# Ch5. Eigenvectors and eigenvalues

## 5.1. • Finding eigenvalues

from characteristic equation

• Finding eigenvectors

from null spaces of matrices $A-\lambda I$

# Chapter 5. Eigenvalues and Eigenvectors

## §5.1 The same: " "

0. The "simplest" matrices are those of type

$$A = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a \end{bmatrix} \quad (A = aI_n)$$

the unspecified entries are ~~understood~~ to be 0 (!)

because $Ax$ is the a-multiple of $x$: $Ax = ax$.

1. The next best thing for a matrix $A$ is for us to know a vector $v \neq 0$ such that $Av$ is a multiple $\lambda v$ of $v$:

$v \neq 0$ & $Av = \lambda v$ ("lambda")

Then we say that

$v$ is an eigenvector of $A$ with eigenvalue $\lambda$. $\boxed{v \neq 0,\ Av = \lambda v}$ for eigenvector $v$

E1. $Av = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6-30 \\ 30-10 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} =$

$A=$, $v=$

$Av$ happens to be $-4v = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix}$.

So: $v = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is an eigenvector of with the eigenvalue $-4$.

E2. $Au = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$ is not a multiple of $u = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. So: $u$ is not an eigenvector of $A$ for any $\lambda$.

Let $A$ be a square matrix of size $n$.

For a given number $\lambda$ denote by $V_\lambda$ the set of all vectors $v \in \mathbb{R}^n$ such that

$$Av = \lambda v.$$

$V_\lambda$ = all eigenvectors with eigenvalue $\lambda$

L1. $V_\lambda$ is a subspace of $V$.

("the eigenspace of $A$ for $\lambda$").

Pf. • For $v_1, v_2 \in V_\lambda$ we want $v_1 + v_2 \in V_\lambda$ (?)

$Av_1 = \lambda v_1$, $Av_2 = \lambda v_2$ $\xrightarrow{\text{add}}$ $A(v_1+v_2) = \lambda(v_1+v_2)$

• For $v \in V_\lambda$ and $c \in \mathbb{R}$ we want $cv \in V_\lambda$.

$Av = \lambda v$, but then: $A(cv) = c(Av) = c \cdot \lambda v = \lambda \cdot cv$ ✓.

• We want $0 \in V_\lambda$ (?)

means $A0 = \lambda \cdot 0$ (?), both have 0 i.e. $0 = 0$ □

We say that a number $\lambda$ is an eigenvalue of $A$ if $V_\lambda \neq \{0\}$

i.e. there exists $v \neq 0$ with $Av = \lambda v$.

if $A$ has eigenvector with eigenvalue $\lambda$

L2. Number $\lambda$ is an eigenvalue of $A$ iff

$$\det(A - \lambda I_n) = 0.$$

Pf (1) $\boxed{Av = \lambda v}$ is the same as $\boxed{(A-\lambda I_n)v = 0}$

since: $\underbrace{(A - \lambda I_n)v}_{0} \overset{v \in V_\lambda}{=} Av - \lambda I_n v = Av - \lambda v$ i.e. $Av = \lambda v$

i.e. $V_\lambda = \mathrm{Nul}(A - \lambda I)$.

② $\boxed{\det(A-\lambda I)=0}$ iff

$\boxed{A-\lambda I}$ is not invertible

which is equivalent to

$Nul(A-\lambda I) \neq \{0\}$

invertible $\Leftrightarrow$ Nul $=0$

i.e. to $V_\lambda \neq \{0\}$ which we state as: $\boxed{\lambda \text{ is an eigenvalue of } A.}$ ✓

So, eigenvalues of A are the solutions of $\boxed{\det(A-\lambda I)=0}$

$\boxed{\text{the characteristic equation of } A.}$

Ex. $A=\begin{bmatrix}1 & 6\\ 5 & 2\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}1-\lambda & 6\\ 5 & 2-\lambda\end{bmatrix}$

$\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$

$\boxed{\det(A-\lambda I)} = (1-\lambda)(2-\lambda)-6\cdot 5=$

$\overset{||}{0}$

$=\lambda^2-2\lambda-\cancel{\lambda}+2-30$

$=\lambda^2-3\lambda-28=(\lambda+4)(\lambda-7)$

$\boxed{\text{Key}}$: factor out to find eigenvalues: $\lambda_1=-4$, $\lambda_2=7$.

Now that we know the eigenvalues what are the corresponding eigenspaces?

For $\lambda_1 = -4$, $A - \overset{-4}{\lambda} I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - (-4)\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}$

equations:

$\boxed{5x_1 + 6x_2 = 0}$,

$x_1 = -\frac{6}{5} x_2$

$x_2 = x_2$

known → span $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6/5 \\ 1 \end{bmatrix} x_2$

So: $V_{-4} = \text{span}\begin{bmatrix} -6/5 \\ 1 \end{bmatrix} = \text{span}\begin{bmatrix} -6 \\ 5 \end{bmatrix}$.

$\boxed{\text{For } \lambda_2 = 7}$: $A - 7I_2 = \begin{bmatrix} 1-7 & 6 \\ 5 & 2-7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$

reduce to $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

and REF: $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

equations: $-6x_1$

↓

$x_1 - x_2 = 0$

i.e. $x_1 = x_2$

$x_2 = x_2$,

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So: $V_7 = \text{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

we have found all eigenvectors!

Actually, ~~ex~~ there is a basis of eigenvectors

$v_1 = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.