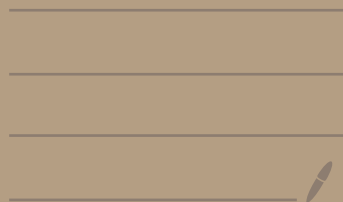


4.3. Bases of vector spaces

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§4.3. Bases of Vector spaces

A set v_1, \dots, v_p of vectors in V is said to be:

4.2.4

(1) a spanning set of V if $\text{Span}(v_1, \dots, v_p) = V$

(2) linearly independent if the ~~only~~ linear combination $c_1 v_1 + \dots + c_p v_p$ which is zero is the trivial one i.e. with $c_1 = c_2 = \dots = c_p = 0$

Equivalently, none of v_i is a linear combination of others

(3) a basis of V if both spanning set & linearly independent

Ex. $V = \mathbb{R}^n$ has standard basis e_1, \dots, e_n

① is a basis ✓ ② we call it standard

The dimension of V is the # of elements of any basis $B = \{b_1, \dots, b_n\}$.

$$\text{dim}(V) = n$$

measures size or complexity of V

if it has a basis with n elements.

$$\text{Ex. } \text{dim}(\mathbb{R}^n) = n$$

$$1 = \text{dim}(\mathbb{R}^1) = \text{dim}(\text{line}) \quad \checkmark$$

$$2 = \text{dim}(\mathbb{R}^2) = \text{dim}(\text{plane})$$

$$3 = \text{dim}(\mathbb{R}^3) = \text{dim}(\text{space})$$

4.2.5 Solve $Ax=0$

TL: let $A \in M_{m \times n}$

Via REF

a) $\dim \{ \text{Col}(A) \} = \# \text{ pivots in } A$

b) $\dim \{ \text{Nul}(A) \} = \# \text{ of free variables}$

Stronger claim:

a) A basis of Col(A) is given by pivotal columns in A.

b) A basis of Nul(A) is found from free variables, these are vectors v_i in solution: $x = x_2 v_1 + x_3 v_2 + \dots + x_n v_{n-1}$

Pf: 1) minimal code

Start $k=1$ $\text{Nul}_k(A) = \text{solutions}$

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ REF $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ free

$x_1 = 0$
 $0 = 0$

basis of Col(A)

all solutions = Nul(A)

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Col(A) = Span(pivotal columns)
Nul(A) = Span(v_1)

x_2 -multiples of $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_1$

free variable \uparrow basis of null space

$B = \{v_1\}$
= " x_2 -multiples of v_1 "

in general

$\text{Nul}(A) = \{ x = x_2 \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix} + x_3 \begin{bmatrix} \vdots \\ \vdots \\ 1 \\ \vdots \end{bmatrix} + x_5 \begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ 1 \\ \vdots \end{bmatrix} \}$ vector with free variable