4.5. Dimension of a vector space

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L.Q. For subsets $\mathcal{S} \subseteq \mathcal{T}$ of V,

(a) $\text{span}(\mathcal{S}) \subseteq \text{span}(\mathcal{T})$.

(b) If $\mathcal{T} \subseteq \text{span}(\mathcal{S})$ then $\text{span}(\mathcal{S}) = \text{span}(\mathcal{T})$.

Proof. (a) $\mathcal{S} \cup \mathcal{R} = \mathcal{T}$ and then ……

(b) $\mathcal{R} = \{u_1, \dots, u_r\}$, $\mathcal{S} = \{v_1, \dots, v_p\}$

and $\sum$ $u_i = c_1^i v_1 + \dots + c_p^i v_p \in \text{span}(\mathcal{S})$

then any linear combination x of $\mathcal{T}$

is $x = \sum_{i=1}^{p} a_i v_i + \sum_{j=1}^{r} b_j u_j$, $u_j \in \mathcal{R}$

$$= \sum_{i=1}^{p} a_i v_i + \sum_{j=1}^{r} b_j \sum_{i=1}^{p} c_i^j v_i \in \text{span}(\mathcal{S})$$

Replacement in a spanning set:

$\mathcal{S}$ $\quad \mathcal{R}\ (v_p)$ $\quad$ 340

L1. If $\mathcal{S} = \{v_1, \ldots, v_p\} \subseteq V$ is a spanning set and u is a vector in V such that

$$u = c_1 v_1 + \cdots + c_p v_p = \sum_{i=1}^{p} c_i v_i$$

& $\boxed{c_p \neq 0}$ $\quad$ sum $\quad (\mathcal{R} \setminus u) = \mathcal{S}_1$

then $\mathcal{S}' = \{v_1, \ldots, v_{p-1}, u\}$ is also a spanning set!

Pf. For each $v \in V$:

$$v = a_1 v_1 + \cdots + a_{p-1} v_{p-1} + a_p v_p$$

solve for v_p

$$v_p = \frac{1}{c_p} u - \frac{c_1 v}{c_p} - \frac{c_2 v_2}{c_p} - \cdots - \frac{c_{p-1} v_{p-1}}{c_p}$$

$$v = a_1 v_1 + \cdots + a_{p-1} v_{p-1}$$

$$+ a_p \left(-\frac{c_1}{c_p} v_1 + \cdots - \frac{c_{p-1}}{c_p} v_{p-1} + \frac{1}{c_p} u \right)$$

$$= \left(a_1 - \frac{c_1 a_p}{c_p}\right) v_1 + (\quad) v_2 + \cdots + (\quad) v_{p-1} + \frac{a_p}{c_p}$$

more replace:

Proposition 2. Let $\mathcal{I} = \{u_1, \ldots, u_q\} \subseteq V$ be linearly independent. Then (in) any spanning set $\mathcal{S}$ q elements can be replaced by $\mathcal{I}$, so that one still has a spanning set.

In other words, $\mathcal{S}$ can be ordered as $\{v_1, \ldots, v_q, v_{q+1}, \ldots, v_n\} = \mathcal{S}$, so that $\{u_1, \ldots, u_q, v_{q+1}, \ldots, v_n\}$ is still a spanning set.

$\{v_1\} \amalg \mathcal{R}_1 = \mathcal{S}$

Proof. $\boxed{\text{Step 1.}}$ We can write u_1 as a linear combination of elements of $\mathcal{S}$. Since $u_1 \neq 0$ there is a vector $v_1 \in \mathcal{S}$ whose coefficient is $\neq 0$. Let $\mathcal{R}_1$ be the remaining vectors in $\mathcal{S}$.

Then by Lemma 1, $\mathcal{S}_1 = \{u_1\} \amalg \mathcal{R}_1$ is still a spanning set.

replace v_1 by u_1

So, we have replaced one element (v_1) of $\mathcal{S}$ by u_1 & we get $\mathcal{S}_1$ which is still a spanning set.

$\mathcal{S}_1 = $ $u_2 / \mathcal{R}_1$

Step 2. Now write u_2 as a linear combination of elements of $\mathcal{S}_1$. Then there is a vector v_2 in $\mathcal{R}_1$ with a non-zero coefficient.

(If the coefficients of all elements of $\mathcal{R}_1$ were zero, then u_2 would be a linear combination of u_1. Then $\mathcal{S}$ would be lin. dependent!)

Now, denote $\mathcal{R}_2 = \mathcal{R}_1 \setminus \{v_2\}$, $\mathcal{R}_1 = \{v_2\} \cup \mathcal{R}_2$

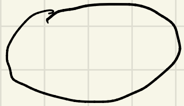
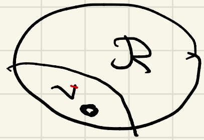
By lemma 1: $\mathcal{S}_2 = \{u_1, u_2\} \sqcup \mathcal{R}_2$ is still a spanning set!

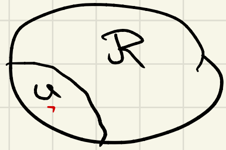
$u_2 | \mathcal{R}$

So, we have replaced 2 vectors in $\mathcal{S}$ by u_1, u_2 & gotten a new spanning set $\mathcal{S}_2$.

Steps 3, 4, ..., 9: repeat the above procedure and replace q elements of $\mathcal{S}$ with $u_1, \ldots, u_q$ to get a spanning set $\mathcal{S}_q$.

Scheme of the proof:

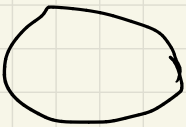
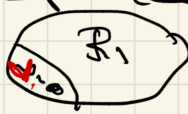
1. 
$\mathcal{S}$ & $u \neq 0$: divide $\mathcal{S}$: 

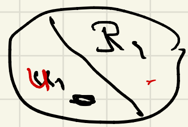
so that $\mathcal{S}_1 =$ 

is a spanning set.

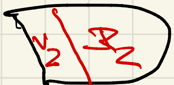
$R =$

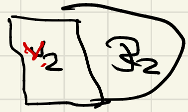
2. Start with $u_1 \in \mathcal{S}$ & do replacement

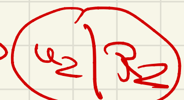
$\mathcal{S} =$  $=$ 

to $\mathcal{S}_1 =$ 

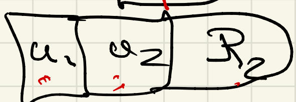
then u_2 leads to

$R_1 =$ 

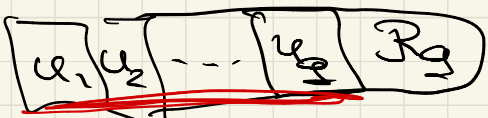
$=$ 

$\longrightarrow$ 

is a spanning set.

and $\mathcal{S}_2 =$ 

⋮

$\mathcal{S}_g =$ 

is a spanning set.

T3. If $S = \{v_1, \dots, v_p\}$ is spanning in V & $\mathcal{U} = \{u_1, \dots, u_q\}$ is linearly independent then $p \geq q$. Pf. $S \ni v_1, \dots, v_q$ that can be replaced by $\mathcal{U}$, so: $p \geq q$.

T4. If $B_1 = \{v_1, \dots, v_p\}$ & $B_2 = \{u_1, \dots, u_q\}$ are two bases of V then $\boxed{p = q}$.

R. So, $\dim(V)$ = size of any basis. is well defined.

Proof of T4: T3 says $p \geq q$

B_1 is spanning, B_2 is ind.

So, also $q \geq p$. Hence $p = q$!!

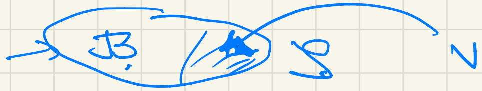
T4: v. spaces $\xrightarrow{}$ finite sets

$V \mapsto B$

$\dim(V)$ size B.

T0. [Spanning set theorem]

Any spanning set S for V contains a subset B which is a basis.



Pf. Let B be any maximal lin. ind. subset of S.

$= \{b_1, \dots, b_d\}$

If $v \in S$ but $v \notin B$, then $v \in \mathrm{Span}(B)$:

① $\underbrace{\{b_1, \dots, b_d}_{B}, v\}$ is not lin. ind!

② So, $c_1 b_1 + \dots + c_d b_d + cv = 0$ for some nontrivial lin. comb.

③ Then $c \neq 0$: otherwise B would be lin. dependent!

④ So, $v = -\frac{c_1}{c} b_1 + \dots - \frac{c_d}{c} b_d$ is in $\mathrm{Span}(B)$.

⑤ Now $S \subseteq \mathrm{span}(B)$ hence $\mathrm{span}(S) \subseteq \mathrm{span}(B)$ by L0.

⑦ ~~This is equality since $S \supseteq B$.~~

More consequences for finite dimensional V:

We say that V is finite dimensional if it has a finite spanning set $\mathcal{S}$.

- Then V has a finite basis $\mathcal{B}$, so $\dim(V)$ is some finite number!

R. If $\mathcal{S} \subseteq V$ is lin. ind. its size is $\leq \dim(V)$!

$\boxed{T5.}$ If V is finite dimensional then any linearly independent subset $\mathcal{S}$ can be completed to a basis $\mathcal{B} \supseteq \mathcal{S}$.

$\mathcal{B}$

$\mathcal{S}$

Pf. Step 1: If $\mathcal{S} = \{u_1, \dots, u_q\}$ does not span V then choose $v \in V$ which is not in span($\mathcal{S}$).

Then $\boxed{\mathcal{S}_1 = \mathcal{S} \cup \{v\}}$ is $\boxed{\text{lin. independent}}$:

If $c_1 u_1 + \dots + c_q u_q + cv = 0$ {lin. comb. for $\mathcal{S}_1$}

① then $\boxed{c = 0}$; ($c \neq 0$ then $v \in \text{Span}(\mathcal{S})$). $\mathcal{S} \cup \{v\} = \mathcal{S}_1$

② but then also $\boxed{c_1 = \dots = c_q = 0}$: ($\mathcal{S}$ l.i.)

Step 2. Repeat (S1) $\mathcal{S} \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \dots \subseteq \mathcal{S}_s$.

by constructing larger independent set

If $\mathcal{S}_s$ is a basis: DONE.

But one of $\mathcal{S}_s$ has to be a basis...

Can repeat: as long as $\mathcal{S}_p$ do not span V.

$$\mathcal{G} \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \cdots \subseteq \mathcal{G}_p \subsetneq \mathcal{G}_{p+1}$$

if $\mathcal{G}_p$ does not span

Can the procedure run to many times?

$\boxed{\text{NO}}$: $\text{size}(\mathcal{G}_p) = \text{size}(\mathcal{G}) + p$ has to be $\underline{\le \dim(V)}_{1.2}$

$= \text{size}(\mathcal{B})$

basis

is spanning

$\mathcal{G}$ ind.

So at some $\mathcal{G}_p$ proc. stops i.e. $\mathcal{G}_p$ spans V.

But then $\mathcal{G}_p$ is a basis: ind, span!

$$\mathcal{G}_p \supseteq \mathcal{G}$$

$\square$

Basis theorem: (Last)

T6. Let $\dim(V) = n$.

a) If $\boxed{\text{independent}}$ & $\boxed{\text{size } n}$ $\Rightarrow$ basis!

b) If $\boxed{\text{spanning}}$ & $\boxed{\text{size } n}$ $\Rightarrow$ basis!

Proof. (a) $\mathcal{Y}$ ind. T5 $\Rightarrow$ $\mathcal{Y} \subseteq B$ basis!

but: $|\mathcal{Y}| = n$ (size) $|B| = \dim(V) = n$

Then $\boxed{\mathcal{Y} = B}$.

(b) $\mathcal{S}$ spanning, some subset B is a basis. Now $|\mathcal{S}| = \dim(V) = |B|$ & $\mathcal{S} \supseteq B$, so $\mathcal{S} = B$.

Q. if $\mathcal{Y} \subseteq V$ is ind. is it a basis?

useful: if size is correct then suffices to prove one of ind & span (!)

Memorize All theorems