### 425.1 The Final Project

Each of the following five problems is worth 20 points. Explain your answers. In particular, I should be able to follow your line of thought. The handwriting should be legible. ${ }^{(1)}$

Write in your own sentences. It is not OK to copy sentences from any source or change the notation from the one here. It is not OK to copy solution from anybody though it is OK to discuss difficulties with others. The idea is that there is much work to do but it should be doable.

We will discuss these problems in the zoom class on Tuesday the 28th (the last day of lectures) and at the Review Sessions (I suggest Sunday the 26th at 4 and Tuesday the 30th at 7). We may possibly start in class explanations earlier - on this Thursday, if there will be time.

The exam is due on MAY 5th at any time. You will submit it on Moodle as the 9th homework.
0.1. Integral over a closed surface. Consider the pyramid $P$ whose base is the square in the xy-plane with corners at $(0,0,0),(1,0,0),(0,1,0),(1,1,0)$, and the vertex of the pyramid is at the point $(0,0,1)$. For the vector field $F=\left\langle x, y^{2}, z\right\rangle$ calculate the integral of its normal component $\int_{\partial P} F \cdot n d A$ over the boundary $\partial P$ of the pyramid $P$. Here, the boundary is oriented inwards (into the pyramid).
0.2. Conservation. (a) Consider a force field $F$ caused by an object at the origin of the coordinate system. Suppose that $F=k \phi(|r|) r$, where

- $r$ is the position vector,
- $\phi$ is some function (which we apply to the length $|r|$ of the position vector) and
- $k$ is a constant.

Show that such force is conservative.
(b) Prove that the vector field $F=\left\langle y e^{x y} \sin (z), x e^{x y} \sin (z), e^{x y} \cos (z)\right\rangle$ is conservative.
(c) Find a potential for the vector field from (b).
(d) For the vector field from (b) calculate the integral $\int_{C} F \cdot d \vec{r}$ for a curve from the point $(1,0,-\pi / 2)$ to the point $(0,-1, \pi / 2)$ which is given by a very complicated formula that is too long to write here.
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[^0]0.3. Gauss law. For guidance on this problem read the subsection of section 8.4 on Gauss's law. (It starts on the page 570 in the book.)

The Gauss law in physics says that
(GLP) The flux (the total flow) of the electric field through a closed surface $S$ is equal to the total electric charge inside $S$.

The electric field $E$ in space is the vector field given by the electric force. Gauss's law is a mathematical expression of the observation that if the incoming and outgoing contributions to the flow of the electric field through a closed surface $S$ do not cancel, then there must be some electric charge inside $S$ that creates the imbalance.

The Gauss's law in mathematics says that for any point $p$ we can decide whether it is inside of a given closed surface $S$ by calculating certain integral over $S$. The integral uses a vector field $F_{p}$ which is defined on $\mathbb{R}^{3}$ (except at the point $p$ ). At a point $q$ it is

$$
F_{p}(q)=\overrightarrow{p q} /|\overrightarrow{p q}|^{3}
$$

(At the point $p$ one would have to divide by zero!) Then the claim is that
(GLM) The integral $\iint_{S} F_{p} \cdot n d A$ is zero if the point $p$ does not lie inside $S$ and it is $4 \pi$ if $p$ is enclosed by $S$.
$\bigcirc$
Answer the questions and justify the claims below.

- (a) Let us choose a coordinate system such that the point $p$ is the origin. Denote the position vector of a point by $r$ and its length by $r=|r|$.

Show that in terms of this coordinate system the vector field $F_{p}$ is $\vec{r} / r^{3}$ Explain that the above statement of GLM is the same as the one in the book.

- (b) Prove GLM when $p$ is not inside $S$.
- (c) Prove GLM when $S$ is a sphere with the center $p$.
- (d) Prove GLM when $S$ is any closed surface that encloses $p$.
- (e) Explain why GLP is an example of GLM.
0.4. Meaning of the div operation. For guidance read the subsection of section 8.4 which is on Divergence as a flux per unite Volume (starting on the page 568) in the book.

Suppose that $V$ is the velocity vector for a flow of some fluid in space. Answer the following questions.

- (a) Consider a closed surface $S$ in space, oriented by means of a unit normal vector n. Explain that the integral $\int_{S} V \cdot d \boldsymbol{S}=\int_{S} V \cdot n d S$ measures the total flow of the fluid through the surface $S$ per unit time and in the direction of $n$.
- (b) At a point $P$ in space consider the balls $B_{r}$ of radius $r$ with center at $P$. Explain the limit formula for the divergence of the vector field $V$ at the point $P$

$$
\operatorname{div}(F)(P)=\lim _{r \rightarrow 0} \frac{1}{\operatorname{Vol}\left(B_{r}\right)} \iint_{\partial B_{r}} F \cdot n d S
$$

- (c) Explain what does it mean that the divergence of $F$ at $P$ measures the rate of creation or destruction of the fluid at $P$.
- (d) Explain why does one say that
(1) The point $P$ is the source of the fluid if $\operatorname{div}(F)(P)>0$ ?
(2) The point $P$ is the sink of the fluid if $\operatorname{div}(F)(P)<0$ ?
(3) The fluid is incompressable if $\operatorname{div}(F)=0$ ?

Remark. Here we found the meaning of the differential operator div on vector fields by considering integrals.
0.5. The meaning of the curl operation. Let $V$ be the velocity vector for a flow of some fluid in space. Consider a surface $S$ in space with compatible orientations of the surface and its boundary $\partial S$. Explain that

- (a) What the integral $\int_{\partial S} V \cdot d r$ measures is the circulation of the flow about the curve $\partial S$ and in the direction of the oriented curve $\partial S$.
- (b) At a point $p$ in $\mathbb{R}^{3}$ consider a direction given by a unit vector $u$ at $p$. Let $L$ be the line through $p$ and in the direction of $u$. Then, the $u$-component $\operatorname{curl}(V)(p) \cdot u$ of the curl vector field $\operatorname{curl}(V)$ at $p$, is exactly the "circulation (at the point $p$ ) of the fluid about the line $L$, viewed from $u$ ".
- (c) The value $\operatorname{curl}(V)(p)$ of the curl of the velocity vector field $V$ at a point $p$ in $\mathbb{R}^{3}$ is the vector which simultaneously measures the circulation of the fluid about the point $p$ in all directions.
- Directions for part (a): The circulation around $\partial C$ is the total amount of rotation of particles of the fluid about $\partial S$ and in the direction of $\partial S$. Your explanation for (a) should use discussion with drawing of the following three special cases:
- (a1) when $V$ is orthogonal to $\partial S$;
- (a2) when $V$ is in the direction of $\partial S$;
- (a3) when $V$ is in the direction opposite of that of $\partial S$.
- Directions for part (b). We will first restate precisely what we mean by "circulation of the fluid about the line $L$, viewed from $u$ ". For this we consider the plane $P$ which is normal to the line $L$ and passes through the point $p$. In this plane consider the disc $D_{R}$ with the center $p$ and radius $R$. Now, vector $u$ is a unit normal vector to the plane $P$ and it supplies the boundary circle $\partial D_{r}$ with an orientation (the one that looks counterclockwise from the tip of $u$ ). This leads to the following precise version of what $\operatorname{curl}(V)$ measures: Now, by "circulation of the fluid about the line $L$, viewed from $u$ " we mean
- the circulation of the fluid around the circle $\partial D_{R}$ (with the stated orientation) when $R$ is very small (i.e., the limit of such circulation as $R$ approaches 0 ).

So, the claim that you really need to explain for (b) is that
Certain measure of the circulation of the flow around the circle $\partial D_{R}$ (with the stated orientation)
converges for $R \rightarrow 0$ to the dot product $(\operatorname{curl}(V)(p) \cdot u$
You should explain the last claim using the Stokes theorem and the constant approximation $V \sim V(p)$ of the vector field $V$ on the disc $D_{R}$ for $R$ small.

- Directions for part (c): Here, you just need to point out that the precise meaning of the claim (c) is the claim (b).


[^0]:    ${ }^{1}$ Reading invisible writing is not one of my superpowers. What I can not read I can not credit.

