field $\Bbbk$.

2. Let $B = \Bbbk[u,v] \xrightarrow{\phi} A = \Bbbk[x,y]$ by $u{\mapsto}x,\ v{\mapsto}xy$. Each $p = (a,b) \in \mathbb{A}^2$ defines $B = \Bbbk[u,v] \xrightarrow{\varepsilon} \Bbbk$ by $u{\mapsto}a,\ v{\mapsto}b$. These two maps can be used to think of A and $\Bbbk$ as B-algebras, and then $A\otimes_B\Bbbk$ is again an algebra over B (hence in particular over $\Bbbk$).

(a) Show that for each p the $\Bbbk$-algebra $A\otimes_B\Bbbk$ is isomorphic to one of the following algebras $\Bbbk[z]$ or $\Bbbk$ or 0.

(b) What is the geometric meaning of this?

[107]Due Thursday March 25

Homework 6

♡[108]

Sheaves. Sheaves are a machinery which addresses an essential problem – the relation between local and global information – so they appear throughout mathematics.

A. Example of a sheaf: smooth functions on $\mathbb{R}$. Let X be $\mathbb{R}$ or any smooth manifold. The notion of smooth functions on X gives the following data:

- for each open $U\subseteq X$ an algebra $C^\infty(U)$ (the smooth functions on U),
- for each inclusion of open subsets $V\subseteq U\subseteq X$ a map of algebras $C^\infty(U) \xrightarrow{\rho_V^U} C^\infty(V)$ (the restriction map);

and these data have the following properties

(1) (*transitivity of restriction*) $\rho_V^U\circ\rho_V^U = \rho_W^U$ for $W\subseteq V\subseteq U$,
(2) (*gluing*) if the functions $f_i \in C^\infty(U_i)$ on open subsets $U_i\subseteq X$, $i \in I$, are compatible in the sense that $f_i = f_j$ on the intersections $U_{ij} \stackrel{\text{def}}{=} U_i \cap U_j$, then they glue into a unique smooth function f on $U = \cup_{i\in I}\, U_i$.

The context of dealing with objects which can be restricted and glued compatible pieces is formalized in the notion of sheaves. The definition is formal (precise) way of saying that a given class $\mathcal{C}$ of objects forms a sheaf if it is *defined by local conditions*, i.e., conditions which can be checked in a neighborhood of each point:

B. Definition of sheaves on a topological space. A sheaf of sets $\mathcal{S}$ on a topological space $(X, \mathcal{T})$ consists of the following data:

- for each open $U\subseteq X$ a set $\mathcal{S}(U)$,
- for each inclusion of open subsets $V\subseteq U\subseteq X$ a map $\mathcal{S}(U) \xrightarrow{\rho_V^U} \mathcal{S}(V)$ (called the restriction map);

and these data are required to satisfy

(1) (*identity*) $\rho_U^U = id_{\mathcal{S}(U)}$.
(2) (*transitivity of restriction*) $\rho_W^V\circ\rho_V^U = \rho_W^U$ for $W\subseteq V\subseteq U$,
(3) $\mathcal{S}(\emptyset) = \emptyset$.
(4) (*gluing*) Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of an open $U\subseteq X$. For a family of elements $f_i \in \mathcal{S}(U_i)$, $i \in I$, compatible in the sense that $\rho_{U_{ij}}^{U_i} f_i = \rho_{U_{ij}}^{U_i} f_j$ in $\mathcal{S}(U_{ij})$ for $i, j \in I$; there is a unique $f \in \mathcal{S}(U)$ such that on the intersections $\rho_{U_{ij}}^{U_i} f = f_i$ in $\mathcal{S}(U_i)$, $i \in I$.

We can equally define sheaves of abelian groups, rings, modules, etc – only the least interesting requirement has to be modified, say in abelian groups we would ask that $\mathcal{S}(\emptyset)$ is the trivial group $\{0\}$.

[108]Due Thursday April 8

1.+2. Examples of sheaves. Which of the following constructions are sheaves? (In all examples bellow we are dealing with functions of some kind and the the restriction operation ρ_V^U is always taken to be the restriction of functions.)

(1) On a topological space X, $C_X(U) \overset{\text{def}}{=}$ continuous functions from U to $\mathbb{R}$.
(2) If X is a smooth manifold, $C_X^\infty(U) \overset{\text{def}}{=}$ smooth functions from U to $\mathbb{R}$.
(3) On a complex manifold, $\mathcal{O}_X^{an} \overset{\text{def}}{=}$ holomorphic functions from U to $\mathbb{C}$.
(4) Let X be a topological space and S a set. Let $S^X(U) \overset{\text{def}}{=}$ the set of constant functions from U to S.
(5) Let X be a topological space and S a set. Let $S_X(U) \overset{\text{def}}{=}$ the set of *locally* constant functions from U to S.
(6) Let $X = \mathbb{R}$ and $C_c(U) \overset{\text{def}}{=}$ continuous functions f from U to $\mathbb{R}$ such that the support is compact. (The support $supp(f)$ can be defined as $U - V$ for the largest open subset $V \subseteq U$ such that $f|V = 0$, or as the closure in U of $\{x \in U;\ f(x) \neq 0\}$).
(7) Let $Y \xrightarrow{\pi} X$ be a continuous map between two topological spaces. For U open in X let $\mathcal{Y}(U)$ be the set of all *continuous sections of the map* π *over* U, i.e., of all continuous maps $s : U \to Y$ such that $\pi \circ s = id_U$.[109]

Global sections functor $\Gamma : \mathcal{S}heaves(X) \to \mathcal{S}ets$**.** Elements of $\mathcal{S}(X)$ are called the sections of a sheaf $\mathcal{S}$ on $U \subseteq X$ [110]. By $\Gamma(X, \mathcal{S})$ we denote the set $\mathcal{S}(X)$ of global sections.[111]

The construction $\mathcal{S} \mapsto \Gamma(\mathcal{S})$ means that we are looking at global objects of a given class $\mathcal{S}$ (for some class of objects $\mathcal{S}$ which defined by local conditions).[112]

For instance, on any smooth manifold X, $\Gamma(C^\infty) = C^\infty(X)$ is huge while on a compact complex manifold M we will see that $\Gamma(M, \mathcal{O}_M^{an}) = \mathbb{C}$, so the holomorphic situation is more subtle.

3. Global functions on $\mathbb{P}^1(\mathbb{C})$**.** Show that $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{an}) = \mathbb{C}$, i.e., all global holomorphic functions are constant.

4. Line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(-1)$ **on** $\mathbb{P}^1$**.** Let $V = \mathbb{C}^2$. Recall that the blow up $\widetilde{V}$ of V lies in the product $\mathbb{P}^1 \times V$, so it comes with the maps $\mathbb{P}^1 \xleftarrow{\pi} \widetilde{V} \xrightarrow{\mu} V$.

(1) Describe natural structures of a complex manifold on $\mathbb{P}^1$ and $\widetilde{V}$.
(2) Show that map π is holomorphic.
(3) Show that if one associates to each open $U \subseteq \mathbb{P}^1$ the set $\mathcal{L}(U)$ of holomorphic sections of π over U, then $\mathcal{L}$ is a sheaf on $\mathbb{P}^1$.[113]

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[109]This is the same as asking that for any $a \in U$, the value $s(a)$ is in the fiber $Y_a \overset{\text{def}}{=} \pi^{-1}(a) \subseteq Y$.

[110]this terminology is from classical geometry

[111]The point of the new notation is psychological: we view $\mathcal{S}$ as a variable in the construction $\Gamma(X, -)$ which assigns something to each sheaf on X.

[112]We will see later that this idea has a hidden part, the cohomology $\mathcal{S} \mapsto H^\bullet(X, \mathcal{S})$ of sheaves on X.

[113]When one views the blow up $\widetilde{V}$ as a space over $\mathbb{P}(V)$, it is called *the tautological line bundle on a projective space* $\mathbb{P}(V)$, or *the tautological line subbundle on a projective space.* The specification *subbundle* is to remind us that $\widetilde{V} \subseteq \mathbb{P}(V) \times V$ is a vector subbundle of the trivial vector bundle $\mathbb{P}(V) \times V$ on $\mathbb{P}(V)$. The sheaf $\mathcal{L}$ of sections of the tautological line subbundle is often denoted $\mathcal{O}_{\mathbb{P}(V)}(-1)$ on $\mathbb{P}(V)$

Homework 7

$\heartsuit$[114]

Cohomology of a sheaf $\mathcal{A}$ with respect to an open cover $\mathcal{U}$ (Čech cohomology). Cohomology of sheaves is a machinery which deals with the subtle ("hidden") part of the the relation between local and global information. The Čech cohomology is the simplest calculational tool for sheaf cohomology.

Calculation of global section via an open cover. The first idea is to find all global sections of a sheaf by examining how one can glue local sections into global sections. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of a topological space X, we will choose a complete ordering on I[115] We will use finite intersections $U_{i_0,\dots,i_p} \overset{\text{def}}{=} U_{i_0} \cap \cdots \cap U_{i_p}$ with $i_0 < \cdots < i_p$.

To a sheaf of abelian groups $\mathcal{A}$ on X we associate a map of abelian groups (e,

- $C^0(\mathcal{U},\mathcal{A}) \overset{\text{def}}{=} \prod_{i\in I} \ \mathcal{A}(U_i)$, its elements are systems $f = (f_i)_{i\in I}$, with one section $f_i \in \mathcal{A}(U_i)$ for each open set U_i,
- $C^1(\mathcal{U},\mathcal{A}) \overset{\text{def}}{=} \prod_{\{(i,j)\in I^2;\ i<j\}} \ \mathcal{A}(U_{ij})$, its elements are systems $g = (g_{ij})_{(i,j)\in I^2}$ of sections $g_{ij} \in \mathcal{A}(U_{ij})$ on all intersections U_{ij}.
- map sends $f = (f_i)_{i\in I} \in C^0$ to $df \in C^1$ with

$$(df)_{ij} \overset{\text{def}}{=} \rho^{U_j}_{U_{ij}} f_j - \rho^{U_i}_{U_{ij}} f_i.$$

Less formally, we usually state it as $(df)_{ij} = \ f_j|U_{ij} - f_i|U_{ij}$.

1. Show that for any sheaf of abelian groups $\mathcal{A}$ on X

$$\Gamma(\mathcal{A}) \xrightarrow{\cong} \mathrm{Ker}[C^0(\mathcal{U},\mathcal{A}) \xrightarrow{d} C^1(\mathcal{U},\mathcal{A})].$$

$\heartsuit$

Čech complex $C^\bullet(\mathcal{U},\mathcal{A})$. Emboldened, we try more of the same. We want to capture more of the relation between local sections by extending the construction into a sequence of maps of abelian groups

$$C^0(\mathcal{U},\mathcal{A}) \xrightarrow{d^0} C^1(\mathcal{U},\mathcal{A}) \xrightarrow{d^1} C^2(\mathcal{U},\mathcal{A}) \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} C^n(\mathcal{U},\mathcal{A}) \xrightarrow{d^n} C^{n+1}(\mathcal{U},\mathcal{A}) \xrightarrow{d^{n+1}} \cdots.$$

Here,

$$C^n(\mathcal{U},\mathcal{A}) \overset{\text{def}}{=} \prod_{i_0<\cdots<i_n} \mathcal{A}(U_{i_0,\dots,i_n})$$

consists of all systems of sections on $(n+1)$-tuple intersections. The map d^n (we call it the n^{th} *differential*), creates from $f = (f_{i_0,\dots,i_n})_{I^n} \in C^n$ an element $d^n(f) \in C^{n+1}$, with

$$d^n(f)_{i_0,\dots,i_{n+1}} = \sum_{s=0}^{n+1} (-1)^s f_{i_0,\dots,i_{s-1},i_{s+1},\dots,i_{n+1}}.$$

[114] Due Thursday April 15

[115] It is not really necessary but it simplifies practical calculations.

From this we construct groups of *n-cocycles* and *n-coboundaries*

$$Z^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} \operatorname{Ker}(C^n \xrightarrow{d^n} C^{n+1}) \subseteq C^n \quad \text{and} \quad B^n(\mathcal{U},\mathcal{A}) \stackrel{\text{def}}{=} Im(C^{n-1} \xrightarrow{d^{n-1}} C^n) \subseteq C^n.$$

2. Čech complex $C^\bullet(\mathcal{U},\mathcal{A})$. (a) Show that d^0 is the same as before.

(b) Show that $(C^\bullet(\mathcal{U},\mathcal{A}), d^\bullet)$ is a complex, i.e., $d^n \circ d^{n-1} = 0$.

(c) Show that $B^n(\mathcal{U},\mathcal{A}) \subseteq Z^n(\mathcal{U},\mathcal{A})$.

♡

Čech cohomology $\check{H}^\bullet_{\mathcal{U}}(X,\mathcal{A})$. It is defined as the cohomology of the Čech complex $C^\bullet(\mathcal{U},\mathcal{A})$, i.e.,

$$\check{H}^n_{\mathcal{U}}(X,\mathcal{A}) \stackrel{\text{def}}{=} Z^n(\mathcal{U};\mathcal{A})/B^n(\mathcal{U};\mathcal{A}), \quad n = 0,1,2,....$$

This construction is a generalization of the global sections of a sheaf since

$$\check{H}^0_{\mathcal{U}}(X,\mathcal{A}) = Z^0(\mathcal{U},\mathcal{A})/B^0(\mathcal{U},\mathcal{A}) = Z^0(\mathcal{U},\mathcal{A}) = \Gamma(\mathcal{A}).$$

3. If the open cover $\mathcal{U}$ consists of two open sets U and V, show that

(1) $\check{H}^0_{\mathcal{U}}(X,\mathcal{A}) = \{(a,b) \in \mathcal{A}(U) \oplus \mathcal{A}(V);\ a = b \text{ on } U \cap V\} \cong \Gamma(X,\mathcal{A})$.
(2) $\check{H}^1_{\mathcal{U}}(X,\mathcal{A}) = \frac{\mathcal{A}(U\cap V)}{\rho^U_{U\cap V}\mathcal{A}(U) + \rho^V_{U\cap V}\mathcal{A}(V)}$.
(3) $\check{H}^i_{\mathcal{U}}(X,\mathcal{A}) = 0$ for $i > 1$.

♡

The True Cohomology of sheaves. There is a general *cohomology theory for sheaves* which associates to any sheaf of abelian groups $\mathcal{A}$ a sequence of groups $H^i(X,\mathcal{A})$ (no dependence on any open cover!). The usefulness of Čech cohomology comes from the fact that *often*, the Čech cohomology $\check{H}^i_{\mathcal{U}}(X,\mathcal{A})$ computes these cohomology groups $H^i(X,\mathcal{A})$.[116] At least there is never a disagreement on the level 0 since always

$$\check{H}^0_{\mathcal{U}}(X,\mathcal{A}) = \Gamma(X,\mathcal{A}) = H^0(X,\mathcal{A}).$$

♡

Divisors and line bundles on a curve. Let X be a complex curve (i.e., a complex manifold of dimension one). The group $Div(X)$ of divisors on X is the free abelian group with a basis given by all points of X. So, any divisor $D \in Div(X)$ can be written as $D = \sum d_i \cdot \alpha_i$ for some distinct points $\alpha_1, ..., \alpha_p$ of X, and some integers $d_1, ..., d_p$.

We can use a divisor $D \in Div(X)$ to modify the sheaf $\mathcal{O}^{an}_X$ of holomorphic (=analytic) functions on X. For any open $U \subseteq X$ we define $\mathcal{O}_X(D)(U) \stackrel{\text{def}}{=}$ all holomorphic functions f on

[116] This means that in practice, for a specific class of sheaves $\mathcal{A}$ one can find the corresponding class of open covers $\mathcal{U}$ such that $\check{H}^i_{\mathcal{U}}(X,\mathcal{A}) = H^i(X,\mathcal{A})$. For instance in algebraic geometry one usually considers the *quasicoherent sheaves* and then it suffices if all U_i are affine.

$U - \{\alpha_1, ..., \alpha_p\}$ such that at each α_i in U, the order of f at α_i[117] is at least $-d_i$, i.e.,

$$ord_{\alpha_i}(f) + d_i \geq 0.$$

4. Sheaves $\mathcal{O}_X(D)$. (a) Show that for any divisor D on a complex curve X, construction $\mathcal{O}_X(D)$ is a sheaf on X.

(b) Let $\mathbf{0}$ be the zero element of $\mathbb{C}$. For $n \in \mathbb{Z}$ consider the sheaf $\mathcal{L} = \mathcal{O}_{\mathbb{C}}(n{\cdot}\mathbf{0})$ on $\mathbb{C}$. Show that for any open $U \subseteq \mathbb{C}$, $\mathcal{L}(U) = z^{-n}{\cdot}\mathcal{O}^{an}(U)$.

(c) We say that a divisor $D = \sum d_i{\cdot}\alpha_i \in Div(X)$ is effective if all multiplicities d_i are ≥ 0. Show that $\mathcal{O}_X(D)$ contains $\mathcal{O}_X$ iff D is effective.

♡

Cohomology of line bundles on $\mathbb{P}^1$. Let $\mathbf{0}$ be the zero in $\mathbb{C} \subseteq \mathbb{P}^1$. For each $n \in \mathbb{Z}$ consider the sheaf $\mathcal{L}_n \stackrel{\text{def}}{=} \mathcal{O}^{an}_{\mathbb{P}^1}(n{\cdot}\mathbf{0})$ on $\mathbb{P}^1$. We will use the open covering $\mathcal{U} = \{U, V\}$ of $\mathbb{P}^1$, with $U = \mathbb{CP}^1 - \{\infty\}$ and $V = \mathbb{P}^1 - \{\mathbf{0}\}$.

5. Find the dimensions of the Čech cohomology vector spaces $\check{H}^i_{\mathcal{U}}(\mathbb{P}^1, \mathcal{L}_n)$, $n \in \mathbb{Z}$, $i \geq 0$.

[117] Let ϕ be a function holomorphic on some open $V \subseteq X$. If $\alpha \in X$ is an isolated singularity of ϕ in the sense that V contains some punctured neighborhood of α, then we can define the order of ϕ at α, $ord_\alpha(\phi) \in \mathbb{Z}$, by using a local chart on X near α.

Homework 8

♡[118]

Calculations with complexes. Calculations with complexes are based on the ideas bellow.

1. Functoriality of cohomology. (a) Show that a cohomology is a functor, i.e., that a nap of complexes $A \xrightarrow{\alpha} B$ gives maps of cohomology groups $H^n(A) \xrightarrow{H^n(\alpha)} H^n(B)$.

(b) Show that a short exact sequence of complexes $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ gives maps of cohomology groups $H^n(C) \xrightarrow{\partial^n} H^{n+1}(A)$. So, one needs for a class $\gamma \in \mathrm{H}^n(C)$ to construct a class $\partial\gamma \in \mathrm{H}^{n+1}$. To do this show that

1. (1) For any choice of $c \in Z^n(C)$ such that the class of a cocycle c is $\gamma = [c]$, there is some $b \in B^n$ such that $\beta(b) = c$.
2. (2) For such b, there is cocycle $a \in Z^{n+1}(A)$ such that $d(b) = \alpha(a)$.
3. (3) The class $[a] \in H^{n+1}(A)$ depends only on γ (not on any choices we made).

2. Long exact sequence of cohomology groups. (b) Show that a short exact sequence of complexes $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ gives a long exact sequence of cohomologies

$$\cdots \xrightarrow{\partial^{n-1}} \mathrm{H}^n(A) \xrightarrow{\mathrm{H}^n(\alpha)} \mathrm{H}^n(B) \xrightarrow{\mathrm{H}^n(\beta)} \mathrm{H}^n(C) \xrightarrow{\partial^n} \mathrm{H}^{n+1}(A) \xrightarrow{\mathrm{H}^{n+1}(\alpha)} \mathrm{H}^{n+1}(B) \xrightarrow{\mathrm{H}^{n+1}(\beta)} \cdots.$$

♡

Resolving singularities by blow-ups. Let $V = \mathbb{C}^n$. Recall that the blow up $\widetilde{V}$ of V lies in the product $\mathbb{P}(V)\times V$, so it comes with the maps $\mathbb{P}(V) \xleftarrow{\pi} \widetilde{V} \xrightarrow{\mu} V$. Let $X\subseteq V$ be an affine subvariety given by some polynomial equations. The *proper transform of* X is the subvariety $\widetilde{X}$ of the blow-up, obtained as the closure $\overline{\mu^{-1}X^*}$ in $\widetilde{V}$ of the inverse of $X^* = X - \{0\}$. It comes with a map $\widetilde{X} \xrightarrow{p} X$ (the restriction of μ). We call $E = \widetilde{X} \cap \mu^{-1}0$ the exceptional locus of the proper transform.

We will say that Y is a hypersurface of a complex manifold M if it is (locally) given by one equation $Y = \{z \in M;\ F(z) = 0\}$ for some holomorphic function F. In this case at a point $p \in Y$, Y is a submanifold of M iff $d_pF \neq 0$.

[118]Due Thursday April 22

3+4. Quadratic singularities in $\mathbb{A}^2$ and $\mathbb{A}^3$. Consider

(1) $X = \{(x,y) \in \mathbb{C}^2;\ xy = 0\} \subseteq \mathbb{C}^2$,
(2) $X = \{(x,y,z) \in \mathbb{C}^3;\ x^2+y^2+z^2 = 0\} \subseteq \mathbb{C}^3$.

In each case

- Show that X has an isolated singularity at 0.
- Describe the exceptional fiber E.
- Show that $\widetilde{X}$ is a submanifold of $\widetilde{V}$.[119]

♡

The strategy of resolving by blow-ups. If X has an *isolated singularity at* 0 (meaning that X is smooth off 0, i.e., X^* is a manifold), usually $\widetilde{X}$ is in some sense *less singular*. One can try to explain it in the following way. In the blow up $\widetilde{V}$, one replaces $0 \in V$ with $\mathbb{P}(V)$, i.e., *with all directions of approaching* 0. If we imagine that the singularity of X at 0 is created by a mess of many unusual ways of approaching this point from X, the blow-up has the effect of separating, pulling apart, these ways and decreasing a mess. This falls under standard idea that singularities are caused by forgetting some relevant data, so they are resolved by adding data, in this case the direction of the approach.

For instance let X be a curve in $V = \mathbb{C}^2$ with a singularity at the origin which come from two branches B_1, B_2 of X (i.e., little pieces of X), meeting at 0. In the proper transforms $\widetilde{B_i}$ of B_i one replaces 0 by the direction of approaching 0, i.e., the tangent lines T_i to B_i at 0. So if the tangent lines to B_i at 0 are different, the exceptional fiber will consist of two different points T_i and $\widetilde{X}$ will be smooth because the branches will no longer meet.

This is the case when B_1, B_2 agree to the 0^{th} order at the origin (i.e., they meet but they are not tangent to each other), and the proper transform kills this 0^{th} order contact: $\widehat{B_i}$'s do not meet. In general, if B_1, B_2 agree to the p^{th} order at the origin the transforms will agree to the $(p-1)^{\text{st}}$ order, so one need to blow up $p+1$ times to resolve singularity. Say, if we have first order contact then $T_1 = T_2$ (i.e., B_i's are tangent), so $\widetilde{B_i}'s$ meet but they are not tangent any more, so the next blow up will do.

Actually, by using slightly more general versions of the blow-up construction one can get rid of any singularity:

Theorem. [Hironaka] Any singularity can be resolved by successive blow-ups.

[119]The problem assumes that $\widetilde{V}$ is a manifold. In dimension 2 this has been checked in a previous homework. In any dimension the proof is similar – one has to extend the standard charts for $\mathbb{P}(V)$ to $\widetilde{V}$.

Homework 9

♡[120]

Sheafification. If when playing with sheaves we get lost and find ourselves in a larger world of *presheaves* (and these are less interesting objects) we need to find the way home. This is the main technical step[121] in making sheaves useful.

By a presheaf we mean a the same structure as a sheaf, except that we do not require the *gluing* property. For instance while locally constant functions are a sheaf, constant functions are just a presheaf. Presheaves are by themselves not so interesting because lack of gluing means that they do not relate local and global information well. Unfortunately presheaves are not avoidable, for instance we will see that applying some basic constructions to sheaves results in presheaves.

Sheafification is a way to improve any presheaf of sets $\mathcal{S}$ into a sheaf of sets $\widetilde{\mathcal{S}}$. We will obtain the sections of the sheaf $\tilde{\mathcal{S}}$ associated to a presheaf $\mathcal{S}$ in two steps:[122]

(1) *add more sections* by gluing systems of local sections s_i which are compatible in the sense that they are <u>*locally*</u> the same, and
(2) *cut down on sections* by identifying two results of such gluing procedures when the local sections in the two families are <u>*locally*</u> the same.

In the first step for each open $U \subseteq X$ we replace $\mathcal{S}(U)$ by a larger set $\hat{\mathcal{S}}(U)$, and in the second by $\widetilde{\mathcal{S}}(U)$ which is a quotient of $\widehat{\mathcal{S}}(U)$ by an equivalence relation $\equiv$. The definitions are

(1) $\widehat{\mathcal{S}}(U)$ consists of all families $(U_i, s_i)_{i\in I}$ such that
   - $(U_i)_{i\in I}$ is an open cover of U and $s_i \in \mathcal{S}(U_i)$ is a section of $\mathcal{S}$ on U_i,
   - sections s_i are *weakly compatible* in the sense that they are locally the same, i.e., for any $i, j \in I$ and any point $x \in U_{ij}$. we ask that sections s_i and s_j are the same near x:

     There is neighborhood W of x in U_{ij} such that $s_i|W = s_j|W$.
(2) We say that two systems $(U_i, s_i)_{i\in I}$ and $(V_j, t_j)_{j\in J}$ are $\equiv$ if for any $i \in I,\ j \in J$, the sections s_i and t_j are weakly equivalent[123]

1 (a) The relation $\equiv$ on $\widehat{\mathcal{S}}(U)$ really says that $(U_i, s_i)_{i\in I} \equiv (V_j, t_j)_{j\in J}$ iff the disjoint union $(U_i, s_i)_{i\in I} \sqcup (V_j, t_j)_{j\in J}$ is again in $\widehat{\mathcal{S}}(U)$.

(b) $\equiv$ is an equivalence relation on $\widehat{\mathcal{S}}(U)$.

(c) $\widetilde{\mathcal{S}}(U)$ is a presheaf.

2 $\widetilde{\mathcal{S}}$ is a sheaf.

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[120] Due Thursday April 29
[121] and the most painful.
[122] You can think that we are imitating the passage from constant functions to locally constant functions.
[123] i.e., for each $x \in U_i \cap V_j$, there is an open set W with $x \in W \subseteq U_i \cap V_j$ such that "$s_i = t_j$ on W" in the sense of restrictions being the same.

♡

Adjointness of sheafification and forgetting. Fortunately, in practice we do not have to recall the specifics of the the sheafification construction. Instead we use an abstract (categorical) characterization of the sheaf $\widetilde{\mathcal{S}}$. First we define the category $preSh(X)$ of presheaves of sets on X: the objects are presheaves and a map of presheaves $\alpha \in \mathrm{Hom}_{preSh(X)}(\mathcal{A},\mathcal{B})$ is a system of maps $\alpha_U : \mathcal{A}(U) \to \mathcal{B}(U)$, one for each open $U \subseteq X$, which is compatible with restrictions, i.e., whenever $V \subseteq U \subseteq X$, the following commutes

$$\begin{array}{ccc} \mathcal{A}(U) & \xrightarrow{\alpha_U} & \mathcal{B}(U) \\ {\scriptstyle \rho^U_V}\downarrow & & {\scriptstyle \rho^U_V}\downarrow \\ \mathcal{A}(V) & \xrightarrow{\alpha_V} & \mathcal{B}(V) \end{array} .$$

The category $Sh(X)$ of sheaves of sets on X has sheaves for objects but the maps are defined the same: any sheaves $\mathcal{A},\mathcal{B}$ are in particular presheaves (we just *forget* that they do satisfy the gluing property), and $\mathrm{Hom}_{Sh(X)}(\mathcal{A},\mathcal{B}) \overset{\text{def}}{=} \mathrm{Hom}_{preSh(X)}(\mathcal{A},\mathcal{B})$.

3 (a) There is a canonical map of presheaves $\mathcal{S} \xrightarrow{\iota} \widetilde{\mathcal{S}}$.

(b) For any presheaf $\mathcal{S}$ and any sheaf $\mathcal{F}$ the map

$$\iota^* : \ \mathrm{Hom}_{\mathcal{S}heaves}(\widetilde{\mathcal{S}},\mathcal{F}) \ \to \ \mathrm{Hom}_{pre\mathcal{S}heaves}(\mathcal{S},\mathcal{F}), \qquad \iota_*\alpha = \ \alpha\circ\iota$$

is a bijection.

♡

The bijection ι^* relates two procedures (i.e. functors): *sheafification* (on the LHS) and *forgetting* (on the RHS). A relation of this type between two functors is called *adjunction*, we say that sheafification is the left adjoint of the forgetful functor (and that the forgetful functor is the right adjoint of sheafification). One can see that two adjoint functors determine each other by using the Yoneda lemma.

4 (a) Show that the adjunction property characterizes the sheafification, i.e., if $\mathcal{S}$ is a presheaf and $\iota_i : \mathcal{S} \to \mathcal{S}_i$ $(i = 1, 2)$, are maps of presheaves such that

(1) $\mathcal{S}_i$ are sheaves,
(2) For any sheaf $\mathcal{F}$ the maps

$$\iota_i^* : \ \mathrm{Hom}_{\mathcal{S}heaves}(\widetilde{\mathcal{S}},\mathcal{F}) \ \to \ \mathrm{Hom}_{pre\mathcal{S}heaves}(\mathcal{S},\mathcal{F}), \qquad \iota_*\alpha = \ \alpha\circ\iota$$

are bijections.

then there is a canonical isomorphism $\mathcal{S}_1 \to \mathcal{S}_2$.

(b) Show that the sheafification of constant functions is given by locally constant functions.

Homework 10

♡[124]

The direct and inverse image of sheaves. Any map of sets $X \xrightarrow{\pi} Y$ defines a linear operator $\mathbb{C}[Y] \xrightarrow{\pi^*} \mathbb{C}[X]$ between spaces of functions, this is the *pull-back* or *inverse image* operation $\pi^* g = \ g{\circ}\pi$. We can also go in the opposite direction with a *direct image* (or "*integration over fibers*) operation $\mathbb{C}[X] \xrightarrow{\pi_*} \mathbb{C}[Y]$ by $(pi_* f)(y) = \ \sum_{x\in\pi^{-1}y}\ f(x)$, provided we resolve some convergence problem, for instance it is fine if the fibers of π are finite.

One can do the same in sheaves and without any convergence problem. Any map of topological spaces $X \xrightarrow{\pi} Y$ defines two operations, the direct and inverse image operations

$$\mathcal{S}heaves(X) \xrightarrow{\pi_*} \mathcal{S}heaves(Y) \quad \text{and} \quad \mathcal{S}heaves(Y) \xrightarrow{\pi^*} \mathcal{S}heaves(X).$$

The direct image is much easier to define while the inverse image is much easier to calculate in practice.

In our thinking, we assumed some analogy between functions and sheaves. This is sound, i.e., one should think of sheaves as more subtle versions of functions. However, one should notice the increased level of subtlety: while functions on X form a vector space, sheaves on X form a category. So our new operations are not going to be linear operators but functors.

1 (a) Let $X \xrightarrow{\pi} Y$ be a map of topological spaces. Show that for any sheaf $\mathcal{M}$ on X, the formula

$$\pi_*(\mathcal{M})\ (V) \overset{\text{def}}{=} \mathcal{M}(\pi^{-1}V),$$

defines a sheaf $\pi_*\mathcal{M}$ on Y, and this gives a functor $\mathcal{S}heaves(X) \xrightarrow{\pi_*} \mathcal{S}heaves(Y)$.

(b) If $X \xrightarrow{\pi} Y \xrightarrow{\tau} Z$ then $\tau_*(\pi_*\mathcal{A}) \cong (\tau{\circ}\pi)_*\mathcal{A}$.

(c) $(1_X)_*\mathcal{A} \cong\ \mathcal{A}$.

(d) (Map to a point.) If $a : X \to pt$ then $a_*(\mathcal{F}) = \Gamma(X, \mathcal{F})$.

♡

2. Examples of maps of sheaves. (a) Consider the sheaf $\mathcal{O}_M$ of holomorphic functions on a complex manifold $M = \mathbb{C}$. Which of the following are maps of sheaves of vector spaces (and why?):

(1) (Differentiation) $\mathcal{O}_M(U) \ni f \mapsto f' \in \mathcal{O}_M(U)$,
(2) (Squaring) $\mathcal{O}_M(U) \ni f \mapsto f^2 \in \mathcal{O}_M(U)$,
(3) (multiplication by a function) $\mathcal{O}_M(U) \ni f \mapsto zf \in \mathcal{O}_M(U)$,
(4) (translation by 1) $\mathcal{O}_M(U) \ni f(x) \mapsto f(x+1) \in \mathcal{O}_M(U)$.

[124] Due Thursday May 6

(b) Circle $S=\{z\in\mathbb{C};\ |z|=1\}$ is clearly a one dimensional real manifold (a restriction of $\mathcal{E}:\mathbb{R}\to S,\ \mathcal{E}(x)=e^{ix}$ to any open interval $I\subseteq\mathbb{R}$ of length $<2\pi$ provides a chart $I\to\mathcal{E}(I)\subseteq S$). Let C_S^∞ be the sheaf of smooth functions on S. Show that

(1) One can define operators $\partial_U : C_S^\infty(U)\to C_S^\infty(U)$ by the formula[125] $\partial f(e^{ix}) \stackrel{\text{def}}{=} \frac{d}{dx} f(e^{ix})$.
(2) Show that ∂ is a map of sheaves $\partial : C_S^\infty \to C_S^\infty$.

♡

Category $\mathcal{A}b\mathcal{S}h(X)$ of sheaves of abelian groups is an abelian category. We want to see that in the category $\mathcal{A}b\mathcal{S}h(X)$ of sheaves of abelian groups on X, we can calculate very much like we calculate with abelian groups or with modules over rings, i.e., the precise meaning of this is:

Category $\mathcal{A}b\mathcal{S}h(X)$ of sheaves of abelian groups is an <u>abelian category.</u>

We will consider a map of sheaves of abelian groups $\mathcal{A}\xrightarrow{\alpha}\mathcal{B}$. We want to see that it has a kernel, image and cokernel. We will take care of the kernel, and indicate3 how to construct the image and the cokernel.

3 (a) Show that the formula $\mathcal{K}_\alpha(U) = Ker[\mathcal{A}(U)\xrightarrow{\alpha_U}\mathcal{B}(U)]$ defines a sheaf $\mathcal{K}_\alpha$ which is a subsheaf[126] of $\mathcal{A}$.

(b) Let $i:\mathcal{K}_\alpha\to\mathcal{A}$ be the inclusion, i.e., Show that $(\mathcal{K}_\alpha, i)$ is the kernel of α, according to the following categorical definition of the kernel

Any map of sheaves $\mathcal{F}\xrightarrow{\phi}\mathcal{A}$ such that $\beta\circ\alpha=0$ factors uniquely through α, i.e., there is a unique map $\mathcal{F}\xrightarrow{\Phi}\mathcal{K}_\alpha$ such that $(\mathcal{F}\xrightarrow{\phi}\mathcal{A}) = (\mathcal{K}_\alpha\xrightarrow{i}\mathcal{A})\circ(\mathcal{F}\xrightarrow{\Phi}\mathcal{K}))$.

4 (a) Show that the formula $I_\alpha(U) = Im[\mathcal{A}(U)\xrightarrow{\alpha_U}\mathcal{B}(U)]$ defines a presheaf I_α which is a subpresheaf of $\mathcal{B}$.

(b) Show that the formula $C_\alpha(U) = Coker[\mathcal{A}(U)\xrightarrow{\alpha_U}\mathcal{B}(U)]$ defines a presheaf C_α.

(c) Consider the the map of sheaves $\partial : C_S^\infty\to C_S^\infty$ defined above. Show that in this case I_∂ and C_∂ are not sheaves.

(d) What are the sheafifications $\mathcal{I}_\partial$ and $\mathcal{C}_\partial$ of presheaves I_∂ and C_∂ from part (c)?

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[125] If you do not like the formula you can identify for $V\subseteq S$ open, $C_S^\infty(V)$ with the $\mathbb{Z}$-periodic functions on $\mathcal{E}^{-1}V$, and then use the operator $\frac{d}{dx}$.

[126] For sheaves $\mathcal{S}$ and $\mathcal{S}'$ we say that $\mathcal{S}'$ is a sub(pre)sheaf of $\mathcal{S}$ if $\mathcal{S}'(U)\subseteq\mathcal{S}(U)$ for each open U and the restriction maps for $\mathcal{S}'$, $\mathcal{S}'(U)\xrightarrow{\rho'}\mathcal{S}'(V)$ are restrictions of the restriction maps for $\mathcal{S}$, $\mathcal{S}(U)\xrightarrow{\rho}\mathcal{S}(V)$. The same definition works for presheaves.

Homework X

THIS HOMEWORK IS DUE BY THE END OF THIS SEMESTER

♡

Monodromy of cycles in curves. For $\lambda \in \mathbb{C}$ consider the affine cubic curve

$$\mathcal{C}_\lambda = \{(x:y) \in \mathbb{A}^2;\ y^2 = x(x-1)(x-\lambda)\} \subseteq \mathbb{A}^2$$

and the corresponding projective cubic curve

$$C_\lambda = \{[x:y:u] \in \mathbb{P}^2;\ y^2u = x(x-u)(x-\lambda u)\} \subseteq \mathbb{P}^2.$$

Circles $\alpha_\lambda, \beta_\lambda$ in C_λ. Let $0 < |\lambda| < 1$. We use the projection to the x-line $\pi : \mathcal{C}_\lambda \to \mathbb{A}^1$, $\pi(x,y) = x$, to define circles

$$\alpha_\lambda = \pi^{-1}[0,\lambda] \quad \text{and} \quad \beta_\lambda = \pi^{-1}[\lambda,1]$$

in $\mathcal{C}_\lambda$, as inverses of segments joining $0, \lambda$ and 1.

♡

1. The vanishing cycle. (a) Draw C_0 (with all explanations).

(b) What happens with the circles $\alpha_\lambda, \beta_\lambda$ as $\lambda \to 0$?

♡

♡ ★ ♡

♡

2. Monodromy. When λ goes around 0 on a small circle, what happens to the circles α_λ and β_λ in the torus C_λ?

♡

Explanation. Start with $\lambda = \frac{1}{2}$ and the circles $\alpha_{\frac{1}{2}} = \pi^{-1}[0,\frac{1}{2}]$ and $\beta_{\frac{1}{2}} = \pi^{-1}[\frac{1}{2},1]$ in $C_{\frac{1}{2}}$. As one rotates $\lambda = \frac{1}{2}$ around the origin by $\lambda_\theta = \frac{1}{2}e^{i\theta}$, $0 \le \theta \le 2\pi$, one needs to choose continuously circles $\alpha(\theta), \beta(\theta)$, in the curves $C_{\frac{1}{2}e^{i\theta}}$, and then the question is what are the circles $\alpha(2\pi), \beta(2\pi)$?

Since there are choices involved, this is not a completely precisely posed question. To eliminate the effect of continuous choices, we can look at circles up to homotopy (i.e., up to continuous deformations). Recall that the homotopy classes $[\alpha_\lambda], [\beta_\lambda]$ form a basis of the group of closed paths up to homotopy: $\pi_1(C_\lambda) = \mathbb{Z}\cdot[\alpha]\oplus\mathbb{Z}\cdot[\beta_\lambda]$, $\lambda \in \mathbb{C} - \{0,1\}$. Now the classes of $\alpha(\theta), \beta(\theta)$ in $\pi_1(C_{\frac{1}{2}e^{i\theta}})$ are well defined, and the question is to calculate $[\alpha(2\pi)], [\beta(2\pi)]$ in $\pi_1(C_{\frac{1}{2}e^{2\pi i}}) = \pi_1(C_{\frac{1}{2}})$, i.e., to find the integer coefficients

$$[\alpha_{2\pi}] = \mu_{11}\cdot[\alpha(0)] + \mu_{12}\cdot[\beta(0)] \quad \text{and} \quad [\beta_{2\pi}] = \mu_{21}\cdot[\alpha(0)] + \mu_{22}\cdot[\beta(0)].$$

A simple way to choose $\alpha(\theta), \beta(\theta)$ is to take the inverses $\alpha(\theta) = {\pi_{\frac{1}{2}e^{i\theta}}}^{-1}\, a(\theta)$, $\beta(\theta) = {\pi_{\frac{1}{2}e^{i\theta}}}^{-1}\, b(\theta)$ in $C_{\frac{1}{2}e^{i\theta}}$, of paths on the x-line $a(\theta), b(\theta) \subseteq \mathbb{C} - \{0,1\}$. Here $a(\theta)$ is a curve from 0 to $\frac{1}{2}e^{i\theta}$ and

$b(\theta)$ from $\frac{1}{2}e^{i\theta}$ to 1. So, one starts with $a(0), b(0)$ which are segments, and then moves them continuously in the family $a(\theta), b(\theta)$.

Appendix A. **Categories**

A.0.8. *Why categories?* The notion of a category is misleadingly elementary. It formalizes the idea that we study certain kind of objects (i.e., endowed with some specified structures) and that it makes sense to go from one such object to another via something (a "morphism") that preserves the relevant structures. Since this is indeed what we usually do, the *language* of categories is convenient.

However, soon one finds that familiar notions and constructions (such as (i) empty set, (ii) union of sets, (iii) product of sets, (iv) abelian group, ...) **categorify**, i.e., have analogues (and often more then one) in general categories (respectively: (i) initial object, final object, zero object; (ii) sum of objects or more generally a direct (inductive) limit of objects; (iii) product of objects or more generally the inverse (projective) limit of objects; (iv) additive category, abelian category; ...). This enriched language of categories was recognized as fundamental for describing various complicated phenomena, and the study of *special kinds of categories* mushroomed to the level of study of functions with various properties in analysis.

A.1. **Categories.** A category $\mathcal{C}$ consists of

1. a class $Ob(\mathcal{C})$ whose elements are called objects of $\mathcal{C}$,
2. for any $a, b \in Ob(\mathcal{C})$ a set $\mathrm{Hom}_{\mathcal{C}}(a, b)$ whose elements are called morphisms ("maps") from a to b,
3. for any $a, b, c \in Ob(\mathcal{C})$ a function $\mathrm{Hom}_{\mathcal{C}}(b, c) \times \mathrm{Hom}_{\mathcal{C}}(a, b) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(a, c)$, called composition,
4. for any $a \in Ob(\mathcal{C})$ an element $1_a \in \mathrm{Hom}_{\mathcal{C}}(a, a)$,

such that the composition is associative and 1_a is a neutral element for composition.

Instead of $a \in \mathrm{Ob}(\mathcal{C})$ we will just say $a \in \mathcal{C}$.

A.1.1. *Examples.*

1. Categories of *sets with additional structures*: $\mathcal{S}ets$, $\mathcal{A}b$, $\mathfrak{m}(\Bbbk)$ for a ring $\Bbbk$ (denoted also $\mathcal{V}ect(\Bbbk)$ if $\Bbbk$ is a field), $\mathcal{G}roups$, $\mathcal{R}ings$, $\mathcal{T}op$, $\mathcal{O}rd\mathcal{S}ets \stackrel{\text{def}}{=}$ category of ordered sets, ...
2. To a category $\mathcal{C}$ one attaches the opposite category $\mathcal{C}^o$ so that objects are the same but the "direction of arrows reverses":
$$\mathrm{Hom}_{\mathcal{C}^o}(a, b) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(b, a).$$
3. Any partially ordered set $(I, \leq)$ defines a category with $Ob = I$ and $\mathrm{Hom}(a, b) =$ point (call this point (a, b)) if $a \leq b$ and $\emptyset$ otherwise.
4. Sheaves of sets on a topological space X, Sheaves of abelian groups on X,...

A.2. **Functors.** The analogue on the level of categories of a function between two sets is a *functor between two categories.*

A functor F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of

- for each object $a\in\mathcal{A}$ an object $F(a)\in\mathcal{B}$,
- for each map $\alpha\in\mathrm{Hom}_{\mathcal{A}}(a',a'')$ in $\mathcal{A}$ a map $F(\alpha)\in\mathrm{Hom}_{\mathcal{B}}(Fa',Fa'')$

such that F preserves compositions and units, i.e., $F(\beta{\circ}\alpha)=F(\beta){\circ}F(\alpha)$ and $F(1_a)=1_{Fa}$.

A.2.1. *Examples.* (1) A *functor* means some construction, say a map of rings $\Bbbk\xrightarrow{\phi}l$ gives

- a pull-back functor $\phi^*:\mathfrak{m}(l)\longrightarrow\mathfrak{m}(\Bbbk)$ where $\phi^*N=N$ as an abelian group, but now it is considered as module for $\Bbbk$ via ϕ.
- a push-forward functor $\phi_*:\mathfrak{m}(\Bbbk)\longrightarrow\mathfrak{m}(l)$ where $\phi_*M\overset{\text{def}}{=}l\otimes_{\Bbbk}M$. This is called "change of coefficients".

To see that these are functors, we need to define them also on maps. So, a map $\beta:N'\longrightarrow N''$ in $\mathfrak{m}(l)$ gives a map $\phi^*(\beta):\phi^*(N')\longrightarrow\phi^*(N'')$ in $\mathfrak{m}(\Bbbk)$ which as a function between sets is really just $\beta:N'\longrightarrow N''$. On the other hand, $\alpha:M'\longrightarrow M''$ in $\mathfrak{m}(\Bbbk)$ gives $\phi_*(\alpha):\phi_*(M')\longrightarrow\phi_*(M'')$ in $\mathfrak{m}(l)$, this is just the map $1_{l\otimes\alpha:\ l\otimes_{\Bbbk}M'\longrightarrow l\otimes_{\Bbbk}M'',\ c\otimes x\mapsto\ c\otimes\alpha(x)}$.

Here we see a general feature:

functors often come in pairs ("adjoint pairs of functors") and usually one of them is stupid and the other one an interesting construction.

(2) For any category $\mathcal{A}$ there is the identity functor $1_{\mathcal{A}}:\mathcal{A}\longrightarrow\mathcal{A}$. Two functors $\mathcal{A}\xrightarrow{F}\mathcal{B}$ and $\mathcal{B}\xrightarrow{G}\mathcal{C}$ can be composed to a functor $\mathcal{A}\xrightarrow{G\circ F}\mathcal{C}$.

(3) An object $a\in\mathcal{A}$ defines two functors, $\mathrm{Hom}_{\mathcal{A}}(a,-):\ \mathcal{A}\longrightarrow\mathcal{S}ets$, and $\mathrm{Hom}_{\mathcal{A}}(-,a):\ \mathcal{A}^o\longrightarrow\mathcal{S}ets$. Moreover, $\mathrm{Hom}_{\mathcal{A}}(-,-)$ is a functor from $\mathcal{A}^o{\times}\mathcal{A}$ to sets!

(4) For a ring $\Bbbk$, tensoring is a functor $-\otimes_{\Bbbk}-:\ \mathfrak{m}^r(\Bbbk){\times}\mathfrak{m}^l(\Bbbk)\longrightarrow\mathcal{A}b$.

A.2.2. *Contravariant functors.* We say that a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is given by assigning to any $a\in\mathcal{A}$ some $F(a)\in\mathcal{B}$, and for each map $\alpha\in\mathrm{Hom}_{\mathcal{A}}(a',a'')$ in $\mathcal{A}$ a map $F(\alpha)\in\mathrm{Hom}_{\mathcal{B}}(Fa'',Fa')$ – notice that we have changed the direction of the map so now we have to require $F(\beta{\circ}\alpha)=F(\alpha){\circ}F(\beta)$ (and $F(1_a)=1_{Fa}$).

This is just a way of talking, not a new notion since a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is the same as a functor F from $\mathcal{A}$ to $\mathcal{B}^o$ (or a functor F from $\mathcal{A}^o$ to $\mathcal{B}$).

A.3. **Natural transformations of functors ("morphisms of functors").** A natural transformation η of a functor $F:\mathcal{A}\longrightarrow\mathcal{B}$ into a functor $G:\mathcal{A}\longrightarrow\mathcal{B}$ consists of maps $\eta_a\in$

$\mathrm{Hom}_{\mathcal{B}}(Fa,Ga)$, $a\in\mathcal{A}$ such that for any map $\alpha: a'\longrightarrow a''$ in $\mathcal{A}$ the following diagram commutes

$$\begin{array}{ccc} F(a') & \xrightarrow{F(\alpha)} & F(a'') \\ {\scriptstyle\eta_{a'}}\downarrow & & {\scriptstyle\eta_{a''}}\downarrow \\ G(a') & \xrightarrow{G(\alpha)} & G(a'') \end{array}\quad,\qquad i.e.,\qquad \eta_{a''}\circ F(\alpha)=G(\alpha)\circ\eta_{a'}.$$

So, η relates values of functors on objects in a way compatible with the values of functors on maps. In practice, any "natural" choice of maps η_a will have the compatibility property.

A.3.1. *Example.* For the functors $\phi_*M=l\otimes_{\Bbbk}M$ and $\phi^*N=N$ from A.2.1(1), there are canonical morphisms of functors

$$\alpha:\ \phi_*\circ\phi^*\longrightarrow 1_{\mathfrak{m}(l)},\qquad \phi_*\circ\phi^*(N)=l\otimes_{\Bbbk}N\xrightarrow{\alpha_N}N=1_{\mathfrak{m}(l)}(N)$$

is the action of l on N and

$$\beta:\ 1_{\mathfrak{m}(\Bbbk)}\longrightarrow\phi^*\circ\phi_*,\qquad \phi^*\circ\phi_*(M)=l\otimes_{\Bbbk}M\xleftarrow{\beta_M}M=1_{\mathfrak{m}(\mathbb{M})}(M)$$

is the map $m\mapsto 1_{l\otimes m}$.

For any functor $F:\mathcal{A}\longrightarrow\mathcal{B}$ one has $1_F: F\longrightarrow F$ with $(1_F)_a=1_{Fa}:Fa\longrightarrow Fa$. For three functors F,G,H from $\mathcal{A}$ to $\mathcal{B}$ one can compose morphisms $\mu:F\longrightarrow G$ and $\nu:G\longrightarrow H$ to $\nu\circ\mu:F\longrightarrow H$

A.3.2. *Lemma.* For two categories $\mathcal{A},\mathcal{B}$, the functors from $\mathcal{A}$ to $\mathcal{B}$ form a category $Funct(\mathcal{A},\mathcal{B})$.

Proof. For $F,G:\mathcal{A}\to\mathcal{B}$ one defines $\mathrm{Hom}(F,G)$ as the set of natural transforms from F to G, then all the structure is routine.

A.4. **Construction (description) of objects via representable functors.** Yoneda lemma bellow says that passing from an object $a\in\mathcal{A}$ to the corresponding functor $\mathrm{Hom}_{\mathcal{A}}(-,a)$ does not loose any information – a can be recovered from the functor $\mathrm{Hom}_{\mathcal{A}}(-,a)$.[127] This has the following applications:

(1) One can describe an object a by describing the corresponding functor $\mathrm{Hom}_{\mathcal{A}}(-,a)$. This turns out to be **the** *most natural* description of a.
(2) One can start with a functor $F:\mathcal{A}^o\longrightarrow\mathcal{S}ets$ and ask whether it comes from some objects of a. (Then we say that a represents F and that F is representable).
(3) Functors $F:\mathcal{A}^o\longrightarrow\mathcal{S}ets$ behave somewhat alike the objects of $\mathcal{A}$, and we can think of their totality as a natural enlargement of $\mathcal{A}$ (like one completes $\mathbb{Q}$ to $\mathbb{R}$).

[127]This is the precise form of the *Interaction Principle* on the level of categories that we used to pass from varieties to spaces and stacks. The interactions of a with all objects of the same kind are encoded in the functor $\mathrm{Hom}_{\mathcal{A}}(-,a)$, so Yoneda says that if you know the interactions of a you know a.

A.4.1. *Category* $\widehat{\mathcal{A}}$. To a category $\mathcal{A}$ one can associate a category

$$\widehat{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{F}unct(\mathcal{A}^o, \mathcal{S}ets)$$

of contravariant functors from $\mathcal{A}$ to sets. Observe that each object $a \in \mathcal{A}$ defines a functor

$$\iota_a = \ \mathrm{Hom}_{\mathcal{A}}(-, a) \in \ \widehat{\mathcal{A}}.$$

The following statement essentially says that one can recover a form the functor $\mathrm{Hom}_{\mathcal{A}}(-, a)$, i.e., that this functor contains all information about a.

A.4.2. *Theorem.* (Yoneda lemma)
(a) Construction ι is a functor $\iota : \mathcal{A} \to \widehat{\mathcal{A}}$.

(b) For any functor $F \in \widehat{\mathcal{A}} = \mathcal{F}unct(\mathcal{A}^o, \mathcal{S}ets)$ and any $a \in \mathcal{A}$ there is a canonical identification

$$\mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, F) \cong \ F(a).$$

Proof. (b) Recall that a map of functors $\eta : \iota_a \to F$ (functors from $\mathcal{A}^o$ to $\mathcal{S}ets$), means for each $x \in \mathcal{A}$ one map of sets $\eta_x : \iota_a(x) = \mathrm{Hom}_{\mathcal{A}}(x, a) \to F(x)$, and this system of maps should be such that for each morphism $y \xrightarrow{\alpha} x$ in $\mathcal{A}$ (i.e., $x \xrightarrow{\alpha} y$ in $\mathcal{A}^o$), the following diagram commutes

$$\begin{array}{ccc} F(x) & \xrightarrow{F(\alpha)} & F(y) \\ {\scriptstyle\eta_x}\uparrow & & {\scriptstyle\eta_y}\uparrow \\ \iota_a(x) & \xrightarrow{\iota_a(\alpha)} & \iota_a(y) \end{array} \quad , \quad \textit{i.e.,} \quad F(\alpha)\circ\eta_x \ = \ \eta_y\circ\iota_a(\alpha).$$

Such η in particular gives $\eta_a : \iota_a \to F(a)$, and since $\iota_a = \mathrm{Hom}_{\mathcal{A}}(a, a) \ni 1_a$ we get an element $\overline{\eta} \stackrel{\text{def}}{=} \eta_a(1_a)$ of $F(a)$.

In the opposite direction, a choice of $f \in F(a)$, gives for any $x \in \mathcal{A}$ the composition of functions

$$\widetilde{f}_x \stackrel{\text{def}}{=} [\iota_a(x) = \mathrm{Hom}_{\mathcal{A}}(x, a) = \mathrm{Hom}_{\mathcal{A}^o}(a, x) \xrightarrow{F}= \mathrm{Hom}_{\mathcal{S}ets}[F(a), F(x)] \xrightarrow{ev_f} F(x)].$$

Now one checks that

- (i) $\widetilde{f}$ is a map of functors $\iota_a \to F$, and
- (ii) procedures $\eta \mapsto \overline{\eta}$ and $f \mapsto \widetilde{f}$ are inverse functions between $\mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, F)$ and $F(a)$.

Corollary. (a) Yoneda functor $\iota : \mathcal{A} \to \widehat{\mathcal{A}}$ is a full embedding of categories, i.e., for any $a, b \in \mathcal{A}$ the map

$$\iota : \mathrm{Hom}_{\mathcal{A}}(a, b) \ \to \ \mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a, \iota_b),$$

given by the functoriality of ι, is an isomorphism.

(b) Functor $\mathrm{Hom}_{\mathcal{A}}(-, a) = \iota_a$ determines a up to a unique isomorphism, i.e., if $\iota_a \cong \iota_b$ in $\widehat{\mathcal{A}}$ then $a \cong b$ in $\mathcal{A}$.

Proof. (a) follows the part (b) of the Yoneda lemma (take $F = \iota_b$). (b) follows from (a).

Remark. We say that a functor $F:\mathcal{B}\to\mathcal{C}$ is a full embedding of categories if for any $a,b\in\mathcal{B}$ the map $\mathrm{Hom}_{\mathcal{A}}(a,b)\xrightarrow{F_{a,b}}\mathrm{Hom}_{\widehat{\mathcal{A}}}(\iota_a,\iota_b)$ given by the functoriality of F, is an isomorphism. The meaning of this is we put $\mathcal{B}$ into a larger category which has objects from $\mathcal{B}$ and maybe also some new objects, but the old objects (from $\mathcal{B}$) relate to each other in $\mathcal{C}$ the same as they used to in $\mathcal{B}$. We also say that F makes $\mathcal{B}$ into a full subcategory of $\mathcal{C}$.

A.5. **Yoneda completion $\widehat{\mathcal{A}}$ of a category $\mathcal{A}$.** Yoneda lemma says that $\mathcal{A}$ lies in a larger category $\widehat{\mathcal{A}}$. The hope is that the category $\widehat{\mathcal{A}}$ may contain many *beauties* that should morally be in $\mathcal{A}$ (but are not). One example will be a way of treating inductive systems in $\widehat{\mathcal{A}}$. In particular we will see inductive systems of infinitesimal geometric objects that underlie the differential calculus.

A.5.1. *Distributions.* This Yoneda completion is a categorical analogue of one of the basic tricks in analysis:

since among functions one can not find beauties like the δ-functions, we extend the notion of of functions by adding distributions.

Remember that the distributions on an open $U\subseteq\mathbb{R}^n$ are the (nice) linear functionals on the vector space of of (nice) functions: $\mathcal{D}(U,\mathbb{C})\subseteq C_c^\infty(U,\mathbb{C})^* = \mathrm{Hom}_{\mathbb{C}}[C_c^\infty(U),\mathbb{C}]$.

A.5.2. *Representable functors.* First we get a feeling for how objects of $\mathcal{A}$ are viewed inside $\widehat{\mathcal{A}}$, i.e., the relation between thinking of $a\in\mathcal{A}$ and the functor ι_a.

We will say that a functor $F\in\widehat{\mathcal{A}}$, i.e., $F:\mathcal{A}^o\to\mathcal{S}ets$, is *representable* if there is some $a\in\mathcal{A}$ and an isomorphism of functors $\eta:\ \mathrm{Hom}_{\mathcal{A}}(-,a)\to F$. Then we say that a represents F. This is the *basic categorical trick for describing an object a up to a canonical isomorphism:* :

instead of describing a directly we describe a functor F isomorphic to $\mathrm{Hom}_{\mathcal{A}}(-,a)$.

A.5.3. *Examples.* (1) *Products.* A product of a and b is an object that represents the functor

$$\mathcal{A}\ni x\mapsto \mathrm{Hom}(x,a)\times\mathrm{Hom}_{\mathcal{A}}(x,b)\in\ \mathcal{S}ets.$$

(2) In the category of $\Bbbk$-varieties, functor

$$X\mapsto\{(f_1,...,f_n);\ f_i\in\mathcal{O}(X)\}\ =\ \mathcal{O}(X)^n$$

represents $\mathbb{A}^n$.

(3) In the category of schemes,

$$X\mapsto\{f\in\mathcal{O}(X);\ f^2=0\}$$

represents the double point scheme $\mathrm{Spec}(\mathbb{Z}[x]/x^2)$.

(4) If $\mathbb{A}^n=\oplus_1^n\ \Bbbk\cdot e_i$, then the set

$$\mathbb{A}^\infty\stackrel{\text{def}}{=}\cup_0^\infty\ \mathbb{A}^n=\oplus_1^\infty\ \Bbbk\cdot e_i$$

is an increasing union of $\Bbbk$-varieties. In analogy with (2), we see that the functor corresponding to this construction should be given by all infinite sequences of functions

$$X \mapsto \{(f_0, f_1, ..., f_n, ...);\ f_i \in \mathcal{O}(X)\} \ = \ Map(\mathbb{N}, \mathcal{O}(X)).$$

However, this functor is *not* representable in $\Bbbk$-varieties, i.e., $\mathbb{A}^\infty$ is not a $\Bbbk$-variety. We may expect that it lives in the larger world of schemes, but even this fails. So, its natural ambient is the category the Yoneda completion $\Bbbk$-$\widehat{Varieties}$ of the category $\Bbbk$-$Varieties$.

A.5.4. *Limits.* One can describe the completion of $\mathcal{A}$ to $\widehat{\mathcal{A}}$ as *adding to $\mathcal{A}$ all limits of* inductive systems *in* $\mathcal{A}$, just as one constructs $\mathbb{R}$ from $\mathbb{Q}$. The simplest kinds of inductive systems in $\mathcal{A}$ are the diagrams $\boldsymbol{a} = (a_0 \to a_1 \to \cdots)$ in $\mathcal{A}$[128] The limit $\lim\limits_{\to} \boldsymbol{a}$ is roughly speaking the object that should naturally appear at the end: $(a_0 \to a_1 \to \cdots \to \lim\limits_{\to\ n}\ a_n)$. It need not exist in $\mathcal{A}$ at least it is easy to see that if $\mathcal{A} = \mathcal{S}ets$ then all inductive limits always exist!

A consequence of this good situation in the category $\mathcal{S}ets$ is that:

even if $\lim\limits_{\to\ n}\ a_n$ *does not exist in $\mathcal{A}$, it always exists in the larger category* $\widehat{\mathcal{A}}$.

An inductive system $\boldsymbol{a}$ defines an object in $\mathcal{A}$ *if* the limit $\lim\limits_{\to}\ a_n$ exists in $\mathcal{A}$, however it *always* defines a functor $\iota_{\boldsymbol{a}} = \ \mathop{\mathbf{lim}}\limits_{\substack{\longrightarrow \\ n}} \iota_{a_n} \in \ \widehat{\mathcal{A}}$, by

$$\iota_{\boldsymbol{a}}(c) \overset{\text{def}}{=} \ \lim_{\to\ n}\ \iota_{a_n}(c) = \ \lim_{\to\ n}\ \mathrm{Hom}_{\mathcal{A}}(c, a_n) \in \ \mathcal{S}ets.$$

(This definition uses the existence of inductive limits in the category $\mathcal{S}ets$!)

This allows us to <u>*think*</u> of the functor $\iota_{\boldsymbol{a}}$ as the limit of the inductive system $\boldsymbol{a}$ that exists in the larger category $\widehat{\mathcal{A}}$. All together, we can think of any inductive system *as if* it were an object $\mathop{\mathbf{lim}}\limits_{\longrightarrow} a_i$ in $\mathcal{A}$ (since we can identify it with $\boldsymbol{a} \in \widehat{\mathcal{A}}$). For this reason an inductive system in $\mathcal{A}$ is called an *ind-object of* $\mathcal{A}$ (while it really gives an object of $\widehat{\mathcal{A}}$). [129]

Examples. The basic example of inductive system is an increasing union. Some infinite increasing unions of $\Bbbk$-schemes are not $\Bbbk$-schemes but they are objects of the category of $\Bbbk(\overset{\text{def}}{=})\Bbbk$-$\widehat{Schemes}$. The most obvious examples are $\mathbb{A}^\infty$ (above) which should be a $\Bbbk$-variety but it is not, and the formal neighborhood of a closed subscheme (bellow).

A.6. **Category of $\Bbbk$-spaces (Yoneda completion of the category of $\Bbbk$-schemes).** This will be our main example of a Yoneda completion of a category. For examples of non-representable functors, i.e., functors which are in $\widehat{\mathcal{A}}$ but not in $\mathcal{A}$.

This is a geometric example. The geometry we use here is the algebraic geometry. Its geometric objects are called schemes and they are obtained by gluing schemes of a somewhat special type,

[128]Here *inductive* means that it stretches to the right, while for instance $(\cdots \leftarrow b_n \leftarrow b_1 \leftarrow b_0)$ would be called a *projective system.*

[129]Similarly one calls projective systems pro-objects of $\mathcal{A}$.

which are called affine schemes (like manifolds are all obtained by gluing open pieces of $\mathbb{R}^n$'s). We start with a brief review.

A.6.1. *Affine* $\Bbbk$*-schemes.* Fix a commutative ring $\Bbbk$.

An affine scheme S over $\Bbbk$ is determined by its algebra of functions $\mathcal{O}(S)$, which is a $\Bbbk$-algebra. Moreover, any commutative $\Bbbk$-algebra A is the algebra of functions on some $\Bbbk$-scheme – the scheme is called the spectrum of A and denoted $\mathrm{Spec}(A)$. So, affine $\Bbbk$-schemes are really the same as commutative $\Bbbk$-algebras, except that a map of affine schemes $X \xrightarrow{\phi} Y$ defines a map of functions $\mathcal{O}(Y) \xrightarrow{\phi^*} \mathcal{O}(X)$ in the opposite direction (the pull-back $\phi^*(f) = f \circ \phi$). The statement

information contained in two kinds of objects is the same but the directions reverse when one passes from geometry to algebra

is stated in categorical terms:

$$\textit{categories } \ \mathcal{A}ff\mathcal{S}ch_{\Bbbk} \ \textit{ and } (\mathcal{C}om\mathcal{A}lg_{\Bbbk})^o \textit{ are equivalent.}$$

[130]

The basic strategy. Our intuition is often geometric. So, one starts by translating geometric ideas into precise statements in algebra. These are then proved in algebra. Once sufficiently many geometric statements are verified in algebra, one can build up on these and do more purely in geometry.

A.6.2. *Formal neighborhood of* $0 \in \mathbb{A}^1$*.* Consider the contravariant functor on $\Bbbk$-*Schemes*

$$\Bbbk\text{-}Schemes \ni X \mapsto F(X) = \ \{f \in \mathcal{O}(X);\ f \text{ is nilpotent}\} \ \in \mathcal{S}ets.$$

It is an increasing union of subfunctors

$$\Bbbk\text{-}Schemes \ni X \mapsto F_n(X) = \ \{f \in \mathcal{O}(X);\ f^{n+1} = 0\} \ \in \mathcal{S}ets.$$

Looking for geometric interpretation of these functor we start with the n^{th} infinitesimal neighborhood $IN^n_{\mathbb{A}^1_{\Bbbk}}(0)$ of the point 0 in the line $\mathbb{A}^1_{\Bbbk} = \mathrm{Spec}(\Bbbk[x])$. This is the $\Bbbk$-scheme defined by the algebra

$$\mathcal{O}\big(IN^n_{\mathbb{A}^1_{\Bbbk}}(0)\big) \stackrel{\text{def}}{=} \Bbbk[x]/x^{n+1}, \quad \text{i.e.,} \quad IN^n_{\mathbb{A}^1}(0) \stackrel{\text{def}}{=} \mathrm{Spec}(\Bbbk[x]/x^{n+1}).$$

For instance, $IN^0_{\mathbb{A}^1}(0) = \{0\}$ is a point while $IN^1_{\mathbb{A}^1}(0)$ is a *double point*, etc.

We see that the functor F_n is representable – it is represented by the scheme $IN^n_{\mathbb{A}^1_{\Bbbk}}(0)$. Therefore, one should think of the functor F as the increasing union of infinitesimal neighborhoods of $0 \in \mathbb{A}^1$. For that reason we call F the *formal neighborhood of* $0 \in \mathbb{A}^1$.

[130] One can simplify this kind of thinking and *define* the category of affine schemes over $\mathbb{C}$ as the the opposite of the category of commutative $\mathbb{C}$-algebras. The part that would be skipped in this approach is how one develops a geometric point of view on affine schemes defined in this way.

A.6.3. *Formal neighborhood of a closed subscheme.* In general if Y is a closed subscheme of a scheme X given by the ideal $I_Y = \{f \in \mathcal{O}(X);\ f|Y = 0\}$, one can again define the n^{th} infinitesimal neighborhood of Y in X as an affine scheme

$$IN^n_X(Y) \stackrel{\text{def}}{=} \operatorname{Spec}(\mathcal{O}(X)/I_Y^{n+1}),$$

and then one defines the formal neighborhood $FN_X(Y)$ as a $\Bbbk$-space which is the union of infinitesimal neighborhoods, i.e., as the functor

$$\Bbbk\text{-}Schemes \ni Z \mapsto \cup_n\ Map[Z, IN^n_X(Y)].$$

A.7. **Groupoids (groupoid categories).** We consider a special class of categories, the *groupoid categories* . We get a new respect for categories when we notice that this special case of categories, is a common generalization of both groups and equivalence relations.

A.7.1. A groupoid category is a category such that all morphisms are invertible (i.e., isomorphisms).

A.7.2. *Example: Group actions and groupoids.* An action of a group G on a set X, produces a category X_G with $Ob(X_G) = X$ and

$$\operatorname{Hom}_{X_G}(a,b) \stackrel{\text{def}}{=} \{(b,g,a);\ g \in G \quad \text{and} \quad b = \ ga\}.$$

Here $1_a = (a,1,a)$ and the composition is given by multiplication in G: $(c,h,b)\circ(b,g,a) \stackrel{\text{def}}{=} (c,hg,a)$. This is a groupoid category: $(b,g^{-1},a)\circ(b,g,a) \stackrel{\text{def}}{=} (a,1,a)$.

A.7.3. *Example: Equivalence relations.* Any equivalence relation $\cong$ on a set X defines a category $X_{\cong}$ with $Ob(X_{\cong}) = X$ and $\operatorname{Hom}_{X_{\cong}}(a,b)$ is a point $\{b,a)\}$ if $a \cong b$ and otherwise $\operatorname{Hom}_{X_{\cong}}(a,b) = \emptyset$. The composition is $(c,b)\circ(b,a) \stackrel{\text{def}}{=} (c,a)$ and $1_a = (a,a)$. This is a groupoid category: $(a,b)\circ(b,a) = 1_a$.

A.7.4. *Lemma.* Let $\mathcal{C}$ be a groupoid category.

(a) A groupoid category $\mathcal{C}$ gives: a set $\pi_0(\mathcal{C})$ of isomorphism classes of objects of $\mathcal{C}$, and (b) for each object $a \in \mathcal{G}$ a group $\pi_1(\mathcal{C},a) \stackrel{\text{def}}{=} \operatorname{Hom}_{\mathcal{C}}(a,a)$.

(b) If $a,b \in \mathcal{C}$ are isomorphic then $\operatorname{Hom}_{\mathcal{C}}(a,b)$ is a *bitorsor* for $(\operatorname{Hom}_{\mathcal{C}}(a,a), \operatorname{Hom}_{\mathcal{C}}(b,b))$, i.e., a torsor for each of the groups $\operatorname{Hom}_{\mathcal{C}}(a,a)$ and $\operatorname{Hom}_{\mathcal{C}}(a,a)$, and the actions of the two groups commute.

(c) A groupoid category on one object is the same as a group.

A.7.5. *Examples.* (1) For the *action groupoid* associated to an action of G on X

$$\pi_0(X_G) = X/G \quad \text{and} \quad \pi_1(X_G,a) = \ G_a.$$

(2) If $X_{\cong}$ is the groupoid given by an equivalence relation $\cong$ on X then

$$\pi_0(X_{\cong}) = X/\cong \quad \text{and} \quad \pi_1(X_{\cong},a) = \ \{1\}.$$

A.7.6. *Remarks.* Passing from a groupoid category $\mathcal{C}$ to the set $\pi_0(\mathcal{C})$ of isomorphism classes in $\mathcal{C}$, the main information we forget is the *automorphism groups* $\mathrm{Hom}_{\mathcal{C}}(a,a) = \mathrm{Aut}_{\mathcal{C}}(a)$ of objects.

To see the importance of this loss, we will blame the formation of singularities in the invariant theory quotients on passing from a groupoid category to the set of isomorphism classes. Remember that when $G = \{\pm 1\}$ acting on $X = \mathbb{A}^2$, one can organize the set theoretic quotient X/G into algebraic variety $X//G$ which has one singular point – the image of $\mathbf{0} = (0, 0$. Recall that $\mathbf{0}$ is the only point in X which has a non-trivial stabilizer, i.e., which has a non-trivial automorphism group $\mathrm{Aut}_{X_{\cong}}(\mathbf{0})$ when we encode the action of G on X as a category structure $X_{\cong}$ on X.

So, the hint we get from this example is:

One may be able to remove some singularities in sets of isomorphism classes by remembering the automorphisms, i.e., remembering the corresponding groupoid category rather then just the set of isomorphism classes of objects.

This is the principle behind the introduction of *stack quotients*.

Appendix B. **Manifolds**

B.1. **Real manifolds.**

B.1.1. *Charts, atlases, manifolds.* A homeomorphism $U \xrightarrow{\phi} V$ with $M \overset{open}{\supseteq} U$ and $V \overset{open}{\subseteq} \mathbb{R}^n$ for some n, is called a *local chart* on the topological space M. Two charts $(U_k \xrightarrow{\phi} V_k)$ on M $(k \in \{i, j\})$, are said to be *compatible* if (for $U_{ij} = U_i \cap U_j$), the *comparison function* (or *transition function*),

$$V_j \supseteq \phi_j(U_{ij}) \xrightarrow{\phi_{ij} \overset{\mathrm{def}}{=} \phi \circ \phi_j{}^{-1}} \phi_i(U_{ij}) \subseteq V_i$$

is a C^∞-map between two open subsets of $\mathbb{R}^n$. An atlas on M is a family of compatible charts on M that cover M.

We say that any atlas defines on M a *structure of a manifold*, and two atlases define the same manifold structure if they are compatible, i.e., if their union is again an atlas.

So, "compatible" is an equivalence relation on atlases, and a structure of a manifold on a topological space M is precisely an equivalence class of compatible atlases on M. On the other hand, if $\mathcal{A}$ is an atlas on M the set $\widetilde{\mathcal{A}}$ of all charts on M that are compatible with the charts in $\mathcal{A}$ is a maximal atlas on M. So, any equivalence class of atlases contains the largest element and we can think of manifold structures on M as *maximal atlases* on M.[131]

B.1.2. *Once again.* A real manifold M of dimension n is a topological space M which is locally isomorphic to $\mathbb{R}^n$ in a smooth way and without contradictions. Here,

[131]Later, we will find a nicer way to describe the manifold structure in terms of *ringed spaces*.

- <u>*Locally isomorphic to* $\mathbb{R}^n$</u> means that we are given an open cover $U_i,\ i \in I$, of M, and for each $i \in I$ a topological identification (*homeomorphism*), $\phi : U_i \xrightarrow{\cong} V_i$ with V_i open in $\mathbb{R}^n$.
- <u>*Smooth way without contradictions*</u> means that for any $i, j \in I$ (and $U_{ij} = U_i \cap U_j$), the transition function

$$V_j \supseteq \phi_j(U_{ij}) \xrightarrow{\phi_{ij} \overset{\text{def}}{=} \phi \circ \phi_j{}^{-1}} \phi_i(U_{ij}) \subseteq V_i$$

is a C^∞-map between two open subsets of $\mathbb{R}^n$.

B.1.3. *The sheaf C^∞_M of smooth functions on a manifold M.* For any open $U \subseteq M$ we define $C^\infty(U, \mathbb{R})$ to consist of all functions $f : U \to \mathbb{R}$ such that for any chart $(U_i \xrightarrow{\phi} V_i)$ the function $f \circ \phi^{-1} :\ \phi_i(U \cap U_i) \to \mathbb{R}$ is C^∞ on the open subset $\phi_i(U \cap U_i) \subseteq V_i \subseteq \mathbb{R}^n$.

Because of the *no-contradiction* policy one does not have to check all charts, but only sufficiently many to cover U.

Lemma. (a) Though the definition of C^∞_M is complicated, locally we get just the usual smooth functions on $\mathbb{R}^n$. If U lies in some chart (U_i, ϕ, V_i) (i.e., in $U \subseteq U_i$), then ϕ_i gives identification $C^\infty(U) \cong C^\infty(\phi_i(U))$ of smooth fonctions on U with smooth functions on an open part of $\mathbb{R}^n$.

(b) C^∞_M is a sheaf of $\mathbb{R}$-algebras on M,, i.e.,

- (0) for each open $U \subseteq X$ $C^\infty(U)$ is an $\mathbb{R}$-algebra,
- (1) for each inclusion of open subsets $V \subseteq U \subseteq X$ the restriction map $C^\infty(U) \xrightarrow{\rho^U_V} C^\infty(V)$ is map of $\mathbb{R}$-algebras

and these data satisfy

- (Sh0) $\rho^U_U = id$
- (Sh1) (Transitivity of restriction) $\rho^U_V \circ \rho^U_V = \ \rho^U_W$ for $W \subseteq V \subseteq U$
- (Sh2) (Gluing) If $(W_j)_{j \in J}$ is an open cover of an open $U \subseteq M$[132], we ask that any family of compatible $f_j \in C^\infty(W_j),\ j \in J$, glues uniquely. This means that if all f_j agree on intersections in the sense that $\rho^{W_i}_{W_{ij}} f_i = \rho^{W_i}_{W_{ij}} f_j$ in $C^\infty(W_{ij})$ for any $i, j \in J$; then there is a unique $f \in C^\infty(U)$ such that $\rho^U_{W_j} f = f_j$ in $C^\infty(W_j),\ j \in J$.
- (Sh3) $C^\infty(\emptyset)$ is $\{0\}$.

Proof. (a) is clear from definitions. The notion of $\mathcal{F}$ is a sheaf", that appears in (b), is really a shorthand for "$\mathcal{F}$ is of local nature", i.e., "$\mathcal{F}$ is defined by some local property". Now C^∞_M is a sheaf because to check that a function f on U is smooth, one only has to check locally, i.e., one has to consider f on a small neighborhood of each point.

[132]We denote $W_{ij} = W_i \cap W_j$ etc.!

B.1.4. *Examples.* The following are real manifolds

(1) $M=\mathbb{R}^n$
(2) M an open subset of $\mathbb{R}^n$
(3) $M=S^1$ or $M=S^n$.
(4) $M=\mathbb{RP}^1$ or $M=\mathbb{RP}^n$.

B.1.5. *Category of real manifolds.* For two real manifolds M',M'' we define the set $\mathrm{Hom}(M',M'')=Map(M',M'')$ of *smooth maps* or *morphisms of manifolds* , to consist of all maps $F:M'\to M''$ which are smooth when checked in local charts.

This means that for each $x\in M'$ there are charts $M'\supseteq U_i \xrightarrow{\phi} V_i\subseteq\mathbb{R}^{m'}$ and $M''\supseteq U''_j \xrightarrow{\phi} V''_j\subseteq\mathbb{R}^{m''}$, such that $x\in U'_i$ and $F(x)\in U''_j$, and the map

$$\begin{array}{ccc} U'_i\cap F^{-1}U''_j & \xrightarrow{F} & U''_j \\ \phi'_i\Big\downarrow & & \phi''_j\Big\downarrow \\ V'_i\supseteq \phi'_i(U'_i\cap F^{-1}U''_j) & \xrightarrow{F_{ij}} & \phi''_j(U''_j)\subseteq V''_j \end{array}$$

is a smooth map between open subsets of $\mathbb{R}^{m'}$ and $\mathbb{R}^{m''}$.

Again, no-contradiction policy implies that if the above is true for one pair of charts at x and $F(x)$, it is true for any pair of charts.

B.1.6. *Examples.*

(1) For any manifold M, $\mathrm{Hom}(M,\mathbb{R}^n)=C^\infty(M,\mathbb{R})^n$.
(2) A smooth map $F\in\mathrm{Hom}(M,N)$ defines for any pair of open subsets $U\subseteq M$ and $V\subseteq N$ the pull-back map $C^\infty_N(V)\xrightarrow{F^*}C^\infty_M(U),\ g\mapsto F^*g=g\circ F|U$.

B.2. **The (co)tangent bundles.**

B.2.1. *Cotangent spaces $T^*_a(M)$.* The cotangent space at a point $m\in M$ is defined by

$$T^*_m(M)\stackrel{\text{def}}{=}\mathfrak{m}_a/\mathfrak{m}_a^2 \quad\text{for}\quad \mathfrak{m}_a\stackrel{\text{def}}{=}\{g\in C^\infty(M);\ g(a)=0\}.$$

For any open $U\subseteq M$ and $f\in C^\infty(U)$, the differential at a of f is defined as the image

$$d_af\stackrel{\text{def}}{=}(f-f(a))+\mathfrak{m}_a^2\in\ T^*_a(M)$$

of $f-f(a)$ in T^*_aM.

B.2.2. *Tangent spaces $T_a(M)$.* The *tangent vectors* at $a\in M$ are the "derivatives at a", i.e., all linear functionals in the tangent space

$$T_m(M)\stackrel{\text{def}}{=}\{\xi\in\mathrm{Hom}_{\mathbb{R}}[C^\infty(M),\mathbb{R}];\ \xi(fg)=\xi(f)\cdot g(a)+f(a)\cdot\xi(g)\}.$$

The *vector fields* on M are all "derivatives on M", i.e., all linear operators in

$$X(M)\stackrel{\text{def}}{=}\{\Xi\in\mathrm{Hom}_{\mathbb{R}}[C^\infty(M),C^\infty(M)];\ \Xi(fg)=\Xi(f)\cdot g+f\cdot\Xi(g)\}.$$

A vector field Ξ defines a tangent vector $\Xi_a \in T_a(M)$ at each point $a \in M$

$$\Xi_a(f) \stackrel{\text{def}}{=} (\Xi f)(a), \qquad f \in C^\infty(M).$$

B.2.3. *Local coordinates.* For any open $U \subseteq M$, we say that functions $x_1, ..., x_n \in C^\infty(U)$ form a coordinate system on U if $\phi = (x_1, ..., x_n) : U \to \mathbb{R}^n$ gives a chart, i.e.,

- $\phi(U)$ is open in $\mathbb{R}^n$,
- $\phi : U \to \phi(U)$ is a bijection, and
- the inverse function is a map of manifolds.

By the Implicit Function Theorem the last condition is equivalent to

For each $a \in U$, the differentials $d_a x_i$ form a basis of $T_a^* M$.

B.2.4. *Vector bundles.* Now we consider how to organize all tangent spaces $T_a M$, $a \in M$, into one manifold TM and what is the natural level of organization (structure) on TM.

Vector bundle is the relative version of the notion of a vector space. First, if M is a <u>set</u> a vector bundle V over M consists of a map $V \xrightarrow{p} M$ and a structure of a vector space on each fiber $V_m = p^{-1}(m)$, $m \in M$. Next, if M is a *topological space*, we also ask that V is a topological space, the map $V \xrightarrow{p} M$ is continuous and the vector space structure of the fibers does not change wildly in the sense that

each $m \in M$ has a neighborhood U such that there exists a homeomorphism $\phi : V|U \to U \times \mathbb{R}^n$ which

- *maps each fiber to a fiber, i.e., the diagram*

$$\begin{array}{ccc} V|U & \xrightarrow{\phi} & U \times \mathbb{R}^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle pr_U} \\ U & \xrightarrow{=} & U \end{array}$$

  commutes,
- *The restriction of ϕ to fibers is an isomorphism of vector spaces.*

Finally, if M is a *manifold*, we ask that V is a manifold, the map $V \xrightarrow{p} M$ is a map of manifolds and the vector space structure on fibers changes smoothly in the sense that

each $m \in M$ has a neighborhood U such that there exists an isomorphism of manifolds $\phi : V|U \to U \times \mathbb{R}^n$, which preserves fibers and the restrictions of ϕ to fibers are isomorphisms of vector spaces.

Examples. (a) For any manifold M,

$$TM \stackrel{\text{def}}{=} \cup_{a\in M}\ T_aM \quad \text{and} \quad T^*M \stackrel{\text{def}}{=} \cup_{a\in M}\ T_a^*M$$

are naturally vector bundles over the manifold M.

(b) For a vector bundle V on M, any map of manifolds $f : N \to M$ can be used to pull-back the vector bundle V to a vector bundle

$$f^*V \stackrel{\text{def}}{=} \ \cup_{n\in N}\ V_{f(n)}$$

on N. So, by definition $(f^*V)_n = V_{f(n)}$, i.e., the fiber of f^*V at $n \in N$ is the same as the fiber of V at $f(n) \in M$.

B.3. **Constructions of manifolds.**

B.3.1. *The differential of manifold maps.* A map of manifolds $f : M \to N$, produces for any open $V\subseteq N$ and $U = \subseteq M$ such that $f(U)\subseteq V$, the pull-back of functions

$$f^* : C^\infty_N(V) \to C^\infty_M(U), \quad \phi \mapsto f^*\phi \stackrel{\text{def}}{=} \phi\circ f|U.$$

For each $a \in M$, $f^*I^N_{f(a)}\subseteq I^M_a$, so we get a linear map

$$d^*_af :\ T^*_{f(a)}(N) = I^N_{f(a)}/(I^N_{f(a)})^2 \to I^M_a/(I^M_a)^2 = T^*_a(M), \quad d_af(d_{f(a)}\phi) \stackrel{\text{def}}{=} d_a(\phi\circ f).$$

In other words,

$$d_af(\ [\phi - \phi(f(a))] + (I^N_{f(a)})^2\ )\ =\ [\phi\circ f - (\phi\circ f)(a))] + (I^M_a)^2.$$

In the opposite (covariant) direction one has the map called the differential of f

$$d_af :\ T_a(M) \to T_{f(a)}(N),\ (d_af\xi)\phi \stackrel{\text{def}}{=} \xi(f^*\phi)\ =\ \xi(\phi\circ f)$$

which is the adjoint of d^f_a. In terms of the local coordinates x_i around $a \in M$ and y_j around $f(a) \in N$,

$$(d_af)\partial_{i,a}\ =\ \sum_j\ \partial_{i,a}(y_j\circ f)\cdot\partial_{j,f(a)}$$

and the matrix $(\partial_{i,a}(y_j\circ f))_{i,j}$ of d_af in the bases $\partial_{i,a}$, $\partial_{j,f(a)}$ is called the Jacobian of f at a.

B.3.2. *Theorem.* Let $f : M \to N$ be a map of manifolds which is of *constant rank* (i.e., all differentials $d_af : T_a(M) \to T_{f(a)}(N)$ have the same rank). Then the fibers $M_b \stackrel{\text{def}}{=} f^{-1}b,\ b \in N$, are naturally manifolds.

This is again a consequence of the Implicit Function Theorem.

B.3.3. *Examples.* Let $f \in C^\infty(M)$ and $b \in \mathbb{R}$ be such that $d_af \neq 0$ for any $a \in M_b$. Then M_b is a submanifold.

Proof. $d_af \neq 0$ for any $a \in M_b$, so the same is true for a in some neighborhood U of M_b. Now, $M_b = f^{-1}b =\ (f|U)^{-1}b$ and on U the rank is 1.

B.4. **Complex manifolds.** A complex manifold M of dimension n is a topological space M which is locally isomorphic to $\mathbb{C}^n$ in a holomorphic way and without contradictions. Here,

- *Locally isomorphic to* $\mathbb{C}^n$ means that we are given an open cover $U_i,\ i\in I$, of M, and for each $i\in I$ a topological identification (*homeomorphism*), $\phi: U_i\xrightarrow{\cong}V_i$ with V_i open in $\mathbb{C}^n$.
- *In a holomorphic way* means that for any $i,j\in I$, the transition function ϕ_{ij} is a holomorphic map between two open subsets of $\mathbb{C}^n$.[133]
- *No contradictions* means that for any $i,j,k\in I$, the two identifications of $\phi_k(U_{ijk})\subseteq V_k$ and $\phi_i(U_{ijk})\subseteq V_i$, are the same: $\phi_{ij}\circ\phi_{jk} \ = \ \phi_{ik}$.

We call each $(U_i \xrightarrow{\phi} V_i)$ a local chart on the manifold. A collection $(U_i \xrightarrow{\phi} V_i)_{i\in I}$, of charts on a topological space is said to be compatible if it satisfies the conditions *smooth way* and *no-contradictions*. A collection of compatible charts that cover M is called an *atlas* on M. We say that any atlas defines on M a structure of a manifold, and two atlases define the same manifold structure if they are compatible, i.e., if their union is again an atlas.

So a structure of a manifold on a topological space M can be viewed is an equivalence class of compatible atlases on M. On the other hand, if $\mathcal{A}$ is an atlas on M the set $\widetilde{\mathcal{A}}$ of all charts on M that are compatible with the charts in $\mathcal{A}$ is a maximal atlas on M. So, any equivalence class of atlases contains the largest element.

B.4.1. *The sheaf* $\mathcal{O}^{an}_M$ *of holomorphic functions on a manifold* M. For any open $U\subseteq M$ we define $\mathcal{O}^{an}(U,\mathbb{R})$ to consist of all functions $f:U\to\mathbb{R}$ such that for any chart $(U_i \xrightarrow{\phi} V_i)$ the function $f\circ\phi^{-1}:\ \phi_i(U\cap U_i)\to\mathbb{R}$ is $\mathcal{O}^{an}$ on the open subset $\phi_i(U\cap U_i)\subseteq V_i\subseteq\mathbb{R}^n$.

Because of the *no-contradiction* policy one does not have to check all charts, but only sufficiently many to cover U.

Lemma. (a) If U lies in some chart U_i then ϕ gives identification $\mathcal{O}^{an}(U)\cong\mathcal{O}^{an}(\phi_i(U))$ of holomorphic fonctions on U with holomorphic functions on an open part of $\mathbb{C}^n$.

(b) $\mathcal{O}^{an}_M$ is a sheaf of $\mathbb{C}$-algebras on M,, i.e.,

- (0) for each open $U\subseteq X$ $\mathcal{O}^{an}(U)$ is a $\mathbb{C}$-algebra,
- for each inclusion of open subsets $V\subseteq U\subseteq X$ the restriction map $\mathcal{O}^{an}(U)\xrightarrow{\rho^U_V}\mathcal{O}^{an}(V)$ is map of $\mathbb{C}$-algebras

and these data satisfy

- (Sh0) $\rho^U_U=id$
- (Sh1) (Transitivity of restriction) $\rho^U_V\circ\rho^U_V = \ \rho^U_W$ for $W\subseteq V\subseteq U$
- (Sh2) (Gluing) If $(W_j)_{j\in J}$ is an open cover of an open $U\subseteq M$ we ask that any family of compatible $f_j\in\mathcal{O}^{an}(W_j),\ j\in J$, glues uniquely.
- (Sh3) $\mathcal{O}^{an}(\emptyset)$ is $\{0\}$.

[133] ...

B.4.2. *Examples.*

(1) $M=\mathbb{C}^n$
(2) M an open subset of $\mathbb{C}^n$
(3) $M=\mathbb{CP}^1$ or $M=\mathbb{CP}^n$.

B.4.3. *Category of complex manifolds.* For two complex manifolds M',M'' we define the set $\mathrm{Hom}(M',M'')=Map(M',M'')$ of *holomorphic maps* or *morphisms of complex manifolds* to consist of all maps $F:M'\to M''$ which are holomorphic when checked in local charts.

B.4.4. *Examples.*

(1) For any manifold M, $\mathrm{Hom}(M,\mathbb{C}^n)=\mathcal{O}^{an}(M,\mathbb{C})^n$.
(2) A holomorphic map $F\in\mathrm{Hom}(M,N)$ defines for any pair of open subsets $U\subseteq M$ and $V\subseteq N$ the pull-back map $\mathcal{O}^{an}_N(V)\xrightarrow{F^*}\mathcal{O}^{an}_M(U)$, $g\mapsto F^*g=g\circ F|U$.

B.5. **Manifolds as ringed spaces.** We will see that a geometric space (for instance a manifold of a certain type) can naturally be thought of as a topological space with a sheaf of rings.

B.5.1. *Ringed spaces.* A ringed space consists of a topological space X and a sheaf of rings $\mathcal{O}$ on X. Usually we call $\mathcal{O}$ the structure sheaf of X and we denote it $\mathcal{O}_X$.

B.5.2. *Real manifolds as ringed spaces.* As we have seen, any real manifold M defines a ringed space (M,C^∞_M). Actually,

Lemma. (a) For a manifold M one can recover the manifold structure on M from the sheaf of rings C^∞_M[134]
(b) Manifolds are the same as ringed spaces $(X,\mathcal{O}_X)$ that are locally isomorphic to $(\mathbb{R}^n,C^\infty_{\mathbb{R}^n})$.[135]

B.5.3. *Complex manifolds as ringed spaces.* The story is the same. Any complex manifold M defines a locally ringed space $(M,\mathcal{O}^{an}_M)$. Actually, complex manifolds are the same as ringed spaces $(X,\mathcal{O}_X)$ that are locally isomorphic to $(\mathbb{C}^n,\mathcal{O}^{an}_{\mathbb{C}^n})$.

B.5.4. *Terminology.* We will speak of a $\Bbbk$-manifold $(M,\mathcal{O}_M)$ where $\Bbbk$ is either $\mathbb{R}$ or $\mathbb{C}$, and we will mean the above notion of a real manifold with $\mathcal{O}_M=C^\infty_M$ if $\Bbbk=\mathbb{R}$, or the above notion of a complex manifold with $\mathcal{O}_M=\mathcal{O}^{an}_M$ if $\Bbbk=\mathbb{C}$.

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[134] The largest atlas for the manifold M consists of all data $M\overset{open}{\supseteq}U\xrightarrow[homeomorphism]{\phi}V\overset{open}{\subseteq}\mathbb{R}^n$, such that for any $g\in C^\infty(V)$ the pull-back $g\circ\phi$ is in $C^\infty_M(U)$.

[135] This means that X can be covered by open sets U such that

(1) there is a homeomorphism $\phi:U\xrightarrow{\cong}V$ with V open in some $\mathbb{R}^n$, with the property that
(2) for any U' open in U, the restriction of ϕ to $U'\to\pi(U')=V'$ identifies $\mathcal{O}_X(U')$ and $C^\infty_{\mathbb{R}^n}(V')$.

B.5.5. *Use of sheaves.* Sheaves are more <u>fundamental</u> for $\mathbb{C}$-manifolds then for $\mathbb{R}$-manifolds because for an $\mathbb{R}$-manifold M, all information is contained in one ring $C^\infty(M)$, while for a $\mathbb{C}$-manifold the global functions need not contain enough information – for instance $\mathcal{O}^{an}(\mathbb{CP}^n) = \mathbb{C}$. This forces one to control all local function rather then just the global functions (i.e., the sheaf $\mathcal{O}_M$ rather then just $\mathcal{O}_M(M)$).

However, the general role of sheaves is that they control the relation between local and global objects, and this make them useful in many a context.

B.6. **Manifolds as locally ringed spaces.** We saw that geometric space can naturally be thought of as a ringed spaces, actually their geometric nature will be reflected in a special property of the corresponding ringed spaces – these are the *locally ringed spaces.*

B.6.1. *Stalks.* The stalk of the sheaf $\mathcal{O}$ at $a \in X$ is intuitively $\mathcal{O}(\mathcal{U})$ for a "very small neighborhood $\underline{U}$ of a". More precisely, if $a \in V \subseteq U$ then $\mathcal{O}(U)$ and $\mathcal{O}(V)$ are related by the restriction map $\mathcal{O}(U) \xrightarrow{\rho^U_V} \mathcal{O}(V)$, and the stalk at a is a certain limit of these restriction maps (called *inductive limit* or *colimit*), i.e.,

$$\mathcal{O}_a \stackrel{\text{def}}{=} \lim_{\substack{\rightarrow \\ U \ni a}} \mathcal{O}(U)$$

of $\mathcal{O}(U)$ over smaller and smaller neighborhoods U of a in X.

The elements of $\mathcal{O}_a$ are called the *germs* of $\mathcal{O}$-functions at a, and $\mathcal{O}_a$ can be described in en elementary way

(1) For any neighborhood U of a point a any $f \in \mathcal{O}(U)$ defines a germ $\underline{f}_a = \underline{(U,f)}_a \in \mathcal{O}_a$, and any germ is obtained in this way.
(2) Two germs $\underline{(U,f)}_a$ and $\underline{(V,g)}_a$ at a, are the same if there is neighborhood $W \subseteq U \cap V$ such that $f = g$ on W.

Then one defines the structure of a ring on $\mathcal{O}_a$ by

$$\underline{(U,f)}_a + \underline{(V,g)}_a \stackrel{\text{def}}{=} \underline{(U\cap V, f+g)}_a \quad \text{and} \quad \underline{(U,f)}_a \cdot \underline{(V,g)}_a \stackrel{\text{def}}{=} \underline{(U\cap V, f\cdot g)}_a .$$

B.6.2. *Local rings.* A commutative ring A is said to be a *local ring* if it has the *largest proper ideal.*

Examples.

(1) Any field is local, the largest ideal is 0.
(2) The ring of formal power series $\Bbbk[[X_1, ..., X_n]]$ over a field $\Bbbk$ is local, the largest ideal $\mathfrak{m}$ consists of series that vanish at 0 (i.e, the constant term is 0).
(3) $\mathbb{C}[x]$ is not at all local.

A commutative ring is local iff it has precisely one maximal ideal (then this is the largest ideal). Remember that maximal ideals correspond to the naive notion of "ordinary" points of a space. So, uniqueness of a maximal ideal in a ring A intuitively means that this ring corresponds to a *space with one ordinary point.*

B.6.3. *Locally ringed spaces.* We say that a ringed space $(X, \mathcal{O})$ is *locally ringed* if all $\mathcal{O}(U)$ are commutative rings and each stalk $\mathcal{O}_a,\ a \in X$, is a *local ring*, i.e., it has the *largest proper ideal.* This ideal is then denoted $\mathfrak{m}_a \subseteq \mathcal{O}_a$.

Example. The stalk of the sheaf of analytic functions $\mathcal{O}^{an}_{\mathbb{C}^n,0}$ consists of all formal series in n variables $f(Z_1, ..., Z_n) = \ sum_I\ f_I \cdot Z^I$ which converge on some ball around $0 \in \mathbb{C}^n$ (think of $\underline{(}U, f)_0$ as the expansion of f at 0). This is a local ring, and the largest ideal is

$$\mathfrak{m}_a \stackrel{\text{def}}{=} \mathcal{O}_a \cap \sum Z_i \cdot \mathbb{C}[[Z_1, ..., Z_n]] = \text{ all germs at } a \text{ of functions that vanish at } a.$$

Remark. Remember that a local ring intuitively corresponds to a space with one ordinary point. Therefore, it makes sense that the stalk $\mathcal{O}_{X,a}$ should be a local ring since $\mathcal{O}_{X,a}$ should only see one ordinary point – the point a.

B.6.4. *Manifolds as locally ringed spaces.* As we have seen, any manifold M (real or complex) defines a ringed space. Actually,

Lemma. The ringed space of any manifold M is a locally ringed space. The largest ideal $\mathfrak{m}_a$ of the stalk at a consists of germs of functions that vanish at a.

Proof. Let $\mathcal{O}$ be the structure sheaf (i.e., C^∞_M or $\mathcal{O}^{an}_M$) and let $\phi \in \mathcal{O}_a$ be the germ $\phi = \underline{\overline{(U, f)}}_a$ of a function at a. If $\phi \notin \mathfrak{m}_a$, i.e., $f(a) \neq 0$ then the restriction of f to the neighborhood $V = f^{-1}\mathbb{k}^*$ (for $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$) of a, is invertible. Therefore ϕ is invertible (so ϕ can not lie in a proper ideal!).

Appendix C. **Abelian categories**

An abelian category is a category $\mathcal{A}$ which has the formal properties of the category $\mathcal{A}b$, i.e., we can do in $\mathcal{A}$ all computations that one can do in $\mathcal{A}b$.

C.1. **Additive categories.** Category $\mathcal{A}$ is additive if

- (A0) For any $a, b \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(a,b)$ has a structure of abelian group such that then compositions are bilinear.
- (A1) $\mathcal{A}$ has a zero object,
- (A2) $\mathcal{A}$ has sums of two objects,
- (A3) $\mathcal{A}$ has products of two objects,

C.1.1. *Lemma.* (a) Under the conditions (A0),(A1) one has (A3) $\Leftrightarrow$ (A4).

(b) In an additive category $a \oplus b$ is canonically the same as $a \times b$,

For additive categories $\mathcal{A}, \mathcal{B}$ a functor $F : \mathcal{A} \to \mathcal{B}$ is additive if the maps $\mathrm{Hom}_{\mathcal{A}}(a', a'') \to \mathrm{Hom}_{\mathcal{B}}(Fa', Fa'')$ are always morphisms of abelian groups.

C.1.2. *Examples.* (1) $\mathfrak{m}(\Bbbk)$, (2) $\mathcal{F}ree(\Bbbk)$, (3) $\mathcal{F}ilt\mathcal{V}ect_{\Bbbk} \stackrel{\text{def}}{=}$ filtered vector spaces over $\Bbbk$.

C.2. **(Co)kernels and (co)images.** In module categories a map has kernel, cokernel and image. To incorporate these notions into our project of defining abelian categories we will find their abstract formulations.

C.2.1. *Kernels: Intuition.* Our intuition is based on the category of type $\mathfrak{m}(\Bbbk)$. For a map of $\Bbbk$-modules $M \xrightarrow{\alpha} N$

- the kernel $\mathrm{Ker}(\alpha)$ is a subobject of M,
- the restriction of α to it is zero,
- and this is the largest subobject with this property

C.2.2. *Categorical formulation.* Based on this, our general definition (in an additive category $\mathcal{A}$), of "k is a kernel of the map $a \xrightarrow{\alpha} b$", is

- we have a map $k \xrightarrow{\sigma} M$ from k to M,
- if we follow this map by α the composition is zero,
- map $k \xrightarrow{\sigma} M$ is universal among all such maps, in the sense that
  - · all maps into a, $x \xrightarrow{\tau} a$, which are killed by α,
  - · factor uniquely through k (i.e., through $k \xrightarrow{\sigma} a$).

  So, all maps from x to a which are killed by α are obtained from σ (by composing it with some map $x \to k$). This is the "universality" property of the kernel.

C.2.3. *Reformulation in terms of representability of a functor.* A compact way to restate the above definition is:

- The kernel of $a \xrightarrow{\alpha} b$ is any object that represents the functor

$$\mathcal{A} \ni x \mapsto {}_{\alpha}\mathrm{Hom}_{\mathcal{A}}(x,a) \stackrel{\text{def}}{=} \{\gamma \in \mathrm{Hom}_{\mathcal{A}}(x,a);\ \alpha\circ\gamma = 0\ \}.$$

One should check that this is the same as the original definition.

We denote the kernel by $\mathrm{Ker}(\alpha)$, but as usual, remember that

- this is not one specific object – it is only determined up to a canonical isomorphism,
- it is not only an object but a pair of an object and a map into a

.

C.2.4. *Cokernels.* In $\mathfrak{m}(\Bbbk)$ the cokernel of $M \xrightarrow{\alpha} N$ is $N/\alpha(M)$. So N maps into it, composition with α kills it, and the cokernel is universal among all such objects. When stated in categorical terms we see that we are interested in the functor

$$x \mapsto \mathrm{Hom}_{\mathcal{A}}(b,x)_{\alpha} \stackrel{\text{def}}{=} \{\tau \in \mathrm{Hom}_{\mathcal{A}}(b,x);\ \tau\circ\alpha = 0\ \},$$

and the formal definition is symmetric to the definition of a kernel:

- The cokernel of f is any object that represents the functor $\mathcal{A} \ni x \mapsto \mathrm{Hom}_{\mathcal{A}}(b,x)_{\alpha}$.

So this object $\mathrm{Coker}(\alpha)$ is supplied with a map $b \longrightarrow \mathrm{Coker}(\alpha)$ which is universal among maps from b that kill α.

C.2.5. *Images and coimages.* In order to define the image of α we need to use kernels and cokernels. In $\mathfrak{m}(\Bbbk)$, $Im(\alpha)$ is a subobject of N which is the kernel of $N \longrightarrow \alpha(M)$. We will see that the categorical translation obviously has a symmetrical version which we call coimage. Back in $\mathfrak{m}(\Bbbk)$ the coimage is $M/\mathrm{Ker}(\alpha)$, hence there is a canonical map $Coim(\alpha) = M/\mathrm{Ker}(\alpha) \longrightarrow Im(\alpha)$, and it is an isomorphism. This observation will be the final ingredient in the definition of abelian categories. Now we define

- Assume that α has cokernel $b \longrightarrow \mathrm{Coker}(\alpha)$, the image of α is $Im(\alpha) \stackrel{\text{def}}{=} \mathrm{Ker}[b \longrightarrow \mathrm{Coker}(\alpha)]$ (if it exists).
- Assume that α has kernel $\mathrm{Ker}(\alpha) \longrightarrow a$, the coimage of α is $Coim(\alpha) \stackrel{\text{def}}{=} \mathrm{Coker}[\mathrm{Ker}(\alpha) \longrightarrow a]$. (if it exists).

C.2.6. *Lemma.* If α has image and coimage, there is a canonical map $Coim(\alpha) \longrightarrow Im(\alpha)$, and it appears in a canonical factorization of α into a composition

$$a \longrightarrow Coim(\alpha) \longrightarrow Im(\alpha) \longrightarrow b.$$

C.2.7. *Examples.* (1) In $\mathfrak{m}(\Bbbk)$ the categorical notions of a (co)kernel and image have the usual meaning, and coimages coincide with images.

(2) In $\mathcal{F}ree(\Bbbk)$ kernels and cokernels need not exist.

(3) In $\mathcal{FV}\stackrel{\text{def}}{=}\mathcal{F}ilt\mathcal{V}ect_{\Bbbk}$ for $\phi\in\mathrm{Hom}_{\mathcal{FV}}(M_*,N_*)$ (i.e., $\phi:M\longrightarrow N$ such that $\phi(M_k)\subseteq N_k,\ k\in\mathbb{Z}$), one has

- $\mathrm{Ker}_{\mathcal{FV}}(\phi)=\ \mathrm{Ker}_{\mathcal{V}ect}(\phi)$ with the induced filtration $\mathrm{Ker}_{\mathcal{FV}}(\phi)_n=\ \mathrm{Ker}_{\mathcal{V}ect}(\phi)\cap M_n$,
- $\mathrm{Coker}_{\mathcal{FV}}(\phi)\ =\ N/\phi(M)$ with the induced filtration $\mathrm{Coker}_{\mathcal{FV}}(\phi)_n\ =$ image of N_n in $N/\phi(M)=\ [N_n+\phi(M)]/\phi(M)\cong\ N_n/\phi(M)\cap N_n$.
- $Coim_{\mathcal{FV}}(\phi)\ =\ M/\mathrm{Ker}(\phi)$ with the induced filtration $Coim_{\mathcal{FV}}(\phi)_n\ =$ image of M_n in $M/\mathrm{Ker}(\phi)=\ M_n+\mathrm{Ker}(\phi)/\mathrm{Ker}(\phi)\cong=\ M_n/M_n\cap\mathrm{Ker}(\phi)$,
- $Im_{\mathcal{FV}}(\phi)=\ Im_{\mathcal{V}ect}(\phi)\subseteq N$, with the induced filtration $Im_{\mathcal{FV}}(\phi)_n=\ Im_{\mathcal{V}ect}(\phi)\cap N_n$.

Observe that the canonical map $Coim_{\mathcal{FV}}(\phi)\longrightarrow Im_{\mathcal{FV}}(\phi)$ is an isomorphism of vector spaces $M/\mathrm{Ker}(\phi)\longrightarrow Im_{\mathcal{V}ect}(\phi)$, however the two spaces have filtrations induced from filtrations on M and N respectively, and these need not coincide.

For instance one may have M and N be two filtrations on the same space V, if $M_k\subseteq N_k$ then $\phi=1_V$ is a map of filtered spaces $M\longrightarrow N$ and $\mathrm{Ker}=0\mathrm{Coker}$ so that $Coim_{\mathcal{FV}}(\phi)=M$ and $Im_{\mathcal{FV}}(\phi)=N$ and the map $Coim_{\mathcal{FV}}(\phi)\longrightarrow Im_{\mathcal{FV}}(\phi)$ is the same as ϕ, but ϕ is an isomorphism iff the filtrations coincide: $M_k=N_k$.

C.3. **Abelian categories.** Category $\mathcal{A}$ is abelian if

- (A0-3) It is additive,
- It has kernels and cokernels (hence in particular it has images and coimages!),
- The canonical maps $Coim(\phi)\longrightarrow Im(\phi)$ are isomorphisms

C.3.1. *Examples.* Some of the following are abelian categories: (1) $\mathfrak{m}(\Bbbk)$ including $\mathcal{A}b=\mathfrak{m}(\mathbb{Z})$. (2) $\mathfrak{m}_{fg}(\Bbbk)$ if $\Bbbk$ is noetherian. (3) $\mathcal{F}ree(\Bbbk)\subseteq\ \mathcal{P}roj(\Bbbk)\subseteq\ \mathfrak{m}(\Bbbk)$. (4) $\mathcal{C}^\bullet(\mathcal{A})$. (5) Filtered vector spaces.

C.4. **Abelian categories and categories of modules.**

C.4.1. *Exact sequences in abelian categories.* Once we have the notion of kernel and cokernel (hence also of image), we can carry over from module categories $\mathfrak{m}(\Bbbk)$ to general abelian categories our homological train of thought. For instance we say that

- a map $i:a\longrightarrow b$ makes a into a subobject of b if $\mathrm{Ker}(i)=0$ (we denote it $a\hookrightarrow b$ or even informally by $a\subseteq b$, one also says that i is a monomorphism or informally that it is an inclusion),
- a map $q:b\longrightarrow c$ makes c into a quotient of b if $\mathrm{Coker}(q)=0$ (we denote it $b\twoheadrightarrow c$ and say that q is an epimorphism or informally that q is surjective),
- the quotient of b by a subobject $a\xrightarrow{i}b$ is $b/a\stackrel{\text{def}}{=}\mathrm{Coker}(i)$,

- a complex in $\mathcal{A}$ is a sequence of maps $\cdots A^n \xrightarrow{d^n} A^{n+1} \rightarrow \cdots$ such that $d^{n+1} \circ d^n = 0$, its cocycles, coboundaries and cohomologies are defined by $B^n = Im(d^n)$ is a subobject of $Z^n = \mathrm{Ker}(d^n)$ and $H^n = Z^n/B^n$;
- sequence of maps $a \xrightarrow{\mu} b \xrightarrow{\nu} c$ is exact (at b) if $\nu \circ \mu = 0$ and the canonical map $Im(\mu) \rightarrow \mathrm{Ker}(\nu)$ is an isomorphism.

Now with all these definitions we are in a familiar world, i.e., they work as we expect. For instance, sequence $0 \rightarrow a' \xrightarrow{\alpha} a \xrightarrow{\beta} a'' \rightarrow 0$ is exact iff a' is a subobject of a and a'' is the quotient of a by a', and if this is true then

$$\mathrm{Ker}(\alpha) = 0, \mathrm{Ker}(\beta) = a', \mathrm{Coker}(\alpha) = a'', \mathrm{Coker}(\beta) = 0, \mathrm{Im}(\alpha) = a', \mathrm{Im}(\beta) = a''.$$

The difference between general abelian categories and module categories is that while in a module category $\mathfrak{m}(\mathbb{k})$ our arguments often use the fact that $\mathbb{k}$-modules are after all abelian groups and sets (so we can think in terms of their elements), the reasoning valid in any abelian category has to be done more formally (via composing maps and factoring maps through intermediate objects). However, this is mostly appearances – if we try to use set theoretic arguments we will not go wrong:

C.4.2. *Theorem.* [Mitchell] Any abelian category is equivalent to a full subcategory of some category of modules $\mathfrak{m}(\mathbb{k})$.

Appendix D. **Abelian category of sheaves of abelian groups**

In this section we fill in some details in the construction of the cohomology of sheaves. We check that the category of sheaves of abelian groups on a given topological space, has all ingredients needed in order to use the homological algebra, i.e., it is an abelian category with enough injectives.

For a topological space X we will denote by $\mathcal{S}h(X) = \mathcal{S}heaves(X, \mathcal{A}b)$ the category of sheaves of abelian groups on X. Since a sheaf of abelian groups is something like an abelian group smeared over X, we hope that $\mathcal{S}h(X)$ is again an abelian category, i.e., that one can do the computations here the same way as one can do in the category $\mathcal{A}b$ of abelian groups. However,

D.0.3. *Presheaves and sheafification.* When we attempt to construct the cokernels of maps, we find that the first idea does not quite work – it produces something like a sheaf but without the gluing property. This forces us to

- (i) generalize the notion of sheaves to a weaker notion of a presheaf,
- (ii) find a canonical procedure that improves a presheaf to a sheaf.

We will also see another example that requires the same strategy: the *pull-back* operation on sheaves.

Now it is easy to check that we indeed have an abelian category. What allows us to compute in this abelian category is the lucky break that one can understand kernels, cokernels, images and exact sequences just by looking at the *stalks* of sheaves.

D.0.4. *Stalks of sheaves.* In order to think of sheaves as a refined notion of functions, we would like to attach to a sheaf of abelian groups $\mathcal{A}$ its "value" $\mathcal{A}_x$ at each point $x \in X$. For that one should consider the groups $\mathcal{A}(U)$ for smaller and smaller neighborhoods of a, and in fact, one can actually pass to the limit of such groups $\mathcal{A}(U)$. The limit group

$$\mathcal{A}_a \stackrel{\text{def}}{=} \varinjlim_{U \ni a} \mathcal{A}(U)$$

is called the *stalk* of $\mathcal{A}$ at a. The collection of all stalks $\mathcal{A}_x,\ x \in X$, does not record all structure of a sheaf but it suffices for some purposes.

D.1. **Categories of (pre)sheaves.** A *presheaf of sets* $\mathcal{S}$ on a topological space $(X, \mathcal{T})$ consists of the following data:

- for each open $U \subseteq X$ a set $\mathcal{S}(U)$,
- for each inclusion of open subsets $V \subseteq U \subseteq X$ a map $\mathcal{S}(U) \xrightarrow{\rho^U_V} \mathcal{S}(V)$ (called the restriction map);

and these data are required to satisfy

- (Sh0)(Transitivity of restriction) $\rho^U_V \circ \rho^U_V = \rho^U_W$ for $W \subseteq V \subseteq U$

D.1.1. *Sheaves.* Now we can define sheaves as a special case of presheaves.

A *sheaf of sets* on a topological space $(X,\mathcal{T})$ is a presheaf $\mathcal{S}$ which also satisfies

- (Sh1) (Gluing) Let $\mathcal{U}=(U_i)_{i\in I}$ be an open cover of an open $U\subseteq X$ (We denote $U_{ij}=U_i\cap U_j$ etc.). We ask that any family of compatible sections $f_i\in\mathcal{S}(U_i),\ i\in I$, glues uniquely. This means that if sections f_i agree on intersections in the sense that $\rho^{U_i}_{U_{ij}}f_i=\rho^{U_i}_{U_{ij}}f_j$ in $\mathcal{S}(U_{ij})$ for any $i,j\in I$; then there is a unique $f\in\mathcal{S}(U)$ such that $\rho^{U}_{U_i}f=f_i$ in $\mathcal{S}(U_i),\ i\in I$.
- $\mathcal{S}(\emptyset)$ is a *point*.

D.1.2. *Remarks.* (1) Presheaves of sets on X form a category $\mathit{pre}\mathcal{S}\mathit{heaves}(X,\mathcal{S}\mathit{ets})$ when $\mathrm{Hom}(\mathcal{A},\mathcal{B})$ consists of all systems $\phi=(\phi_U)_{U\subseteq X\ open}$ of maps $\phi_U:\mathcal{A}(U)\to\mathcal{B}(U)$ which are compatible with restrictions, i.e., for $V\subseteq U$

$$\begin{array}{ccc}\mathcal{A}(U) & \xrightarrow{\phi_U} & \mathcal{B}(U)\\ \rho^U_V\big\downarrow & & \rho^U_V\big\downarrow\\ \mathcal{A}(V) & \xrightarrow{\phi_V} & \mathcal{B}(V)\end{array}.$$

(One reads the diagram above as : "the diagram ... commutes".) The sheaves form a full subcategory $\mathit{pre}\mathcal{S}\mathit{heaves}(X,\mathcal{S}\mathit{ets})$ of $\mathcal{S}\mathit{heaves}(X,\mathcal{S}\mathit{ets})$.

(2) We can equally define categories of sheaves of abelian groups, rings, modules, etc. For a sheaf of abelian groups we ask that all $\mathcal{A}(U)$ are abelian groups, all restriction morphisms are maps of abelian groups, and we modify the least interesting requirement (Sh2): $\mathcal{S}(\phi)$ is the trivial group $\{0\}$. In general, for a category $\mathcal{A}$ one can define categories $\mathit{pre}\mathcal{S}\mathit{heaves}(X,\mathcal{A})$ and $\mathcal{S}\mathit{heaves}(X,\mathcal{A})$ similarly (the value on $\emptyset$ should be the final object of $\mathcal{A}$).

D.2. **Sheafification of presheaves.** We will use the wish to pull-back sheaves as a motivation for a procedure that improves presheaves to sheaves.

D.2.1. *Functoriality of sheaves.* Recall that for any map of topological spaces $X\xrightarrow{\pi}Y$ one wants a pull-back functor $\mathcal{S}\mathit{heaves}(Y)\xrightarrow{\pi^{-1}}\mathcal{S}\mathit{heaves}(X)$.[136] The natural formula is

$$\underline{\pi^{-1}}(\mathcal{N})\ (U)\overset{\text{def}}{=}\lim_{\substack{\to\\ V\supseteq\pi(U)}}\ \mathcal{N}(V),$$

where limit is over open $V\subseteq Y$ that contain $\pi(U)$, and we say that $V'\le V''$ if V'' better approximates $\pi(U)$, i.e., if $V''\subseteq V'$.

D.2.2. *Lemma.* This gives a functor of presheaves $\mathit{pre}\mathcal{S}\mathit{heaves}(X)\xrightarrow{\pi^{-1}}\mathit{pre}\mathcal{S}\mathit{heaves}(Y)$.

Proof. For $U'\subseteq U$ open, $\underline{\pi^{-1}}\mathcal{N}(U')=\lim_{\to\ V\supseteq\pi(U')}\mathcal{N}(V)$ and $\underline{\pi^{-1}}\mathcal{N}(U)=\lim_{\to\ V\supseteq\pi(U)}\mathcal{N}(V)$ are limits of inductive systems of $\mathcal{N}(V)$'s, and the second system is a subsystem of the first one, this gives a canonical map $\underline{\pi^{-1}}\mathcal{N}(U)\to\underline{\pi^{-1}}\mathcal{N}(U')$.

[136]A special case of this, when we pull-back to a point, will be the notion of a *stalk* of a sheaf.

D.2.3. *Remarks.* Even if $\mathcal{N}$ is a sheaf, $\underline{\pi^{-1}}(\mathcal{N})$ need not be sheaf.

For that let $Y = pt$ and let $\mathcal{N} = S_Y$ be the constant sheaf of sets on Y given by a set S. So, $S_Y(\emptyset) = \emptyset$ and $S_Y(Y) = S$. Then $\underline{\pi^{-1}}(S_Y)\ (U) = \begin{cases} \emptyset & if\ U = \emptyset, \\ S & U \neq \emptyset \end{cases}$. We can say: $\underline{\pi^{-1}}(S_Y)\ (U) =$ constant functions from U to S. However, we have noticed that constant functions do not give a sheaf, so we need to correct the procedure $\underline{\pi^{-1}}$ to get sheaves from sheaves. For that remember that for the presheaf of constant functions there is a related sheaf S_X of *locally constant* functions.

Our problem is that the presheaf of constant functions is defined by a global condition (constancy) and we need to change it to a local condition (local constancy) to make it into a sheaf. So we need the procedure of

D.2.4. *Sheafification.* This is a way to improve any presheaf of sets $\mathcal{S}$ into a sheaf of sets $\widetilde{\mathcal{S}}$. We will imitate the way we passed from constant functions to locally constant functions. More precisely, we will obtained the sections of the sheaf $\tilde{\mathcal{S}}$ associated to the presheaf $\mathcal{S}$ in two steps:

(1) we glue systems of local sections s_i which are compatible in the weak sense that they are <u>locally</u> the same, and
(2) we identify two results of such gluing if the local sections in the two families are <u>locally</u> the same.

Formally these two steps are performed by replacing $\mathcal{S}(U)$ with the set $\widetilde{\mathcal{S}}(U)$, defined as the set of all equivalence classes of systems $(U_i, s_i)_{i\in I}$ where

(1) Let $\widehat{\mathcal{S}}(U)$ be the class of all systems $(U_i, s_i)_{i\in I}$ such that
   - $(U_i)_{i\in I}$ is an open cover of U and s_i is a section of $\mathcal{S}$ on U_i,
   - sections s_i are *weakly compatible* in the sense that they are locally the same, i.e., for any $i', i'' \in I$ sections $s_{i'}$ and $s_{i''}$ are the same near any point $x \in U_{i'i''}$. (Precisely, this means that there is neighborhood W such that $s_{i'}|W = s_{i''}|W$.)
(2) We say that two systems $(U_i, s_i)_{i\in I}$ and $(V_j, t_j)_{j\in J}$ are $\equiv$, iff for any $i \in I,\ j \in J$ sections s_i and t_j are weakly equivalent (i.e., for each $x \in U_i \cap V_j$, there is an open set W with $x \in W \subseteq U_i \cap V_j$ such that "$s_i = t_j$ on W" in the sense of restrictions being the same).

D.2.5. *Remark.* The relation $\equiv$ on $\widehat{\mathcal{S}}(U)$ really says that $(U_i, s_i)_{i\in I} \equiv (V_j, t_j)_{j\in J}$ iff the disjoint union $(U_i, s_i)_{i\in I} \sqcup (V_j, t_j)_{j\in J}$ is again in $\widehat{\mathcal{S}}(U)$.

D.2.6. *Lemma.* (a) $\equiv$ is an equivalence relation.

(b) $\widetilde{\mathcal{S}}(U)$ is a presheaf and there is a canonical map of presheaves $\mathcal{S} \xrightarrow{q} \widetilde{\mathcal{S}}$.

(c) $\widetilde{\mathcal{S}}$ is a sheaf.

Proof. (a) is obvious.

(b) The restriction of a system $(U_i, s_i)_{i\in I}$ to $V \subseteq U$ is the system $(U_i \cap V, s_i|U_i \cap V)_{i\in I}$. The weak compatibility of restrictions $s_i|U \cap V$ follows from the weak compatibility of sections s_i. Finally, restriction is compatible with $\equiv$, i.e., if $(U_i', s_i')_{i\in I}$ and $(U_j'', s_j'')_{j\in J}$ are $\equiv$, then so are $(U_i' \cap V, s_i'|U_i' \cap V)_{i\in I}$ and $(U_j'' \cap V, s_j''|U_j'' \cap V)_{j\in J}$.

The map $\mathcal{S}(U) \to \widetilde{\mathcal{S}}(U)$ is given by interpreting a section $s \in \mathcal{S}(U)$ as a (small) system: open cover of $(U_i)_{i\in\{0\}}$ is given by $U_0 = U$ and $s_0 = s$.

(c') Compatible systems of sections of the presheaf $\widetilde{\mathcal{S}}$ glue. Let $V^j,\ j \in J$, be an open cover of an open $V\subseteq X$, and for each $j \in J$ let $\sigma^j = [(U_i^j, s_i^j)_{i\in\ I_j}]$ be a section of $\widetilde{\mathcal{S}}$ on V_j. So, σ^j is an equivalence class of the system $(U_i^j, s_i^j)_{i\in\ I_j}$ consisting of an open cover $U_i^j,\ i \in\ I_j$, of V_j and weakly compatible sections $s_j^i \in \mathcal{S}(U_j^i)$.

Now, if for any $j, k \in J$ sections $\sigma^j =\ [(U_p^j, s_p^j)_{p\in\ I_j}]$ and $\sigma^k =\ [(U_q^k, s_q^k)_{q\in\ I_k}]$ of $\widetilde{\mathcal{S}}$ on V^j and V^k, agree on the intersection V^{jk}. This means that for any j, k $\sigma^j|V^{jk} = \sigma^k|V^{jk}$, i.e.,

$$(U_p^j \cap V^{jk}, s_p^j|U_p^j \cap V^{jk})_{p\in\ I_j} \quad\equiv\quad (U_q^k \cap V^{jk}, s_q^k|U_q^k \cap V^{jk})_{q\in\ I_k}.$$

This in turn means that for $j, k \in J$ and any $p \in I_j,\ q \in I_k$, sections s_p^j and s_q^k are weakly compatible. Since all sections $s_p^j,\ j \in J,\ p \in I_j$ are weakly compatible, the disjoint union of all systems $(U_i^j, s_i^j)_{i\in\ I_j},\ j \in J$ is a system in $\widehat{\mathcal{S}}(V)$. Its equivalence class σ is a section of $\widetilde{\mathcal{S}}$ on V, and clearly $\sigma|V^j = \sigma^j$.

(c") Compatible systems of sections of the presheaf $\widetilde{\mathcal{S}}$ glue uniquely. If $\tau \in \widetilde{\mathcal{S}}(V)$ is the class of a system $(U_i, s^i)_{i\in I}$ and $\tau|V^j = \sigma^j$ then σ's are compatible with all s_p^j's, hence $(U_i, s^i)_{i\in I} \equiv \sqcup_{j\in J}(U_i^j, s_i^j)_{i\in\ I_j}$, hence $\tau = \sigma$.

D.2.7. *Sheafification as a left adjoint of the forgetful functor.* As usual, we have not invented something new: it was already there, hidden in the more obvious forgetful functor

D.2.8. *Lemma.* Sheafification functor $pre\mathcal{S}heaves \ni \mathcal{S}\mapsto\widetilde{\mathcal{S}} \in \mathcal{S}heaves$, is the left adjoint of the inclusion $\mathcal{S}heaves\subseteq pre\mathcal{S}heaves$, i.e, for any presheaf $\mathcal{S}$ and any sheaf $\mathcal{F}$ there is a natural identification

$$\mathrm{Hom}_{\mathcal{S}heaves}(\widetilde{\mathcal{S}}, \mathcal{F}) \xrightarrow{\cong} \mathrm{Hom}_{pre\mathcal{S}heaves}(\mathcal{S}, \mathcal{F}).$$

Explicitly, the bijection is given by $(\iota_{\mathcal{S}})_*\alpha =\ \alpha\circ\iota_{\mathcal{S}}$, i.e., $(\widetilde{\mathcal{S}} \xrightarrow{\alpha} \mathcal{F})\mapsto (\mathcal{S} \xrightarrow{\iota_{\mathcal{S}}} \widetilde{\mathcal{S}} \xrightarrow{\alpha} \mathcal{F})$.

D.3. **Inductive limits (or "colimits") of abelian groups.** Remember that we want to define the stalk of a presheaf $\mathcal{A}$ at a point x as the limit over (diminishing) neighborhoods U of x

$$\mathcal{A}_x \stackrel{\text{def}}{=} \varinjlim\ \mathcal{A}(U).$$

This will mean that

(1) any $s \in \mathcal{A}(U)$ with $U \ni x$ defines an element s_x of the stalk,
(2) all elements of $\mathcal{A}_x$ arise in this way, and
(3) For $s' \in \mathcal{A}(U')$ and $s'' \in \mathcal{A}(U'')$ one has $s'_x = s''_x$ iff for some neighborhood W of x in $U' \cap U''$ one has $s' = s''$ on W.

This can be achieved in the following way:

D.3.1. *Lemma.* (a) The relation $\sim$ defined on the disjoint union

$$\sqcup_{U\ni x}\ \mathcal{A}(U) \overset{\text{def}}{=} \cup_U\ \ \mathcal{A}(U)\times\{U\},$$

by

$$(a,U)\sim(b,V)\ \ \textit{(for } a\in\mathcal{A}(U),\ \ b\in\mathcal{A}(V)\textit{), if there is some } W{\subseteq}U\cap V\ \textit{ such that}$$
$$\textit{“}a=b\ \textit{ in } \mathcal{A}(W)\textit{”, i.e., if } \rho_W^U a=\rho_W^V b\ ,$$

is an equivalence relation.

(b) The quotient $\lim\limits_{\to\ U\ni x}\ \mathcal{A}(U) \overset{\text{def}}{=} [\sqcup_U\ \mathcal{A}(U)]/\sim$, has a canonical structure of an abelian group, and it satisfies the above properties (1-3).

D.3.2. *Inductive limits.* One can skip the remainder of this subsection. We just gibe the categorical framework of the above construction of a limit. An *inductive system* of objects in a category $\mathcal{C}$, over a partially ordered set $(I,\le)$, consists of

- objects $a_i\in\mathcal{C},\ i\in I$; and
- maps $\phi_{ji}: a_i\longrightarrow a_j$ for all $i\le j$ in I;

such that

$$\phi_{ii}=1_{a_i},\ i\in I \quad \text{and} \quad \phi_{kj}{\circ}\phi_{ji}=\phi_{ki},\quad i\le j\le k.$$

Its *limit* is a pair $(a,(\rho_i)_{i\in I})$ of $a\in\mathcal{C}$ and maps $\rho_i: a_i\longrightarrow a$ such that

(1) $\rho_j{\circ}\phi_{ji}=\rho_i$ for $i\le j$, and moreover
(2) $(a,(\rho_i)_{i\in I})$ is universal with respect to this property in the sense that for any $(a',(\rho'_i)_{i\in I})$ that satisfies $\rho'_j{\circ}\phi_{ji}=\rho'_i$ for $i\le j$, there is a unique map $\rho: a\longrightarrow a'$ such that $\rho'_i=\rho{\circ}\rho_i,\ i\in I$.

Informally, we write: $\lim\limits_{\to\ I,\le}\ a_i=\ a$.

D.3.3. *Limits in sets, abelian groups, modules and such.* In each of the categories $\mathcal{S}ets,\mathcal{A}b,\mathfrak{m}(\Bbbk)$ inductive limits exist and are calculated in the following way

Lemma. Let $(I,\le)$ be a partially ordered set such that for any $i,j\in I$ there is some $k\in I$ such that $i\le k\ge j$. Let the family of sets $(A_i)_{i\in I}$ and maps $(\phi_{ji}: A_i\longrightarrow A_j)_{i\le j}$ be an inductive system of sets.

(1) The relation $\sim$ defined on the disjoint union $\sqcup_{i\in I}\ A_i\overset{\text{def}}{=}\cup_{i\in I}\ \ A_i\times\{i\}$ by
$$(a,i)\sim(b,j)\ \ \textit{(for } a\in A_i,\ \ b\in A_j\textit{), if there is some } k\ge i,j\ \textit{ such that}$$
$$\textit{“}a=b\ \textit{ in } A_k\textit{”, i.e., if } \phi_{ki}a=\phi_{kj}b\ ,$$
is an equivalence relation.
(2) $\lim\limits_{\to} A_i$ is the quotient $[\sqcup_{i\in I}\ A_i]/\sim$ of the disjoint union by the above equivalence relation.

Corollary. (a) For an inductive system of abelian groups (or sets) A_i over $(I,\leq)$, inductive limit $\varinjlim A_i$ can be described by

- for $i\in I$ any $a\in A_i$ defines an element $\overline{a}$ of $\varinjlim A_i$,
- all elements of $\varinjlim A_i$ arise in this way, and
- for $a\in A_i$ and $b\in A_j$ one has $\overline{a}=\overline{b}$ iff for some $k\in I$ with $i\leq k\geq j$ one has $a=b$ in A_k.

(b) For a subset $K\subseteq I$ one has a canonical map $\varinjlim_{i\in K} A_i \longrightarrow \varinjlim_{i\in I} A_i$.

D.4. **Stalks.**

D.4.1. *Stalks of a sheaf.* We want to restrict a sheaf of sets $\mathcal{F}$ on a topological space X to a point $a\in X$. The restriction $\mathcal{F}|a$ is a sheaf on a point, so it just one set $\mathcal{F}_a \overset{\text{def}}{=} (\mathcal{F}|a)(\{a\})$ called the stalk of $\mathcal{F}$ at a. What should $\mathcal{F}_a$ be? It has to be related to all $\mathcal{F}(U)$ where $U\subseteq X$ is is open and contains a, and $\mathcal{F}(U)$ should be closer to $\mathcal{F}_a$ when U is a smaller neighborhood. A formal way to say this is that

- (i) the set $\mathcal{N}_a$ of neighborhoods of a in X is partially ordered by $U\leq V$ if $V\subseteq U$,
- (ii) the values of $\mathcal{F}$ on neighborhoods $(\mathcal{F}(U))_{U\in\mathcal{N}_a}$ form an inductive system,
- (iii) we define the stalk by $\mathcal{F}_a \overset{\text{def}}{=} \varinjlim_{U\in\mathcal{N}_a} \mathcal{F}(U)$.

Example. The stalk at the origin of a the sheaf $\mathcal{H}_{\mathbb{C}}$ of holomorphic functions on $\mathcal{C}$ is canonically identified with the ring of convergent power series. ("Convergent" means that the series converges on *some* disc around the origin.)

D.4.2. *Lemma.* For a presheaf $\mathcal{S}$, the canonical map $\mathcal{S}\longrightarrow\widetilde{\mathcal{S}}$ is an isomorphism on stalks.

Proof. We consider a point $a\in X$ as a map $\text{pt}=\{a\}\xrightarrow{i} X$, so that $\mathcal{A}_x=i^{-1}\mathcal{A}$. For a sheaf B on the point

$$\text{Hom}_{\mathcal{S}h(pt)}(i^{-1}\widetilde{\mathcal{S}},\mathcal{B})\cong \text{Hom}_{\mathcal{S}h(X)}(\widetilde{\mathcal{S}},i_*\mathcal{B})\cong \text{Hom}_{pre\mathcal{S}h(X)}(\mathcal{S},i_*\mathcal{B})$$
$$\cong \text{Hom}_{pre\mathcal{S}h(pt)}(i^{-1}\mathcal{S},\mathcal{B}) = \text{Hom}_{\mathcal{S}h(pt)}(i^{-1}\mathcal{S},\mathcal{B}).$$

D.4.3. *Germs of sections and stalks of maps.* For any neighborhood U of a point x we have a canonical map $\mathcal{S}(U)\longrightarrow \varinjlim_{V\ni x} \mathcal{S}(V)\overset{\text{def}}{=}\mathcal{S}_x$ (see lemma D.3.3.b), and we denote the image of a section $s\in\Gamma(U,\mathcal{S})$ in the stalk $\mathcal{S}_x$ by s_x, and we call it the *germ* of the section at x. The germs of two sections are the same at x iff the sections are the same on some (possibly very small) neighborhood of x (this is again by the lemma D.3.3.b).

A map of sheaves $\phi:\mathcal{A}\to\mathcal{B}$ defines for each $x\in M$ a map of stalks $\mathcal{A}_x\to\mathcal{B}_x$ which we denote ϕ_x. It comes from a map of inductive systems given by ϕ, i.e., from the system of maps $\phi_U:\mathcal{A}(U)\longrightarrow\mathcal{B}(U),\ U\ni x$; and on germs it is given by $\phi_x(a_x)=\ [\phi_U(a)]_x,\ a\in\mathcal{A}(U)$.

Example. For instance, let $\mathcal{A} = \mathcal{H}_{\mathbb{C}}$ be the sheaf of holomorphic functions on $\mathbb{C}$. Remember that the stalk at $a \in \mathbb{C}$ can be identified with all convergent power series in $z - a$. Then the germ of a holomorphic function $f \in \mathcal{H}_{\mathbb{C}}(U)$ at a can be thought of as the power series expansion of f at a. An example of a map of sheaves $\mathcal{H}_{\mathcal{C}} \xrightarrow{\Phi} \mathcal{H}_{\mathbb{C}}$ is the multiplication by an entire function $\phi \in \mathcal{H}_{\mathbb{C}}(\mathbb{C})$, its stalk at a is the multiplication of the the power series at a by the power series expansion of ϕ at a.

D.4.4. The following lemma shows how much the study of sheaves reduces to the study of their stalks.

Lemma. (a) Maps of sheaves $\phi, \psi : \mathcal{A} \rightarrow \mathcal{B}$ are the same iff the maps on stalks are the same, i.e., $\phi_x = \psi_x$ for each $x \in M$.

(b) Map of sheaves $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is an isomorphism iff ϕ_x is an isomorphism for each $x \in M$.

D.5. **Inverse and direct images of sheaves.**

D.5.1. *Pull back of sheaves (finally!)* Now we can define for any map of topological spaces $X \xrightarrow{\pi} Y$ a pull-back functor

$$\mathcal{S}heaves(Y) \xrightarrow{\pi^{-1}} \mathcal{S}heaves(X), \qquad \pi^{-1}\mathcal{N} \overset{\text{def}}{=} \widetilde{\underline{\pi^{-1}}\mathcal{N}}.$$

D.5.2. *Examples.* (a) A point $a \in X$ can be viewed as a map $\{a\} \xrightarrow{\rho} X$. Then $\rho^{-1}\mathcal{S}$ is the stalk $\mathcal{S}_a$.

(b) Let $a : X \rightarrow \text{pt}$, for any set S one has $S_X = a^{-1}S$.

D.5.3. *Direct image of sheaves.* Besides the pull-back of sheaves which we defined in D.5.1, there is also a much simpler procedure of the push-forward of sheaves:

D.5.4. *Lemma.* (Direct image of sheaves.) Let $X \xrightarrow{\pi} Y$ be a map of topological spaces. For a sheaf $\mathcal{M}$ on X, formula

$$\pi_*(\mathcal{M})\ (V) \overset{\text{def}}{=} \mathcal{M}(\pi^{-1}V),$$

defines a sheaf $\pi_*\mathcal{M}$ on Y, and this gives a functor $\mathcal{S}heaves(X) \xrightarrow{\pi_*} \mathcal{S}heaves(Y)$.

D.5.5. *Adjunction between the direct and inverse image operations.* The two basic operations on sheaves are related by adjunction:

Lemma. For sheaves $\mathcal{A}$ on X and $\mathcal{B}$ on Y one has a natural identification

$$\mathrm{Hom}(\pi^{-1}\mathcal{B},\mathcal{A}) \cong \mathrm{Hom}(\mathcal{B},\pi_*\mathcal{A}).$$

Proof. We want to compare $\beta\in\mathrm{Hom}(\mathcal{B},\pi_*\mathcal{A})$ with α in

$$\mathrm{Hom}_{\mathcal{S}h(X)}(\pi^{-1}\mathcal{B},\mathcal{A}) = \mathrm{Hom}_{\mathcal{S}h(X)}(\widetilde{\underline{\pi^{-1}}\mathcal{B}},\mathcal{A}) \cong \mathrm{Hom}_{pre\mathcal{S}h(X)}(\underline{\pi^{-1}}\mathcal{B},\mathcal{A}).$$

α is a system of maps

$$\varinjlim_{V\supseteq\pi(U)} \mathcal{B}(V) = \underline{\pi^{-1}}\mathcal{B}(U) \xrightarrow{\alpha_U} \mathcal{A}(U), \quad \text{for } U \text{ open in } X,$$

and β is a system of maps

$$\mathcal{B}(V) \xrightarrow{\beta_V} \mathcal{A}(\pi^{-1}V), \quad \text{for } V \text{ open in } Y.$$

Clearly, any β gives some α since

$$\varinjlim_{V\supseteq\pi(U)} \mathcal{B}(V) \xrightarrow{\varinjlim \beta_V} \varinjlim_{V\supseteq\pi(U)} \mathcal{A}(\pi^{-1}V) \to \mathcal{A}(U),$$

the second map comes from the restrictions $\mathcal{A}(\pi^{-1}V)\to\mathcal{A}(U)$ defined since $V\supseteq\pi(U)$ implies $\pi^{-1}V\supseteq U$.

For the opposite direction, any α gives for each V open in Y, a map $\varinjlim_{W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W) = \underline{\pi^{-1}}\mathcal{B}(\pi^{-1}V) \xrightarrow{\alpha_{\pi^{-1}V}} \mathcal{A}(\pi^{-1}V)$. Since $\mathcal{B}(V)$ is one of the terms in the inductive system we have a canonical map $\mathcal{B}(V)\to \varinjlim_{W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W)$, and the composition with the first map $\mathcal{B}(V)\to \varinjlim_{W\supseteq\pi(\pi^{-1}V)} \mathcal{B}(W) \xrightarrow{\alpha_{\pi^{-1}V}} \mathcal{A}(\pi^{-1}V)$, is the wanted map β_V.

D.5.6. *Lemma.* (a) If $X \xrightarrow{\pi} Y \xrightarrow{\tau} Z$ then

$$\tau_*(\pi_*\mathcal{A}) \cong (\tau\circ\pi)_*\mathcal{A} \quad \text{and} \quad \tau_*(\pi_*\mathcal{A}) \cong (\tau\circ\pi)_*\mathcal{A}.$$

(b) $(1_X)_*\mathcal{A} \cong \mathcal{A} \cong (1_X)^{-1}\mathcal{A}$.

Proof. The statements involving direct image are very simple and the claims for inverse image follow by adjunction.

D.5.7. *Corollary.* (Pull-back preserves the stalks) For $a\in X$ one has $(\pi^{-1}\mathcal{N})_a \cong \mathcal{N}_{\pi(a)}$.

This shows that the pull-back operation which was difficult to define is actually very simple in its effect on sheaves.

D.6. **Abelian category structure.** Let us fix a map of sheaves $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ since the nontrivial part is the construction of (co)kernels. Consider the example where the space is the circle $X=\{z\in\mathbb{C},\ |z|=1\}$ and $\mathcal{A}=\mathcal{B}$ is the sheaf $\mathcal{C}_X^\infty$ of smooth functions on X, and the map α is the differentiation $\partial = \frac{\partial}{\partial\theta}$ with respect to the angle θ. For $U\subseteq X$ open, $\mathrm{Ker}(\partial_U):\ \mathcal{C}_X^\infty(U)\to\mathcal{C}_X^\infty(U)$ consists of locally constant functions and the cokernel $\mathcal{C}_X^\infty(U)/\partial_U\mathcal{C}_X^\infty(U)$ is

- zero if $U\neq X$ (then any smooth function on U is the derivative of its indefinite integral defined by using the exponential chart $z=e^{i\theta}$ which identifies U with an open subset of $\mathbb{R}$),
- one dimensional if $U=X$ – for $g\in C^\infty(X)$ one has $\int_X\ \partial g=0$ so say constant functions on X are not derivatives (and for functions with integral zero the first argument applies).

So by taking kernels at each level we got a sheaf but by taking cokernels we got a presheaf which is not a sheaf (local sections are zero but there are global non-zero sections, so the object is not controlled by its local properties).

D.6.1. *Subsheaves.* For (pre)sheaves $\mathcal{S}$ and $\mathcal{S}'$ we say that $\mathcal{S}'$ is a sub(pre)sheaf of $\mathcal{S}$ if $\mathcal{S}'(U)\subseteq\mathcal{S}(U)$ and the restriction maps for $\mathcal{S}'$, $\mathcal{S}'(U)\xrightarrow{\rho'}\mathcal{S}'(V)$ are restrictions of the restriction maps for $\mathcal{S}$, $\mathcal{S}(U)\xrightarrow{\rho}\mathcal{S}(V)$.

D.6.2. *Lemma.* (Kernels.) Any map $\mathcal{A}\xrightarrow{\alpha}\mathcal{B}$ has a kernel and $\mathrm{Ker}(\alpha)(U)=\mathrm{Ker}[\mathcal{A}(U)\xrightarrow{\phi(U)}\mathcal{B}(U)]$ is a subsheaf of $\mathcal{A}$.

Proof. First, $\mathcal{K}(U)\overset{\text{def}}{=}\mathrm{Ker}[\mathcal{A}(U)\xrightarrow{\phi(U)}\mathcal{B}(U)]$ is a sheaf, and then a map $\mathcal{C}\xrightarrow{\mu}\mathcal{A}$ is killed by α iff it factors through the subsheaf $\mathcal{K}$ of $\mathcal{A}$.

Lemma. (Cokernels.) Any map $\mathcal{A}\xrightarrow{\alpha}\mathcal{B}$ defines a presheaf $C(U)\overset{\text{def}}{=}\mathcal{B}(U)/\alpha_U(\mathcal{A}(U))$, the associated sheaf $\mathcal{C}$ is the cokernel of α.

Proof. For a sheaf $\mathcal{S}$ one has

$$\mathrm{Hom}_{\mathcal{S}heaves}(\mathcal{B},\mathcal{S})_\alpha\cong\ \mathrm{Hom}_{pre\mathcal{S}heaves}(C,\mathcal{S})\cong\ \mathrm{Hom}_{\mathcal{S}heaves}(\mathcal{C},\mathcal{S}).$$

The second identification is the adjunction. For the first one, a map $\mathcal{B}\xrightarrow{\phi}\mathcal{S}$ is killed by α, i.e., $0=\phi\circ\alpha$, if for each U one has $0=(\phi\circ\alpha)_U\mathcal{A}(U)=\ \phi_U(\alpha_U\mathcal{A}(U))$; but then it gives a map $C\xrightarrow{\overline{\phi}}\mathcal{S}$, with $\overline{\phi}_U:C(U)=\mathcal{B}(U)/\alpha_U\mathcal{A}(U)\to SS(U)$ the factorization of ϕ_U. The opposite direction is really obvious, any $\psi:C\to\mathcal{S}$ can be composed with the canonical map $\mathcal{B}\to C$ (i.e., $\mathcal{B}(U)\to\mathcal{B}(U)/\alpha_U\mathcal{A}(U)$) to give map $\mathcal{B}\to\mathcal{S}$ which is clearly killed by α.

D.6.3. *Lemma.* (Images.) Consider a map $\mathcal{A}\xrightarrow{\alpha}\mathcal{B}$.

(a) It defines a presheaf $I(U)\overset{\text{def}}{=}\ \alpha_U(\mathcal{A}(U))\subseteq\ \mathcal{B}(U)$ which is a subpresheaf of $\mathcal{B}$. The associated sheaf is the image of α.

(b) It defines a presheaf $c(U)\overset{\text{def}}{=}\ \mathcal{A}(U)/\mathrm{Ker}(\alpha_U)$, the associated sheaf $\mathcal{I}$ is the coimage of α.

(c) The canonical map $Coim(\alpha)\to Im(\alpha)$ is isomorphism.

Proof. (a) $Im(\alpha)\overset{\text{def}}{=}\ \mathrm{Ker}[\mathcal{B}\to\mathrm{Coker}(\alpha)]$ is a subsheaf of $\mathcal{B}$ and $b\in\mathcal{B}(U)$ is a section of $Im(\alpha)$ iff it becomes zero in $\mathrm{Coker}(\alpha)$. But a section $b+\alpha_U\mathcal{A}(U)$ of C on U is zero in $\mathcal{B}$ iff it is locally zero in C, i.e., there is a cover U_i of U such that $b|U_i\in\alpha_{U_i}\mathcal{A}(U_i)$. But this is the same as saying that b is locally in the subpresheaf I of $\mathcal{B}$, i.e., the same as asking that b is in the corresponding presheaf $\mathcal{I}$ of $\mathcal{B}$.

(b) The coimage of α is by definition $Coim(\alpha) \stackrel{\text{def}}{=} \text{Coker}[\text{Ker}(\alpha) \to \mathcal{A}]$, i.e., the sheaf associated to the presheaf $U \mapsto \mathcal{A}(U)/\text{Ker}(\alpha)(U) = c(U)$.

(c) The map of sheaves $Coim(\alpha) \to Im(\alpha)$ is associated to the canonical map of presheaves $c \to I$, however already the map of presheaves is an isomorphism: $c(U) = \mathcal{A}(U)/\text{Ker}(\alpha)(U) \cong \alpha_U \stackrel{\text{def}}{=} \mathcal{A}(U) = I(U)$.

D.6.4. *Stalks of kernels, cokernels and images; exact sequences of sheaves.*

D.6.5. *Lemma.* For a map of sheaves $\mathcal{A} \xrightarrow{\alpha} \mathcal{B}$ and $x \in X$

- (a) $\text{Ker}(\mathcal{A} \xrightarrow{\alpha} \mathcal{B})_x = \text{Ker}(\alpha_x : \mathcal{A}_x \to \mathcal{B}_x)$,
- (b) $\text{Coker}(\mathcal{A} \xrightarrow{\alpha} \mathcal{B})_x = \text{Coker}(\alpha_x : \mathcal{A}_x \to \mathcal{B}_x)$,
- (c) $Im(\mathcal{A} \xrightarrow{\alpha} \mathcal{B})_x = Im(\alpha_x : \mathcal{A}_x \to \mathcal{B}_x)$.

Proof. (a) Let $x \in U$ and $a \in \mathcal{A}(U)$. The germ a_x is killed by α_x if $0 = \alpha_x(a_x) \stackrel{\text{def}}{=} (\alpha_U(a))_x$, i.e., iff $\alpha_U(a) = 0$ on some neighborhood U' of x in U. But this is the same as saying that $0 = \alpha_U(a)|U' = \alpha_{U'}(a|U')$, i.e., asking that some restriction of a to a smaller neighborhood of x is a section of the subsheaf $\text{Ker}(\alpha)$. And this in turn, is the same as saying that the germ a_x lies in the stalk of $\text{Ker}(\alpha)$.

(b) Map $\mathcal{B} \xrightarrow{q} \text{Coker}(\alpha)$ is killed by composing with α, so the map of stalks $\mathcal{B}_x \xrightarrow{q_x} \text{Coker}(\alpha)_x$ is killed by composing with α_x.

To see that q_x is surjective consider some element of the stalk $\text{Coker}(\alpha)_x$. It comes from a section of a presheaf $U \mapsto \mathcal{B}(U)/\alpha_U \mathcal{A}(U)$, so it is of the form $[b + \alpha_U(\mathcal{A}(U))]_x$ for some section $b \in \mathcal{B}(U)$ on some neighborhood U of x. Therefore it is the image $\alpha_x(b_x)$ of an element b_x of $\mathcal{B}_x$.

To see that q_x is injective, observe that a stalk $b_x \in \mathcal{B}_x$ (of some section $b\mathcal{B}(U)$), is killed by q_x iff its image $\alpha_x(b_x) = [b + \alpha_U(\mathcal{A}(U))]_x$ is zero in $\text{Coker}(\alpha)$, i.e., iff there is a smaller neighborhood $U' \subseteq U$ such that the restriction $[b + \alpha_U(\mathcal{A}(U))]|U' = b|U' + \alpha_{U'}(\mathcal{A}(U'))$ is zero, i.e., $b|U'$ is in $\alpha_{U'}\mathcal{A}(U')$. But the existence of such U' is the same as saying that b_x is in the image of α_x.

(c) follows from (a) and (b) by following how images are defined in terms of kernels and cokernels.

D.6.6. *Corollary.* A sequence of sheaves is exact iff at each point the corresponding sequence of stalks of sheaves is exact.

D.7. **Injective resolutions of sheaves.** We state the last ingredient need in order to use the homological algebra in the category of sheaves:

D.7.1. *Theorem.* The category $\mathcal{S}h\mathcal{A}b(X)$ of sheaves of abelian groups on X has enough injectives, i.e., any sheaf of abelian groups is a subsheaf of an injective sheaf of abelian groups.

D.8. **Appendix: Sheafifications via the etale space of a presheaf.** One can skip this subsection. We will once again construct the sheafification of a presheaf $\mathcal{S}$. This approach is more elegant and less explicit (it is more abstract and we use the notion of stalks). The main idea is that to a presheaf $\mathcal{S}$ over X one can attach a map of topological spaces $\overset{\bullet}{\mathcal{S}} \to X$. Here, $\overset{\bullet}{\mathcal{S}}$ is called the *etale space* of the presheaf. Then the sheafification of $\mathcal{S}$ is obtained using the following idea:

D.8.1. *Sheaf of sections of a map.* If $Y \xrightarrow{p} X$ is a continuous map, let us attach to each open $U \subseteq X$ the set

$$\mathcal{Y}(U) \overset{\text{def}}{=} \{s : U \longrightarrow Y,\ s \text{ is continuous and } p \circ s = 1_u\}.$$

its the elements are called the (continuous) sections of p over U.

Lemma. $\mathcal{Y}$ is a sheaf of sets (the *sheaf of sections of* p).

D.8.2. *The* etale space *of a presheaf.* To apply this construction we need a space $\overset{\bullet}{\mathcal{S}}$ that maps to X:

- Let $\overset{\bullet}{\mathcal{S}}$ be the union of all stalks $\mathcal{S}_m,\ m \in X$.
- Let $p : \overset{\bullet}{\mathcal{S}} \to X$ be the map such that the fiber at m is the stalk at m.
- For any pair (U, s) with U open in X and $s \in \mathcal{S}(U)$, define a section $\tilde{s}$ of p over U by

$$\tilde{s}(x) \overset{\text{def}}{=} s_x \in \mathcal{S}_x \subset \overset{\bullet}{\mathcal{S}}, \qquad x \in U.$$

D.8.3. *Lemma.* (a) If for two sections $s_i \in \mathcal{S}(U_i),\ i = 1, 2;$ of $\mathcal{S}$, the corresponding sections $\tilde{s}_1$ and $\tilde{s}_2$ of p agree at a point then they agree on some neighborhood of of this point.[137]

(b) All sets $\tilde{s}(U)$ (for $U \subseteq X$ open and $s \in \mathcal{S}(U)$), form a basis of a topology on $\overset{\bullet}{\mathcal{S}}$.

(c) Map $p : \overset{\bullet}{\mathcal{S}} \to M$ is continuous. Moreover, it is *etale*[138]

(d) Let Σ be the sheaf of continuous sections of p over U. Then there is a canonical map of presheaves $i : \mathcal{S} \to \Sigma$.

D.8.4. *Lemma.* The canonical map of presheaves $i : \mathcal{S} \to \Sigma$, is the sheafification of $\mathcal{S}$.

Proof. Sections of p over $U \subseteq X$ are the same as the equivalence classes of systems $\widehat{\mathcal{S}}/\equiv$ defined in 9.10.7.

[137] If $\tilde{s}_1(x) = \tilde{s}_2(x)$ for some $x \in U_{12} \overset{\text{def}}{=} U_1 \cap U_2$, we claim that there is a neighborhood $W \subseteq U_{12}$ of x, such that $\tilde{s}_1 = \tilde{s}_2$ on W.

[138] "Etale" means "locally an isomorphism", i.e., for each point $\sigma \in \overset{\bullet}{\mathcal{S}}$ there are neighborhoods $\sigma \in W \subseteq \overset{\bullet}{\mathcal{S}}$ and $p(\sigma) \subseteq U \subseteq X$ such that $p|W$ is a homeomorphism $W \xrightarrow{\cong} U$.

Appendix E. **Multilinear Algebra**

This is mostly the exposition in S.Lang's *Algebra* written as a sequence of problems. A more advanced text is N.Bourbaki's *Algebra*.

Tensor product of modules over a ring

Let A be a ring with a unit 1.

1. Tensor product of A-modules. *For a left A-module U and a right A-module V, we define a free abelian group $F = F_{U,V}$, with a basis $U\times V$: $F = \oplus_{u\in U,\ v\in V}\ \mathbb{Z}\cdot(u,v)$. The tensor product of U and V is the abelian group $U\underset{A}{\otimes}V$ defined as a quotient $U\underset{A}{\otimes}V \overset{def}{=} F/R$ of F by the subgroup R generated by the elements of one of the following forms (here $u, u_i \in U,\ v, v_i \in V,\ a \in A$):*

(1) $(u_1+u_2, v) - (u_1, v) - (u_2, v)$, (2) $(u, v_1+v_2) - (u, v_1) - (u, v_2)$, (3) $(u\cdot a, v) - (u, a\cdot v),$.

The image of $(u,v) \in F$ in $U\underset{A}{\otimes}V$ is denoted $u\otimes v$. Let $\pi : U\times V \to U\underset{A}{\otimes}V$ be the composition of maps $U\times V \hookrightarrow F \to U\underset{A}{\otimes}V$, so that $\pi(u,v) = u\otimes v$.

(a) Show that

(a_1) $(u_1+u_2)\otimes v = u_1\otimes v + u_2\otimes v$,

(a_2) $u\otimes(v_1+v_2) = u\otimes v_1 + u\otimes v_2$,

(a_3) $(u\cdot a)\otimes v = u\otimes(a\cdot v)$.

(b) Show that each element of $U\underset{A}{\otimes}V$ is a finite sum of the form $\sum_{i=1}^{n}\ u_i\otimes v_i$, for some $u_i \in U,\ v_i \in V$.

2. The universal property of the tensor product $\underset{A}{\otimes}$. *We say that a map $\phi : U\times V \to H$ with values in an abelian group H, is A-balanced if it satisfies the conditions $\phi(u_1+u_2, v) = \phi(u_1, v) + \phi(u_2, v)$, $\phi(u, v_1+v_2) = \phi(u, v_1) + \phi(u, v_2)$, $\phi(u\cdot a, v) = \phi(u, a\cdot v)$, for $u, u_i \in U,\ v, v_i \in V,\ a \in A$.*

Show that the balanced maps $\phi : U\times V \to H$ are in a one-to-one correspondence with the morphisms of abelian groups $\psi : U\underset{A}{\otimes}V \to H$, by $\psi \mapsto \phi \overset{\text{def}}{=} \psi\circ\pi$.

[*The above notion of "balanced maps" is a version of the notion of bilinear maps which makes sense even for non-commutative rings A. One direction of the one-to-one correspondence above says that the tensor product reformulates balanced maps in terms of linear maps. In the opposite direction, one constructs maps from a tensor product $U\underset{A}{\otimes}V$ by constructing balanced maps from the product $U\times V$.*]

3. Functoriality. Let $U_0 \xrightarrow{\alpha} U_1$ be a map of right A-modules and $V_0 \xrightarrow{\beta} V_1$ be a map of left A-modules.

(a) A map of abelian groups $U_0\otimes_A V_0 \xrightarrow{\gamma} U_1\otimes_A V_1$ is well defined by $\gamma(u\otimes v) = \alpha(u)\otimes\beta(v),\ u \in U_0,\ v \in V_0$. [This map is usually denoted by $\alpha\otimes\beta$ though one need not think of it as an element of some tensor product.]

(b) $1_U\otimes 1_V = 1_{U\otimes V}$.

(c) If one also has maps $U_1 \xrightarrow{\alpha'} U_2$ of right modules and $V_1 \xrightarrow{\beta'} V_2$ of left A-modules, then

$$(\alpha'\otimes\beta')\circ(\alpha\otimes\beta) = (\alpha'\circ\alpha)\otimes(\beta'\circ\beta).$$

4. Additivity. Let $V = \oplus_{i\in I} V_i$ be a direct sum of left A-modules, then $U\underset{A}{\otimes}V = \oplus_{i\in I} U\underset{A}{\otimes}V_i$.

5. Free Modules. If V is a free left A-module with a basis $v_i,\ i \in I$, then $U\underset{A}{\otimes}V \cong U^I$, i.e., it is a sum of I copies of U.

6. Cancellation. Show that the map $U\underset{A}{\otimes}A \xrightarrow{\alpha} U,\ \alpha(\sum\ u_i\otimes a_i) = \sum u_i{\cdot}a_i$, is (i) well defined, (ii) an isomorphism of abelian groups.

7. Quotient by relations interpreted as tensoring. For each left ideal I in A, map $U\underset{A}{\otimes}A/I \xrightarrow{\alpha} U/U{\cdot}I,\ \alpha[\sum\ u_i\otimes(a_i+I)] = (\sum u_i{\cdot}a_i)+U{\cdot}I$; is (i) well defined, (ii) an isomorphism of abelian groups.

8. Tensoring of bimodules. If U is a bimodule for a pair of rings (R, A) and V is a bimodule for a pair of rings (A, S), show that $U\underset{A}{\otimes}V$ is a bimodule for (R, S).

9. Right exactness of tensor products. (a) Let A be a ring, L a right A-module and $0{\to}M'{\to}M{\to}M''{\to}0$ a short exact sequence of left A-modules. Show that there is a short exact sequence of abelian groups $L\underset{A}{\otimes}M'{\to}L\underset{A}{\otimes}M{\to}L\underset{A}{\otimes}M''{\to}0$. Find an example when the sequence $0{\to}L\underset{A}{\otimes}M'{\to}L\underset{A}{\otimes}M{\to}L\underset{A}{\otimes}M''{\to}0$ is not exact, i.e., $L\underset{A}{\otimes}M' \not\subseteq L\underset{A}{\otimes}M$.

(b) Let A be a ring, L a right A-module and M a left A-module. Show that any algebra morphism $\phi : B{\to}A$, gives a surjective map $L\underset{B}{\otimes}M{\to}L\underset{A}{\otimes}M$, with the kernel generated by elements of the form $x{\cdot}a\otimes y - x\otimes a{\cdot}y,\ x \in L,\ y \in M,\ a \in A$.

Tensoring over commutative rings

1. Tensoring over a commutative ring. (a) If A is a commutative ring then the left and right modules coincide, say a right A-module U becomes a left A-module with the action defined by $a{\cdot}u \overset{\text{def}}{=} u{\cdot}a$.

(b) In that case the above general construction gives the following construction of tensoring of two left modules U, V:

$$U\underset{A}{\otimes}V \overset{\text{def}}{=} \oplus_{u\in U,\ v\in V}\ \mathbb{Z}{\cdot}(u, v)/R$$

where R is the subgroup generated by the elements of the form (1), (2) and (3′) $(a{\cdot}u, v)-(u, a{\cdot}v)$.

(c) The tensoring of left modules over a commutative ring satisfies properties $(a_1), (a_2)$ above, as well as

$$(a'_3)\ (a{\cdot}u){\otimes}v) = u{\otimes}(a{\cdot}v),$$

and the analogue of the above universal property.

2. Module structure. For a commutative ring A, $U\underset{A}{\otimes}V$ is again an A-module with the action

$$a{\cdot}\ (u{\otimes}v)\overset{\text{def}}{=}(a{\cdot}\ u){\otimes}v = u{\otimes}(a{\cdot}v).$$

3. Tensoring over a field. If A is a field then A-modules are vector spaces over A. Let $u_i, i \in I$, $v_j, j \in J$, be bases of U and V, show that $u_i{\otimes}v_j,\ i \in I, j \in J$; is a basis of $U\underset{A}{\otimes}V$ and $\dim(U\underset{A}{\otimes}V) = \dim(U){\cdot}\dim(V)$.

4. Tensoring of finite abelian groups over $\mathbb{Z}$. Show that $\mathbb{Z}_n\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_m \cong \mathbb{Z}_k$ for some k and calculate k.

5. Tensoring of algebras. Let A be a commutative ring and let B and C be A-algebras. Then $B{\otimes}_A C$ has a canonical structure of an algebra such that

$$(b_1{\otimes}c_1){\cdot}(b_2{\otimes}c_) =\ b_1b_2{\otimes}c_1c_2.$$

Multiple tensor products,

Let M_i be an (A_{i-1}, A_i)-bimodule for $i = 1, ..., n$.

The tensor product $M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$ is defined as a quotient of a free abelian group F with a basis $M_1 \times \cdots \times M_n$, by the subgroup $\mathcal{A}$ generated by the elements of one of the following forms ($m_i \in M_i,\ a_i \in A_i$):

$$(1)\quad (m_1, ..., m_i' + m_i'', ..., m_n) - (m_1, ..., m_i', ..., m_n) - (m_1, ..., m_i'', ..., m_n),$$

$$(2)\quad (m_1, ..., m_{i-1} \cdot a_i, ..., m_n) - (m_1, ..., a_i \cdot m_i, ..., m_n).$$

The image of $(m_1, ..., m_n) \in F$ in $U \underset{A}{\otimes} V$ is denoted $m_1 \otimes \cdots \otimes m_n$. Let $\pi : M_1 \times \cdots \times M_n \to M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$ be the composition $M_1 \times \cdots \times M_n \hookrightarrow F \to M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$, so that $\pi(m_1, ..., m_n) = m_1 \otimes \cdots \otimes m_n$.

1. Multiple tensor products. (a) Show that in $M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$

$$(1)\quad m_1 \otimes \cdots \otimes m_i' + m_i'' \otimes \cdots \otimes m_n = m_1 \otimes \cdots \otimes m_i' \otimes \cdots \otimes m_n + m_1 \otimes \cdots \otimes m_i'' \otimes \cdots \otimes m_n,$$

$$(2)\quad m_1 \otimes \cdots \otimes m_{i-1} \cdot a_i \otimes m_i \otimes \cdots \otimes m_n = m_1 \otimes \cdots \otimes m_{i-1} \otimes a_i \cdot m_i \otimes \cdots \otimes m_n.$$

(b) Each element of $M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$ is a finite sum of the form $\sum_{k=1}^{p}\ m_{k,1} \otimes \cdots \otimes m_{k,n}$.

2. Universal property. Formulate and prove the universal property of multiple tensor products.

3. Bimodule structure. Show that $M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$ is an (A_0, A_n)-bimodule.

4. Associativity. This definition is associative in the sense that there are canonical isomorphisms

$$(M_1 \underset{A_1}{\otimes} \cdots \underset{A_{p_1-1}}{\otimes} M_{p_1}) \underset{A_{p_1}}{\otimes} (M_{p_1+1} \underset{A_{p_1+1}}{\otimes} \cdots \underset{A_{p_1+p_2-1}}{\otimes} M_{p_1+p_2}) \underset{A_{p_1+p_2}}{\otimes} \cdots \otimes (M_{p_1+\cdots+p_{k-1}+1} \otimes \cdots \otimes M_{p_1+\cdots+p_k})$$

$$\cong M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{p_1+\cdots+p_k-1}}{\otimes} M_{p_1+\cdots+p_k}.$$

5. Two factors. For $n = 2$ this notion of a tensor product agrees with the one introduced previously.

6. Commutative rings. If the rings A_i are all commutative explain how the above construction defines a multiple tensoring operation $M_1 \underset{A_1}{\otimes} M_2 \underset{A_2}{\otimes} \cdots \underset{A_{n-1}}{\otimes} M_n$ for when each M_i is a left module for A_{i-1} and A_i, and these two actions commute.

Tensor algebras of modules over commutative rings

Let M be a module for a commutative ring A with a unit. We will denote the n-tuple tensor product $M \underset{A}{\otimes} \cdots \underset{A}{\otimes} M$ by $T^n_A(M) = \overset{n}{\otimes}_A M \overset{def}{=} M^{\otimes n}$. For $n = 0$ this is - by definition - A itself (so it does not depend on M). For $n = 1$ this is the module M.

1. Tensor algebra $T_A(M)$. (a) Show that $T(M) \overset{\text{def}}{=} \sum_{n \geq 0} T^n(M)$ has a unique structure of an associative A-algebra, such that for all $p, q \geq 0$ and $m_i, n_j \in M$,

$$(m_1 \otimes \cdots \otimes m_p) \cdot (n_1 \otimes \cdots \otimes n_q) = m_1 \otimes \cdots \otimes m_p \otimes n_1 \otimes \cdots \otimes n_q.$$

For this algebra structure structure $m_1 \otimes \cdots \otimes m_p$ is the product $m_1 \cdots m_p$ of $m_i \in M^{\otimes 1} = M$.

2. Universal property of tensor algebras. For each A-algebra B restriction

$$\mathrm{Hom}_{assoc.\ A-alg.\ with\ 1}(TM, B) \ni \phi \mapsto \phi | M \in \mathrm{Hom}_{A-modules}(M, B),$$

is a bijection.

Remark. We say that $T(M)$ is that A-algebra defined by the A-module M, or that $T(M)$ is universal among A-algebras B endowed with a map of A-modules $M \to B$.

3. Algebras generated by generators and relations. To an A-module M and a set of relations $\mathcal{R}$ we will associate the *universal* algebra $A(M, \mathcal{R})$ in which these relations are satisfied.

What we mean by algebraic relations between elements of an A-module M are intuitively the conditions of type

$$(*) \quad \sum_{i=1}^{k} a_i \cdot m_{i,1} \cdots m_{i,n_i} = 0$$

for some $a_i \in A,\ m_{i,j} \in M$. The precise meaning of that is that the expression on the left hand side defines an element $r = \sum_{i=1}^k a_i \cdot m_{i,1} \otimes \cdots \otimes m_{i,n_i}$ of the tensor algebra $T(M)$. So any set of such relations defines

- (i) a subset $\mathcal{R} \subseteq T(M)$,
- (ii) an A-algebra $A(M, \mathcal{R}) \overset{\text{def}}{=} T(M)/ < \mathcal{R} >$ where $< \mathcal{R} >$ denotes the 2sided ideal in $T(M)$ generated by $\mathcal{R}$, with
- (iii) a canonical map of A-modules $\iota \overset{\text{def}}{=} [M \subseteq T(M) \twoheadrightarrow A(M, \mathcal{R})]$.

(a) Show that for each A-algebra B, $\mathrm{Hom}_{assoc.\ A-alg.\ with\ 1}[A(M, \mathcal{R}), B]$ is naturally identified with the set of all $\beta \in \mathrm{Hom}_{A-modules}(M, B)$, such that for all $r = \sum_{i=1}^k a_i \cdot m_{i,1} \otimes \cdots \otimes m_{i,n_i}$ in $\mathcal{R}$, the following relation between (images of) elements of M holds in B: $\sum_{i=1}^k a_i \cdot \beta(m_{i,1}) \cdots \beta(m_{i,n_i}) = 0$.

Remark. Therefore, an algebraic relation of type $(*)$ between elements of M acquires meaning in any algebra B supplied with a map of A-modules $M \to B$. Algebra $A(M,\mathcal{R})$ is universal among all such A-algebras B that satisfy relations from $\mathcal{R}$.

Exterior and Symmetric algebras of modules over commutative rings

1. Exterior algebra of an A-module. *Let M be a module for a commutative ring A with a unit. The exterior algebra $\overset{\bullet}{\wedge} M = \underset{A}{\overset{\bullet}{\wedge}} M$ is the associative A-algebra with 1 generated by M and by anti-commutativity relations $\mathcal{R} = \{x \otimes y + y \otimes x,\ x, y \in M\}$. The multiplication operation in $\overset{\bullet}{\wedge} M$ is denoted $\wedge$, so that the image of $m_1 \otimes \cdots \otimes m_n \in T(M)$ in $\overset{\bullet}{\wedge} M$ is denoted $m_1 \wedge \cdots \wedge m_n \in \overset{n}{\wedge} M$.*

(a) $\overset{\bullet}{\wedge} M$ is a graded algebra. Show that (a_0) the ideal $I = <\mathcal{R}>$ in $T(M)$ is homogeneous, i.e., $I = \oplus_{n \geq 0} I^n$ for $I^n \overset{\text{def}}{=} I \cap T^n M$. Show that the quotient algebra $\overset{\bullet}{\wedge} M = T(M)/I$ satisfies (a_1) $\overset{\bullet}{\wedge} M \cong \oplus_{n \geq 0} \overset{n}{\wedge} M$ for $\overset{n}{\wedge} M \overset{\text{def}}{=} T^n(M)/I^n$; and (a_2) $\overset{\bullet}{\wedge} M$ is a graded algebra, i.e., $\overset{n}{\wedge} M \cdot \overset{m}{\wedge} M \subseteq \overset{n+m}{\wedge} M,\ n, m \geq 0$.

2. Universal property of the exterior algebra. Show that for each A-algebra B, $\mathrm{Hom}_{assoc.\ A-alg.\ with\ 1}[\overset{\bullet}{\wedge} M, B]$ can be identified with a set of all $\phi : \mathrm{Hom}_{A-moduli}(M, B)$, such that the ϕ-images of elements of M anti-commute in B, i.e., $\phi(y)\phi(x) = -\phi(x)\phi(y),\ x, y \in M$.

3. Basic properties of exterior algebras. (a) Low degrees. $\overset{0}{\wedge} M = T^0(M) = A$ and $\overset{1}{\wedge} M = T^1(M) = M$.

(b) Bilinear forms extend to exterior algebras. For any A-modules L and M, and any A-bilinear map $<\ ,\ >: L \times M \to A$ (i.e., linear in each variable); there is a unique A-bilinear map $<\ ,\ >: \overset{n}{\wedge} L \times \overset{n}{\wedge} M \to A$, such that $< l_1 \wedge \cdots l_n, m_1 \wedge \cdots \wedge m_n >= \det(< l_i, m_j >)$.

(c) Free modules If M is a free A-module with a basis $e_1, ..., e_d$, then $\overset{k}{\wedge} M$ is a free A-module with a basis $e^J = e_{j_1} \wedge \cdots \wedge e_{j_k}$, indexed by all subsets $J = \{j_1 < \cdots < j_n\} \subseteq I$ with k elements.

(d) Dimension. $\dim(\overset{\bullet}{\wedge}\, \mathbb{C}^n) = 2^n$.

4. Symmetric algebra of an A-module. *Let M be a module for a commutative ring A with a unit. The symmetric algebra $S(M) = S_A(M)$ of M is the algebra generated by M and the relations $\mathcal{R} = \{x \otimes y - y \otimes x,\ x, y \in M\}$.*

(a) $S(M)$ is a graded algebra. Show that

- (a_0) the ideal $I = <\mathcal{R}>$ in $T(M)$ is homogeneous and $S(M) = T(M)/I$ satisfies
- (a_1) $S(M) \cong \oplus_{n \geq 0} S^n(M)$ for $S^n(M) \overset{\text{def}}{=} T^n(M)/I^n$; and
- (a_2) $S(M)$ is a graded algebra.

5. Universal property of the symmetric algebra. (a) Show that (i) $S(A)$ is commutative, (ii) for any commutative A-algebra B, $\mathrm{Hom}_{assoc.\ A-alg.\ with\ 1}[S(M), B]$ can be identified with $\mathrm{Hom}_{A-modules}(M, B)$.

6. Basic properties of symmetric algebras. (a) $T^0(M) = S^0(M) = A$ and $T^1(M) = S^1(M) = M$.

(b) If M is a free A-module with a basis $e_i,\ i \in I$, then $S^n(M)$ is a free A-module with a basis $e^J = \prod_{i\in I}\ e_i^{J_i}$, indexed by all maps $J : I \to \mathbb{N}$ with the integral n.

(c) The algebra of polynomials $\mathbb{C}[x_1, ..., x_n]$ is isomorphic to the symmetric algebra $S(\mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n)$.