In the next section, we shall use Riemann sums to rigorously define the double integral for a large class of functions of two variables without recourse to the notion of volume. Although we shall drop the requirement that $f(x, y) \geq 0$, equations (1) and (2) will remain valid. Therefore, the iterated integral will again provide the key to computing the double integral. In Section 5.3, we treat double integrals over regions more general than rectangles.

Finally, we remark that it is common to delete the brackets in iterated integrals such as equations (1) and (2) and write

$$\int_a^b \int_c^d f(x, y)\, dy\, dx \qquad \text{in place of} \qquad \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

and

$$\int_c^d \int_a^b f(x, y)\, dx\, dy \qquad \text{in place of} \qquad \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy.$$

EXERCISES

1. Evaluate the following iterated integrals:

(a) $\int_{-1}^1 \int_0^1 (x^4 y + y^2)\, dy\, dx$

(b) $\int_0^{\pi/2} \int_0^1 (y \cos x + 2)\, dy\, dx$

(c) $\int_0^1 \int_0^1 (xye^{x+y})\, dy\, dx$

(d) $\int_{-1}^0 \int_1^2 (-x \log y)\, dy\, dx$

2. Evaluate the integrals in Exercise 1 by integrating with respect to x and then with respect to y. [The solution to part (b) only is in the Study Guide to this text.]

3. Use Cavalieri's principle to show that the volumes of two cylinders with the same base and height are equal (see Figure 5.1.10).

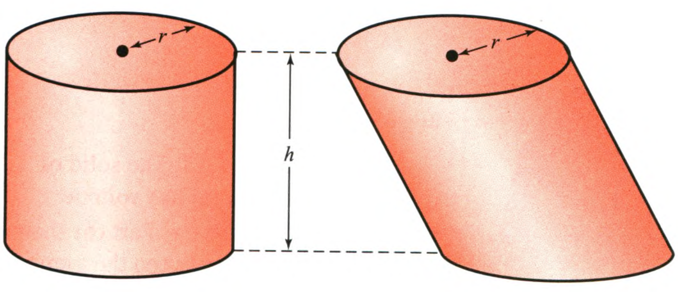

Figure 5.1.10 Two cylinders with the same base and height have the same volume.

4. Using Cavalieri's principle, compute the volume of the structure shown in Figure 5.1.11; each cross section is a rectangle of length 5 and width 3.

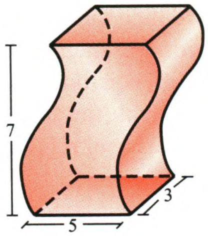

Figure 5.1.11 Compute this volume.

5. A lumberjack cuts out a wedge-shaped piece W of a cylindrical tree of radius r obtained by making two saw cuts to the tree's center, one horizontally and one at an angle θ. Compute the volume of the wedge W using Cavalieri's principle. (See Figure 5.1.12.)

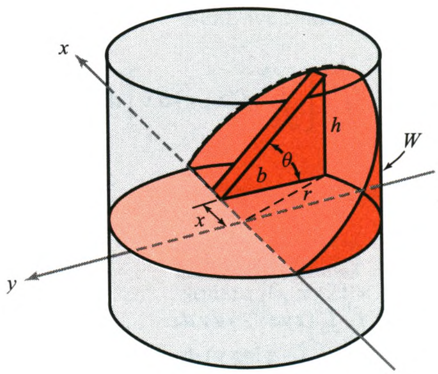

Figure 5.1.12 Find the volume of W.

6. (a) Show that the volume of the solid of revolution shown in Figure 5.1.13(a) is

$$\pi \int_a^b [f(x)]^2\, dx.$$

(b) Show that the volume of the region obtained by rotating the region under the graph of the parabola $y = -x^2 + 2x + 3$, $-1 \le x \le 3$, about the x axis is $512\pi/15$ [see Figure 5.1.13(b)].

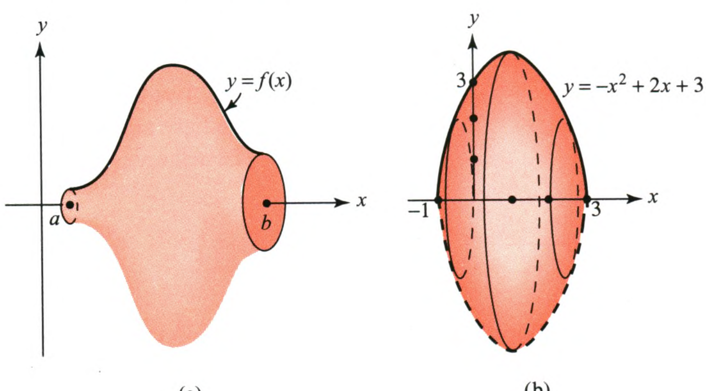

Figure 5.1.13 The solid of revolution (a) has volume $\pi \int_a^b [f(x)]^2\, dx$. Part (b) shows the region between the graph of $y = -x^2 + 2x + 3$ and the x axis rotated about the x axis.

Evaluate the double integrals in Exercises 7 to 9, where R is the rectangle $[0, 2] \times [-1, 0]$.

7. $\displaystyle\iint_R (x^2y^2 + x)\, dy\, dx$

8. $\displaystyle\iint_R \left(|y| \cos \frac{1}{4}\pi x\right) dy\, dx$

9. $\displaystyle\iint_R \left(-xe^x \sin \frac{1}{2}\pi y\right) dy\, dx$

10. Find the volume bounded by the graph of $f(x, y) = 1 + 2x + 3y$, the rectangle $[1, 2] \times [0, 1]$, and the four vertical sides of the rectangle R, as in Figure 5.1.1.

11. Repeat Exercise 10 for the function $f(x, y) = x^4 + y^2$ and the rectangle $[-1, 1] \times [-3, -2]$.

5.2 The Double Integral Over a Rectangle

We are ready to give a rigorous definition of the double integral as the limit of a sequence of sums. This will then be used to *define* the volume of the region under the graph of a function $f(x, y)$. We shall not require that $f(x, y) \geq 0$; but if $f(x, y)$ assumes negative values, we shall interpret the integral as a signed volume, just as for the area under the graph of a function of one variable. In addition, we shall discuss some of the fundamental algebraic properties of the double integral and prove Fubini's theorem, which states that the double integral can be calculated as an iterated integral. To begin, let us establish some notation for partitions and sums.

Definition of the Integral

Consider a closed rectangle $R \subset \mathbb{R}^2$; that is, R is a Cartesian product of two intervals: $R = [a, b] \times [c, d]$. By a ***regular partition*** of R of order n we mean the two ordered collections of $n + 1$ equally spaced points $\{x_j\}_{j=0}^n$ and $\{y_k\}_{k=0}^n$, that is, the points satisfying

$$a = x_0 < x_1 < \cdots < x_n = b, \qquad c = y_0 < y_1 < \cdots < y_n = d$$

and

$$x_{j+1} - x_j = \frac{b - a}{n}, \qquad y_{k+1} - y_k = \frac{d - c}{n}$$

(see Figure 5.2.1).

A function $f(x, y)$ is said to be ***bounded*** if there is a number $M > 0$ such that $-M \leq f(x, y) \leq M$ for all (x, y) in the domain of f. A continuous function on a *closed* rectangle is always bounded, but, for example, $f(x, y) = 1/x$ on $(0, 1] \times [0, 1]$ is continuous but is not bounded, because $1/x$ becomes arbitrarily large for x near 0. The rectangle $(0, 1] \times [0, 1]$ is not closed, because the endpoint 0 is missing in the first factor.

(called *Fourier series*), Riemann needed a clear, precise definition of the integral, which he presented in a paper in 1854. In this paper he defines his integral and gives necessary and sufficient conditions for a bounded function f to be integrable over an interval $[a, b]$.

In 1876, the German mathematician Karl J. Thomae generalized Riemann's integral to apply to functions of several variables, as we do in this chapter. We further develop this approach in the Internet supplement.

In the first half of the nineteenth century, Cauchy had observed that for continuous function of two variables, Fubini's theorem was valid. But Cauchy also gave an example of an unbounded function of two variables for which the iterated integrals were not equal. In 1878, Thomae gave the first example of a bounded function of two variables where one iterated integral exists and the other does not. In these examples, the functions were not "Riemann integrable" in the sense described in this section. Cauchy and Thomae's examples demonstrated that one must apply caution and not necessarily assume that iterated integrals are always equal.

In 1902, the French mathematician Henri Lebesgue developed a truly sweeping generalization of the Riemann integral. Lebesgue's theory allowed integration of vastly more functions than did Riemann's approach. Perhaps, unforeseen by Lebesgue, his theory was to have a profound impact on the development of many areas of mathematics in the twentieth century—in particular the theory of partial differential equations. Mathematics students go into more depth about the Lebesgue integral in their first year of graduate study.

In 1907, the Italian mathematician Guido Fubini used the Lebesgue integral to state the most general form of the theorem on the equality of iterated integrals, the form that is studied today and used by working mathematicians and scientists in their research.

EXERCISES

1. Evaluate each of the following integrals if $R = [0, 1] \times [0, 1]$.

(a) $\iint_R (x^3 + y^2)\, dA$

(b) $\iint_R ye^{xy}\, dA$

(c) $\iint_R (xy)^2 \cos x^3\, dA$

(d) $\iint_R \ln[(x + 1)(y + 1)]\, dA$

2. Evaluate each of the following integrals if $R = [0, 1] \times [0, 1]$.

(a) $\iint_R (x^m y^n)\, dx\, dy$, where $m, n > 0$

(b) $\iint_R (ax + by + c)\, dx\, dy$

(c) $\iint_R \sin(x + y)\, dx\, dy$

(d) $\iint_R (x^2 + 2xy + y\sqrt{x})\, dx\, dy$

3. Compute the volume of the region over the rectangle $[0, 1] \times [0, 1]$ and under the graph of $z = xy$.

4. Compute the volume of the solid bounded by the xz plane, the yz plane, the xy plane, the planes $x = 1$ and $y = 1$, and the surface $z = x^2 + y^4$.

5. Let f be continuous on $[a, b]$ and g continuous on $[c, d]$. Show that

$$\iint_R [f(x)g(y)]\,dx\,dy = \left[\int_a^b f(x)\,dx\right]\left[\int_c^d g(y)\,dy\right],$$

where $R = [a, b] \times [c, d]$.

6. Compute the volume of the solid bounded by the surface $z = \sin y$, the planes $x = 1$, $x = 0$, $y = 0$, and $y = \pi/2$, and the xy plane.

7. Compute the volume of the solid bounded by the graph $z = x^2 + y$, the rectangle $R = [0, 1] \times [1, 2]$, and the "vertical sides" of R.

8. Let f be continuous on $R = [a, b] \times [c, d]$; for $a < x < b, c < y < d$, define

$$F(x, y) = \int_a^x \int_c^y f(u, v)\,dv\,du.$$

Show that $\partial^2 F/\partial x\,\partial y = \partial^2 F/\partial y\,\partial x = f(x, y)$. Use this example to discuss the relationship between Fubini's theorem and the equality of mixed partial derivatives.

9. Let $f\colon [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 1 & x \text{ rational} \\ 2y & x \text{ irrational.} \end{cases}$$

Show that the iterated integral $\int_0^1 \left[\int_0^1 f(x, y)\,dy\right] dx$ exists but that f is not integrable.

10. Express $\iint_R \cosh xy\,dx\,dy$ as a convergent sequence, where $R = [0, 1] \times [0, 1]$.

11. Although Fubini's theorem holds for most functions met in practice, one must still exercise some caution. This exercise gives a function for which it fails. By using a substitution involving the tangent function, show that

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\,dy\,dx = \frac{\pi}{4}, \quad \text{yet} \quad \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\,dx\,dy = -\frac{\pi}{4}.$$

Why does this not contradict Theorem 3 or 3′?

12. Let f be continuous, $f \geq 0$, on the rectangle R. If $\iint_R f\,dA = 0$, prove that $f = 0$ on R.

5.3 The Double Integral Over More General Regions

Our goal in this section is twofold: First, we wish to define the double integral of a function $f(x, y)$ over regions D more general than rectangles; second, we want to develop a technique for evaluating this type of integral. To accomplish this, we shall define three special types of subsets of the xy plane, and then extend the notion of the double integral to them.

Elementary Regions

Suppose we are given two continuous real-valued functions $\phi_1\colon [a, b] \to \mathbb{R}$ and $\phi_2\colon [a, b] \to \mathbb{R}$ that satisfy $\phi_1(x) \leq \phi_2(x)$ for all $x \in [a, b]$. Let D be the set of all points (x, y) such that $x \in [a, b]$ and $\phi_1(x) \leq y \leq \phi_2(x)$. This region D is said to be ***y-simple***. Figure 5.3.1 shows various examples of y-simple regions. The curves and straight-line segments that bound the region together constitute the ***boundary*** of D, denoted ∂D. We use the phrase y-simple because the region is described in a relatively simple way, using y as a function of x.

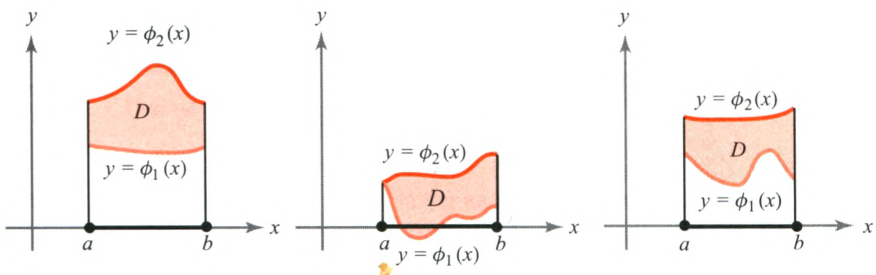

Figure 5.3.1 Some y-simple regions.

We say that a region D is ***x-simple*** if there are continuous functions ψ_1 and ψ_2 defined on $[c, d]$ such that D is the set of points (x, y) satisfying

$$y \in [c, d] \qquad \text{and} \qquad \psi_1(y) \leq x \leq \psi_2(y)$$

where $\psi_1(y) \leq \psi_2(y)$ for all $y \in [c, d]$. Again, the curves that bound the region D constitute its boundary ∂D. Some examples of x-simple regions are shown in Figure 5.3.2. In this situation, x is the distinguished variable, given as a function of y. Thus, the phrase x-simple is appropriate.

Finally, a ***simple*** region is one that is both x- and y-simple; that is, a simple region can be described as both an x-simple region and a y-simple region. An example of a simple region is a unit disk (see Figure 5.3.3).

Sometimes we will refer to any of the regions as ***elementary regions***. Note that the boundary ∂D of an elementary region is the type of set of discontinuities of a function allowed in Theorem 2.

Thus, $A(D)$ is the limit of the areas of the rectangles "circumscribing" D. The reader should draw a figure to accompany this discussion. ▲

The methods for treating x-simple regions are entirely analogous. Specifically, we have the following.

THEOREM 4′: Iterated Integrals for x-Simple Regions Suppose that D is the set of points (x, y) such that $y \in [c, d]$ and $\psi_1(y) \leq x \leq \psi_2(y)$. If f is continuous on D, then

$$\iint_D f(x, y)\, dA = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx \right] dy. \tag{2}$$

To find the area of D, we substitute $f = 1$ in formula (2); this yields

$$\iint_D dA = \int_c^d (\psi_2(y) - \psi_1(y))\, dy.$$

Again, this result for area agrees with the results of single-variable calculus for the area of a region between two curves.

Either the method for y-simple or the method for x-simple regions can be used for integrals over simple regions.

It follows from formulas (1) and (2) that $\iint_D f\, dA$ is independent of the choice of the rectangle R enclosing D used in the definition of $\iint_D f\, dA$, because, if we had picked another rectangle enclosing D, we would have arrived at the same formula (1).

EXERCISES

1. Evaluate the following iterated integrals and draw the regions D determined by the limits. State whether the regions are x-simple, y-simple, or simple.

(a) $\displaystyle\int_0^1 \int_0^{x^2} dy\, dx$

(b) $\displaystyle\int_1^2 \int_{2x}^{3x+1} dy\, dx$

(c) $\displaystyle\int_0^1 \int_1^{e^x} (x + y)\, dy\, dx$

(d) $\displaystyle\int_0^1 \int_{x^3}^{x^2} y\, dy\, dx$

2. Evaluate the following integrals and sketch the corresponding regions.

(a) $\displaystyle\int_{-3}^2 \int_0^{y^2} (x^2 + y)\, dx\, dy$

(b) $\displaystyle\int_{-1}^1 \int_{-2|x|}^{|x|} e^{x+y}\, dy\, dx$

(c) $\displaystyle\int_0^1 \int_0^{(1-x^2)^{1/2}} dy\, dx$

(d) $\displaystyle\int_0^{\pi/2} \int_0^{\cos x} y \sin x\, dy\, dx$

(e) $\displaystyle\int_0^1 \int_{y^2}^{y} (x^n + y^m)\, dx\, dy, \quad m, n > 0$

(f) $\displaystyle\int_{-1}^0 \int_0^{2(1-x^2)^{1/2}} x\, dy\, dx$

3. Use double integrals to compute the area of a circle of radius r.

4. Using double integrals, determine the area of an ellipse with semiaxes of length a and b.

5. What is the volume of a barn that has a rectangular base 20 ft by 40 ft, vertical walls 30 ft high at the front (which we assume is on the 20-ft side of the barn), and 40 ft high at the rear? The barn has a flat roof. Use double integrals to compute the volume.

6. Let D be the region bounded by the positive x and y axes and the line $3x + 4y = 10$. Compute

$$\iint_D (x^2 + y^2)\, dA.$$

7. Let D be the region bounded by the y axis and the parabola $x = -4y^2 + 3$. Compute

$$\iint_D x^3 y\, dx\, dy.$$

8. Evaluate $\int_0^1 \int_0^{x^2} (x^2 + xy - y^2)\, dy\, dx$. Describe this iterated integral as an integral over a certain region D in the xy plane.

9. Let D be the region given as the set of (x, y) where $1 \le x^2 + y^2 \le 2$ and $y \ge 0$. Is D an elementary region? Evaluate $\iint_D f(x, y)\, dA$ where $f(x, y) = 1 + xy$.

10. Use the formula $A(D) = \iint_D dx\, dy$ to find the area enclosed by one period of the sine function $\sin x$, for $0 \le x \le 2\pi$, and the x axis.

11. Find the volume of the region inside the surface $z = x^2 + y^2$ and between $z = 0$ and $z = 10$.

12. Set up the integral required to calculate the volume of a cone of base radius r and height h.

13. Evaluate $\iint_D y\, dA$ where D is the set of points (x, y) such that $0 \le 2x/\pi \le y$, $y \le \sin x$.

14. From Exercise 5, Section 5.2, $\int_a^b \int_c^d f(x)g(y)\, dy\, dx = \left(\int_a^b f(x)\, dx\right)\left(\int_c^d g(y)\, dy\right)$. Is it true that $\iint_D f(x)g(y)\, dx\, dy = \left(\int_a^b f(x)\, dx\right)\left(\int_{\phi_1(a)}^{\phi_2(b)} g(y)\, dy\right)$ for y-simple regions?

15. Let D be a region given as the set of (x, y) with $-\phi(x) \le y \le \phi(x)$ and $a \le x \le b$, where ϕ is a nonnegative continuous function on the interval $[a, b]$. Let $f(x, y)$ be a function on D such that $f(x, y) = -f(x, -y)$ for all $(x, y) \in D$. Argue that $\iint_D f(x, y)\, dA = 0$.

16. Use the methods of this section to show that the area of the parallelogram D determined by two planar vectors $\mathbf{a}$ and $\mathbf{b}$ is $|a_1 b_2 - a_2 b_1|$, where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$.

17. Describe the area $A(D)$ of a region as a limit of areas of inscribed rectangles, as in Example 3.

Dividing through inequality (5) by $A(D)$, we get

$$m \leq \frac{1}{A(D)} \iint_D f(x, y)\, dA \leq M. \tag{6}$$

Because a continuous function on D takes on every value between its maximum and minimum values (this is the two-variable *intermediate value theorem* proved in advanced calculus; see also Review Exercise 32), and because the number $[1/A(D)] \iint_D f(x, y)\, dA$ is, by inequality (6), between these values, there must be a point $(x_0, y_0) \in D$ with

$$f(x_0, y_0) = \frac{1}{A(D)} \iint_D f(x, y)\, dA,$$

which is precisely the conclusion of Theorem 5. ■

EXERCISES

1. In the following integrals, change the order of integration, sketch the corresponding regions, and evaluate the integral both ways.

(a) $\displaystyle\int_0^1 \int_x^1 xy\, dy\, dx$

(b) $\displaystyle\int_0^{\pi/2} \int_0^{\cos\theta} \cos\theta\, dr\, d\theta$

(c) $\displaystyle\int_0^1 \int_1^{2-y} (x+y)^2\, dx\, dy$

(d) $\displaystyle\int_a^b \int_a^y f(x, y)\, dx\, dy$ (express your answer in terms of antiderivatives).

2. Find

(a) $\displaystyle\int_{-1}^1 \int_{|y|}^1 (x+y)^2\, dx\, dy$

(b) $\displaystyle\int_{-3}^1 \int_{-\sqrt{(9-y^2)}}^{\sqrt{(9-y^2)}} x^2\, dx\, dy$

(c) $\displaystyle\int_0^4 \int_{y/2}^2 e^{x^2}\, dx\, dy$

(d) $\displaystyle\int_0^1 \int_{\tan^{-1} y}^{\pi/4} (\sec^5 x)\, dx\, dy$

3. If $f(x, y) = e^{\sin(x+y)}$ and $D = [-\pi, \pi] \times [-\pi, \pi]$, show that

$$\frac{1}{e} \leq \frac{1}{4\pi^2} \iint_D f(x, y)\, dA \leq e.$$

4. Show that

$$\frac{1}{2}(1 - \cos 1) \leq \iint_{[0,1]\times[0,1]} \frac{\sin x}{1 + (xy)^4}\, dx\, dy \leq 1.$$

5. If $D = [-1, 1] \times [-1, 2]$, show that

$$1 \leq \iint_D \frac{dx\, dy}{x^2 + y^2 + 1} \leq 6.$$

6. Using the mean value inequality, show that

$$\frac{1}{6} \le \iint_D \frac{dA}{y - x + 3} \le \frac{1}{4},$$

where D is the triangle with vertices $(0, 0)$, $(1, 1)$, and $(1, 0)$.

7. Compute the volume of an ellipsoid with semiaxes a, b, and c. (HINT: Use symmetry and first find the volume of one half of the ellipsoid.)

8. Compute $\iint_D f(x, y)\,dA$, where $f(x, y) = y^2\sqrt{x}$ and D is the set of (x, y) where $x > 0$, $y > x^2$, and $y < 10 - x^2$.

9. Find the volume of the region determined by $x^2 + y^2 + z^2 \le 10$, $z \ge 2$. Use the disk method from one-variable calculus and state how the method is related to Cavalieri's principle.

10. Evaluate $\iint_D e^{x-y}\,dx\,dy$, where D is the interior of the triangle with vertices $(0, 0)$, $(1, 3)$, and $(2, 2)$.

11. Evaluate $\iint_D y^3(x^2 + y^2)^{-3/2}\,dx\,dy$, where D is the region determined by the conditions $\frac{1}{2} \le y \le 1$ and $x^2 + y^2 \le 1$.

12. Given that the double integral $\iint_D f(x, y)\,dx\,dy$ of a positive continuous function f equals the iterated integral $\int_0^1 \left[\int_{x^2}^{x} f(x, y)\,dy\right] dx$, sketch the region D and interchange the order of integration.

13. Given that the double integral $\iint_D f(x, y)\,dx\,dy$ of a positive continuous function f equals the iterated integral $\int_0^1 \left[\int_{y}^{\sqrt{2-y^2}} f(x, y)\,dx\right] dy$, sketch the region D and interchange the order of integration.

14. Prove that $2\int_a^b \int_x^b f(x)f(y)\,dy\,dx = \left(\int_a^b f(x)\,dx\right)^2$. [HINT: Notice that $\left(\int_a^b f(x)\,dx\right)^2 = \iint_{[a,b]\times[a,b]} f(x)f(y)\,dx\,dy$.]

15. Show that (see Exercise 27, Section 2.5)

$$\frac{d}{dx}\int_a^x \int_c^d f(x, y, z)\,dz\,dy = \int_c^d f(x, y, z)\,dz + \int_a^x \int_c^d f_x(x, y, z)\,dz\,dy.$$

5.5 The Triple Integral

Triple integrals are needed for many physical problems. For example, if the temperature inside an oven is not uniform, determining the average temperature involves "summing" the values of the temperature function at all points in the solid region enclosed by the oven walls and then dividing the answer by the total volume of the oven. Such a sum is expressed mathematically as a triple integral.

EXAMPLE 6 Evaluate

$$\int_0^1 \int_0^x \int_{x^2+y^2}^2 dz\,dy\,dx.$$

Sketch the region W of integration and interpret.

SOLUTION

$$\int_0^1 \int_0^x \int_{x^2+y^2}^2 dz\,dy\,dx = \int_0^1 \int_0^x (2 - x^2 - y^2)\,dy\,dx$$
$$= \int_0^1 \left(2x - x^3 - \frac{x^3}{3}\right) dx = 1 - \frac{1}{4} - \frac{1}{12} = \frac{2}{3}.$$

This integral is the volume of the region sketched in Figure 5.5.8. ▲

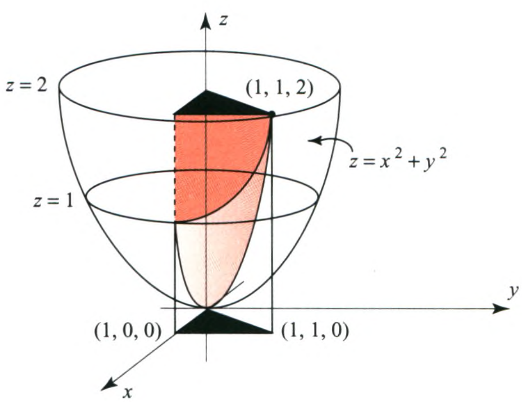

Figure 5.5.8 The region W lies between the paraboloid $z = x^2 + y^2$ and the plane $z = 2$, and above the region D.

EXERCISES

In Exercises 1 to 4, perform the indicated integration over the given box.

1. $\iiint_B x^2\,dx\,dy\,dz$, $B = [0, 1] \times [0, 1] \times [0, 1]$
2. $\iiint_B e^{-xy}\,y\,dx\,dy\,dz$, $B = [0, 1] \times [0, 1] \times [0, 1]$
3. $\iiint_B (2x + 3y + z)\,dx\,dy\,dz$, $B = [0, 2] \times [-1, 1] \times [0, 1]$
4. $\iiint_B ze^{x+y}\,dx\,dy\,dz$, $B = [0, 1] \times [0, 1] \times [0, 1]$

In Exercises 5 to 8, describe the given region as an elementary region.

5. The region between the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$.

6. The region cut out of the ball $x^2 + y^2 + z^2 \leq 4$ by the elliptic cylinder $2x^2 + z^2 = 1$, that is, the region inside the cylinder and the ball.

7. The region inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$.

8. The region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 4$, and $x = z - y - 1$.

Find the volume of the region in Exercises 9 to 12.

9. The region bounded by $z = x^2 + y^2$ and $z = 10 - x^2 - 2y^2$.

10. The solid bounded by $x^2 + 2y^2 = 2$, $z = 0$, and $x + y + 2z = 2$.

11. The solid bounded by $x = y$, $z = 0$, $y = 0$, $x = 1$, and $x + y + z = 0$.

12. The region common to the intersecting cylinders $x^2 + y^2 \leq a^2$ and $x^2 + z^2 \leq a^2$.

Evaluate the integrals in Exercises 13 to 21.

13. $\displaystyle\int_0^1 \int_1^2 \int_2^3 \cos[\pi(x + y + z)]\, dx\, dy\, dz$

14. $\displaystyle\int_0^1 \int_0^x \int_0^y (y + xz)\, dz\, dy\, dx$

15. $\displaystyle\iiint_W (x^2 + y^2 + z^2)\, dx\, dy\, dz$; W is the region bounded by $x + y + z = a$ (where $a > 0$), $x = 0$, $y = 0$, and $z = 0$.

16. $\displaystyle\iiint_W z\, dx\, dy\, dz$; W is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 = 1$, with $x \geq 0$, $y \geq 0$.

17. $\displaystyle\iiint_W x^2 \cos z\, dx\, dy\, dz$; W is the region bounded by $z = 0$, $z = \pi$, $y = 0$, $y = 1$, $x = 0$, and $x + y = 1$.

18. $\displaystyle\int_0^2 \int_0^x \int_0^{x+y} dz\, dy\, dx$

19. $\displaystyle\iiint_W (1 - z^2)\, dx\, dy\, dz$; W is the pyramid with top vertex at $(0, 0, 1)$ and base vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$.

20. $\displaystyle\iiint_W (x^2 + y^2)\, dx\, dy\, dz$; W is the same pyramid as in Exercise 19.

21. $\displaystyle\int_0^1 \int_0^{2x} \int_{x^2+y^2}^{x+y} dz\, dy\, dx.$

22. (a) Sketch the region for the integral $\displaystyle\int_0^1 \int_0^x \int_0^y f(x, y, z)\, dz\, dy\, dx.$

(b) Write the integral with the integration order $dx\, dy\, dz$.

For the regions in Exercises 23 to 26, find the appropriate limits $\phi_1(x)$, $\phi_2(x)$, $\gamma_1(x, y)$, and $\gamma_2(x, y)$, and write the triple integral over the region W as an iterated integral in the form

$$\iiint_W f\,dV = \int_a^b \left\{ \int_{\phi_1(x)}^{\phi_2(x)} \left[\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)\,dz \right] dy \right\} dx.$$

23. $W = \{(x, y, z) \mid \sqrt{x^2 + y^2} \le z \le 1\}$

24. $W = \{(x, y, z) \mid \frac{1}{2} \le z \le 1 \text{ and } x^2 + y^2 + z^2 \le 1\}$

25. $W = \{(x, y, z) \mid x^2 + y^2 \le 1, z \ge 0 \text{ and } x^2 + y^2 + z^2 \le 4\}$

26. $W = \{(x, y, z) \mid |x| \le 1, |y| \le 1, z \ge 0 \text{ and } x^2 + y^2 + z^2 \le 1\}$

27. Show that the formula using triple integrals for the volume under the graph of a positive function $f(x, y)$, on an elementary region D in the plane, reduces to the double integral of f over D.

28. Let W be the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$, and $z = x + y$.

(a) Find the volume of W.
(b) Evalute $\iiint_W x\,dx\,dy\,dz$.
(c) Evalute $\iiint_W y\,dx\,dy\,dz$.

29. Let f be continuous and let B_ε be the ball of radius ε centered at the point (x_0, y_0, z_0). Let vol (B_ε) be the volume of B_ε. Prove that

$$\lim_{\varepsilon \to 0} \frac{1}{\text{vol}\,(B_\varepsilon)} \iiint_{B_\varepsilon} f(x, y, z)\,dV = f(x_0, y_0, z_0).$$

REVIEW EXERCISES FOR CHAPTER 5

Evaluate the integrals in Exercises 1 to 4.

1. $\displaystyle\int_0^3 \int_{-x^2+1}^{x^2+1} xy\,dy\,dx$

2. $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 (x + y)^2\,dy\,dx$

3. $\displaystyle\int_0^1 \int_{e^x}^{e^{2x}} x \ln y\,dy\,dx$

4. $\displaystyle\int_0^1 \int_1^2 \int_2^3 \cos\,[\pi(x + y + z)]\,dx\,dy\,dz.$

Reverse the order of integration of the integrals in Exercises 5 to 8 and evaluate.

5. The integral in Exercise 1.

6. The integral in Exercise 2.

7. The integral in Exercise 3.

8. The integral in Exercise 4.

9. Evaluate the integral $\int_0^1 \int_0^x \int_0^y (y + xz)\, dz\, dy\, dx$.

10. Evaluate $\int_0^1 \int_y^{y^2} e^{x/y}\, dx\, dy$.

11. Evaluate $\int_0^1 \int_0^{(\arcsin y)/y} y \cos xy\, dx\, dy$.

12. Change the order of integration and evaluate

$$\int_0^2 \int_{y/2}^1 (x+y)^2\, dx\, dy.$$

13. Show that evaluating $\iint_D dx\, dy$, where D is a y-simple region, reproduces the formula from one-variable calculus for the area between two curves.

14. Change the order of integration and evaluate

$$\int_0^1 \int_{y^{1/2}}^1 (x^2 + y^3 x)\, dx\, dy.$$

15. Let D be the region in the xy plane inside the unit circle $x^2 + y^2 = 1$. Evaluate $\iint_D f(x, y)\, dx\, dy$ in each of the following cases:

(a) $f(x, y) = xy$ (b) $f(x, y) = x^2 y^2$ (c) $f(x, y) = x^3 y^3$

16. Find $\iint_D y[1 - \cos(\pi x/4)]\, dx\, dy$, where D is the region in Figure 5.R.1.

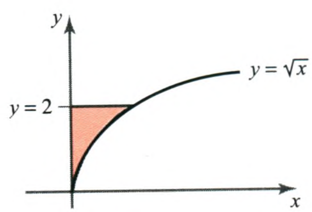

Figure 5.R.1 The region of integration for Exercise 16.

Evaluate the integrals in Exercises 17 to 24. Sketch and identify the type of the region (corresponding to the way the integral is written).

17. $\displaystyle\int_0^{\pi} \int_{\sin x}^{3 \sin x} x(1+y)\, dy\, dx$

18. $\displaystyle\int_0^1 \int_{x-1}^{x \cos(\pi x/2)} (x^2 + xy + 1)\, dy\, dx$

19. $\displaystyle\int_{-1}^1 \int_{y^{2/3}}^{(2-y)^2} \left(\frac{3}{2}\sqrt{x} - 2y\right) dx\, dy$

20. $\displaystyle\int_0^2 \int_{-3(\sqrt{4-x^2})/2}^{3(\sqrt{4-x^2})/2} \left(\frac{5}{\sqrt{2+x}} + y^3\right) dy\, dx$

21. $\displaystyle\int_0^1 \int_0^{x^2} (x^2 + xy - y^2)\, dy\, dx$

22. $\displaystyle\int_2^4 \int_{y^2-1}^{y^3} 3\, dx\, dy$

23. $\displaystyle\int_0^1 \int_{x^2}^{x} (x+y)^2\, dy\, dx$

24. $\displaystyle\int_0^1 \int_0^{3y} e^{x+y}\, dx\, dy$

In Exercises 25 to 27, integrate the given function f over the given region D.

25. $f(x, y) = x - y$; D is the triangle with vertices (0, 0), (1, 0), and (2, 1).

26. $f(x, y) = x^3 y + \cos x$; D is the triangle defined by $0 \le x \le \pi/2, 0 \le y \le x$.

27. $f(x, y) = x^2 + 2xy^2 + 2$; D is the region bounded by the graph of $y = -x^2 + x$, the x axis, and the lines $x = 0$ and $x = 2$.

In Exercises 28 and 29, sketch the region of integration, interchange the order, and evaluate.

28. $\displaystyle\int_1^4 \int_1^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$

29. $\displaystyle\int_0^1 \int_{1-y}^{1} (x + y^2)\, dx\, dy$

30. Show that

$$4e^5 \le \iint_{[1,3]\times[2,4]} e^{x^2+y^2}\, dA \le 4e^{25}.$$

31. Show that

$$4\pi \le \iint_D (x^2 + y^2 + 1)\, dx\, dy \le 20\pi,$$

where D is the disk of radius 2 centered at the origin.

32. Suppose W is a ***path-connected region***, that is, given any two points of W there is a continuous path joining them. If f is a continuous function on W, use the intermediate-value theorem to show that there is at least one point in W at which the value of f is equal to the average of f over W, that is, the integral of f over W divided by the volume of W. (Compare this with the mean-value theorem for double integrals.) What happens if W is not connected?

33. Prove: $\int_0^x [\int_0^t F(u)\, du]\, dt = \int_0^x (x - u) F(u)\, du$.

Evaluate the integrals in Exercises 34 to 36.

34. $\displaystyle\int_0^1 \int_0^z \int_0^y xy^2 z^3\, dx\, dy\, dz$

35. $\displaystyle\int_0^1 \int_0^y \int_0^{x/\sqrt{3}} \frac{x}{x^2 + z^2}\, dz\, dx\, dy$

36. $\displaystyle\int_1^2 \int_1^z \int_{1/y}^{2} yz^2\, dx\, dy\, dz$

37. Write the iterated integral $\int_0^1 \int_{1-x}^1 \int_x^1 f(x, y, z)\, dz\, dy\, dx$ as an integral over a region in $\mathbb{R}^3$ and then rewrite it in five other possible orders of integration.

EXERCISES

1. Let $S^* = (0, 1] \times [0, 2\pi)$ and define $T(r, \theta) = (r\cos\theta, r\sin\theta)$. Determine the image set S. Show that T is one-to-one on S^*.

2. Define

$$T(x^*, y^*) = \left(\frac{x^* - y^*}{\sqrt{2}}, \frac{x^* + y^*}{\sqrt{2}}\right).$$

Show that T rotates the unit square, $D^* = [0, 1] \times [0, 1]$.

3. Let $D^* = [0, 1] \times [0, 1]$ and define T on D^* by $T(u, v) = (-u^2 + 4u, v)$. Find the image D. Is T one-to-one?

4. Let D^* be the parallelogram bounded by the lines $y = 3x - 4$, $y = 3x$, $y = \frac{1}{2}x$, and $y = \frac{1}{2}(x + 4)$. Let $D = [0, 1] \times [0, 1]$. Find a T such that D is the image of D^* under T.

5. Let $D^* = [0, 1] \times [0, 1]$ and define T on D^* by $T(x^*, y^*) = (x^*y^*, x^*)$. Determine the image set D. Is T one-to-one? If not, can we eliminate some subset of D^* so that on the remainder T is one-to-one?

6. Let D^* be the parallelogram with vertices at $(-1, 3)$, $(0, 0)$, $(2, -1)$, and $(1, 2)$, and D be the rectangle $D = [0, 1] \times [0, 1]$. Find a T such that D is the image set of D^* under T.

7. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the spherical coordinate mapping defined by $(\rho, \phi, \theta) \mapsto (x, y, z)$, where

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi.$$

Let D^* be the set of points (ρ, ϕ, θ) such that $\phi \in [0, \pi]$, $\theta \in [0, 2\pi]$, $\rho \in [0, 1]$. Find $D = T(D^*)$. Is T one-to-one? If not, can we eliminate some subset of D^* so that, on the remainder, T will be one-to-one?

In Exercises 8 and 9, let $T(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix.

8. Show that T is one-to-one if and only if the determinant of A is not zero.

9. Show that $\det A \neq 0$ if and only if T is onto.

10. Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear and is given by $T(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix. Show that if $\det A \neq 0$, then T takes parallelograms onto parallelograms. [Hint: The general parallelogram in $\mathbb{R}^2$ can be described by the set of points $\mathbf{q} = \mathbf{p} + \lambda\mathbf{v} + \mu\mathbf{w}$ for $\lambda, \mu \in (0, 1)$ where $\mathbf{p}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^2$ with $\mathbf{v}$ not a scalar multiple of $\mathbf{w}$.]

11. Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is as in Exercise 10 and that $T(P^*) = P$ is a parallelogram. Show that P^* is a parallelogram.

12. Consider the map $T\colon D \to D$, where D is the unit disk in the plane, given by

$$T(r\cos\theta, r\sin\theta) = (r^2\cos 2\theta, r^2\sin 2\theta).$$

Using complex notation, $z = x + iy$, the map T can be written as $T(z) = z^2$. Show that the Jacobian determinant of T vanishes only at the origin. Thus, away from the origin, T is locally one-to-one. However, show that T is not globally one-to-one on $\mathbb{R}^2$ minus the origin.

6.2 The Change of Variables Theorem

Given two regions D and D^* in $\mathbb{R}^2$, a differentiable map T on D^* with image D, that is, $T(D^*) = D$, and any real-valued integrable function $f\colon D \to \mathbb{R}$, we would like to express $\iint_D f(x, y)\, dA$ as an integral over D^* of the composite function $f \circ T$. In this section we shall see how to do this.

Assume that D^* is a region in the uv plane and that D is a region in the xy plane. The map T is given by two coordinate functions:

$$T(u, v) = (x(u, v), y(u, v)) \qquad \text{for} \qquad (u, v) \in D^*.$$

At first, one might conjecture that

$$\iint_D f(x, y)\, dx\, dy \stackrel{?}{=} \iint_{D^*} f(x(u, v), y(u, v))\, du\, dv, \tag{1}$$

where $f \circ T(u, v) = f(x(u, v), y(u, v))$ is the composite function defined on D^*. However, if we consider the function $f\colon D \to \mathbb{R}^2$ where $f(x, y) = 1$, then equation (1) would imply

$$A(D) = \iint_D dx\, dy \stackrel{?}{=} \iint_{D^*} du\, dv = A(D^*). \tag{2}$$

But equation (2) will hold for only a few special cases and not for a general map T. For example, define T by $T(u, v) = (-u^2 + 4u, v)$. Restrict T to the unit square; that is, to the region $D^* = [0, 1] \times [0, 1]$ in the uv plane (see Figure 6.2.1). Then, as in Exercise 3, Section 6.1, T takes D^* onto $D = [0, 3] \times [0, 1]$. Clearly, $A(D) \neq A(D^*)$, and so formula (2) is *not valid.*

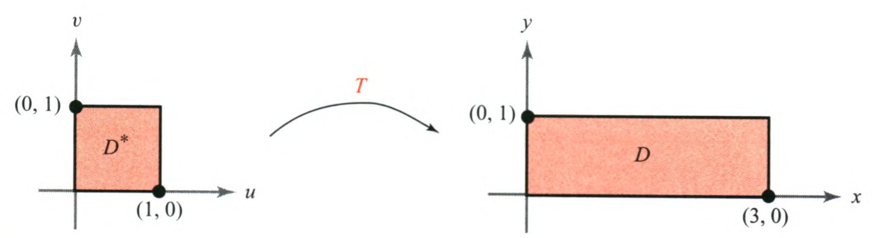

Figure 6.2.1 The map $T\colon (u, v) \mapsto (-u^2 + 4u, v)$ takes the square D^* onto the rectangle D.

EXAMPLE 7 Let W be the ball of radius R and center $(0, 0, 0)$ in $\mathbb{R}^3$. Find the volume of W.

SOLUTION The volume of W is $\iiint_W dx\,dy\,dz$. This integral may be evaluated by reducing it to iterated integrals or by regarding W as a volume of revolution, but let us evaluate it here by using spherical coordinates. We get

$$\begin{aligned}\iiint_W dx\,dy\,dz &= \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{R^3}{3}\int_0^{\pi}\int_0^{2\pi} \sin\phi\, d\theta\, d\phi \\ &= \frac{2\pi R^3}{3}\int_0^{\pi} \sin\phi\, d\phi = \frac{2\pi R^3}{3}\{-[\cos(\pi) - \cos(0)]\} = \frac{4\pi R^3}{3},\end{aligned}$$

which is the standard formula for the volume of a solid sphere. ▲

EXERCISES

1. Let D be the unit disk: $x^2 + y^2 \leq 1$. Evaluate

$$\iint_D \exp(x^2 + y^2)\, dx\, dy$$

by making a change of variables to polar coordinates.

2. Let D be the region $0 \leq y \leq x$ and $0 \leq x \leq 1$. Evaluate

$$\iint_D (x + y)\, dx\, dy$$

by making the change of variables $x = u + v$, $y = u - v$. Check your answer by evaluating the integral directly by using an iterated integral.

3. Let $T(u, v) = (x(u, v), y(u, v))$ be the mapping defined by $T(u, v) = (4u, 2u + 3v)$. Let D^* be the rectangle $[0, 1] \times [1, 2]$. Find $D = T(D^*)$ and evaluate

(a) $\displaystyle\iint_D xy\, dx\, dy$ (b) $\displaystyle\iint_D (x - y)\, dx\, dy$

by making a change of variables to evaluate them as integrals over D^*.

4. Repeat Exercise 3 for $T(u, v) = (u, v(1 + u))$.

5. Evaluate

$$\iint_D \frac{dx\, dy}{\sqrt{1 + x + 2y}},$$

where $D = [0, 1] \times [0, 1]$, by setting $T(u, v) = (u, v/2)$ and evaluating an integral over D^*, where $T(D^*) = D$.

6. Define $T(u, v) = (u^2 - v^2, 2uv)$. Let D^* be the set of (u, v) with $u^2 + v^2 \leq 1$, $u \geq 0$, $v \geq 0$. Find $T(D^*) = D$. Evaluate $\iint_D dx\, dy$.

7. Let $T(u, v)$ be as in Exercise 6. By making a change of variables, "formally" evaluate the "improper" integral

$$\iint_D \frac{dx\,dy}{\sqrt{x^2+y^2}}.$$

[NOTE: This integral (and the one in the next exercise) is *improper*, because the integrand $1/\sqrt{x^2+y^2}$ is neither continuous nor bounded on the domain of integration. (The theory of improper integrals is discussed in Section 6.4.)]

8. Calculate $\iint_R \frac{1}{x+y}\,dy\,dx$, where R is the region bounded by $x = 0$, $y = 0$, $x + y = 1$, $x + y = 4$, by using the mapping $T(u, v) = (u - uv, uv)$.

9. Evaluate $\iint_D (x^2+y^2)^{3/2}\,dx\,dy$ where D is the disk $x^2 + y^2 \le 4$.

10. Let D^* be a v-simple region in the uv plane bounded by $v = g(u)$ and $v = h(u) \le g(u)$ for $a \le u \le b$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation given by $x = u$ and $y = \psi(u, v)$, where ψ is of class C^1 and $\partial\psi/\partial v$ is never zero. Assume that $T(D^*) = D$ is a y-simple region; show that if $f\colon D \to \mathbb{R}$ is continuous, then

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(u, \psi(u,v)) \left|\frac{\partial\psi}{\partial v}\right| du\,dv.$$

11. Use double integrals to find the area inside the curve $r = 1 + \sin\theta$.

12. (a) Express $\int_0^1 \int_0^{x^2} xy\,dy\,dx$ as an integral over the triangle D^*, which is the set of (u, v) where $0 \le u \le 1$, $0 \le v \le u$. (HINT: Find a one-to-one mapping T of D^* onto the given region of integration.)

(b) Evaluate this integral directly and as an integral over D^*.

13. Integrate $ze^{x^2+y^2}$ over the cylinder $x^2 + y^2 \le 4$, $2 \le z \le 3$.

14. Let D be the unit disk. Express $\iint_D (1 + x^2 + y^2)^{3/2}\,dx\,dy$ as an integral over $[0, 1] \times [0, 2\pi]$ and evaluate.

15. Using polar coordinates, find the area bounded by the *lemniscate* $(x^2 + y^2)^2 = 2a^2\,(x^2 - y^2)$.

16. Redo Exercise 11 of Section 5.3 using a change of variables and compare the effort involved in each method.

17. Calculate $\iint_R (x+y)^2 e^{x-y}\,dx\,dy$ where R is the region bounded by $x + y = 1$, $x + y = 4$, $x - y = -1$, and $x - y = 1$.

18. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(u, v, w) = (u\cos v\cos w,\ u\sin v\cos w,\ u\sin w).$$

(a) Show that T is onto the unit sphere; that is, every (x, y, z) with $x^2 + y^2 + z^2 = 1$ can be written as $(x, y, z) = T(u, v, w)$ for some (u, v, w).

(b) Show that T is not one-to-one.

19. Integrate $x^2 + y^2 + z^2$ over the cylinder $x^2 + y^2 \le 2$, $-2 \le z \le 3$.

20. Evaluate $\int_0^\infty e^{-4x^2}\, dx$.

21. Let B be the unit ball. Evaluate

$$\iiint_B \frac{dx\, dy\, dz}{\sqrt{2 + x^2 + y^2 + z^2}}$$

by making the appropriate change of variables.

22. Evaluate $\iint_A [1/(x^2 + y^2)^2]\, dx\, dy$ where A is determined by the conditions $x^2 + y^2 \le 1$ and $x + y \ge 1$.

23. Evaluate $\iiint_W \frac{dx\, dy\, dz}{(x^2 + y^2 + z^2)^{3/2}}$, where W is the solid bounded by the two spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, where $0 < b < a$.

24. Evaluate $\iint_D x^2\, dx\, dy$ where D is determined by the two conditions $0 \le x \le y$ and $x^2 + y^2 \le 1$.

25. Integrate $\sqrt{x^2 + y^2 + z^2}\, e^{-(x^2+y^2+z^2)}$ over the region described in Exercise 23.

26. Evaluate the following by using cylindrical coordinates.

(a) $\iiint_B z\, dx\, dy\, dz$ where B is the region within the cylinder $x^2 + y^2 = 1$ above the xy plane and below the cone $z = (x^2 + y^2)^{1/2}$.

(b) $\iiint_W (x^2 + y^2 + z^2)^{-1/2}\, dx\, dy\, dz$ where W is the region determined by the conditions $\frac{1}{2} \le z \le 1$ and $x^2 + y^2 + z^2 \le 1$.

27. Evaluate $\iint_B (x + y)\, dx\, dy$ where B is the rectangle in the xy plane with vertices at $(0, 1)$, $(1, 0)$, $(3, 4)$, and $(4, 3)$.

28. Evaluate $\iint_D (x + y)\, dx\, dy$ where D is the square with vertices at $(0, 0)$, $(1, 2)$, $(3, 1)$, and $(2, -1)$.

29. Let E be the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \le 1$, where a, b, and c are positive.

(a) Find the volume of E.

(b) Evaluate $\iiint_E [(x^2/a^2) + (y^2/b^2) + (z^2/c^2)]\, dx\, dy\, dz$. (HINT: Change variables and then use spherical coordinates.)

30. Using spherical coordinates, compute the integral of $f(\rho, \phi, \theta) = 1/\rho$ over the region in the first octant of $\mathbb{R}^3$, which is bounded by the cones $\phi = \pi/4$, $\phi = \arctan 2$ and the sphere $\rho = \sqrt{6}$.

31. The mapping $T(u, v) = (u^2 - v^2, 2uv)$ transforms the rectangle $1 \leq u \leq 2$, $1 \leq v \leq 3$ of the uv plane into a region R of the xy plane.

(a) Show that T is one-to-one.
(b) Find the area of R using the change of variables formula.

32. Let R denote the region inside $x^2 + y^2 = 1$, but outside $x^2 + y^2 = 2y$ with $x \geq 0, y \geq 0$.

(a) Sketch this region.
(b) Let $u = x^2 + y^2$, $v = x^2 + y^2 - 2y$. Sketch the region D in the uv plane, which corresponds to R under this change of coordinates.
(c) Compute $\iint_R xe^y \, dx \, dy$ using this change of coordinates.

33. Let D be the region bounded by $x^{3/2} + y^{3/2} = a^{3/2}$, for $x \geq 0$, $y \geq 0$, and the coordinate axes $x = 0$, $y = 0$. Express $\iint_D f(x, y) \, dx \, dy$ as an integral over the triangle D^*, which is the set of points $0 \leq u \leq a$, $0 \leq v \leq a - u$. (Do not attempt to evaluate.)

34. Show that $S(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$, the spherical change-of-coordinate mapping, is one-to-one except on a set that is a union of finitely many graphs of continuous functions.

6.3 Applications

In this section, we shall discuss average values, centers of mass, moments of inertia, and the gravitational potential as applications.

Averages

If $x_1, \ldots, x_n$ are n numbers, their ***average*** is defined by

$$[x_i]_{\text{av}} = \frac{x_1 + \cdots + x_n}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

Notice that if all the x_i happen to have a common value c, then their average, of course, also equals c.

This concept leads one to define the average values of functions as follows.

Average Values The ***average value*** of a function of one variable on the interval $[a, b]$ is defined by

$$[f]_{\text{av}} = \frac{\int_a^b f(x)\, dx}{b - a}.$$

Likewise, for functions of two variables, the ratio of the integral to the area of D,

$$[f]_{\text{av}} = \frac{\iint_D f(x, y)\, dx\, dy}{\iint_D dx\, dy}, \tag{1}$$