4.4. Coordinates on vector spaces

$\boxed{4.26}$

§4.4. Basis of V gives coordinates on the vector space V

Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a $\boxed{\text{basis}}$ of V.

Th. For each vector $v \in V$ there are unique numbers $c_1, \dots, c_n$ such that

$$v = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$$

unique LinComb presentation.

Pf. $v \in V = \mathrm{Span}(b_1, \dots, b_n)$

so v is a lin. comb. of $\mathcal{B}$ i.e.

$$v = c_1 b_1 + \cdots + c_n b_n.$$

If $v = d_1 b_1 + \cdots + d_n b_n$

then : want $c_1 = d_1, \dots, c_n = d_n$.

$$0 = v - v = (c_1 - d_1) b_1 + (c_2 - d_2) b_2 + \cdots + (c_n - d_n) b_n$$

Since $\mathcal{B}$ a basis, it is lin. ind. so the coeffs must be zeros. So: $c_1 - d_1 = 0, \dots$

$c_1 = d_1, c_2 = d_2, \dots$

$\boxed{4.27.}$

Use: for v form $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [v]_{\mathcal{B}}$

using $v = c_1 b_1 + \dots + c_n b_n$

the coordinate vector for v in basis $\mathcal{B}$ (!)

Ex. $V = \mathbb{R}^2$, $b_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $b_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Then:

- $\mathcal{B} = \{b_1, b_2\}$ is a basis!! of $\mathbb{R}^2$:

Proof

Any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ is $c_1 b_1 + c_2 b_2$ — a spanning set

$\mathrm{Span}(b_2) = \{c b_2 ; c \in \mathbb{R}\}$

- lin. ind.

b_1 not a lin. comb. of b_2

b_2 — " — b_1

b_1, b_2 are not multiples

Obvious ✓

- **For the basis $\mathcal{B}$ we ask questions:**

Q1. If $[u]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, **what is u?**

$u = -1 \cdot b_1 + 3 b_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

Q2. **If** $v = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $[v]_{\mathcal{B}} = ?$

what are the $\mathcal{B}$-coordinates of v?

$b_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ 4.2.8

To find coordinates:

$[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ s.t. $v = c_1 b_1 + c_2 b_2$

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} c_1 \cdot 1 + c_2 2 \\ c_1 \cdot 2 + c_2 \cdot 3 \end{bmatrix}$$

?

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

!

matrix whose columns are vectors in $\mathcal{B}$

$$[b_1\, b_2]\, [v]_{\mathcal{B}} = v$$

$$P_{\mathcal{B}} \cdot [v]_{\mathcal{B}} = v \qquad \boxed{[v]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \cdot v}$$

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad P_{\mathcal{B}}^{-1} = \frac{1}{-1} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$[v]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \cdot v = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

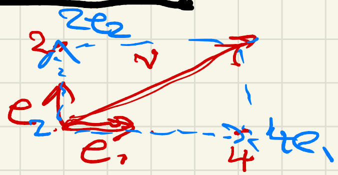
$$\begin{bmatrix} 1 \\ 5 \end{bmatrix} = v = 7 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

If $\mathcal{B}$ is a basis of $\mathbb{R}^n$ then

1. One finds the coordinate vector $[v]_{\mathcal{B}}$ of v as the solution of $P_{\mathcal{B}} x = v$.

Pf. Calculation is as in the above example!

Geometrical meaning of coordinates



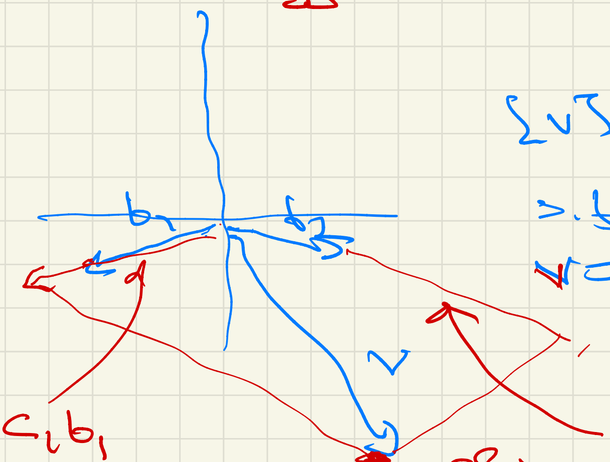
$[v]_{\mathcal{B}} = \begin{bmatrix}4\\2\end{bmatrix} = v$

$\mathcal{B} = \{e_1, e_2\}$ $\qquad v_1 = \begin{bmatrix}4\\2\end{bmatrix} \Leftrightarrow v_1 = 4e_1 + 2e_2$

$v = \begin{bmatrix}4\\2\end{bmatrix}$, $[v]_{\mathcal{B}} = v = \begin{bmatrix}4\\2\end{bmatrix}$

R. $\mathcal{B} = \{e_1, \dots, e_n\}$ in $\mathbb{R}^n$

$[v]_{\mathcal{B}} = v$!!!



$[v]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix}$

s.t.

$v = c_1 b_1 + c_2 b_2$

parallelogram

$\mathcal{B}$-coordinates of v describe how to decompose v into contributions in directions of vectors in $\mathcal{B}$

Last Time:

From 4.2: • Matrix $A \in M_{mn}$ produces subspace

$Nul(A) \subseteq \mathbb{R}^n$, $Col(A) \subseteq \mathbb{R}^m$

• Linear transform $T: U \to V$ produces

$Ker(T) \subseteq U$, $Ran(T) \subseteq V$

When $T(x) = Ax$: $Ker(T) = Nul(A)$

$Ran(T) = Col(A)$

From 4.3 $v_1, \dots, v_n$ in V: is a

• spanning set: $Span(v_1, \dots, v_n) = V$ describes V

• linearly independent:

• basis of V: both.

maximal spanning set

$\dim(V) = n$ if V has a basis of size n

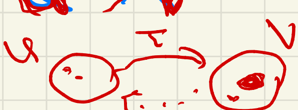
Eg. $e_1, \dots, e_n$ is a basis of $\mathbb{R}^n$, $\dim(\mathbb{R}^n) = n$.

New: The inverse of a linear transform $T: U \to V$ is a linear transform $S: V \to U$ such that $S(T(u)) = u$, $u \in U$

$T(S(v)) = v$, $v \in V$.

Inverse: T^{-1}

L1. Linear Transform $T: U \to V$ has an inverse iff it is a 1-1 correspondence i.e., T is onto & one-to-one.



Then $T^{-1}(v) = \{$unique $u \in U$ such that $T(u) = v\}$

$v \in V$

Ex. If matrix A $(n\times n)$ has inverse A^{-1} then the corresponding linear transformation $T(x) = Ax$ has inverse T^{-1} given by $T^{-1}(y) = A^{-1}y$.

So: $A \in M_{n}$ is invertible iff $T(x) = Ax$ is invertible.

R0. An Invertible Linear Transform $T: U \to V$ is also called An isomorphism of vector spaces $T: U \to V$ means: " the same shape "

R1. The idea is that T allows us to match what is happening in U & V. Say:

$\boxed{\text{L0. } T(u) = 0 \text{ iff } u = 0.}$

L1. Let $T: U \to V$ be an invertible linear transform.

Then for $u_1, \dots, u_p$ in U:

(a) $u_1, \dots, u_p$ is Lin Ind iff $T(u_1), \dots, T(u_p)$ is LI

(b) $u_1, \dots, u_p$ spans U iff $T(u_1), \dots, T(u_p)$ spans V

(c) The same for $\boxed{\text{basis}}$

Proof of LI. a

u_i are ind iff $T(u_i)$ are ind!

a) If $u = c_1u_1 + \dots + c_pu_p$.

then T $T(u) = c_1 T(u_1) + \dots + c_p T(u_p)$

use 2LO

Now: linear combination $c_1u_1 + \dots + c_pu_p$ is zero if and only if the combination $c_1T(u_1) + \dots + c_pT(u_p)$ is zero!

Proof of LO. • If $\boxed{u = 0}$ then we know $T(u) = T(0) = 0$.

• If $\boxed{T(u) = 0}$ then

apply T^{-1}

$u = T^{-1}(T(u)) = T^{-1}0 = 0$. So, $u = 0$. $\square$

Last Time: If $A \in M_{m \times n}$ then

$\boxed{T2}$. (a) A basis of $\boxed{Col(A)}$ is given by $\boxed{\text{pivotal columns}}$

b) A basis of $Nul(A)$ comes from free variables $y_1, \dots, y_q$

If the solution is $\Big\}$ $\boxed{x} = y_1 v_1 + \dots + y_q v_q$

Then $v_1, \dots, v_q$ is a basis of ~~Nul(A)~~ Nul(A).

Proof: inv. lin. tr.

Recollections

From 4.4: A basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ of V

gives coordinates of any vector

v : $[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

coordinate vector

$\mathcal{B} = \{v_1, \ldots, v_n\}$

where: $v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$

Finding coordinates

If $V = \mathbb{R}^n$, $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis

$[v]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} v$

$P_{\mathcal{B}} = [b_1 \ldots b_n]$ invertible

$\Leftarrow$ thus

$P_{\mathcal{B}} \cdot [v]_{\mathcal{B}} = v$ or $[v]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} v$.

Ex. $\mathbb{P}$ = polynomials in T (x)

It is a vector space

$= a\ell\ell\ p_0 + p_1 T + p_2 T^2 \ldots + p_n T^n$

$p_i \in \mathbb{R}$.

Let $\mathbb{P}_n = \mathrm{Span}\{1, T, \ldots, T^n\}$, $1 = T^0$

$=$ polynomials of degree $\leq n$.

L.I. $B = \{1, t, \ldots, t^n\}$ is basis of P_n

Pf. Span$(B) = P_n$ is just the def. of P_n

Why is B linearly indep.?

Consider lin. comb. which is zero.

$$\boxed{c_0 + c_1 t + c_2 t^2 + \cdots} + c_n t^n = 0$$

$= 0$

Diff: $0 = 0' = c_1 + 2c_2 t + \cdots + n c_n t^{n-1}$

Diff: $0 = 2c_2 + \cdots + (n-1) n c_n t^{n-2}$

⋮

n times: $0 = c_n\, n(n-1)(n-2)\cdots$ $\quad t^0 = 1$

$0 = c_n\, n!$

$0 = c_n$

$n-1$ times: $c_{n-1} = 0$

⋮

all $c_i = 0$. $\square$

R. If $p_1, \dots, p_k$ are k polynomials in $\mathbb{P}_n$

$$p_1 = c_{01}T^0 + c_{11}T^1 + \dots + c_{n1}T^n$$

$$p_2 = c_{02}T^0 + c_{12}T^1 + \dots + c_{n2}T^n$$

$$\vdots$$

$$p_k = \quad \vdots$$

the coefficients of a polynomial

$$\underbrace{p} = c_0 T^0 + c_1 T + \dots + c_n T^n$$

$$\begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = [p]_{\mathcal{B}} \qquad \mathcal{B} = \{T^0, T^1, \dots, T^n\}$$

are just the coord. vector of p in basis

Reducing questions to coordinates:

$p_1, \dots, p_k$ are linearly ind. in $\mathbb{P}_n$ iff the coeff. vectors $[p_1]_{\mathcal{B}}, \dots, [p_k]_{\mathcal{B}}$ (in $\mathbb{R}^{n+1}$) are lin. ind!

Good: reduces computations from $\mathbb{P}_n$ to $\mathbb{R}^{n+1}$. True?

It will follow from:

L3. If $\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis of a space V then:

The $\left\{\begin{matrix}\text{coordinate} \\ \text{map}\end{matrix}\right\}$ $V \xrightarrow{[\;]_B} \mathbb{R}^n$

is an $\boxed{\text{isom. of v. spaces}}$ ✓

i.e. the map is

(1) linear &

(2) it has inverse.

$\boxed{\text{Pf.}}$ (1) If $[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $[u] = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

then

$$[v+u]_B \overset{?}{=} [v]_B + [u]_B = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix}$$

since: $\underline{v+u} = \underbrace{a_1v_1 + \cdots + a_nv_n}_{v} + \underbrace{b_1v_1 + \cdots + b_nv_n}_{u}$

$$= (a_1+b_1)v_1 + \cdots + (a_n+b_n)v_n$$

(2) $T : V \longrightarrow \mathbb{R}^n$ $\quad \boxed{T = [\;]_B}$

$T^{-1} : \mathbb{R}^n \longrightarrow V$

$$T^{-1}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \overset{T^{-1}}{=} a_1v_1 + \cdots + a_nv_n \in V.$$

$$\begin{matrix} \| & & \| \\ [v]_B & \xleftarrow{T} & v \end{matrix}$$

∎

coord map is an isom. of vector spaces $[\ ]_B : V \to \mathbb{R}^n$

but then deciding whether

$v_1, \dots, v_k$ is a $\left\{\begin{array}{l}\text{independent}\\ \text{spanning}\\ \text{basis}\end{array}\right\}$ in V

$\Updownarrow$

$[v_1]_B, \dots, [v_k]_B$ is a $\left\{\begin{array}{l}\text{i}\\ \text{s}\\ \text{b}\end{array}\right\}$ in $\mathbb{R}^n$.

Ex. $V = P_n$

$B = \{1, T, \dots, T^n\}$.