5.2. Similiar matrices

have the same eigenvalues,
their multiplicities and
the dimensions of eigenvectors

Recollections from Last Time:

We say that number $\boxed{\lambda}$ is an eigenvalue of matrix $\boxed{A}$ if there exists a nonzero vector v such that $Av = \lambda v$.

[Then v is said to be an eigenvector of A with eigenvalue λ.]

T1. The eigenvalues of A are the solutions of the characteristic equation $\det(A - \lambda I) = 0$.

① eigenvalues are found by factoring the polynomial $\det(A - \lambda I)$.

② For each eigenvalue λ, its eigenvectors are the nonzero vectors in the subspace

$V_\lambda = \mathrm{Nul}(A - \lambda I)$: solutions of $(A - \lambda I)x = 0$.

L2. If A is a triangular matrix then its eigenvalues are exactly the diagonal entries!

Ex. $A = \begin{pmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 8 & 5 & 4 \end{pmatrix}$ eigenvalues are $2, -1, 4$.

Proof of L2.

$$A = \begin{bmatrix} a_1 & & * \\ & a_2 & \\ 0 & & \ddots \\ & & & a_n \end{bmatrix}, \quad \det(A-\lambda I)$$

$$\| $$

$$= \det \begin{bmatrix} a_1-\lambda & & * \\ & a_2-\lambda & \\ 0 & & \ddots \\ & & & a_n-\lambda \end{bmatrix}$$

$$\boxed{\text{eigenvalues of } A} = (a_1-\lambda)(a_2-\lambda)\cdots(a_n-\lambda)$$

$\|$

solutions: $a_1, a_2, \cdots, a_n$.

$\blacksquare$

A. Multiplicities of eigenvalues

If $\det(A-\lambda I) = (\lambda-2)^2 (\lambda+1)^5 (\lambda+7)^1$

then the eigenvalues 2, -1, -7

have multiplicities 2, 5, 1.

So one finds the multiplicities by factoring the characteristic equation!

B. Eigenvectors for different eigenvalues

Next:

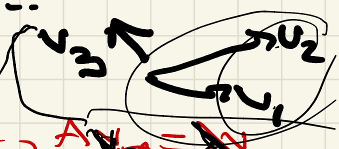
We will notice that the eigenvectors with different eigenvalues are linearly independent.

This means that they point in genuinely different directions.

This fact can be viewed as understanding some kind of geometry of the vector space V by using one linear transform T on V (that corresponds to a matrix A).

Geometry for A: different eigenvalues: different direction

12. If $v_1, \dots, v_p$ are eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_p$ which are all different then $v_1, \dots, v_n$ are linearly independent!



Pf. We have $Av_1 = \lambda_1 v_1, \dots, Av_p = \lambda_p v_p$.

- If $v_1, \dots, v_p$ are not independent we can $\boxed{\text{Choose } q}$ so that $v_1, \dots, v_q$ are linearly independent and $v_1, \dots, v_{q+1}$ are not: lin. dependent
- For a nontrivial combination $c_1 v_1 + \dots + c_{q+1} v_{q+1} = 0$, we find that $\boxed{c_{q+1} \neq 0}$: if $c_{q+1} = 0$ {$v_1, \dots, v_q$ are depend.}
- We solve for v_{q+1}: $v_{q+1} = \boxed{a_1 v_1 + \dots + a_q v_q}$ *
  $a_1 = -c_1/c_{q+1}$
- Apply A:
  $$\lambda_{q+1} v_{q+1} = \boxed{Av_{q+1}} = a_1 Av_1 + \dots + a_q Av_q$$
  $$= a_1 \lambda_1 v_1 + \dots + a_q \lambda_q v_q$$
- Compare with *
  $$\lambda_{q+1} v_{q+1} = \lambda_{q+1}(a_1 v_1 + \dots + a_q v_q)$$
- Subtract the two to get:
  $$0 = (\lambda_{q+1} - \lambda_1) a_1 v_1 + \dots + (\lambda_{q+1} - \lambda_q) a_q v_q \;!$$
- Since $v_1, \dots, v_q$ are lin. independent coefficient $(\lambda_{q+1} - \lambda_i) a_i$ are zero!

for $i=1,\dots,q$ we know that

So: $a_i(\lambda_{q+1}-\lambda_i)=0$,

but

$\lambda_{q+1}\neq\lambda_i$

hence $a_i=0$.

so we get that $v_{q+1}=\underbrace{a_1}_{=0}v_1+\dots+\underbrace{a_q}_{=0}v_q=0$.

However this is impossible since v_{q+1} is an eigenvector, so it is $\neq 0$.

- **Conclusion**: it is impossible that eigenvectors with different eigenvalues are linearly dependent!

$\square$

C. Similiar matrices behave similarly

Start with a matrix A.

If matrix P is invertible

then $\boxed{PAP^{-1}}$ is called the

$\boxed{\text{P-conjugate of A}}$ $\qquad A \longrightarrow PAP^{-1}$

We say that A, B are $\boxed{\text{similiar}}$ if

B is a conjugate of A.

Announcement of theorem 2:

$\boxed{\text{T2.}}$ If $B = PAP^{-1}$ then:

(a) $A v = \lambda v$ ($v \in V_\lambda^A$) implies that for

$\boxed{u = Pv}$ one has $Bu = \lambda u$. ($u \in V_\lambda^B$) !!!

(b) A, B have the same characteristic polynomial hence the same eigenvalues with the same multiplicities.

(a') P gives isomorphism of V_λ^A and V_λ^B of vector spaces, i.e. have

$V_\lambda^A = Nul(A - \lambda I)$ = v with $Av = \lambda v$

$V_\lambda^B = Nul(B - \lambda I)$ = u with $Bu = \lambda u$

$T_\lambda : V_\lambda^A \longrightarrow V_\lambda^B$ a linear transformation which is invertible.

Such T_λ is $T_\lambda(v) = Pv$.

Detailed Version of T2:

T2. (a) If $B = PAP^{-1}$ then one gets a linear transformation $\underline{T_\lambda : V_\lambda^A \longrightarrow V_\lambda^B}$, by

$$\boxed{T_\lambda(v) = Pv}$$

(a') T_λ is an isomorphism of vector spaces.

$\boxed{\text{Pf.}}$ (a) Need is check that if $\underbrace{v \in V_\lambda^A}$ then $\underbrace{Pv \in V_\lambda^B}$

If $Av = \lambda v$ then $B(Pv) = \lambda Pv$

but: $B(Pv) = \underline{PAP^{-1}}\,\underline{Pv}$

$= PAv = P(\lambda v)$

$= \lambda \cdot Pv.$

T_λ is linear; mult. by a matrix P.

(a') claims that T_λ is invertible.

This is true since its inverse comes from the inverse of the matrix P

$T_\lambda^{-1} u = P^{-1} u.$

$\square$

Main consequence of T2:

C3. If A, B $\boxed{\text{similar}}$ then the following are the same: (S2)

① characteristic polynomials

② eigenvalues & their multiplicities

Pf. ① $\det(B-\lambda I) \stackrel{?}{=} \det(A-\lambda I)$

Yes:

$$\det(B-\lambda I) = \det[PAP^{-1} - \lambda PIP^{-1}]$$

$$= \det[P(A-\lambda I)P^{-1}]$$

$$= \det(P) \cdot \det(A-\lambda I) \cdot \det(P^{-1})$$

$$= \det(A-\lambda I) \quad \det(P \cdot P^{-1}) = \det(I) = 1.$$

$$= \det(A-\lambda I).$$

② follows. $\square$

Remark:

Also: λ-Eigenspaces of A & B

are isomorphic, so they have the same dimension.