THE ONLINE PART OF MATH 425 (ADVANCED MULTIVARIABLE CALCULUS) SPRING 2020

Contents

Intro: Integrals over shapes

The online part of the course really starts with chapter 8 but I will also supply notes in chapter 7 for continuity. In chapter 7 we have learned how to calculate integrals over shapes in $\mathbb{R}^n$ using parameterizations. In chapter 8 we will learn various tricks that simplify these calculations.

All these tricks come from the familiar Fundamental Theorem of Calculus for functions of one variable (here the geometric object is just an interval $[a, b]$). At the very end we will find a generalization of this theorem to a *Fundamental Theorem of Multivariable Calculus* which concerns integrals of functions (or vector fields) of any number of variables over more complicated objects: shapes in $\mathbb{R}^n$'s.

However, most of our work will be on shapes in $\mathbb{R}^2$ and $\mathbb{R}^3$. One reason is that the calculus in dimensions 2 and 3 is the language of classical physics. Even in this special cases the subject is very rich, so we will learn the meaning of the above general FTC result in a variety of situations,

Date:

7. **Integration over paths and surfaces (Chapter 7.)**

Here we learn to integrate over various geometric objects in $\mathbb{R}^2$. $\mathbb{R}^3$ or any $\mathbb{R}^n$. I will call these *shapes.*

7.1. **Shapes.**

So far we have been integrating over regions (=domains) in $\mathbb{R}^n$. Regions were described by inequalities.

Now we learn how to integrate over more general geometric pieces in $\mathbb{R}^n$ which are given by equations and inequalities. These I will call *subspaces of* $\mathbb{R}^n$ (or *shapes in* $\mathbb{R}^n$*, or* submanifolds in $\mathbb{R}^n$).

Example. Inequality $x_1^2 + \cdots + x_n^2 \le R^2$ describes a region in $\mathbb{R}^n$ denoted B_R^n. We call it a ball of radius R, and the super index n reminds us that this is is ball in $\mathbb{R}^n$.

Equality $x_1^2 + \cdots + x_n^2 = R^2$ describes a shape in $\mathbb{R}^n$ denoted S_R^n. We call it a sphere of radius R and the super index n is there to remind us that this is is shape inside $\mathbb{R}^n$.

7.1.1. *Dimension of regions and shapes in* $\mathbb{R}^n$*.* Recall the intuitive notion of dimension: we say that X has dimension d (we write $\dim(X) = d$, if on X there are d independent directions of motion. For instance $\mathbb{R}^n$ has dimension n since here one can move in the direction of x_1-axis, x_2-axis, x_3-axis, ... all the way up to x_n-axis, We think of $\dim(X)$ as a measure of how complicated X is.

The big difference between shapes in $\mathbb{R}^n$ and regions in $\mathbb{R}^n$ is that a region R in $\mathbb{R}^n$ has the dimension n, the same as the surrounding space $\mathbb{R}^n$, while a shape in $\mathbb{R}^n$ has dimension d which is less then n.

The difference $n - d$ is usually the number of equations we used to define the shape S!, we call it the *codimension* of S in the n-dimensional space $\mathbb{R}^n$.

Example. We have $\dim(B_R^n) = n$ while $\dim(S_R^n) = n - 1$ since we use one equation to describe a sphere.

Question. Draw B_R^2 and S_R^2 to see that there are more independent directions in the ball than in the sphere.

7.1.2. *Parameterizations of shapes.* The calculations (differentiation and integration) for a region R in $\mathbb{R}^n_{x_1,\ldots,x_n}$ naturally use variables $x_1, ..., x_n$.

One generally can not calculate on a shape S in $\mathbb{R}^n_{x_1,\ldots,x_n}$ directly. One first needs to match the shape S with a region R in some $\mathbb{R}^d_{u_1,\ldots,u_d}$, where the number d of variables is the dimension $d = \dim(S)$ of the shape S. What we mean by "matching' is really called a *parameterization* of S.

- A *parameterization* of S is given by a choice of a region R in some $\mathbb{R}^d$ and a mapping $\Gamma : R \to S$ which is a 1-1 correspondence (outside of a subset of volume zero).

Example. The sphere S^2_R in $\mathbb{R}^2_{x,y}$ is just the circle of radius R. Its dimension is clealr one, so its parameterization will use just one variable u_1. A natural choice of a parameterization of a circle is to use the variable θ whose *geometric meaning* is the angle with the positive x-axis. Then the parameterization matches a number θ with the point (x, y) on the circle corresponding to angle θ. Its coordinates are $x = R\cos(\theta)$ and $y = R\sin(\theta)$. One thinks of this parameterization as a mapping $\gamma : \mathbb{R}^1_\theta \to \mathbb{R}^2_{x,y}$ by $\gamma(\theta) = \left(\begin{smallmatrix} x(\theta) \\ y(\theta) \end{smallmatrix}\right) = \left(\begin{smallmatrix} R\cos(\theta) \\ R\sin(\theta) \end{smallmatrix}\right)$.

7.1.3. *The precise notion of dimension.* We say that a shape S has *dimension* d if it has a parameterization $\Gamma : R \to S$ by a region R in $\mathbb{R}^d$.

This means that a piece of S looks as a piece of $\mathbb{R}^d$. In particular the parameterization Γ transports d independent directions of motion in $\mathbb{R}^d$ to d independent directions of motion in S. So, S has d such directions as in the above intuitive notion of dimension.

7.2. **Integration over shapes: basic ingredients.** In 7.2.1 we recall the meaning of integrals and their approximate and exact calculations. In the remainder we list some aspects of integration in higher dimension: parameterization of shapes 7.2.2, orientation of shapes (7.2.4 and 7.2.5) and what kind of integrals will appear 7.2.3.

7.2.1. *The meaning and calculation of integrals.* As in the preceding chapters there are two levels to integration.

(1) *What integrals* mean. This will be essential for thinking about integrals and how to use them in applications.
(2) *Computation*, i.e., finishing the problem by actually calculating a number.

(1) The basic *meaning* of integration over a space S is that the the integral $\int_S f$ is the the total of a quantity Q which is spread over S with density f. One can describe it as gathering together the local contributions to Q which are described by the density function f.

(2) In principle, this integral is an abstract object – the limit of its approximations, as our method of approximating gets better and better. This method of approximations can be useful in practice, for instance one may be able to calculate approximations with computers.

However, when possible, one prefers computations of integrals that produce formulas, as these carry more information. So, the main content of this course is methods ("tricks") for exact calculation of integrals.

7.2.2. *Parameterizations of shapes.* We will compute integrals over shapes by using the familiar Change of Variables method. Previously the geometric meaning of our change of variable was a mapping Γ from a region R in $\mathbb{R}^n_{u_1,\dots,u_n}$ to a region S in $\mathbb{R}^n_{x_1,\dots,x_n}$ (which had to be a 1-1 correspondence – at least up to a possible error on a subset of volume zero).

Now, our change of variables will be a parameterization of a shape S, i.e., a mapping Γ from a region R in $\mathbb{R}^p_{u_1,\dots,u_d}$ to a shape S in $\mathbb{R}^n_{x_1,\dots,x_n}$ (of dimension $d \leq n$), Again, it will have to be a 1-1 correspondence up to a possible "volume zero error".

7.2.3. *Kinds of integrals that we will consider (diversity of integrals).* Over a shape S we will consider integrals of the form $\int_S \phi\ d\psi$. This means an integral of a quantity "ϕ" with respect to a quantity "ψ". Here "ϕ" will be a function or a vector field.

- When ϕ is a function then the other ingredient "ψ" will be the d-dimensional volume V^d on the shape S. This will meaning length for $d = 1$, area for $d = 2$ and ordinary volume for $d = 3$.
- When ϕ is a vector field then ψ will also be some vector valued quantity.
  (1) If the shape S is a curve then Ψ is the position vector r (sometimes denoted $\vec{r}$).
  (2) If S is a surface then Ψ will be a "vector valued area" $\vec{S}$. Then its value $d\vec{S}$ for a small piece of surface will be the multiple $n\ dA$ of a unit normal vector n for a surface (sometimes denoted $\vec{n}$). Here, dA is the area of this small piece.

7.2.4. *Orientations of a shape S.* Some types of integrals over a shape S will require a choice of an *orientation* of S.

A. Orientation of curves. Let us consider this phenomenon in the simple case when the shape S is just a curve C in some $\mathbb{R}^n$. (For instance an interval between the numbers a and b in $\mathbb{R}^1 = \mathbb{R}$.)

For curves, an orientation means a choice of a *direction* of the curve. Graphically we would indicate it by an arrow drawn on the path.

Example. (0) If C between is curve between two different points A and B we say that the boundary of C is the set $\partial C \stackrel{\text{def}}{=} \{A, B\}$ consisting of the two boundary points. Then there are two choices for orientation of C – from A to B or from B to A.

(1) However, there are *closed curves* like a circle, then the boundary ∂C is the empty set $\emptyset$. They still have two orientations (the directions of traveling on the curve) but there are no end points.

B. Orientation of surfaces. Now let the shape S be 2-dimensional, i.e., a surface. A choice of an orientation of S can be described at any point p of S as a choice of direction of *making a circle around* p!

Example. (0) If S is a surface in $\mathbb{R}^3$ then at a given point p of S there are two choices of a unit normal vector n to S. The choice of such normal vector n defines an orientation on S – the one that looks counter-clockwise from the tip of n.

(1) If surface S has a boundary curve ∂S then an orientation of the curve ∂S gives an orientation of the surface S since a direction of ∂S can be though of as a direction of circling around any point of S !

(2) However, some surface like a sphere are *closed*, meaning that their boundary ∂S is empty. If S is a closed surface in $\mathbb{R}^3$ then it divides $\mathbb{R}63$ in two regions – the inside of S (finite in size) and the outside of S (infinite). Now a choice of a unit normal vector n can be described by *inwards* (if it points inside S) or *outwards*.

Remark. We will later see an an elegant mathematical definition of orientation of shapes S of any dimension which reduces to these two cases when the dimension of S is 1 (a curve) or 2 (a surface)!

7.2.5. *Which integrals need orientated shapes.* Clearly, some questions do not involve orientation. For instance if you want to know the length of s curvexxx C or the area of surface S, the orientation is irrelevant.

In other considerations orientation is essential. For instance if you want to travel the a path C you need to decide the direction of your travel and the work that you do while traveling will depend on the direction (lie the difference of walking uphill or downhill).

Let us consider the same in terms of integrals over a curve C which we will just take to be an interval on the real line. The dependence of integrals on the direction is then seen in the formula $\int_b^a \ f(x)\ dx = -\int_a^b \ f(x)\ dx$, i.e., integrals from a to b or from b to a have opposite values.

On the other hand the interpretation of $\int_a^b f(x)\ dx$ as the area A of the region beneath the graph of a function $y = f(x)$ does not use orientation – but it is correct only when $a \leq b$ and $f(x) \geq 0$.

In general we will encounter two kinds of integrals, depending on whether the orientation is relevant or not. An approximate rule that usually works is that orientation does not matter for integrals of positive functions but it is important for integrals of (components of) vector fields.

8. **Integration theorems of Vector Analysis (Chapter 8.)**

By *vector analysis* or *vector calculus* we mean the calculus in arbitrary dimension, i.e., with any number of variables. The reason for terminology is that this essentially uses vectors.

In preceding chapters we have learned how to integrate over various shapes – curves and surfaces etc. The main technique we developed was the formula describing the way a Change of Variables affects integrals.[1] For this we had to recall how one differentiates mappings between $\mathbb{R}^m$'s and $\mathbb{R}^n$'s.

This last chapter covers another basic tool for calculating integrals, the *Fundamental Theorem of calculus* ("FTC") for Multivariable Functions. The phrase "Integral Theorems" refers to a group of theorems about integrals which are all generalizations to higher dimension of the FTC for functions of one variable.

8.0. **What is the Fundamental Theorem of Calculus ("FTC")?.** Here we start with the known FTC for intervals and use it to reinvent FTC in general, i.e., to guess how FTC should look like in any dimension. This part is not really necessary since we will in any given specific situation explain what FTC means there. Its role is just to see how all cases of FTC below have a common logic.

8.0.1. *FTC for functions of one variable.* This theorem is very familiar, it tells us how to calculate integrals of a function $g(x)$, when we know that this function is a derivative $g = f'$ of another function $f(x)$. Precisely, FTC says that

$$\int_a^b f'(x)\ dx \ = \ f(b) - f(a).$$

Idea. The idea is very simple. Integral $\int_a^b f'(x)\ dx$ is the total of a quantity Q smeared over the interval $[a, b]$ with density $f'(x)$. So, the local contribution at x (= the density at x) is $f'(x)$ which is the rate of change of the function f at x. Therefore, the quantity Q whose local contribution at x is the rate of change of f at x must be the total change of f on $[a, b]$, i.e., $f(b) - f(a)$.

8.0.2. *General FTC: a guess for the LHS.* In general we will integrate over an arbitrary shape S instead of the interval $[a, b]$ and this will force us to think more geometrically. Moreover, we expect that as in the original FTC above, the integrand ("what we integrate") on LHS, should again be some kind of a derivative of somemthing. Let us call this quantity Ψ so that the LHS should look like

$$\int_S \text{derivative of } \psi.$$

[In the 1-dimensional case we have $S = [a, b]$, ψ is a function f and we use the usual derivative f'.]

[1] We have first developed it for mappings between two regions in $\mathbb{R}^n$'s (chapter 6) and later for mappings from a region R to a shape S – these mappings were called parameterizations of the shape (chapter 7).

8.0.3. *General FTC: a guess for the RHS.* Now consider the RHS. It should involve the original quantity ψ (just as $f(b)-f(a)$ features f). For the geometric object on the RHS we first recall for an interval $S=[a,b]$ the RHS is computed using the points a,b. Their geometric meaning is that this is the boundary of the interval: $\partial[a,b]=\{a,b\}$. So, in general the ψ on the RHS should be calculated using the boundary ∂S of the shape S.

What kind of calculation can that be? If S is a surface its boundary ∂S is a curve. Then the only natural way to get a number from ψ and ∂S is to integrate, so the RHS should be

$$\int_{\partial S} \psi.$$

Let us check this in the case of an interval $S=[a,b]$ and $\psi=f$ a function. Since $\partial S=\{a,b\}$ then integral over the boundary $\int_{\partial S} f$ should be $f(a)+f(b)$ since integral gathers together all local contributions and these contributions are $f(a)$ at a and $f(b)$ at b.

However, the RHS of FTC is really $f(b)-f(a)$.

8.0.4. *Orientations in FTC.* Why does FTC use the two boundary points a,b in a different way? The reason is that the original FTC involves a hidden ingredient – an orientation of the interval $[a,b]$.

It turns out that for $a<b$ the integral $\int_a^b g(x)\ dx$ uses a hidden orientation of $[a,b]$. It is given by the direction from left to right, i.e., from a to b.[2] So, the LHS $\int_a^b f'(x)\ dx$ is really an integral over the <u>oriented version of interval $[a,b]$</u> with the <u>standard orientation</u>.

What is the boundary of the *oriented interval* $[a,b]$? The orientation of the interval from a to b says that the boundary points a and b have different roles: a is the beginning point and b is the end point. This difference will be expressed by saying that

The oriented version of the boundary $\partial[a,b]$ of $[a,b]$, consists of two points a,b with <u>signs</u>: b with the sign $+$ and a with the sign $-$.

So, an orientation of $[a,b]$ is a direction of this curve and an orientation of the boundary of $[a,b]$ is a choice of different signs attached to the two boundary points a and b. Moreover, the orientation of $S=[a,b]$ from a to b gives and orientation of the boundary ∂S given by b with $+$ and a with $-$.

8.0.5. *FTC guess.* This leads to our final guess for the general formulation of FTC.

[2] Since an interval $[a,b]$ has a natural orientation we have have to choose its orientation and so we usually do not mention the orientation of $[a,b]$ at all!

[**FTC Guess.**] A choice of an orientation of a shape S determines a compatible choice of orientation of the boundary ∂S. Then

$$\int_S \text{derivative of } \psi \;=\; \int_{\partial S} \psi.$$

Remarks. (1) For each case of FTC we will have to consider some specific class of shapes S and we will have to make appropriate choices of

(1) the meaning of the quantity Ψ (a function or a vector field);
(2) the meaning of the derivative of Ψ (gradient, curl or divergence);
(3) the meaning of the integral, i.e., it could be with respect to the volume V, the position vector r or the "vector valued area $\vec{S}$.

(2) Also, in each case we will also have to explain:

- which orientations of S and ∂S are compatible;
- why the formula is true;
- what is the "real life meaning" of the formula.

We will see that in each case the proof of FTC in a given situation will reduce to the known case of FTC for intervals. So, all complicated cases of FTC always reduce to the simplest case of FTC.

(3) One could also prove the that our "FTC guess" is true in all cases. Moreover, the reason it is true is the idea exapleined in : gathering local rates of change yields the global change.

8.0.6. *The cases of FTC for various kinds of shapes.* Here shape S could a curve, surface, a solid or something in a higher dimension.

In each of these case the FTC has a particular name

- When X is an interval $[a, b]$ this is (as we will see) the "classical" FTC that we use any time we calculate integrals.
- For a curves C from A to B the FTC is the known formula for the line integral of a gradient:
$$\int_C \nabla f \cdot dr \;=\; f(B) - f(A).$$
- For surfaces in $\mathbb{R}^3$ it is called the *Stokes theorem.* (The particular case of a region in $\mathbb{R}^2$ is called *Green's theorem.*
- For solids in $\mathbb{R}^3$ the FTC is called the *Gauss theorem.*

8.0.7. *Various kinds of derivatives.* Recall that the relevant quantity Ψ can be a function of a vector field. The meaning of the derivative of Ψ in FTC will depend on both the quantity Ψ and on the shape S over which we integrate.

- If Ψ is a function f then by "derivative of f" we will mean the gradient vector field ∇f.
- If Ψ is a vector field $\vec{F}$ then the derivative could be the *curl* or the *divergence* of the vector field, defined by

$$curl(F) \stackrel{\text{def}}{=} \nabla\times\vec{F} \quad \text{and} \quad div(F) \stackrel{\text{def}}{=} \nabla\cdot\vec{F}.$$

8.0.8. *Gradient, curl, divergence.* Here, the gradient operation ∇ will be considered as a vector $\nabla = \langle\partial_x, \partial_y, \partial_z\rangle$ with three components which are the three partial derivative operations.

So, the curl of a vector field $F = \langle P, Q, R\rangle$ will mean

$$curl(\vec{F}) \stackrel{\text{def}}{=} \nabla\times F \;=\; det\begin{pmatrix} I & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{pmatrix} \;=\; +i\ det\begin{pmatrix} \partial_y & \partial_z \\ Q & R \end{pmatrix} - j\ det\begin{pmatrix} \partial_x & \partial_z \\ P & R \end{pmatrix} + k\ det\begin{pmatrix} \partial_x & \partial_y \\ P & Q \end{pmatrix}.$$

This is $\langle\partial_y R - \partial_z Q,\ \partial_z P - \partial_x R,\ \partial_x Q - \partial_y P\rangle$ and it remains to apply derivatives to functions to get the final formula:

$$curl(\vec{F}) \;=\; \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\rangle.$$

Similarly, the divergence is

$$div(\vec{F}) \stackrel{\text{def}}{=} \nabla\cdot\vec{F} \;=\; \langle\partial_x, \partial_y, \partial_z\rangle cd\langle P, Q, R\rangle \;=\; P_x + Q_y + R_z.$$

Example. [Curl for planar vector fields.] Let us consider the special case of a *planar* vector field. This means a vector field $\vec{F}$ in a plane, so $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ for its component functions P, Q.

We will extend it to a vector field in space by making it constant in the z direction, then we get

$$\vec{F}(x,y,z) \;=\; \langle P(x,y), Q(x,y), 0\rangle.$$

In order to calculate the curl for a planar vector field $\vec{F}(x,y)$ we apply the above formula for the curl to the extension $F(x,y,z)$. The formula simplifies as the third component R is zero and the first two components P, Q only depend on x, y so that their z derivatives are zero. Therefore,

$$curl(\vec{F}) \;=\; \langle 0,\ 0,\ Q_x - P_y\rangle \;=\; (Q_x - P_y)k.$$

Similarly, the divergence is

$$div(\vec{F}) \;=\; P_x + Q_y.$$

8.0.9. *The classical 1-dimensional FTC on an interval.* Here X is an interval $X = [a, b]$ and the quantity Ψ is a function $f(x)$ on $X = [a, b]$. In this situation the FTC is a familiar formula:

Theorem. [FTC] $\int_a^b f'(x) = f(b) - f(a)$. $\square$

This is of course a standard thing, the trick we use any time we calculate integrals.

The meaning of this FTC. Here we will review its *meaning*. This is instructive because all more complicated cases of FTC that we will encounter later have analogous meaning.

Remember that the meaning of any integral $\int_X \phi(x)\, dV$ is the *total amount* of a quantity Q which is spread over X with density $\phi(x)$ at the point x in X. (The integral gathers together the local contributions to the total of Q and the local contribution to Q at a point X is the density $\phi(x)$ of Q at x.)

In our case the integral on the LHS of FTC is an integral over X which is the interval $[a, b]$. What is the quantity Q? Its density is $f'(x)$, i.e., the rate of change of the function f at the point x.

The phrase "rate of change of f at x" tells us about the intensity of *change of function f at x. This we can also call the "density of change of f at x". Therefore, the original quantity Q must be the* change of function f".

Now, the meaning of the integral is the total amount of the quantity "change of f" on X. This is indeed the total change $f(b) - f(a)$ of f on the interval $[a, b]$.

8.0.10. *The first example of FTC: the case of an interval.* Again, here FTC says that

$$\int_a^b f'(x) = f(b) - f(a).$$

The shape S that we are integrating over on the LHS is an interval $S = [a, b]$. Also, the "quantity Ψ" will be just a function f, so that the phrase "derivative of Ψ" just means $f'(x)$.

It turns out that for $a < b$ the interval $[a, b]$ has a natural orientation given by the direction from left to right, i.e., from a to b. (Therefore we do not have to choose its orientation and so we usually do not mention it all!)

We said that for FTC we want its boundary $\partial[a, b] = \{a, b\}$ to have a compatible orientation. The orientation of the interval from a to b says that the boundary points a and b have different roles: a is the beginning or starting point and b is the end point, i.e., the goal of the motion. This difference will be expressed by saying that

The oriented version of the boundary $\partial[a, b]$ of $[a, b]$, consists of two points a, b with signs: b with the sign $+$ and a with the sign $-$.

So, an orientation of $[a, b]$ is a direction of this curve. An orientation of the boundary of $[a, b]$ is given a choice of different signs attached to the boundary points a and b.

Now we can say what is the meaning of $\int_{\partial[a,b]} f$. The integral over the boundary gathers together all local contributions and these contributions are $f(a)$ at a and $f(b)$ at b.

However, because in the oriented version of the boundary point b is counted with $+$ and a with $-$, the meaning of gathering local contributions is really

$$\int_{\partial[a,b]} f \overset{\text{def}}{=} +f(b) \ + \ -f(a) \ = \ f(b) - f(a).$$

Now the standard FTC $\int_a^b f' \ = \ f(b) - f(a)$ can be really written as a case of the abstract formulation of FTC:

$$\int_{[a,b]} \text{derivative of } f \quad = \quad \int_{\partial[a,b]} f.$$

8.1. **FTC for curves.** This is the theorem 3 in section 7.2. of the book.

Theorem. [*Line integrals of gradient vector fields.*] For a curve C oriented so that it starts at point A and ends at point B

$$\int_C \nabla f \cdot dr \ = \ f(B) - f(A).$$

Remark. When the curve C is an interval $[a,b]$ this is just the usual FTC formula $\int_a^b f'(x)\ dx = \ f(b) - f(a)$ for intervals.

Proof. We choose a parameterization $\gamma:\ [a,b] \to C$ of the curve C with three component functions $\gamma(t) = \langle x(t), y(t), z(t)\rangle$. We will see that γ will reduce the above FTC claim for the curve C to the well know FTC for the interval $[a,b]$.

In terms of the parameterization γ the position vector is now $r = \gamma(t)$, hence $dr = \gamma'(t)dt = \ \langle x'(t), y'(t), z'(t)\rangle dt$.

$$\int_C \nabla f \cdot dr \ = \ \int_a^b \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle \ dt \ = \ \int_a^b \ \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} +, \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}\ dt.$$

The sub-integral function is just chain rule formula for for the derivative of f with respect t, i.e., where $f(x,y,z)$ becomes a function $f(\gamma(t)) = f\left({\scriptstyle x(t), y(t), z(t)} \right)$. So, the integral simplifies to an integral of a derivative

$$= \int_a^b \ \frac{\partial f(\gamma(t))}{\partial t}\ dt = \ f(\gamma(b)) - f(\gamma(a))$$

and this is exactly $f(B) - f(A)$. $\square$

8.2. **FTC for surfaces.**

8.2.1. *Compatible orientations.* Recall that an orientation of a surface S in $\mathbb{R}^3$ is given by the choice of the unit normal vector n at any point of S. Then the boundary curve ∂S inherits an orientation, i.e., a choice of a direction from the surface. This direction is such that when one goes around the boundary ∂S in this direction the surface stays on the *left* of the curve – if one views the process from the tip of the normal vector n to the surface. (See figure 8.2.5. in the book.) We say that such pairs of orientations of S and ∂S are *compatible.*

8.2.2. *The curl of a vector field F.* Recall that it is a new vector field given by $curl(F) \stackrel{\text{def}}{=} \nabla\times F$. If the vector field F has component functions F_i, i.e., $F = \langle F_1, F_2, F_3\rangle$, then one easily calculates the curl to be

$$curl(F) \;=\; \langle (F_3)_y - (F_2)_z,\ (F_1)_z - (F_3)_x,\ (F_2)_x - (F_1)_y\rangle.$$

8.2.3. *Expression $d\vec{S} = \ ndA$.* Once an orientation n of a surface S (i.e., a unit normal vector field) has been chosen, one can introduce the symbol $d\vec{S} \stackrel{\text{def}}{=} \ n\ dA$. At a point p of S, the symbol dA means the are of a small piece of the surface. then ndA is a vector which goes in the normal direction given by n but has length dA.

Now, for a vector field F the meaning of the integral $\int_S \ F{\cdot}d\vec{S}$ of F with respect to $\vec{S}$ as just the integral $\int_S \ F{\cdot}n\ dA$.

Here, $F{\cdot}n$ is the component of the vector field F in the direction n. So, $\int_S \ F{\cdot}d\vec{S} \ \stackrel{\text{def}}{=}\ \int_S \ F{\cdot}n\ dA$ is just the integral over S and with respect to area A, of the normal component of F.

Remarks. (0) One can think that are has been upgraded from a number to a vector.

(1) The phrase "normal component of F" means normal to the surface S and measured in the direction of the orientation n.

Example. When F is the velocity vector field of some flow in space, then the value of the normal component $F{\cdot}n$ at a point p of the surface measures the velocity with which the flow passes through the surface at p. Then $\int_S \ F{\cdot}n\ dA$ is the *total flow* through the whole surface S, also called the *flux* through the surface.

This flux is a number. Its sign depends on the orientation n. The flux is positive if the flow is dominantly in the direction of the orientation vector n.

8.2.4. *Stokes theorem.* It calculates the flux of vector fields of a special kind, the vector fields that are themselves the curls $curl(F) = \nabla\times F$ of some other vector field.

Theorem. Let S be an oriented surface with a boundary curve ∂S and let ∂S be endowed with a compatible orientation. Then for a vector field F in $\mathbb{R}^3$

$$\iint_S curl(F) \cdot n \ dA \ = \ \int_{\partial S} F \ dr.$$

Remark. So, the total flow of $curl(F)$ through the surface S can be calculated on the boundary ∂S and it is given by the line integral $\int_{\partial S} F \ dr$.

8.2.5. *Gauss theorem.* Let R be a region in the plane $\mathbb{R}^2_{xy}$ bounded by the curve ∂P. Let $F = \langle P, Q \rangle$ be a planar vector field with two component functions P and Q.

Theorem.

$$\iint_S Q_x - P_y \ = \ \int_{\partial S} F \ dr$$

This is a simpler statement and

8.2.6. *The scheme of the proof of the Stokes theorem.*

(1) We prove the Gauss theorem in the case when the region S is a rectangle with sides parallel to x and y axes.
   This special case will be verified using the Fundamental Theorem of 1-variable Calculus: $\int_a^b f'(x) \ dx \ = \ [f]_a^b \ = \ f(b) - f(a)$.
(2) We prove the Gasses theorem for all regions S.
   In order to reduce the claim for any surface S to the case (1) we will cut the surface S into small rectangles.
(3) We prove Stokes theorem. (We will deduce the Stokes theorem from Gauss's theorem by calculating integrals using parameterization of the surface.)

rem