Advanced Calculus 425, Homework 5 (The 1^{st} Sample Exam).

Due Wednesday, at the exam.

The exam will be something like 4 of the following problems. One of these problem has to be the theoretical problem. Each problem is worth 25 points.

0.1. Calculate the following double integral by reversing the order of integration

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{1+y^5}\, dy\, dx.$$

0.2. Let $0 < a < b$ and let S be the solid given by the following five inequalities:

$$a^2 \leq \; x^2 + y^2 + z^2 \; \leq b^2 \quad \text{and} \quad x \geq 0,\; y \geq 0,\; z \geq 0.$$

Calculate the integral

$$\int\int\int_S \frac{dx\; dy\; dz}{(x^2 + y^2 + z^2)^{3/2}}.$$

Solution: $?\pi \log(b/a)$.

0.3. Answer the following questions. [This will be discussed in class.]

- **(a).** Why is it important that the change of variable mapping T from R' to R be close to a 1-1 correspondence?
- **(b).** What does the stretching function $\mathcal{J}_T(b)$ for a mapping T measure?
- **(c).** Why does the stretching function appear in the change of variables formula?
- **(d).** The stretching function $\mathcal{J}_T(b)$ for a mapping T and at a point b is the absolute value of the Jacobian determinant $\frac{\partial(x_1,\ldots,x_n)}{\partial(y_1,\ldots,y_n)}$. What are the key steps in explaining this equality?

Your answers can be short as long as they address the question. It is essential that the answers be written in complete meaningful sentences (so that anybody can follow your reasoning) and legible handwriting (for instance your pen should leave a clearly visible trace on the paper).

♡

Volume in any dimension

The n-dimensional ball of radius R is the set of all points $x = (x_1, ..., x_n)$ in $\mathbb{R}^n$ such that $x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1$. Denote it by B_R^n.

0.4. Let $a_1, ..., a_n \geq 0$ and define the mapping T from $\mathbb{R}^n_{u_1,...,u_n}$ to $\mathbb{R}^n_{x_1,...,x_n}$ by

$$T(u) \;=\; \begin{pmatrix} T_1(u) \\ \vdots \\ T_n(u) \end{pmatrix} \;=\; \begin{pmatrix} a_1 u_1 \\ \vdots \\ a_n u_n \end{pmatrix}.$$

(Here the component functions are $T_i(u) = a_i u_i$ for $i = 1, ..., n$.)

(a) Find the Jacobian of T at any point in $\mathbb{R}^n_{u_1,...,u_n}$. [Hint: the determinant of a diagonal matrix C with entries $c_1, c_2, ..., c_n$ on the diagonal is the product $c_1 \cdots c_n$.]

(b) Show that when $a_1, .., a_n$ all equal R then T maps the n-dimensional ball B_1^n of unit radius in $\mathbb{R}^n_{u_1,...,u_n}$, to the ball B_R^n in $\mathbb{R}^n_{x_1,...,x_n}$ of radius R. In other words, if $u_1^2 + u_2^2 + \cdots + u_n^2 \leq 1$, then $T_1(u)^2 + T_2(u)^2 + \cdots + T_n(u)^2 \leq R^2$.

(c) Use the change of variable given by the mapping T in (b) to show that

$$Vol[B_R^n] \;=\; R^n \; Vol[B_1^n].$$

In other words, for balls in dimension n if the radius of the ball grows R times larger, its the volume becomes R^n times larger.

Do the problems from the the book

- Page 366-7: 10, 14, 16, 36.
- Page 418: 12, 15, 17.