4.5. Rank of a matrix

4.6. Rank(A)

$A \in M_{m \times n}$

This is $\dim[\mathrm{Col}(A)] \overset{\text{def}}{=}$ # pivots

- $\mathrm{Col}(A) \subseteq \mathbb{R}^m$
- $\mathrm{Row}(A) \overset{\text{def}}{=}$ span of all rows of $A \subseteq \mathbb{R}^n$

Calculate Row(A)

Th (a) If A, B are row equivalent then $\mathrm{Row}(A) = \mathrm{Row}(B)$

(b) If B is REF of A then a ~~nonzero element~~ basis of $\mathrm{Row}(A)$ is given by $\neq 0$ rows of B.

Pf. ① Such B is obtained from A by elementary row operations. Then the rows of B are linear combinations of rows of A.

So:

rows of $B \subseteq$ Span (rows of A). Hence:

$$\underbrace{\mathrm{span}\{\text{rows of } B\}}_{=\mathrm{Row}(B)} \subseteq \underbrace{\mathrm{span}\{\text{rows of } A\}}_{\mathrm{Row}(A)}.$$

② But one can also recover A from B by elementary operations

So: $\mathrm{Row}(B) \supseteq \mathrm{Row}(A)$, hence

③ they are equal. $\square$

(b) Let B be the REF of A. Then by (a) we know that:

$\text{Row}(A) = \text{Row}(B)$ → which is the span of $\neq 0$ rows of B.

$$\underbrace{\begin{array}{c}\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}\end{array}\begin{array}{l} R_1 \\ R_2 \\ \\ R_3 \end{array}}_{\text{picture of } B}$$

Call these rows $R_1, R_2, \ldots, R_p$

From the picture we can see that these rows are independent: (due to "staircase" form of REF!)

In a linear combination $C = c_1R_1 + \cdots + c_pR_p$ the 1st entry of C is c_1.

So, if $C = 0$ then $c_1 = 0$.

This can be repeated: now $C = [0 \cdots 0\ c_2\ * \cdots *]$, hence $c_2 = 0$.

Etc.

Now we know that the $\neq 0$ rows of B span $\text{Row}(A)$, and they are linearly independent.

So, they are a basis of $\text{Row}(A)$! $\square$

Recall
Row space has a basis by $\neq 0$ rows of REF

So:
L2. $\dim[\mathrm{Row}(A)] = \#$ pivots $= \dim[\mathrm{Col}(A)]$.
$\mathrm{rank}(A) = \dim[\mathrm{Row}(A)] = \dim[\mathrm{Col}(A)]$. $\square$

Example. For a matrix A we will find bases of Row(A) and Col(A).
This will show that **rank(A) = 2**.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

REF of A. Hence:

- A basis of Col(A) $= \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$.
  "pivotal columns"
- A basis of Row(A) $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.
  "the $\neq 0$ rows of REF"

Recollections from the Last Time:

Tuesday April 14.

From 4.5: $\boxed{\text{Bases}}$ of vect spaces.

T1 (a) Any spanning set $\mathcal{S}$ contains a basis $\mathcal{B}$.

(b) Any independent set $\mathcal{I}$ can be completed to a basis $\mathcal{B}$; provided that V is finite dimensional i.e. that it has some finite spanning set.

T2. Any two bases of V have the same size (called the dimension of V).

T3. If $\dim(V) = n$ for a subset $\mathcal{A}$ of size n, the following

- $\mathcal{A}$ is a basis
- $\mathcal{A}$ is linearly indep.
- $\mathcal{A}$ is a spanning set

are equivalent.

From 4.6. : Row space Row(A) & Rank

For a matrix A of type $m \times n$

- $\mathrm{Col}(A) = \mathrm{span}(\text{columns of } A) \subseteq \mathbb{R}^m$
- $\mathrm{Row}(A) = \mathrm{span}(\text{rows of } A) \subseteq \mathbb{R}^n$

T1 (a) $\dim[\mathrm{Col}(A)] = \dim[\mathrm{Row}(A)]$
$= \#$ of pivots.

[This # is called rank of A = rank(A).]

(b) "$\mathrm{Col}(A)$ has a basis by pivotal columns of A",

(c) "$\mathrm{Row}(A)$ has a basis by $\neq 0$ rows of REF of A"

New: Consider a matrix A of type $m \times n$

T2. [Rank Theorem.]

$$\boxed{\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = \dim(\mathbb{R}^n) = n}$$

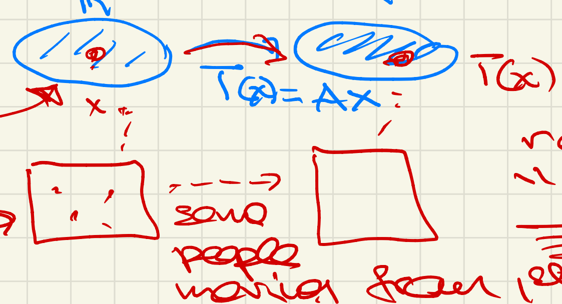
Pf.

$\dim[\mathrm{Col}(A)]$ = # of pivots + # of free variables = n = # of columns of A $\square$

Meaning

n = dim of source = size

$\mathbb{R}^n$ $\qquad$ $\mathbb{R}^m$



reasons are "bases"

T3. [Invertible matrix theorem]

For a square matrix A of type $n \times n$ the following is equivalent:

(1) A is $\boxed{\text{invertible}}$

(2) $\text{rank}(A) = n$

(3) $\dim \text{Nul}(A) = 0$ ie. $\text{Nul}(A) = \{0\}$.

Pf. (2) $\Leftrightarrow$ (3) ("equivalent") because

$\text{rank}(A) + \dim \text{Nul}(A) = n$ (!)

(!) We also know that A is invertible iff

(2') columns of A span $\mathbb{R}^n$

(3') $Ax = 0$ has only the trivial solution $x = 0$

So, we know (1) $\Leftrightarrow$ (2') $\Leftrightarrow$ (3'). $\text{Nul}(A) = \{0\}$ solutions

but:

$\Updownarrow$ $\Updownarrow$

(2) (3). $\dim \text{Nul}(A) = 0$

$\text{rank}(A) = n$

$\underbrace{\dim [\text{Col}(A)]}$

span of columns

(2) says that for $\boxed{\text{Col}(A)} \subseteq \boxed{\mathbb{R}^n}$ the dimension is $n = \dim(\mathbb{R}^n)$.

Since the dimensions are the same: $\boxed{\text{Col}(A) = \mathbb{R}^n}$ $\square$

rank(A)
"

dim (Col(A)) = " # of people who moved from room 1 to room 2"

"basis $e_1 \to e_n$ of $\mathbb{R}^n$"

e_i is moved to

$T(e_i)$ = i^{th} column of A

dim [Nul(A)] = " # of people who stayed in 1st room"

$T(x) = 0$

i.e.

$Ax = 0$

Intuitive picture:

Long intermission:

Lemma. If U is a subspace of V

(a) then $\dim(U) \leq \dim(V)$

(b) $\dim(U) = \dim(V)$ iff $U = V$.

Proof (a) $\dim(U)$ = size of some basis $\mathcal{A}$ of U.

Now (a) $\mathcal{A} \subseteq U \subseteq V$

(b) $\mathcal{A}$ is lin. ind.

basis! of U

Now $\mathcal{A}$ can be completed to a basis B of V. $\mathcal{A} \cup R = B$

then (a) $\underset{\dim U}{\text{size}(\mathcal{A})} \leq \underset{\dim V}{\text{size}(B)}$

(b) if $U = V \Rightarrow \dim U = \dim V$.

On the other hand if

$\dim U = \dim V$

$\text{size } \mathcal{A} \quad \text{size } B$

so: $R = \emptyset$ & $\mathcal{A} = B$

$U = \text{span}(\mathcal{A}) = \text{span}(B) = V$ $\circ$

2nd intermission

basis of $\{0\}$ is $\emptyset$

$\mathcal{L}$. a) $\dim(\{0\}) = 0$

b) $\dim(V) = 0 \Rightarrow V = \{0\}$.

Pf. a) $V = \{0\}$ a basis

$\mathcal{B}$ of $\{0\}$ is

$\mathcal{B} = \emptyset$

$\dim\{0\} = \text{size } \emptyset = 0$.

(b) $\dim(V) = 0$ then every $v \in V$ is 0. If not, if $v \neq 0$ then $S = \{v\}$. It would be independent! Then $\dim V \geq \text{size}(S) = 1$.

$\dim V$ = size of basis

Contradict $\dim V = 0$.

$\square$