# 5.5. Eigenvalues which are complex numbers

- complex eigenvalues come in pairs $\lambda, \bar{\lambda}$ with eigenvectors $v, \bar{v}$
- so all eigenvalues can be divided into $p$ real eigenvalues $r_1, \ldots, r_p$ and $2q$ complex eigenvalues $\lambda_1, \ldots, \lambda_q, \bar{\lambda}_1, \ldots, \bar{\lambda}_q$. (So, $p+2q$ = size of the matrix)
- If $r_1, r_2, \ldots, r_p, \lambda_1, \ldots, \lambda_q, \bar{\lambda}_1, \ldots, \bar{\lambda}_q$ are all distinct, then $A$ has a "block diagonalization" where each $\lambda_k$ contributes one $2\times 2$ block of the form:

$$\begin{pmatrix} a_k & -b_k \\ b_k & a_k \end{pmatrix},$$ where $a_k = \mathrm{Re}(\lambda_k)$, $b_k = \mathrm{Im}(\lambda_k)$.

# § 5.5. Complex eigenvalues

So far, in ~~chapter~~ 5:

- we consider the case when a matrix $A$ acts on a vector $v$ as a number ("scalar") $\lambda$ i.e.

$$Av = \lambda v.$$

Then $v$ was said to be an eigenvector of $A$ with eigenvalue $\lambda$. (provided $v \neq 0$)

- The best we can hope for $A$ is that there is a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ of $\mathbb{R}^n$ consisting of eigenvector of $A$, i.e.,

$$Ab_i = \lambda_i b_i.$$

Such basis gives a diagonalization of $A$, i.e.

$$A = PDP^{-1}$$

where:

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$ is the diagonal matrix of eigenvalues of $A$,

and

$$P = [b_1 \cdots b_n]$$ is the invertible matrix whose columns are given by eigenvector basis $\mathcal{B}$.

- One case when this is guaranteed to happen is when $A$ has $n$ distinct eigenvalues which are real numbers.

- **However**, eigenvalues are solutions of $0 = \det(A - \lambda I)$. So What happens when they are not real numbers?!

# §5.5 Eigenvalues which are complex numbers

## A. Example

Consider $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

then the characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{bmatrix} = \lambda^2 + 1.$$

The eigenvalues are solutions of $\lambda^2 + 1 = 0$

i.e. $\lambda^2 = -1$

so $\lambda_1 = i$ and $\lambda_2 = -i$.

[ Complex numbers are symbols $a+bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.

- Equation $\lambda^2 + 1 = 0$ has no solutions in $\mathbb{R}$ but it has solution $i$ (i.e. $0 + 1i$) and then also $-i$ (i.e. $0 + (-1)i$), in the set $\mathbb{C}$ of all complex numbers. ]

## B. From $\mathbb{R}^n$ to $\mathbb{C}^n$:

- **$\mathbb{C}$ is just** a new system of numbers with addition $(a+bi) + (a'+b'i) = (a+a') + (b+b')i$ and multiplication based on $i \cdot i = i^2 = -1$:

$$(a+bi)\cdot(a'+b'i) = aa' + ba'i + ab'i + bb'i^2$$

$$= (aa' - bb') + (ab' + a'b)i.$$

- **One can** calculate with complex numbers $\mathbb{C}$ the same as with real numbers $\mathbb{R}$.

In particular we can form

$\mathbb{C}^n$ = all n-tuples $(z_1, \dots, z_n)$ with $z_i$ some complex #s

and one can consider matrices with complex entries of type $m \times n$, denoted $M_{mn}(\mathbb{C})$.

Remark 1. All of Linear Algebra works as well with complex numbers (even better)!

Remark 2. The improvement from $\mathbb{R}$ to $\mathbb{C}$ that any polynomial has a solution which is a complex number, but some polynomials, say $\lambda^2+1$ have no solutions in real numbers!

## C. Complex eigenvectors: For $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

we found eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$.

An eigenvector v for $\lambda = i$. So, $v = \begin{bmatrix} x \\ y \end{bmatrix}$ which is killed by $A - iI = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$ i.e.

$$\begin{cases} -i\,x + 1y = 0 \\ -1\cdot x - i\,y = 0 \end{cases}.$$

The $1^{st}$ equation is $y = ix$

The second equation is irrelevant: $i \cdot \text{I} + \text{II}$ gives $0x + 0y = 0$!

So, the solution is $v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ix \end{bmatrix} = x \begin{bmatrix} 1 \\ i \end{bmatrix}$.

So, an eigenvector for $\lambda_1 = i$ is $v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

# D. Conjugation of complex #s

This is the operation that takes $z$ to $\bar{z}$ by changing $i$ to $-i$, i.e. $\overline{a+bi} = a-bi$.

Ex. • $\overline{7-3i} = 7+3i$,
• $\bar{i} = -i$.

It has beatiful properties:

(a) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$, (b) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$.

(b) For a polynomial $P(x) = p_0 + p_1 x + \cdots$ whose coefficients are real numbers $p_0, p_1, \ldots, p_n \in \mathbb{R}$ one has: $\overline{P(z)} = P(\bar{z})$.

(c) If $\lambda \in \mathbb{C}$ is an eigenvalue of a matrix $A$ of real numbers then $\bar{\lambda}$ is also an eigenvalue!

(d) Moreover, if $v \in \mathbb{C}^n$ $= (v_1, \ldots, v_n)$ is a $\lambda$-eigenvector of such matrix $A$ then $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_n)$ is a $\bar{\lambda}$-eigenvector of the same $A$.

Conclusion • Complex eigenvalues of a real matrix $A$ appear in conjugate pairs $\lambda, \bar{\lambda}$.

• An eigenvector $v$ with eigenvalue $\lambda$ gives an eigenvector $\bar{v}$ with eigenvalue $\bar{\lambda}$.

**E. Example: The conjugate eigenvector:**

For $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ we found eigenvalue $\lambda_1 = i$ with eigenvector $v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

Then we know that
$$v_2 = \overline{v_1} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$
is an eigenvector with eigenvalue
$$\lambda_2 = \overline{\lambda_1} = \overline{i} = -i.$$

Since we have a $2 \times 2$ matrix $A$ with two different eigenvalues $\lambda_1, \lambda_2$ the corresponding eigenvectors form an eigenvector basis of $\mathbb{C}^2$!

Explain:

R. The ideas we have developed for real eigenvalues work the same for complex eigenvalues.

- For instance since $\boxed{\lambda_1 \neq \lambda_2}$ i.e. $i \neq -i$, eigenvectors $v_1, v_2$ are independent.
  Two independent vectors in $\mathbb{C}^2$ form a basis of $\mathbb{C}^2$!
  when we think of $\mathbb{C}^2$ as a 2-dimensional vector space over complex numbers $\mathbb{C}$.

**R.** $\mathbb{C}^n$ is also a vector space over $\mathbb{R}$.
Then its basis is $e_1, \ldots, e_n, ie_1, \ldots, ie_n$.
So, dimension over $\mathbb{R}$ is $2n$.

# F. More complicated $2\times 2$ matrices with complex eigenvalues.

**E1.** $R = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$, Then: $\det(R-\lambda I)$

{ Rotation of $\mathbb{R}^2$ by angle $\varphi$ ! }

$$= \det\begin{pmatrix} \cos\varphi-\lambda & -\sin\varphi \\ \sin\varphi & \cos\varphi-\lambda \end{pmatrix}$$

$$= (\cos\varphi-\lambda)^2+\sin^2\varphi$$

equals:
$$= \cos^2\varphi - 2\cos(\varphi)\,\lambda + \lambda^2 + \sin^2\varphi$$
$$= \lambda^2 - 2\cos(\varphi)\,\lambda + 1.$$
$$= (\lambda-\lambda_1)\,(\lambda-\lambda_2)$$

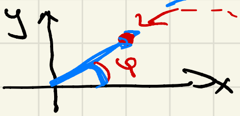
where: Here: $\lambda_2 = \overline{\lambda_1}$.

$$\lambda_1 = \cos\varphi + i\sin\varphi$$
$$\lambda_2 = \cos\varphi - i\sin\varphi$$

**E2.** $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for some real numbers $a, b$.

Let $\boxed{r = \sqrt{a^2+b^2}}$ and write $C = r\begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix}$.

Then




$\left(\frac{a}{r}, \frac{b}{r}\right)$ is a point on the unit circle:

$$\left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = \frac{a^2+b^2}{r^2} = 1$$ !

So: $\frac{a}{r} = \cos\varphi$, $\frac{b}{r} = \sin\varphi$

for some number $\varphi$ (the angle of the point with the x-axis!)

Now: $C = r\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$.

So by E1, eigenvalues of $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

are
$$\begin{cases} \lambda_1 = r(\cos\varphi + i\sin\varphi), \\ \lambda_2 = r(\cos\varphi - i\sin\varphi) = \overline{\lambda_1}. \end{cases}$$

# G. 2×2 matrices with complex eigenvalues are similar to E2!

**T1.** Let $A$ be a 2×2 matrix with a complex eigenvalue $\boxed{\lambda = a+bi}$ (we will assume that $b \neq 0$ so that $\lambda$ is not real!) Then

a) $A$ is $\boxed{\text{similar}}$ to the matrix $\boxed{C = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}}$

b) More precisely one can **find** a matrix $P$ such that $A = P C P^{-1}$, once one knows an eigenvector $v$ for $A$, with eigenvalue $\lambda$. The formula is

$$P = [\operatorname{Re}(v) \ \operatorname{Im}(v)].$$

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**Explanation:** For a complex number $z = a+bi$ its real part $\boxed{\operatorname{Re}(z)}$ is $a$, its imaginary part $\operatorname{Im}(z)$ is $b$.

- We can extend real & imaginary parts to vectors: $\boxed{\operatorname{Re}\begin{pmatrix} 2-7i \\ 3i \end{pmatrix}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\boxed{\operatorname{Im}\begin{pmatrix} 2-7i \\ 3i \end{pmatrix}} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$.

**Ex.** $\operatorname{Re}(2-3i) = 2$

$\operatorname{Im}(2-3i) = -3$

R. $i$ is called "imaginary unit".
[Historically it was believed that complex #s do not exist in nature, that we only imagine that they do. **WRONG.**]

**G. Once again**

**Th. Let** $A$ be a $2\times 2$ matrix with a complex eigenvalue $\lambda$ and the corresponding eigenvector $v$.

**Then** $A = PDP^{-1}$

where ① $D = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for $a = \mathrm{Re}(\lambda)$, $b = \mathrm{Im}(\lambda)$.

② $P = [\mathrm{Re}(v) \ \ \mathrm{Im}(v)]$

**Pf.** $Pe_1 = \mathrm{Re}(v)$, $Pe_2 = \mathrm{Im}(v)$.

So, $P(e_1 + ie_2) = \mathrm{Re}(v) + i\,\mathrm{Im}(v) = v$.

**Now,** $\underline{P(e_1+ie_2) = v}$ gives $\underline{P^{-1}v = e_1 + ie_2}$.

So, $(PDP^{-1})v = PD(e_1+ie_2) = P\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix}$

$= P\begin{bmatrix} a+ib \\ -b+ia \end{bmatrix} = P\begin{bmatrix} \lambda \\ i\lambda \end{bmatrix} =$

$= P\left(\lambda\begin{bmatrix} 1 \\ i \end{bmatrix}\right) = \lambda P\underbrace{\begin{bmatrix} 1 \\ i \end{bmatrix}}_{e_1+ie_2} = \lambda v$
$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad \parallel \; Av.$

So, we have found that $PDP^{-1}v = Av$.

We conjugate to get $PDP^{-1}\bar{v} = \overline{PDP^{-1}v}$

$= \overline{PDP^{-1}v} = \overline{Av} = A\bar{v}$.

Here, $Av = \lambda v$ implies that
$A\bar{v} = \bar{\lambda}\bar{v}$

So, $v$ and $\bar{v}$
are eigenvectors of $A$ with different eigenvalues $\lambda, \bar{\lambda}$.

Hence, $v$ and $\bar{v}$ forms a basis of $\mathbb{C}^2$.

② Matrices $A$ and $B = PDP^{-1}$ act the same on $v, \bar{v}$:

$$Bv = Av, \quad B\bar{v} = A\bar{v}.$$

So them acts the same on all vectors in $\mathbb{C}^2$.

In particular $\underbrace{Be_1 = Ae_1}_{\substack{1^{st} \text{ columns of} \\ B \text{ and } A \\ \text{are the same}}}$, $\underbrace{Be_2 = Ae_2}_{\substack{2^{nd} \text{ columns} \\ \text{of } B \text{ and } A \\ \text{are the same.}}}$.

So, $B = A$, i.e., $PDP^{-1} = A$. $\square$

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# Comparison with the book

In section 5.5, theorem 9 says the same as the above Th.

It seems to me that the above Th is less confusing - easier to use.

# H. Block diagonalization of Matrices A of any size n $\}^{n\times n}$

I. (a) A has n eigenvalues which we can group into: p real #s: $r_1, \dots, r_p$

and q pairs of complex #s: $\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \dots, \lambda_q, \bar{\lambda}_q$.

Here, $p+2q=n$.

(b) If the eigenvalues

$r_1, \dots, r_p$, $\lambda_1, \lambda_2, \dots, \lambda_q$, $\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_q$

are all different then the corresponding eigenvectors are of the form

$u_1, \dots, u_p$, $v_1, \dots, v_q$, $\bar{v}_1, \dots, \bar{v}_q$

where $u_i$'s are real eigenvectors and $v_1, \dots, v_q$ are complex eigenvectors.

This is basis of $\mathbb{C}^n$ consisting of A-eigenvectors.

$\boxed{(C)}$ If so, then A is similar to a block diagonal matrix,

$$\begin{bmatrix} r_1 & & & & & \\ & r_2 & & & & \\ & & \ddots & & & \\ & & & r_p & & \\ & & & & C_1 & \\ & & & & & \ddots \\ & & & & & & C_q \end{bmatrix}$$

where $C_k$'s are $\boxed{2\times 2 \text{ blocks}}$:

$$C_k = \begin{bmatrix} a_k & -b_k \\ b_k & a_k \end{bmatrix}$$

for $\lambda_k = a_k + b_k i$. $\quad a_k = \mathrm{Re}\,\lambda_k$, $b_k = \mathrm{Im}\,\lambda_k$ ?

"Complex" analogue of diagonalization !!! | i.e. when we have complex eigenvalues.

## I. Examples: matrices with $n$ distinct eigenvalues.

$n=2$: $\begin{bmatrix} r_1 & \\ & r_2 \end{bmatrix}$ or $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ & $b \neq 0$.

$n=3$: $\begin{bmatrix} r_1 & & \\ & r_2 & \\ & & r_3 \end{bmatrix}$ or $\begin{bmatrix} r_1 & & \\ & \boxed{\begin{matrix} a & -b \\ b & a \end{matrix}} \end{bmatrix}$ A has a real eigenvalue

$n=4$: $\begin{bmatrix} r_1 & & & \\ & r_2 & & \\ & & r_3 & \\ & & & r_4 \end{bmatrix}$ or $\begin{bmatrix} r_1 & & \\ & r_2 & \\ & & \boxed{\begin{matrix} a & b \\ -b & a \end{matrix}} \end{bmatrix}$.

or $\begin{bmatrix} \begin{matrix} \alpha & -\beta \\ \beta & \alpha \end{matrix} & \\ & \begin{matrix} a & b \\ -a & b \end{matrix} \end{bmatrix}$

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## J. The game: ① Find eigenvalues from factoring

$\det(A-\lambda I) = \underbrace{(\lambda - r_1)\cdots(\lambda - r_p)}_{\text{real}} \cdot$

$(\lambda - \lambda_1)(\lambda - \bar{\lambda}_1)$ complex

$\vdots$

$(\lambda - \lambda_q)(\lambda - \bar{\lambda}_q)$

② Find eigenspace $V_\lambda$ for each eigenvalue $\lambda$.

③ If possible find a basis of eigenvectors of A.

(4) Then Block-diagonalize A as above!