8.3. Conservative Vector Fields

§ 8.3 Which vector fields F are gradients?

A. Motivation: If $F = \nabla f$ we say that f is a potential for F. Then it is easy to integrate over curves:

$$\int_C \nabla f \cdot dr \overset{(*)}{=} f(B) - f(A) \quad \text{for } A \xrightarrow{C} B$$

F

Remark 1. This formula from Chapter 7, is the **FTC** for curves:

Recall the general form of FTC for a shape S & quantity Y:

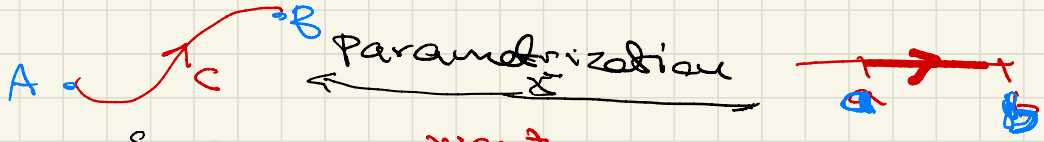
$$\int_{\text{shape } S} \text{certain derivative of } Y = \int_{\partial S} Y$$

Here the shape S is a curve C, quantity Y is a function f and the derivative in question is the gradient ∇.

Also when curve C is oriented from A to B:

$$\partial C = +(B) - A$$

Remark 2. The proof of $(*)$ was by reducing the claim for any curve C to the known case for an interval $[a,b]$ in $\mathbb{R}$.



Then $\int_C \nabla f \cdot d\mathbf{r}$ = $f(B) - f(A)$ (want)

$\int_a^b [f(\gamma(x))]' dx$ = $f(\gamma(b)) - f(\gamma(a))$ (know!)

Remark 3. This is exactly how we have reduced the Stokes theorem to Green's thm - by parametrization of a surface S by a region R.

- **Conclusion:** when $F = \nabla f$ it is easy to integrate vector field F over curves

- **Question:** for a given vector field F:

  a) Is f a gradient of some f?

  If so. b) find a potential f for F.

- These are the things that would make it easy to integrate F over curves!

Closed curves:

We denote $\oint_C$ by $\oint_C$ when C is closed.

B. The big theorem:

Th. For a vector field F on $\mathbb{R}^3$ the following are **equivalent**:

(i) $\oint_C F \cdot dr = 0$ over closed curves C.

(ii) $\int_A^B F \cdot dr$ is path independent, i.e. if two curves go from A to B the integrals are the same: $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$.

(iii) F is a gradient vector field, i.e., $F = \nabla f$ for some function f.

(iv) $\operatorname{curl}(F) = 0$

[Then we say that F is "irrotational"]

Pf: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i)

We now know F is gradient if $\operatorname{curl}(F) = 0$.

Use (iv) to check whether F is gradient. $\operatorname{curl} F = 0$

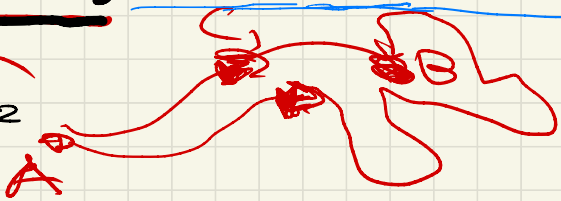
Use: if $\operatorname{curl}(F) = 0$ then find f: (use [ii?])

conservative

C. Proof of i ⇒ ii:

Suppose that for F: $\oint_C F \cdot dr = 0$ for closed curves C.

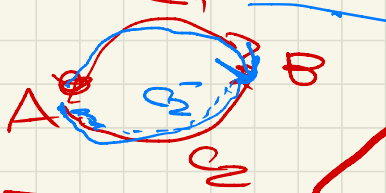
- Consider 2 curves C_1, C_2 from A to B:



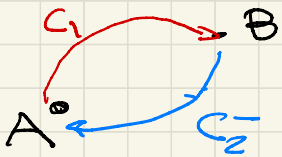
- Want: $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$?

- Consider the difference:

$$\int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2^-} F \cdot dr = \int_C F \cdot dr$$



for the closed curve C that consists of C_1 and C_2^-:



- Since C is closed

$\int_{C^-} F \cdot dr = 0$. Therefore

$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$$

□

Remark: Property (ii) of F is called: "independence of the path" i.e. independence of the choice of a path from A to B.

Proof of 2. ii ⇒ iii

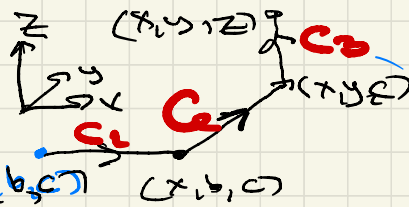
Suppose that $\int_C F \cdot dr$ is defined for F

We want: $F \stackrel{?}{=} \nabla f$ ⇒ We need to construct such f.

We will choose a point (a,b,c) and a path (= curve) C from (a,b,c) to any point (x,y,z). Then we will define

D1. $f(x,y,z) = \int_{(a,b,c)}^{(x,y,z)} F \cdot dr = \int_C F \cdot dr$

D2. In order to calculate we choose a curve C in a simple way:
It moves first in x-direction, then the y-direction and then the z-direction:



When we parametrize each leg of C we get a sum of three integrals

$f(x,y,z) = \int_C F \cdot dr$ > equals if $F = \langle P,Q,R \rangle$

$$= \int_{C_1} + \int_{C_2} + \int_{C_3} \quad P\,dx + Q\,dy + R\,dz$$

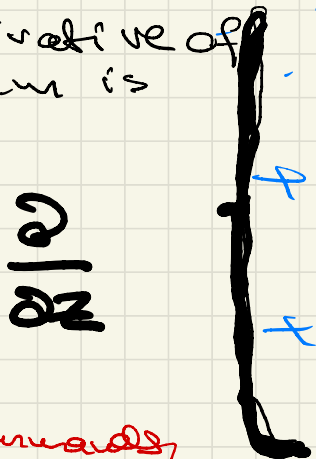
$$= \int_a^x P(\tilde{x},b,c)\,d\tilde{x} + \int_b^y Q(x,\tilde{y},c)\,d\tilde{y} + \int_c^z R(x,y,\tilde{z})\,d\tilde{z}$$

once we use parameterizations of C_1, C_2, C_3!

For our choice of f we need $\nabla f = F$ i.e. $\begin{cases} f_x = P, \\ f_y = Q, \\ f_z = R. \end{cases}$

P3 We will calculate f_z:

This z derivative of the sum is



$$= \frac{\partial}{\partial z}\int_a^x P(x,b,c)\,dx + \frac{\partial}{\partial z}\int_b^y Q(x,y,c)\,dy + \frac{\partial}{\partial z}\int_c^z R(x,y,z)\,dz$$

The 1st two summands do not have any z's so we get:

$$= \frac{\partial}{\partial z}\int_c^z R(x,y,z)\,dz$$

However, this is just $= R(x,y,z)$

because: $\frac{d}{dx}\int_a^x g(u)\,du = g(x)$.

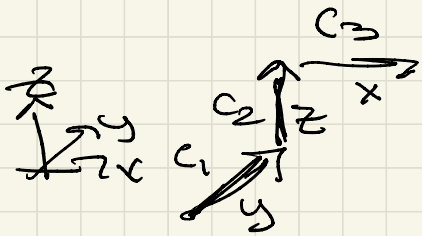
P4: **Calculate f_x, f_y to be P, Q:**

① This cannot be done from the above formula for f!

② However, if we choose another path: 1st y-direction, 2nd z-direction, 3rd x-direction we get another formula for f

$$f(x,y,z) = \int_{(a,b,c)}^{(x,y,z)} F \cdot dr$$

Now, it is easy to calculate $f_x = P$.



The reason is that now x-direction is the last one (so it works the same as for f_z with 1st two terms having zero derivatives).

□

Remark. In order to check that

$$f(x,y,z) = \int_{(a,b,c)}^{(x,y,z)} F \, dr$$

satisfies

$$\nabla f = F,$$

we really needed the property of F that the integral does not depend on the choice of path between two points.

E. Proof of (iii) ⇒ (iv)

Suppose $F = \nabla f$.

We want $\boxed{\text{curl}(F) = 0.}$

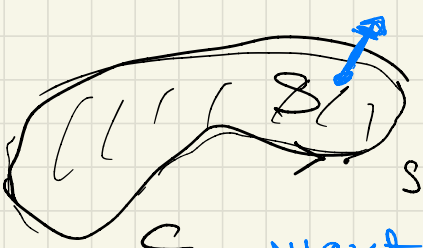
Let $F = \langle P, Q, R \rangle$, so $P = f_x$, $Q = f_y$, $R = f_z$.

$$\text{curl}(F) = \nabla \times F = \left\langle \quad , \quad , \quad (Q_x - P_y) \right\rangle = 0$$

(the third component $Q_x - P_y$ equals 0)

i.e., I will only consider the 3rd component $Q_x - P_y = (f_y)_x - (f_x)_y = 0$

F. Proof of (iv) ⇒ (i):



- Assume $\boxed{\text{curl}(F) = 0}$
- Want $\oint_C F \cdot dr = 0$

Since we know curl we want to use Stokes theorem!

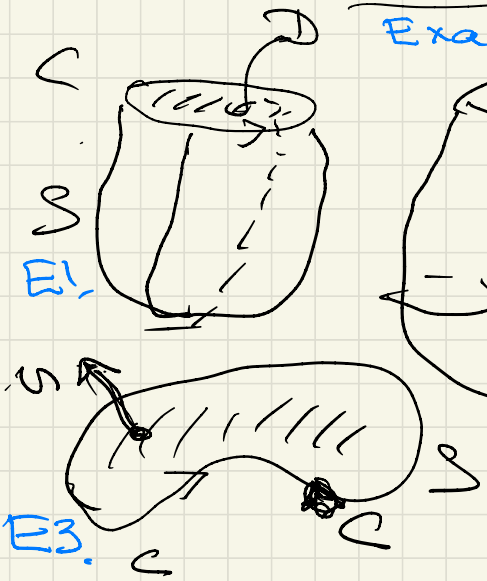
For this, as in the picture, we choose any surface whose boundary is S. On S we put a compatible orientation n.

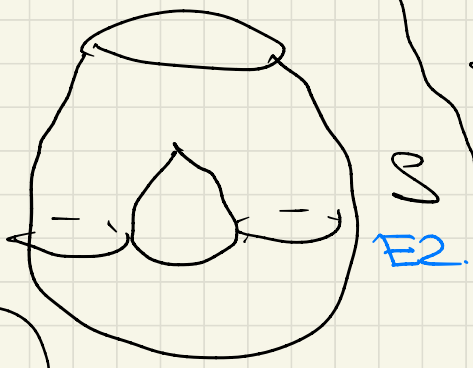
Then $\oint_C F \cdot dr = \iint_S \underbrace{\text{curl}(F) \cdot d\vec{S}}_{=0} = 0.$

□

Remark. There are many choices of S, such that $\partial S = C$

Examples:





G. Formula for the potential:

Let F be a vector field on $\mathbb{R}^3$ & suppose we know it is conservative i.e. potential f exists.

The construction of the potential in the proof of **ii ⇒ iii**, leads to the following simple formula:

$$f = M + N + K + C \quad \text{where:}$$

$$\begin{cases} M = \int P(x,y,z)\,dx \\ N = \int Q - M_y\,dy \\ K = \int R - M_z - N_z\,dz \\ C \text{ is any constant.} \end{cases}$$

So, we are integrating P in the x-direction, Q in y-direction and R in the z-direction as in the proof of (ii)⇒(iii).

[We have extra terms because we are now using indefinite integrals!]

H. The same in 2 variables

Here: $F(x,y) = \langle P(x,y), Q(x,y) \rangle$

Thm (a) F has a potential iff

$$Q_x = P_y$$

(b) then $f = M + N + C$ check.

$$M = \int P\,dx, \quad N = \int Q - M_y\,dy$$

Pf. (a) $F = \nabla f$ then $P = f_x$, $Q = f_y$

$$Q_x = (f_y)_x, \quad P_y = (f_x)_y$$

$$Q_x = P_y$$

If $Q_x = P_y$ then can define

$$f(x,y) = \int_a^x P(\tilde{x},y)\,d\tilde{x} + \int_b^y Q(c,\tilde{y})\,d\tilde{y}$$

just like in the 3d case. Then the proof of $\nabla f = F$ is again as in 3d.

(b) • We are looking for a function f such that $\nabla f = F$ i.e. $f_x = P$ and $f_y = Q$.

• We will use the formula $\int \frac{df}{dx}\,dx = f(x) + C$

• When we keep y as a constant it gives $f(x,y) = \int \frac{\partial f}{\partial x}\,dx + C$ except that there is one constant for each y, so it is really:

$$f(x,y) = \int \frac{\partial f}{\partial x}\,dx + C(y) = M + C(y).$$

• Now, differentiate with respect to y to get

$$Q = \frac{\partial f}{\partial y} = M_y + C'(y).$$

So,

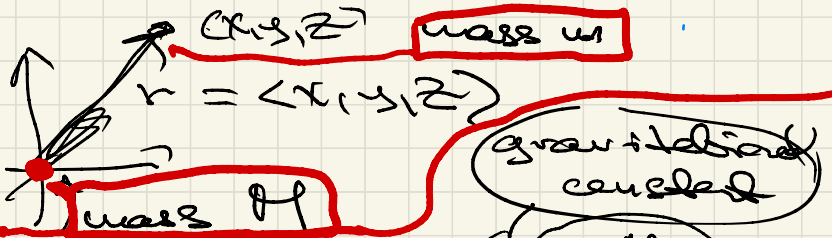
$$C'(y) = Q - M_y$$

hence

$$C(y) = \int C'(y)\,dy = \int Q - M_y\,dy = N$$ □

• Therefore $f = M + C(y) = M + N$.

Example: Gravity



- We use position vector r
- Newton: grav. force $= G\frac{mM}{\text{dist}^2}$ (gravitational constant)
- Precise version as a vector field: (has force)

$$F(x,y,z) = |F(x,y,z)| \cdot u$$

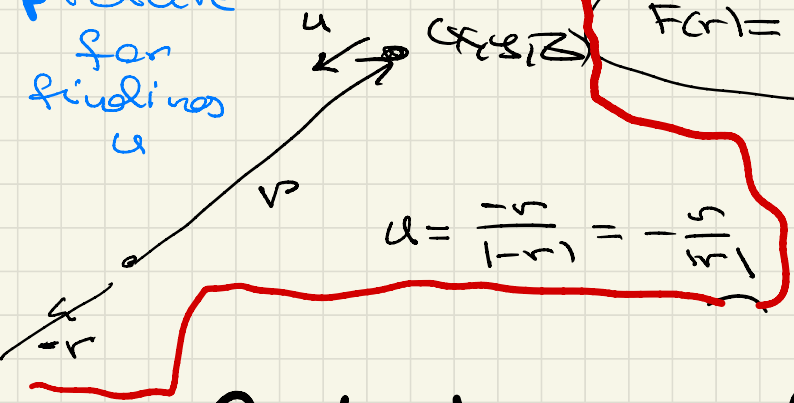
(size; direction; unit vector)

$$F(r) = G\cdot\frac{mM}{|r|^2}\cdot\frac{-r}{|r|}$$

$$F(r) = -G\,mM\,\frac{r}{|r|^3}$$

Final formula for gravitational force.

Picture for finding u



$$u = \frac{-r}{|-r|} = -\frac{r}{|r|}$$

Q. • Is it conservative?
• What is its potential?

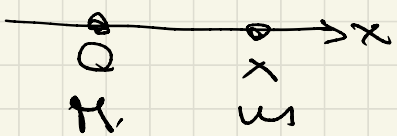
I. Gravitational potential

Instead of computing potential we will guess what it is!

We start with the gravity on a line. Here $\nabla f = f'$ (one variable only!), so $F = \nabla f$ means that $F = f'$ so it is solved by integration:

$$f = \int F(x)\,dx.$$

On a line gravitational force is given by Newton's formula: $F = GmM \cdot \frac{1}{x^2}$ where the picture is:



So, $f(x) = \int F(x)\,dx = \int GmM \frac{dx}{x^2}$

i.e. $f(x) = GmM \frac{-1}{x}$.

Back to 3: We make the guess that here $f = GmM \frac{1}{|r|}$.

This is our candidate for the potential. We have to compute its gradient.

We will consider a more general function $f = C|r|^n$.

For us $C = GmM$ and $n = -1$.

So, let $f = C|r|^n = C(x^2+y^2+z^2)^{\frac{n}{2}}$. Then:

$$f_x = C \cdot \frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \cdot 2x.$$

$$= nC\left(\sqrt{x^2+y^2+z^2}\right)^{n-2} \cdot x$$

hence:

$$\nabla f = Cn \underbrace{\langle x, y, z\rangle}_{r} \cdot |r|^{n-2}$$

$$= nC \frac{r}{|r|^{2-n}}.$$

In the particular case when $n=-1$ we have found that:

$$\nabla\left(\frac{C}{|r|}\right) = (-1)C \cdot \frac{r}{|r|^{2-(-1)}}$$

$$= -C\frac{r}{|r|^3}.$$

So, for $C = GmM$ we have:

$$f = \frac{GMm}{|r|}, \quad \nabla f = -GmM\frac{r}{|r|^3}$$

$$= F_{grav}.$$

We have found that the gravitational force is conservative (since it does have a potential.) □

Conservative forces

Forces F we find in the world are conservative as vector fields. There is a reason for that!

Remember that F being conservative means that the integrals over closed curves vanish:

$$\oint_C F \cdot dr = 0.$$

For a force F, integral over a curve C means the **work** that the force does along the curve C.

It is related to the energy that we posses. Integral $\oint_C F \cdot dr$ being zero means that the total energy is preserved along a closed curve!

Ex. When we go up we loose energy as it is invested into the work of climbing. However, when we go down by falling we gain velocity hence kinetic energy. So, coming down to the starting point we have not gained or lost energy.