4.3. Bases of vector spaces

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$\boxed{4.2.4}$

A set $v_1, \dots, v_p$ of vectors in V ($e_1, \dots, e_n$; $\mathbb{R}^n$)
is said to be:

(1) a spanning set of V if $\mathrm{Span}(v_1, \dots, v_p) = V$

(2) linearly independent
if the only linear combination
$c_1 v_1 + \dots + c_p v_p$ which is zero
is
the trivial one i.e. with $c_1 = c_2 = \dots = c_p = 0$

Equivalently,
none of v_i is a linear combination
of others

(3) a basis of V:
if both spanning set & linearly independent.

E1. $V = \mathbb{R}^n$ has standard basis $e_1, \dots, e_n$ e
① is a basis ✓ ② we call it standard

The dimension of V is the # of elements
of any basis $\mathcal{B} = \{b_1, \dots, b_n\}$.

$\boxed{\dim(V) = n.}$

measures size or complexity of V

if it has a basis with
n elements.

E1. $\dim(\mathbb{R}^n) = n$

$1 \overset{\checkmark}{=} \dim(\mathbb{R}^1) = \dim(\text{line})$ ✓

$2 = \dim(\mathbb{R}^2) = \dim(\text{plane})$

$3 = \dim(\mathbb{R}^3) = \dim(\text{space})$

$\boxed{4.2.5}$ Solve $Ax=0$ via REF

Th. Let $A \in M_{m\times n}$.

a) $\dim\{Col(A)\} = \#$ pivots in A

b) $\dim[Nul(A)] = \#$ of free variables.

Stronger claim:

a) A basis of $Col(A)$ is given by pivotal columns(!) in A.

b) A basis of $Nul(A)$ is found from free variables. These are vectors $\vec{v}_i$ in solution: $x = x_2 v_1 + x_3 v_2 + x_5 v_3$

Pf. 10 minute break

Start 4:10

$Nul(A)$ = solutions

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ basis of $Col(A)$

REF $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ free

$x_1 = 0$
$0 = 0$

all solutions $= Nul(A)$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

x_2-multiples of $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_1$

$Col(A) = Span$(pivotal columns)

$Nul(A) = Span(v_1)$

free variable, basis of null space $B = \{v_1\}$

= "x_2-multiples of v_1"

"vector with free variable"

in general $Nul(A)$ = all $x = x_2 \begin{bmatrix} \vdots \end{bmatrix} + x_3 \begin{bmatrix} \vdots \end{bmatrix} + x_5 \begin{bmatrix} \vdots \end{bmatrix}$