8.2. Green and Stokes Theorems II

Recall formulations of Stokes and Green theorems $\boxed{8.2.1}$

A. We guessed that **FTC** for an oriented shape S should say

$$\int_S \text{derivative of } \varphi = \int_{\partial S} \varphi$$

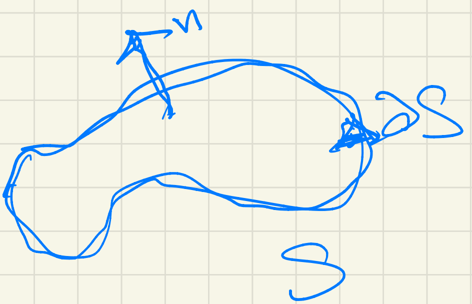
B. The case when S is a surface is

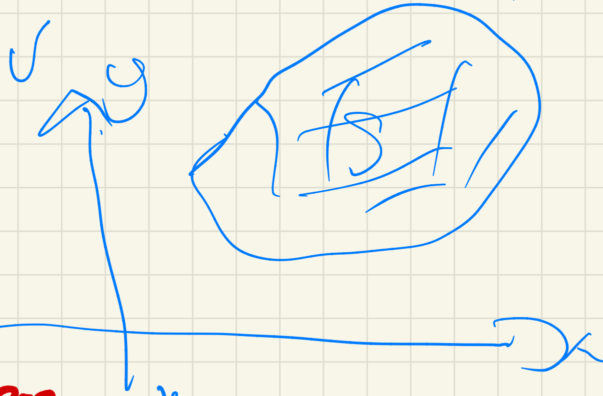
[Stokes theorem] For a vector field F

$$\boxed{\iint_S \operatorname{curl}(F)\, d\vec{S} = \int_{\partial S} F \cdot dr}$$

φ is F, derivative of φ is $\operatorname{curl}(F)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ " $\nabla \times F$

When the surface S is a region in plane and the vector field is $F(x,y,z) = \langle P(x,y), Q(x,y), 0\rangle$ (corresponding to a planar vector field $\langle P(x,y), Q(x,y)\rangle$) then the Stokes





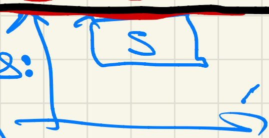
In this case **Green** theorem says

$\mathbf{F} = \langle P(x,y), Q(x,y), 0\rangle$ $\boxed{8.2.2}$

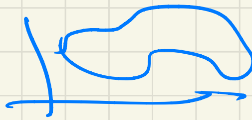
[Green] $\underbrace{\iint_S Q_x - P_y \, dA}_{\iint \mathrm{curl}(\mathbf{F}) \cdot d\vec{S}} = \underbrace{\oint_{\partial S} P\,dx + Q\,dy}_{\mathbf{F}\cdot d\mathbf{r}}$

C. 3 steps in the Proof of Stokes theorem:

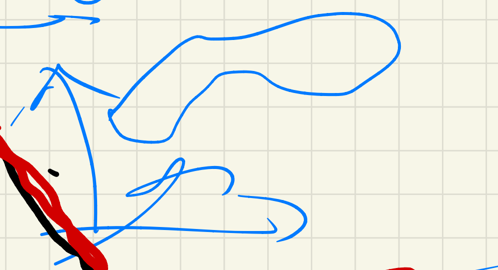
1. Green for rectangles:



2. Green:



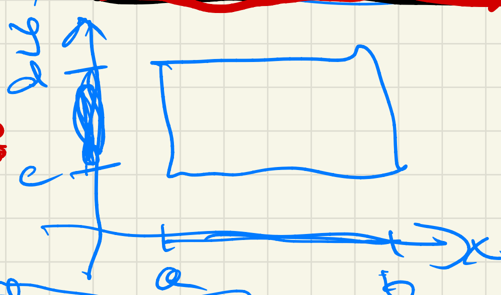
3. Stokes:



D. Green's theorem (Proof of)

when the region S is a rectangle:

Trick: use different order of integration



• The LHS of Green theorem:

$$\iint_S Q_x - P_y \, dA$$

$$= \int_{y=c}^{d}\int_{x=a}^{b} \underbrace{Q_x}_{=\frac{\partial Q}{\partial x}} \, dx\,dy - \int_{x=a}^{b}\int_{y=c}^{d} P_y \, dy\,dx.$$

Now use FTC:

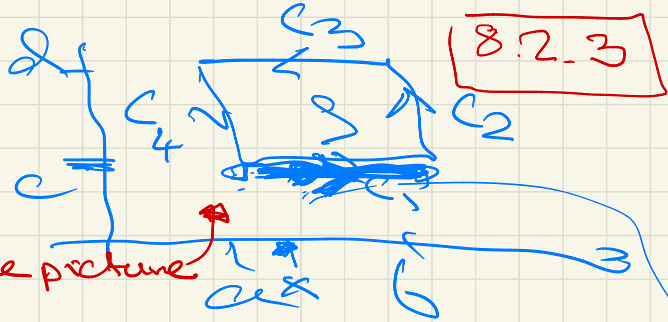
$$= \int_c^d [Q]_{x=a}^{x=b} \, dy - \int_a^b [P]_{y=c}^{d} \, dx$$

$$= \int_c^d Q(b,y) - Q(a,y)\, dy - \int_a^b P(x,d) - P(x,c)\, dx$$

8.2.3

The RHS of Green's:

$$\int_{\partial S} P\,dx + Q\,dy$$



∂S has 4 pieces C_1, C_2, C_3, C_4 as in the picture

$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} + \int_{C_3} + \int_{C_4}.$$

We start with the parametrization of C_1

Points of C_1 are of the form (x, c)

for $a \le x \le b$.

We choose the parameter u as $u = x$, so, $\left\{\begin{array}{l} x = u \\ y = c \end{array}\right\}$ $[a \le u \le b]$.

Notice $dy = 0\,du$, $dx = du$.

Now: $$\int_{C_1} P\,dx + \underbrace{Q\,dy}_{=0} = \int_{u=a}^{b} P(u, c)\,du$$

$$\gamma(u) = \begin{bmatrix} u \\ c \end{bmatrix}$$

Next, we rewrite it using x instead of u:

$$= \boxed{\int_a^b P(x, c)\,dx}$$

• Comparison of LHS & RHS of Green's T.:

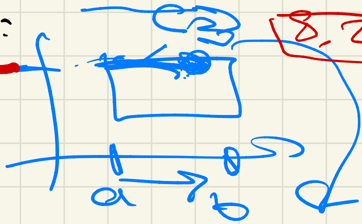
① Each has 4 terms

② We just matched one term, similarly for others.

③ One subtlety: the LHS has some minuses!

However, these will also appear on the RHS due to the "backward" orientation of C_3, C_4:

Here is calculation for C_3: 8.24



To parametrize C_3 we notice that the points are of the form (x,d) for $a \le x \le b$.

So we try parameter $u = x$ and parameterization: $\gamma(u) = \begin{bmatrix} u \\ d \end{bmatrix} = \begin{bmatrix} x \\ d \end{bmatrix}$ with $a \le u \le b$.

However this is really a parametrization of C_3^-, i.e. C_3 as in the picture but with the wrong direction!

C_3^-

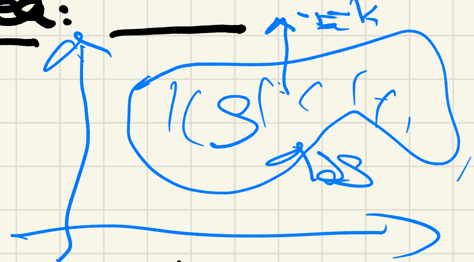
Finally:

$$\int_{C_3} = -\int_{C_3^-} = -\int_{x=a}^{b} dx$$

- In this way all 4 terms on LHS & RHS match exactly. $\square$

We interrupt the proof to bring you **8.2.5**

one application: **E. Computation of the area:**



1. $\underset{\text{area}}{A(S)} = \int_{\partial S} x\,dy$

Proof: Think of $x\,dy$ as $\underset{\substack{\shortparallel \\ 0}}{P\,dx} + \underset{\substack{\shortparallel \\ x}}{Q\,dy}$, so that $P=0$ and $Q=x$.

So:

$$\text{RHS} \overset{\text{Green}}{=} \iint_S \underbrace{Q_x}_{\substack{\shortparallel \\ x_x}} - P_y\, dA = \iint_S \underbrace{1-0}_{=1}\, dA = \iint_S 1\, dA = A(S). \quad \square$$

2.

$$A(S) = \int_{\partial S} x\,dy = -\int_{\partial S} \underbrace{y\,dx}_{P\,dx + Q\,dy}$$

$$= \frac{1}{2} \int_{\partial S} x\,dy - y\,dx$$

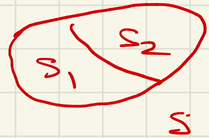
Proof. This corollary gives 3 formulas for the area. The 1^{st} is from the lemma. The 2^{nd} is checked in the same way. The 3^{rd} is the average of 1^{st} & 2^{nd}. $\square$

Now we come back to the proof of Green's theorem, based on the special case of rectangles (which we have checked).

E1, Gluing principle

8.2.6.

L1. If region S is cut into 2 pieces S_1, S_2



Then Stokes for S_1, S_2 implies Stokes for S.

Proof: We want

$$\iint_S \operatorname{curl}(F)\, dS \overset{?}{\underset{\text{Stokes}}{=}} \int_{\partial S} F \cdot dr$$

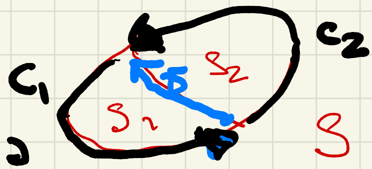
Now LHS is clearly:

$$\iint_S = \iint_{S_1} + \iint_{S_2}.$$

Since Stokes holds for S_1 and for S_2 this is

$$= \int_{\partial S_1} F \cdot dr + \int_{\partial S_2} F \cdot dr.$$

Now notice how ∂S decomposes



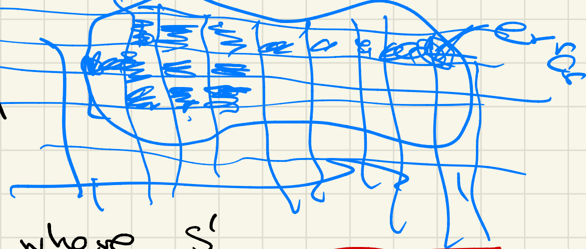
$$= \int_{C_1} + \int_{\gamma} + \int_{C_2} + \int_{\gamma^-} = \int_{C_1} + \int_{C_2}$$

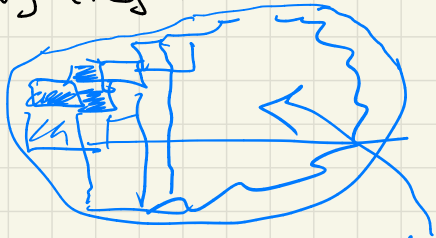
These two cancel since $\int_{\gamma^-} = -\int_{\gamma}$.

$$= \int_{\partial S}. \quad \square$$

E2. Proof of Green:

We approximate a region S by a region S'

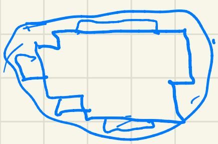




S where S' is obtained by gluing rectangles! $\boxed{8.2.7}$

S' is a union of rectangles!

By the gluing principle Green's theorem holds for S'!



Now we will take the limit of the Green's theorem for S':

$$\iint_{S'} Q_x - P_y \, dA = \oint_{\partial S'} P\,dx + Q\,dy$$

limit as one passes to better approximations of S by S' and of ∂S by ∂S'.

limit

In the limit we get:

$$\iint_{S} Q_x - P_y \, dA \qquad \oint_{\partial S} P\,dx + Q\,dy$$

and so these two are the same! $\square$

Recollections: We want to prove for any oriented surface $\vec{S}$:

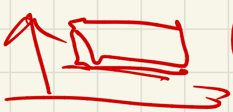
[Stokes] $\displaystyle\int_S \operatorname{curl}(F) \times d\vec{S} = \int F \cdot d\vec{r}$

So, far we have calculated what is the special case of the Stokes theorem when surface lies in the xy plane. This is called Green's theorem. We have proved it in 2 steps

• [Green] For a region R in $\mathbb{R}^2$

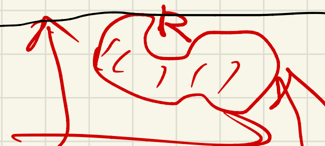
$$\iint_{R} Q_x - P_y \, dA = \int_{\partial R} P\,dx + Q\,dy$$

The 2 steps in the proof:

① 

FTC

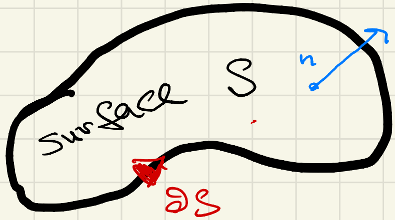
$$\int_a^b f'(x)\,dx = [f]_a^b$$

② 

• **Remarks:** Prove that the Stokes theorem is a consequence of

① Green theorem

F. Reduction of Stokes to Green's theorem



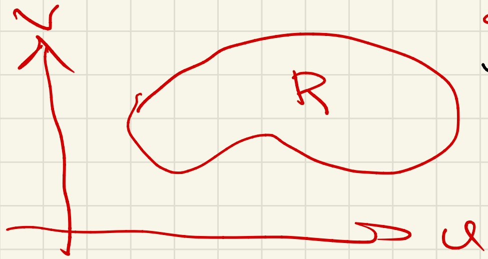
$$\iint_S \mathrm{curl}(F)\cdot d\vec{S}$$

$\|$ Stokes

$$\int_{\partial S} F\cdot dr$$

In the space we consider a surface S with an orientation n and the oriented boundary ∂S.

In order to compute we choose a parameterization Φ of S by a region R in the u,v plane:



So, S & R are matched by a parameterization

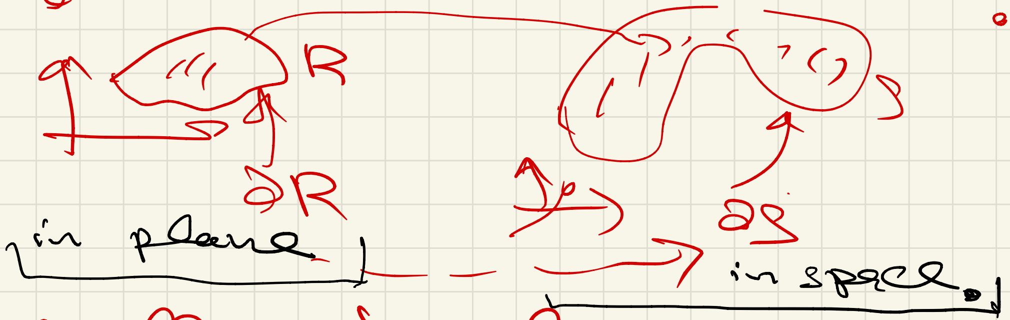
$$\Phi : R \longrightarrow S$$

which is $\boxed{1-1 \text{ correspondence}}$

[Here we disallow any error on the parameterization being a 1-1 correspondence: errors which are "small for S" i.e. of area zero could happen on all of ∂S!]

When $\Phi : R \to S$ is a 1-1 corr.
then it restricts to a 1-1 corr. of boundaries

$\Phi_\partial : \partial R \to \partial S$ (the restriction of Φ).



This allows to calculate using parametrizations for both S & ∂S!

So:

$$\iint_S \ldots \underset{\text{CofV}}{\overset{\Phi}{=}} \iint_R \overset{?}{\ldots}$$

$$\int_{\partial S} \ldots \underset{\text{CofV}}{\overset{\Phi_\partial}{=}} \int_{\partial R} \overset{??}{\ldots}$$

Want Stokes: $\iint_S \quad \int_{\partial S}$

This translates Stokes:
to a claim of the form "$\iint_R = \int_{\partial R}$"

Know Green.

$\blacksquare$

G. In the proof we have skipped calculating what is integrated over R & ∂R, i.e. what are ? and ?? that appear after our changes of variables.

$$\iint_S \nabla \times F \cdot \vec{dS} \overset{?}{\underset{\text{Stokes}}{=}} \int_{\partial S} F \cdot dr$$

$$\iint_R ? \underset{\text{Green Theorem}}{=} \int_{\partial R} ?? \quad P\,dx + Q\,dy$$

Claim: equality of the the last two is just **Green's** theorem!!

The claim is checked by showing that for some P, Q one has

$? = Q_x - P_y$ and $?? = P\,dx + Q\,dy$

Actually, we have already checked this in planar cases!

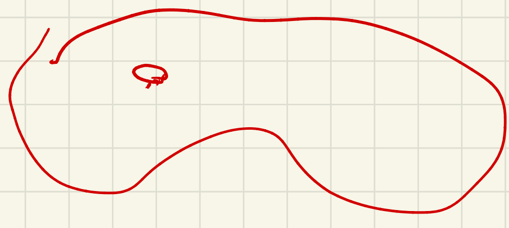
We really checked already that the Stokes theorem when specialized to dS from a plane is Green theorem

4. Remarks: 1. In the book the conditions that are needed for Stokes theorem are stated more precisely

- S is a parametrized surface
  had to be 1–1 corr.?
- F is required to be C^1 on S, i.e., with continuous 1st partial derivatives.

2. Stokes theorem may fail if these conditions are not satisfied, say if F is bad at some point: • if it is not defined or its partials are not defined or they are not continuous:

If have F which is bad



S

then theorem may fail.

See Q12 in 8.2