4.2. Linear Transforms

Chapter 4. Vector Spaces

4.1.1B

A. From §4.1:

Vector space is a set V in which we can do linear combinations $c_1v_1 + \cdots + c_pv_p$ for $v_i \in V$, $c_i \in \mathbb{R}$. Can do $v_1 + v_2$, $c \cdot v$.

B. A subspace U of V is a subset which is closed under linear combinations i.e. under $u_1 + u_2$ and cu, $c \in \mathbb{R}$. Then U is also a vector space.

$u_i \in U \Downarrow u_1 + u_2 \in U$

$U \ni 0$.

4.2.1B

- Standard constructions of subspaces:
  - $v_1, \ldots, v_p \in V$ give $\mathrm{Span}(v_1, \ldots, v_p) \subseteq V$
    all linear combs.
  - A matrix $A \in M_{m \times n}$ gives
    - $\mathrm{Nul}(A) =$ all $x \in \mathbb{R}^n$ with $Ax = 0$ $\subseteq \mathbb{R}^n$
      null space
    - $\mathrm{Col}(A) = \mathrm{Span}\{\underbrace{A_1, \ldots, A_n}_{\text{column of } A}\}$ $\subseteq \mathbb{R}^m$
      column space

New:

§4.2.

C. $\boxed{\text{Linear transform}}$ $T: U \to V$ between vector spaces U and V is a function = mapping = transform from U to V $\quad U \ni u \longmapsto T(u) \in V$ such that it preserves sums & products

$$\left.\begin{aligned} T(u_1 + u_2) &= T(u_1) + T(u_2) \\ T(cu) &= c\,T(u) \end{aligned}\right\}$$

i.e. it preserves ("respects") linear combinations:

$$T(c_1v_1 + \cdots + c_pv_p) = c_1T(v_1) + \cdots + c_pT(v_p).$$

Ex. If $U = \mathbb{R}^n$ & $V = \mathbb{R}^m$ then any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ comes from a matrix A type $m \times n$:

$$T(u) = Au.$$

4.2.2

E2. Consider $T: C^\infty[a,b] \to C^\infty[a,b]$ given by differentiation:

$T(f) = f'$. $f \in C^\infty[a,b]$ "smooth"

It is a linear transform! $(f+g)' = f' + g'$

$(cf)' = c \cdot f'$

L3. Let $\boxed{T: U \to V}$ be a linear transform between vector spaces U & V.

It defines set of all u in U, s.t. $T(u) = 0$

(a) $\text{Ker}(T) = \{u \in U : T(u) = 0\} \subseteq U$

Kernel of T

(b) $\text{Ran}(T) = \{T(u) : u \in U\} \subseteq V$

Range = Image of T "set of values of T"

Then (a) $\text{Ker}(T)$ is a subspace of U,

(b) $\text{Ran}(T)$ is a subspace of V.

for Nul A

Pf. (a) $u_1, u_2 \in \text{Ker}(T)$ means that $T(u_1) = 0 = T(u_2)$

want $u_1 + u_2 \in \text{Ker}(T)$ ✓

means $0 \stackrel{?}{=} T(u_1+u_2) = T(u_1) + T(u_2) = 0 + 0 = 0$ ✓

$u_1 + u_2 \in \text{Ker}(T)$

(b) $v_1, v_2 \in \text{Ran}(T)$ means that

$v_1 = T(u_1)$ for some $u_1 \in U$,

$v_2 = T(u_2)$ for some $u_2 \in U$.

But then:

$v_1 + v_2 = T(u_1) + T(u_2) \stackrel{\text{linear}}{=} T(u_1 + u_2)$

so, $v_1 + v_2 \in \text{Ran}(T)$.

• $cv_1 = c\,T(u_1) \stackrel{\text{linear}}{=} T(cu_1) \in \text{Ran}(T)$.

means that $\boxed{0 = T(u)}$ $u \in U$

• Also $0 \in \text{Ran}(T)$ since $T(0) = 0$ for any linear transform: (in U, in V)

Proof: $T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0_0$ ✓

$\in U$; number, vector in U; in U; in V; in V

$0 = 0 \cdot 0$

$\square$

E4. Let $\boxed{T : \mathbb{R}^n \to \mathbb{R}^m}$ be $\boxed{T(u) = Au}$ for some $A \in M_{m \times n}$.

T linear

Then

a) $\ker(T) = \mathrm{Nul}(A)$

b) $\boxed{\mathrm{Ran}(T)} = \boxed{\mathrm{Col}(A)}$.

$\boxed{4.2.3}$

Pf. a) $\ker(T) = \left\{ \begin{array}{l} \text{all } u \in \mathbb{R}^n \\ \text{s.t. } T(u) = 0 \end{array} \right\}$

$T(u) = Au$, so $T(u) = 0$ means $Au = 0$

$= \text{all } u \in \mathbb{R}^n$ s.t. $Au = 0$

$= \mathrm{Nul}(A)$ (!!)

b) $\mathrm{Col}(A) = \mathrm{Span}(A_1, A_2, \dots, A_n)$

$A_i = A e_i$

$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix}$ ith $= \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \end{bmatrix} = A_i$

$a_{11} \quad a_{12} \; \cdots \; a_{1i} \; \cdots \; a_{1n}$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\boxed{\mathrm{Col}(A)} = \mathrm{Span}(Ae_1, \dots, Ae_n)$

$= \mathrm{Span}(T(e_1), \dots, T(e_n))$

$= \text{all } \underbrace{c_1T(e_1) + c_2T(e_2) + \cdots + c_nT(e_n)}$

$= \text{all } T(c_1e_1 + \cdots + c_ne_n)$ for all c_i

$\begin{bmatrix} c_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ c_n \end{bmatrix} = T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) =$ all values of T