Advanced Calculus 425

Homework 6

Due Thursday March 12.

Integration over curves

A. Functions f with respect to length. This is Section 7.1. The length of the curve is denoted l (or s). is on calculation of an integral One calculates integrals over a curve C by using a parameterization $\gamma(t) = \langle x(t), y(t), z(t)\rangle$ of the curve C by an interval $[a, b]$.

$$\int_C f\ ds \;=\; \int_a^b f(\gamma(t))\ |\gamma'(t)|\ dt \;=\; \int_a^b f(\gamma(t))\ \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\ dt.$$

B. Vector fields with respect to position vector. In this situation the orientation of the curve is important.

This is Section 7.2. One writes such integrals in three forms

$$\int_C \vec{F}\cdot\ d\vec{r} \;=\; \int_C P\ dx \;+\; Q\ dy \;+\; R\ dz \;=\; \int_C \vec{F}\cdot T\ dl.$$

In the first formula $\vec{r}$ is the *position vector field* – at the point with coordinates (x, y, z) its value is just the position vector $\vec{r}(x, y, z) = \langle x, y, z\rangle$. So, this is the integral of the *vector field $\vec{F}$ with respect to position.*

We get the second formula when we write the vector field $\vec{F}$ in terms of its component functions P, Q, R as $\vec{F} = \langle P, Q.R\rangle$. We call this the *integral of functions P, Q, R with respect to the variables x, y, z.*

Finally, when we choose a direction of the curve we get the *unit tangent vector $T(x, y, z)$* at each point (x, y, z) of the curve. Since the dot product $\vec{F}\cdot T$ is the component of $\vec{F}$ in the direction of the tangential vector T, the third formula is called the *integral of the tangential component of a vector field $\vec{F}$* (with respect to the length l).

So, all three integrals are the same. The way one calculates them is in terms of a parameterization $\gamma(t) = \langle x(t), y(t), z(t)\rangle$ of the curve C, we get two formulas (the second one may be simpler) for these integrals as

$$\int_a^b \vec{F}\cdot\gamma'(t)\ dt \;=\; \int_a^b Px'(t) \;+\; Qy'(t) \;+\; Rz'(t)\ dt.$$

Remarks. (0) When the vector field $\vec{F}$ is a *force* field then the meaning of the integral $\int_C \vec{F}\cdot\ d\vec{r}$ is the *work* that the force $\vec{F}$ does as it moves an object along the curve C.

(1) These are also called *line integrals.*

Surfaces

C. Parameterizations of surfaces. These are introduced in Section 7.3. (The book also calculates the tangent plane at a point of a surface but we are only interested in two particular "basic tangent vectors" denoted $T_u = \frac{\partial T}{\partial u}$ and $T_v = \frac{\partial T}{\partial v}$ which are as the notation suggests the partial derivatives of the mapping T in the direction of u, v variables.

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The following two surfaces are already described in terms of parameterizations $T(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$. For these two surfaces calculate :

(a) The basic tangent vectors $T_u = \langle x_u, y_u, z_u\rangle$ and $T_v = \langle x_v, y_v, z_v\rangle$.

(b) The cross product $T_u \times T_v$.

(b) The length $|T_u \times T_v|$.

6.1. $x = 2u,\ y = u^2 + v,\ z = v^2.$

6.2. $x = u^2,\ y = e\sin(e^v),\ z = \frac{1}{3}u\cos(e^v).$

The book

Read sections 7.1-7.4.

Problems.

- Section 7.1, problems 6, 7a, 13.
- Section 7.2, problems 1ab, 2d, 12.
- Page 514, problem 3a.