EXAMPLE 7 Let W be the ball of radius R and center $(0, 0, 0)$ in $\mathbb{R}^3$. Find the volume of W.

SOLUTION The volume of W is $\iiint_W dx\,dy\,dz$. This integral may be evaluated by reducing it to iterated integrals or by regarding W as a volume of revolution, but let us evaluate it here by using spherical coordinates. We get

$$\iiint_W dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{R^3}{3}\int_0^{\pi}\int_0^{2\pi} \sin\phi\, d\theta\, d\phi$$
$$= \frac{2\pi R^3}{3}\int_0^{\pi} \sin\phi\, d\phi = \frac{2\pi R^3}{3}\{-[\cos(\pi) - \cos(0)]\} = \frac{4\pi R^3}{3},$$

which is the standard formula for the volume of a solid sphere. ▲

EXERCISES

1. Let D be the unit disk: $x^2 + y^2 \le 1$. Evaluate

$$\iint_D \exp(x^2 + y^2)\, dx\, dy$$

by making a change of variables to polar coordinates.

2. Let D be the region $0 \le y \le x$ and $0 \le x \le 1$. Evaluate

$$\iint_D (x + y)\, dx\, dy$$

by making the change of variables $x = u + v$, $y = u - v$. Check your answer by evaluating the integral directly by using an iterated integral.

3. Let $T(u, v) = (x(u, v), y(u, v))$ be the mapping defined by $T(u, v) = (4u, 2u + 3v)$. Let D^* be the rectangle $[0, 1] \times [1, 2]$. Find $D = T(D^*)$ and evaluate

(a) $\iint_D xy\, dx\, dy$ (b) $\iint_D (x - y)\, dx\, dy$

by making a change of variables to evaluate them as integrals over D^*.

4. Repeat Exercise 3 for $T(u, v) = (u, v(1 + u))$.

5. Evaluate

$$\iint_D \frac{dx\, dy}{\sqrt{1 + x + 2y}},$$

where $D = [0, 1] \times [0, 1]$, by setting $T(u, v) = (u, v/2)$ and evaluating an integral over D^*, where $T(D^*) = D$.

6. Define $T(u, v) = (u^2 - v^2, 2uv)$. Let D^* be the set of (u, v) with $u^2 + v^2 \le 1$, $u \ge 0$, $v \ge 0$. Find $T(D^*) = D$. Evaluate $\iint_D dx\, dy$.

7. Let $T(u, v)$ be as in Exercise 6. By making a change of variables, "formally" evaluate the "improper" integral

$$\iint_D \frac{dx\, dy}{\sqrt{x^2 + y^2}}.$$

[NOTE: This integral (and the one in the next exercise) is *improper*, because the integrand $1/\sqrt{x^2 + y^2}$ is neither continuous nor bounded on the domain of integration. (The theory of improper integrals is discussed in Section 6.4.)]

8. Calculate $\iint_R \frac{1}{x + y}\, dy\, dx$, where R is the region bounded by $x = 0$, $y = 0$, $x + y = 1$, $x + y = 4$, by using the mapping $T(u, v) = (u - uv, uv)$.

9. Evaluate $\iint_D (x^2 + y^2)^{3/2}\, dx\, dy$ where D is the disk $x^2 + y^2 \le 4$.

10. Let D^* be a v-simple region in the uv plane bounded by $v = g(u)$ and $v = h(u) \le g(u)$ for $a \le u \le b$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation given by $x = u$ and $y = \psi(u, v)$, where ψ is of class C^1 and $\partial\psi/\partial v$ is never zero. Assume that $T(D^*) = D$ is a y-simple region; show that if $f\colon D \to \mathbb{R}$ is continuous, then

$$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(u, \psi(u, v))\left|\frac{\partial\psi}{\partial v}\right| du\, dv.$$

11. Use double integrals to find the area inside the curve $r = 1 + \sin\theta$.

12. (a) Express $\int_0^1 \int_0^{x^2} xy\, dy\, dx$ as an integral over the triangle D^*, which is the set of (u, v) where $0 \le u \le 1$, $0 \le v \le u$. (HINT: Find a one-to-one mapping T of D^* onto the given region of integration.)

(b) Evaluate this integral directly and as an integral over D^*.

13. Integrate $ze^{x^2+y^2}$ over the cylinder $x^2 + y^2 \le 4$, $2 \le z \le 3$.

14. Let D be the unit disk. Express $\iint_D (1 + x^2 + y^2)^{3/2}\, dx\, dy$ as an integral over $[0, 1] \times [0, 2\pi]$ and evaluate.

15. Using polar coordinates, find the area bounded by the *lemniscate* $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$.

16. Redo Exercise 11 of Section 5.3 using a change of variables and compare the effort involved in each method.

17. Calculate $\iint_R (x + y)^2 e^{x-y}\, dx\, dy$ where R is the region bounded by $x + y = 1$, $x + y = 4$, $x - y = -1$, and $x - y = 1$.

18. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(u, v, w) = (u \cos v \cos w, u \sin v \cos w, u \sin w).$$

(a) Show that T is onto the unit sphere; that is, every (x, y, z) with $x^2 + y^2 + z^2 = 1$ can be written as $(x, y, z) = T(u, v, w)$ for some (u, v, w).

(b) Show that T is not one-to-one.

19. Integrate $x^2 + y^2 + z^2$ over the cylinder $x^2 + y^2 \leq 2$, $-2 \leq z \leq 3$.

20. Evaluate $\int_0^\infty e^{-4x^2}\, dx$.

21. Let B be the unit ball. Evaluate

$$\iiint_B \frac{dx\, dy\, dz}{\sqrt{2 + x^2 + y^2 + z^2}}$$

by making the appropriate change of variables.

22. Evaluate $\iint_A [1/(x^2 + y^2)^2]\, dx\, dy$ where A is determined by the conditions $x^2 + y^2 \leq 1$ and $x + y \geq 1$.

23. Evaluate $\displaystyle\iiint_W \frac{dx\, dy\, dz}{(x^2 + y^2 + z^2)^{3/2}}$, where W is the solid bounded by the two spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, where $0 < b < a$.

24. Evaluate $\iint_D x^2\, dx\, dy$ where D is determined by the two conditions $0 \leq x \leq y$ and $x^2 + y^2 \leq 1$.

25. Integrate $\sqrt{x^2 + y^2 + z^2}\, e^{-(x^2+y^2+z^2)}$ over the region described in Exercise 23.

26. Evaluate the following by using cylindrical coordinates.

(a) $\iiint_B z\, dx\, dy\, dz$ where B is the region within the cylinder $x^2 + y^2 = 1$ above the xy plane and below the cone $z = (x^2 + y^2)^{1/2}$.

(b) $\iiint_W (x^2 + y^2 + z^2)^{-1/2}\, dx\, dy\, dz$ where W is the region determined by the conditions $\frac{1}{2} \leq z \leq 1$ and $x^2 + y^2 + z^2 \leq 1$.

27. Evaluate $\iint_B (x + y)\, dx\, dy$ where B is the rectangle in the xy plane with vertices at $(0, 1)$, $(1, 0)$, $(3, 4)$, and $(4, 3)$.

28. Evaluate $\iint_D (x + y)\, dx\, dy$ where D is the square with vertices at $(0, 0)$, $(1, 2)$, $(3, 1)$, and $(2, -1)$.

29. Let E be the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \leq 1$, where a, b, and c are positive.

(a) Find the volume of E.

(b) Evaluate $\iiint_E [(x^2/a^2) + (y^2/b^2) + (z^2/c^2)]\, dx\, dy\, dz$. (Hint: Change variables and then use spherical coordinates.)

30. Using spherical coordinates, compute the integral of $f(\rho, \phi, \theta) = 1/\rho$ over the region in the first octant of $\mathbb{R}^3$, which is bounded by the cones $\phi = \pi/4$, $\phi = \arctan 2$ and the sphere $\rho = \sqrt{6}$.

The factor $(4\pi/3)(\rho_2^3 - \rho_1^3)$ equals the volume of W. Putting back the constants G, m, and the mass density, we find that *the gravitational potential* is $-GmM/R$, *where* M *is the mass of* W. *Thus,* V *is just as it would be if all the mass of* W *were concentrated at the central point.*

Case 2. If $R \le \rho_1$ [that is, if (x_1, y_1, z_1) is inside the hole], then $|\rho - R| = \rho - R$ for ρ in $[\rho_1, \rho_2]$, and so

$$-V(0,0,R) = (Gm)\frac{2\pi}{R}\int_{\rho_1}^{\rho_2} \rho[\rho + R - (\rho - R)]\,d\rho = (Gm)4\pi\int_{\rho_1}^{\rho_2} \rho\,d\rho$$
$$= (Gm)2\pi(\rho_2^2 - \rho_1^2).$$

The result is independent of R, and so the potential V is *constant* inside the hole. Because the gravitational force is minus the gradient of V, we conclude that *there is no gravitational force inside a uniform hollow planet*!

We leave it to the reader to compute $V(0, 0, R)$ for the case $\rho_1 < R < \rho_2$.

A similar argument shows that the gravitational potential outside any *spherically symmetric* body of mass M (even if the density is variable) is $V = GMm/R$, where R is the distance to its center (which is its center of mass).

EXAMPLE 8 Find the gravitational potential acting on a unit mass of a spherical star with a mass $M = 3.02 \times 10^{30}$ kg at a distance of 2.25×10^{11} m from its center ($G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2).

SOLUTION The negative potential is

$$-V = \frac{GM}{R} = \frac{6.67 \times 10^{-11} \times 3.02 \times 10^{30}}{2.25 \times 10^{11}} = 8.95 \times 10^8 \ \mathrm{m^2/s^2}. \ \blacktriangle$$

EXERCISES

1. Find the average of $f(x, y) = y \sin xy$ over $D = [0, \pi] \times [0, \pi]$.

2. Find the average of $f(x, y) = e^{x+y}$ over the triangle with vertices (0, 0), (0, 1), and (1, 0).

3. Find the center of mass of the region between $y = x^2$ and $y = x$ if the density is $x + y$.

4. Find the center of mass of the region between $y = 0$ and $y = x^2$, where $0 \le x \le \frac{1}{2}$.

5. A sculptured gold plate D is defined by $0 \le x \le 2\pi$ and $0 \le y \le \pi$ (centimeters) and has mass density $\delta(x, y) = y^2 \sin^2 4x + 2$ (grams per square centimeter). If gold sells for \$7 per gram, how much is the gold in the plate worth?

6. In Exercise 5, what is the average mass density in grams per square centimeter?

7. (a) Find the mass of the box $[0, \frac{1}{2}] \times [0, 1] \times [0, 2]$, assuming the density to be uniform.
   (b) Same as part (a), but with a mass density $\delta(x, y, z) = x^2 + 3y^2 + z + 1$.

8. Find the mass of the solid bounded by the cylinder $x^2 + y^2 = 2x$ and the cone $z^2 = x^2 + y^2$ if the density is $\delta = \sqrt{x^2 + y^2}$.

9. Find the center of mass of the region bounded by $x + y + z = 2$, $x = 0$, $y = 0$, and $z = 0$, assuming the density to be uniform.

10. Find the center of mass of the cylinder $x^2 + y^2 \le 1$, $1 \le z \le 2$ if the density is $\delta = (x^2 + y^2)z^2$.

11. Find the average value of $\sin^2 \pi z \cos^2 \pi x$ over the cube $[0, 2] \times [0, 4] \times [0, 6]$.

12. Find the average value of e^{-z} over the ball $x^2 + y^2 + z^2 \le 1$.

13. A solid with constant density is bounded above by the plane $z = a$ and below by the cone described in spherical coordinates by $\phi = k$, where k is a constant $0 < k < \pi/2$. Set up an integral for its moment of inertia about the z axis.

14. Find the moment of inertia around the y axis for the ball $x^2 + y^2 + z^2 \le R^2$ if the mass density is a constant δ.

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16. Find the gravitational force exerted on a 70-kg object at the position in Exercise 15.

17. A body W in xyz coordinates is called *symmetric with respect to a given plane* if for every particle on one side of the plane there is a particle of equal mass located at its mirror image through the plane.

   (a) Discuss the planes of symmetry for an automobile shell.

   (b) Let the plane of symmetry be the xy plane, and denote by W^+ and W^- the portions of W above and below the plane, respectively. By our assumption, the mass density $\delta(x, y, z)$ satisfies $\delta(x, y, -z) = \delta(x, y, z)$. Justify the following steps:

$$\bar{z} \cdot \iiint_W \delta(x,y,z)\,dx\,dy\,dz = \iiint_W z\delta(x,y,z)\,dx\,dy\,dz$$
$$= \iiint_{W^+} z\delta(x,y,z)\,dx\,dy\,dz + \iiint_{W^-} z\delta(x,y,z)\,dx\,dy\,dz$$
$$= \iiint_{W^+} z\delta(x,y,z)\,dx\,dy\,dz + \iiint_{W^+} -w\delta(u,v,-w)\,du\,dv\,dw$$
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The factor $(4\pi/3)(\rho_2^3 - \rho_1^3)$ equals the volume of W. Putting back the constants G, m, and the mass density, we find that *the gravitational potential* is $-GmM/R$, *where M is the mass of W. Thus, V is just as it would be if all the mass of W were concentrated at the central point.*

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$$= 0.$$

   (c) Explain why part (b) proves that if a body is symmetrical with respect to a plane, then its center of mass lies in that plane.

5. Find the volume inside the surfaces $x^2 + y^2 = z$ and $x^2 + y^2 + z^2 = 2$.

6. Find the volume enclosed by the cone $x^2 + y^2 = z^2$ and the plane $2z - y - 2 = 0$.

7. A cylindrical hole of diameter 1 is bored through a sphere of radius 2. Assuming that the axis of the cylinder passes through the center of the sphere, find the volume of the solid that remains.

8. Let C_1 and C_2 be two cylinders of infinite extent, of diameter 2, and with axes on the x and y axes, respectively. Find the volume of their intersection, $C_1 \cap C_2$.

9. Find the volume bounded by $x/a + y/b + z/c = 1$ and the coordinate planes.

10. Find the volume determined by $z \leq 6 - x^2 - y^2$ and $z \geq \sqrt{x^2 + y^2}$.

11. The *tetrahedron* defined by $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$ is to be sliced into n segments of equal volume by planes parallel to the plane $x + y + z = 1$. Where should the slices be made?

12. Let E be the solid ellipsoid $E = \{(x, y, z) \mid (x^2/a^2) + (y^2/b^2) + (z^2/c^2) \leq 1\}$ where $a > 0, b > 0$, and $c > 0$. Evaluate

$$\iiint xyz\, dx\, dy\, dz$$

(a) over the whole ellipsoid; and
(b) over that part of it in the first quadrant:

$$x \geq 0, \quad y \geq 0, \quad \text{and} \quad z \geq 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

13. Find the volume of the "ice cream cone" defined by the inequalities $x^2 + y^2 \leq \frac{1}{5}z^2$, and $0 \leq z \leq 5 + \sqrt{5 - x^2 - y^2}$.

14. Let ρ, θ, ϕ be spherical coordinates in $\mathbb{R}^3$ and suppose that a surface surrounding the origin is described by a continuous positive function $\rho = f(\theta, \phi)$. Show that the volume enclosed by the surface is

$$V = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} [f(\theta, \phi)]^3 \sin\phi\, d\phi\, d\theta.$$

15. Using an appropriate change of variables, evaluate

$$\iint_B \exp\,[(y - x)/(y + x)]\, dx\, dy$$

where B is the interior of the triangle with vertices at $(0, 0)$, $(0, 1)$, and $(1, 0)$.

16. Suppose the density of a solid of radius R is given by $(1 + d^3)^{-1}$ where d is the distance to the center of the sphere. Find the total mass of the sphere.

17. The density of the material of a spherical shell whose inner radius is 1 m and whose outer radius is 2 m is $0.4d^2$ g/cm^3, where d is the distance to the center of the sphere in meters. Find the total mass of the shell.

18. If the shell in Exercise 17 were dropped into a large tank of pure water, would it float? What if the shell leaked? (Assume that the density of water is exactly 1 g/cm^3.)

19. The temperature at points in the cube $C = \{(x, y, z) \mid -1 \leq x \leq 1, -1 \leq y \leq 1$, and $-1 \leq z \leq 1\}$ is $32d^2$, where d is the distance to the origin.

(a) What is the average temperature?
(b) At what points of the cube is the temperature equal to the average temperature?

20. Use cylindrical coordinates to find the center of mass of the region defined by

$$y^2 + z^2 \leq \frac{1}{4}, \qquad (x - 1)^2 + y^2 + z^2 \leq 1, \qquad x \geq 1.$$

21. Find the center of mass of the solid hemisphere

$$V = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq a^2 \text{ and } z \geq 0\}$$

if the density is constant.

22. Evaluate $\iint_B e^{-x^2-y^2}\, dx\, dy$ where B consists of those (x, y) satisfying $x^2 + y^2 \leq 1$ and $y \leq 0$.

23. Evaluate

$$\iiint_S \frac{dx\, dy\, dz}{(x^2 + y^2 + z^2)^{3/2}},$$

where S is the solid bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, where $a > b > 0$.

24. Evaluate $\iiint_D (x^2 + y^2 + z^2)xyz\, dx\, dy\, dz$ over each of the following regions.

(a) The sphere $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq R^2\}$
(b) The hemisphere $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq R^2 \text{ and } z \geq 0\}$
(c) The octant $D = \{(x, y, z) \mid x \geq 0, y \geq 0, z \geq 0, \text{ and } z^2 + y^2 + z^2 \leq R^2\}$

25. Let C be the cone-shaped region $\{(x, y, z) \mid \sqrt{x^2 + y^2} \leq z \leq 1\}$ in $\mathbb{R}^3$ and evaluate the integral $\iiint_C (1 + \sqrt{x^2 + y^2})\, dx\, dy\, dz$.

26. Find $\iiint_{\mathbb{R}^3} f(x, y, z)\, dx\, dy\, dz$ where $f(x, y, z) = \exp\,[-(x^2 + y^2 + z^2)^{3/2}]$.

27. The *flexural rigidity EI* of a uniform beam is the product of its Young's modulus of elasticity E and the moment of inertia I of the cross section of the beam with respect to a

9. Evaluate the integral $\int_0^1 \int_0^x \int_0^y (y + xz)\, dz\, dy\, dx$.

10. Evaluate $\int_0^1 \int_y^{y^2} e^{x/y}\, dx\, dy$.

11. Evaluate $\int_0^1 \int_0^{(\arcsin y)/y} y \cos xy\, dx\, dy$.

12. Change the order of integration and evaluate

$$\int_0^2 \int_{y/2}^1 (x + y)^2\, dx\, dy.$$

13. Show that evaluating $\iint_D dx\, dy$, where D is a y-simple region, reproduces the formula from one-variable calculus for the area between two curves.

14. Change the order of integration and evaluate

$$\int_0^1 \int_{y^{1/2}}^1 (x^2 + y^3 x)\, dx\, dy.$$

15. Let D be the region in the xy plane inside the unit circle $x^2 + y^2 = 1$. Evaluate $\iint_D f(x, y)\, dx\, dy$ in each of the following cases:

(a) $f(x, y) = xy$ (b) $f(x, y) = x^2 y^2$ (c) $f(x, y) = x^3 y^3$

16. Find $\iint_D y[1 - \cos(\pi x/4)]\, dx\, dy$, where D is the region in Figure 5.R.1.

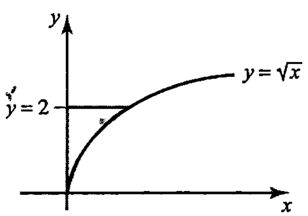

Figure 5.R.1 The region of integration for Exercise 16.

Evaluate the integrals in Exercises 17 to 24. Sketch and identify the type of the region (corresponding to the way the integral is written).

17. $\displaystyle\int_0^\pi \int_{\sin x}^{3 \sin x} x(1 + y)\, dy\, dx$

18. $\displaystyle\int_0^1 \int_{x-1}^{x \cos(\pi x/2)} (x^2 + xy + 1)\, dy\, dx$

19. $\displaystyle\int_{-1}^1 \int_{y^{2/3}}^{(2-y)^2} \left(\frac{3}{2}\sqrt{x} - 2y\right) dx\, dy$

20. $\displaystyle\int_0^2 \int_{-3(\sqrt{4-x^2})/2}^{3(\sqrt{4-x^2})/2} \left(\frac{5}{\sqrt{2+x}} + y^3\right) dy\, dx$

21. $\displaystyle\int_0^1 \int_0^{x^2} (x^2 + xy - y^2)\, dy\, dx$

22. $\displaystyle\int_2^4 \int_{y^2-1}^{y^3} 3\, dx\, dy$

23. $\displaystyle\int_0^1 \int_{x^2}^x (x + y)^2\, dy\, dx$

24. $\displaystyle\int_0^1 \int_0^{3y} e^{x+y}\, dx\, dy$

In Exercises 25 to 27, integrate the given function f over the given region D.

25. $f(x, y) = x - y$; D is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(2, 1)$.

26. $f(x, y) = x^3 y + \cos x$; D is the triangle defined by $0 \le x \le \pi/2$, $0 \le y \le x$.

27. $f(x, y) = x^2 + 2xy^2 + 2$; D is the region bounded by the graph of $y = -x^2 + x$, the x axis, and the lines $x = 0$ and $x = 2$.

In Exercises 28 and 29, sketch the region of integration, interchange the order, and evaluate.

28. $\displaystyle\int_1^4 \int_1^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$

29. $\displaystyle\int_0^1 \int_{1-y}^1 (x + y^2)\, dx\, dy$

30. Show that

$$4e^5 \le \iint_{[1,3]\times[2,4]} e^{x^2+y^2}\, dA \le 4e^{25}.$$

31. Show that

$$4\pi \le \iint_D (x^2 + y^2 + 1)\, dx\, dy \le 20\pi,$$

where D is the disk of radius 2 centered at the origin.

32. Suppose W is a ***path-connected region***, that is, given any two points of W there is a continuous path joining them. If f is a continuous function on W, use the intermediate-value theorem to show that there is at least one point in W at which the value of f is equal to the average of f over W, that is, the integral of f over W divided by the volume of W. (Compare this with the mean-value theorem for double integrals.) What happens if W is not connected?

33. Prove: $\int_0^x [\int_0^t F(u)\, du]\, dt = \int_0^x (x - u) F(u)\, du$.

Evaluate the integrals in Exercises 34 to 36.

34. $\displaystyle\int_0^1 \int_0^z \int_0^y xy^2 z^3\, dx\, dy\, dz$

35. $\displaystyle\int_0^1 \int_0^y \int_0^{x/\sqrt{3}} \frac{x}{x^2 + z^2}\, dz\, dx\, dy$

36. $\displaystyle\int_1^2 \int_1^z \int_{1/y}^2 yz^2\, dx\, dy\, dz$

37. Write the iterated integral $\int_0^1 \int_{1-x}^1 \int_x^1 f(x, y, z)\, dz\, dy\, dx$ as an integral over a region in $\mathbb{R}^3$ and then rewrite it in five other possible orders of integration.