Advanced Calculus 425 Homework 4

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Due Thursday February 20, in class.

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Higher dimensions

4.1. Find the 4-dimensional volume of the pyramid P_4 in $\mathbb{R}^4$ as $\int_{P_4} 1\ dx_1\ dx_2\ dx_3\ dx_4$ (compute it by iterated integrals).

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4.2. Consider the projection π from $\mathbb{R}^4_{x_1,...,x_4}$ to the line $\mathbb{R}_{x_4}$ by $\pi(x_1, ..., x_4) = x_4$.

(a) Show that the image of the pyramid $P(1,1,1,1)$ under this projection π is the interval $[0,1]$ in $\mathbb{R}_{x_4}$.

(b) Find the fiber F_a of the pyramid $P_4 = P_4(1, ..., 1)$ at the point a in this interval $[0,1]$ in $\mathbb{R}_{x_4}$.

[[By $\mathbb{R}^m_{y_1,...,y_m}$ we denote the space $\mathbb{R}^m$ with coordinates $y_1, ..., y_m$.]]

Change of variables

For the following problem, use the prescription on the next page:

4.3. Let $a, b, c > 0$ and let S be the solid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$. The elliptic change of coordinates

$$x = \rho a\ \sin(\phi)\ \cos(\theta), \quad y = \rho b\ \sin(\phi)\ \sin(\theta), \quad z = \rho c\ \cos(\phi)$$

gives a mapping T from the box S' in coordinates ρ, ϕ, θ given by

$$0 \le \rho \le 1, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi$$

to the solid S. This map is a 1-1 correspondence $T : S' \to S$ (up to an error which happens on a volume zero subset).

(a) Compute the Jacobian of T, i.e., the determinant

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \det[(\nabla T)(a)].$$

(b) Calculate the volume of the solid S using the above change of variables.

Change of variables summary

A. Gradient of a mapping. Consider a mapping $T = \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix}$ from $\mathbb{R}^n_{u_1,...,u_m}$ to $\mathbb{R}^n_{x_1,...,x_n}$. Here, for $i = 1, 2, ..., n$, T_i are the component functions $x_i = T_i(u_1, ..., u_m)$ of T. The *gradient* $\nabla T(u)$ of the the mapping T at u is the matrix given by arranging all partial derivatives of component functions T_i of the mapping T into

$$(\nabla T)(u) \;=\; \begin{pmatrix} \frac{\partial T_1}{\partial u_1}(u) & \cdots & \frac{\partial T_1}{\partial u_m}(u) \\ \vdots & \cdots & \vdots \\ \frac{\partial T_n}{\partial u_1}(u) & \cdots & \frac{\partial T_n}{\partial u_m}(u) \end{pmatrix}$$

B. Jacobian of a mapping. When $m = n$ one defines the Jacobian of a mapping T ss the determinant of the gradient matrix of T

$$\mathcal{J}_T \overset{\text{def}}{=} \frac{\partial(v_1, ..., v_n)}{\partial(u_1, ..., u_n)} = \det[(\nabla T)].$$

C. Comparison of integrals over regions $\mathcal{R}$ and $\mathcal{R}'$. Consider an integral $\int_{\mathcal{R}} f(x_1, ..., x_n)\; dx_1 \cdots dx_n$ over a region $\mathcal{R}$ in $\mathbb{R}^n_{x_1,...,x_n}$. In order to compute this integral by a change of variable we need another region $\mathcal{S}$ in $\mathbb{R}^n_{u_1,...,u_n}$ and a mapping $T : \mathcal{S} \to \mathcal{R}$ that relates the old region $\mathcal{R}$ with the new region $\mathcal{S}$. [Actually, this mapping has to be a *1-1 correspondence up to volume zero.*[1]]

D. The change of variables formula. It reduces calculation of an integral over $\mathcal{R}$ to another integral over $\mathcal{S}$:

$$\int_{\mathcal{R}} f \; dV \;=\; \int_{\mathcal{R}'} f{\circ}T \cdot \mathcal{J}_T \; dV.$$

In more details we write it as

$$\int_{\mathcal{R}} f(x_1, ..., x_n)\; dx_1 \cdots dx_n \;=\; \int_{\mathcal{R}'} f(T(u_1, ..., u_n)) \cdot \mathcal{J}_T(u_1, ..., u_n)\; du_1 \cdots du_n.$$

Here, the Jacobian $\mathcal{J}_T$ appears as the stretching factor for the mapping T, i.e., $\mathcal{J}_T(u)$ measures how much the the mapping stretches the volume near a point u in $\mathcal{R}$.

[1]This condition has been discussed in class.