8.4. Gauss Divergence Theorem

(the FTC for solids)

Topics: A. GDT and its proof.

B. Examples.

C. The meaning of divergence

D. Moving the surface of integration!

§ 8.4. Gauss Divergence Theorem

It is about solids in space:

A. GDT:

T1. Let W be a region in $\mathbb{R}^3$ (= solid), enclosed by a closed surface $S = \partial W$. Then

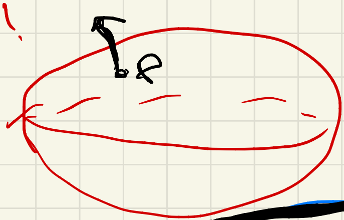
$$\int_W \operatorname{div}(F)\, dV = \int_{\partial W} F \cdot d\vec{S}$$

where:

- F is a vector field
- surface $S = \partial W$ has outwards orientation

Examples:

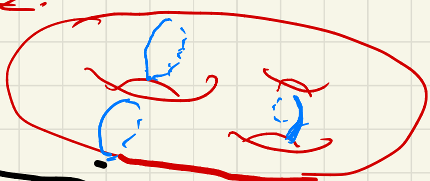
E1.



S = sphere $= \partial W$

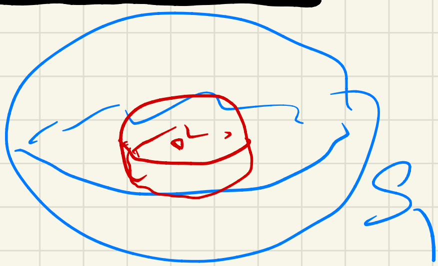
W = ball

E2.



$= S$

E3. Solid W between two concentric spheres S_R, S_r of radii $r < R$.

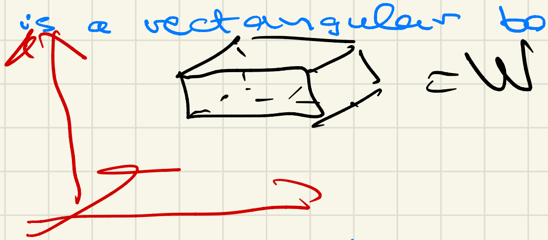


$S = \partial W$, W = space between two spheres.

Proof of the Gauss Divergence theorem:

It works alike the other cases of FTC.

Step 1. When W is a rectangular box with sides parallel to the coordinate planes,



calculate both sides of the theorem. Here, ∂W consists of six rectangles. Each of calculations yields 6 terms, and those happen to match.

Step 2. Approximate solid W by a solid W' which is obtained by gluing boxes together:

Theorem holds for such W' by the Gluing Principle (because it holds for each box by Step 1).

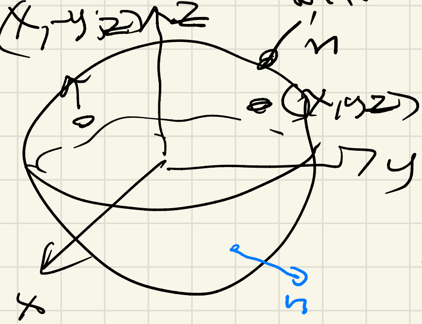
Step 3. W and its boundary ∂W are limits of W' and $\partial W'$ as one passes to better and better approximations W' (by using a grid with smaller and smaller boxes).

So, the claim for W holds because it holds for solids W' glued from boxes.

$\square$

B. Examples E1: Solid W is the unit ball and the vector field is: $F = \langle 2x, y^2, z^2 \rangle = \langle P, Q, R \rangle$

So, $S = \partial W$ is given by: $x^2+y^2+z^2=1$, with orientation outwards:



The Gauss Divergence theorem says that:

$$\iint_{\partial W = S} F \cdot \underbrace{d\vec{S}}_{= n\,dA} = \iiint_{W} \underbrace{div(F)}_{\text{function}} \underbrace{dV}_{\text{volume}}$$

Compute divergence:

$$div(F) = P_x + Q_y + R_z$$
$$= (2x)_x + (y^2)_y + (z^2)_z = 2 + 2y + 2z$$

So, the RHS is:

$$\iiint_{W = \text{ball}} 2 + 2y + 2z \, dV.$$

By symmetry of ball in y & $-y$

$$\iiint_W y\,dV = 0$$
$$\iiint_W z\,dV = 0$$

Also:

$$\iiint_W 2\,dV = 2\,Vol(W) = 2 \cdot \tfrac{4}{3}\pi \cdot 1^3.$$

So, GDT says that the flow of F through the unit sphere S equals $\frac{8}{3}\pi$. $\square$

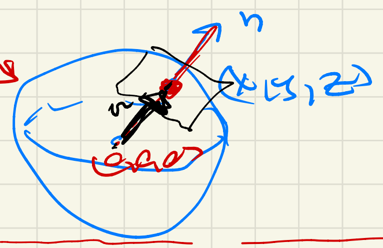
Example 2. Over the unit sphere S we want to integrate a function

$$\iint_S \underbrace{x^2+y+z}_{f}\, dA = ?$$

In order to use the Gauss DT we need to interpret f as

$f = F \cdot n$ since one side of GDT is: $\iint_S F \cdot d\vec{S} = \iint_S F \cdot n\, dA$.

For this we put on S the outwards orientation n. For a sphere n is in the direction of the position vector r, so: as in the picture below:



So, $n = \frac{r}{|r|}$. But, since this is the unit sphere we have $|r| = 1$, hence $n = r = \langle x, y, z \rangle$.

Now, we can see that which vector field $F = \langle P, Q, R \rangle$ works:

we want to equal

$$f = x^2 + y + z$$

$$F \cdot n = \langle P, Q, R \rangle \cdot \langle x, y, z \rangle = xP + yQ + zR.$$

So, we can choose $P = x$, $Q = 1$, $R = 1$.

So, for $F = \langle P, Q, R \rangle = \langle x, 1, 1 \rangle$
we have that:

$$\iint_{\partial W = S} x^2 + y + z \, dA = \iint_S F \cdot n \, dA \overset{GDT}{=} \iiint_W \underset{\underset{P_x + Q_y + R_z = 1}{||}}{\mathrm{div}(F)} \, dV = \iiint_W 1 \, dV$$

$$\underset{\underset{1}{||}}{(x)_x} \quad \underset{0}{\overset{||}{1}} \quad 0$$

so we got:

$= \mathrm{vol}(W)$

$= \frac{4}{3}\pi 1^3 = \frac{4}{3}\pi$.

$\square$

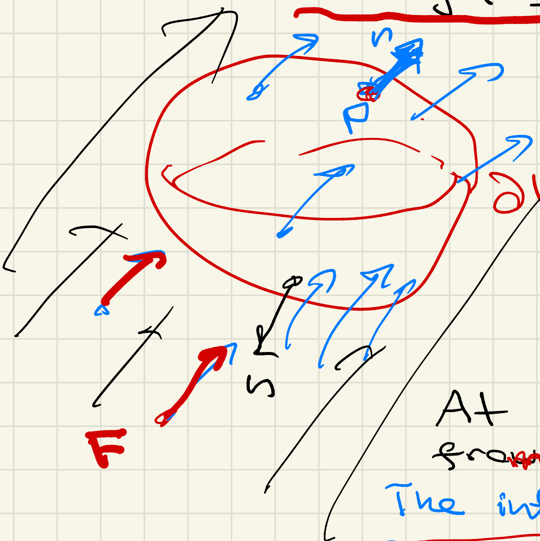
Part C. The meaning of divergence

The meaning of the divergen operation is not clear from its definition

$$\mathrm{div}\langle P, Q, R \rangle = P_x + Q_y + R_z.$$

Since divergence appears in the GDT, we will use GDT to understand the meaning of divergence!

Cl. Idea Let us think of F as velocity vector for the flow of some quantity Q through space. Then $\iint_{\partial W} F\cdot d\vec{S}$ is the flux, i.e. total flow through the surface S.



$\partial W = S$

At some points P on the surface Q flows in into $S=\partial W$ i.e. it enters the solid W inside S.

At some points Q exits from W.

The integral $\iint_{\partial W} F\cdot d\vec{S}$ gathers all such local contributions at all points P of ∂W. So this integral is a sum of positive contributions (where Q flows out) and negative contributions (where it flows in). This sum is then the total $\overset{\text{change}}{\text{loss}}$ of the amount of Q in W.

Since by GDT this total change in W can be represented as $\boxed{\iiint_W \operatorname{div}(F)\, dV}$

We see that the divergence div(F) should be related to change of concentration of Q in W. So we guess that div(F)(x) is the rate of change of concentration of Q at x.

C2. Why would concentration change?

Such change can be of different nature in different situations:

- increase may come from some source of Q inside W

Ex. If Q is heat such source of heat may be some heating device, such as a stove, positioned inside.

- increase may also be accumulation of Q inside W

Ex. For a flow of a gas, such as air, the flow itself may force the gas to accumulate at some places (producing increased pressure as in weather systems).

- decrease may come from a sink inside W i.e. a place where Q gets lost.

Ex. For a flow of water in a river, a sink would be a place where water seeps under the ground or where city draws water.

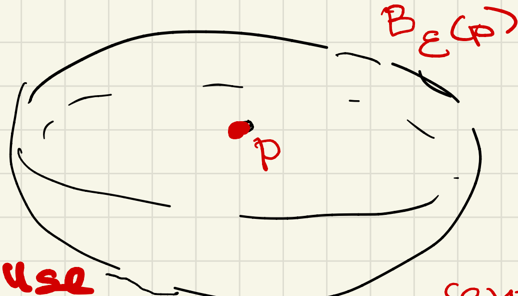
- decrease may also be due to the flow pushing Q to decrease concentration

Ex. Lowering of pressure.

C3. Making the argument precise:

So far, we understood that the integral of divergence over W measures the total change (really loss) of the amount of Q inside W.

To understand the value of div(F) at a single point p we will have to zoom in from all of W to the point p.



This involves choosing some small ε and replacing the whole solid W with a ball $B_\varepsilon(p)$ with center p and radius ε!

Use C4. GDT for a small ball:

The GDT for this ball says that

$$\iiint_{B_\varepsilon(p)} \operatorname{div}(F)\, dV = \iint_{\partial B_\varepsilon(p)} F \cdot d\vec{S} = RHS$$

and we have seen that the RHS is total change of amount of Q in the ball $B_\varepsilon(p)$.

C5. Value $f(P)$ at a point P $\boxed{f = \text{div} F}$ and integrals over small balls

On a small ball $B_\varepsilon(p)$ a given function $f(x)$ is likely not to change much. So we can approximate $f(x)$ by a constant which is the value $f(p)$ at the center of the ball. This gives approximation of the integral:

$$\iiint_{B_\varepsilon(p)} f(x)\, dV \approx \iiint_{B_\varepsilon(p)} f(p)\, dV$$

$$= f(p) \iiint_{B_\varepsilon(p)} 1\, dV =$$

$$= f(p) \cdot \text{Vol}[B_\varepsilon(p)]$$

$$= f(p)\ \tfrac{4}{3}\pi\, \varepsilon^3.$$

- So, we can approximate the value at p using the integral over a small ball:

$$f(p) \approx \frac{1}{\text{Vol}[B_\varepsilon(p)]} \iiint_{B_\varepsilon(p)} f(x)\, dV.$$

- As the radius ε of the ball $B_\varepsilon(p)$ gets smaller, the approximation $f(x) \approx f(p)$ on the ball gets better and better. So, in the limit we get the exact value at p:

$$\boxed{f(p)} = \lim_{\varepsilon \to 0} \frac{\iiint_{B_\varepsilon(p)} f(x)\, dV}{\text{Vol}[B_\varepsilon(p)]}.$$

C6. Conclusion:

We apply formula C5 to the function $f = \mathrm{div}(F)$. We get:

C5.
$$\mathrm{div}(F)(p) = \lim_{\varepsilon \to 0} \left[\iiint_{B_\varepsilon(p)} \mathrm{div}(F)\, dV\right] / \mathrm{Vol}[B_\varepsilon(p)]$$

$$= \lim_{\varepsilon \to 0} \frac{\text{rate change of the amount of } Q \text{ in } B_\varepsilon(p)}{\mathrm{Vol}[B_\varepsilon(p)]}$$

So:

The meaning of the value of the divergence $\mathrm{div}(F)$ at a point p is the rate of change of the amount of Q near P, compared to the volume of the ball where we measure the change.

We can restate it as:

Function $\mathrm{div}(F)$ describes the rate of change of the quantity Q near a given point P where
- rate of change is calculated per unit volume!

or as:

- $\mathrm{div}(F)$ describes the rate of creation/annihilation (change) of quantity Q near each point (when calculated per unit volume).

$\square$