4.1. Subspaces

with some of 4.2–4.3.

§4.1. "Vector Spaces"

①

Vector spaces: alike $\mathbb{R}^n$

4.1.1

Ⓐ A **vector space** is a set V with operations of
- addition of vectors $u+v$
- multiplication by a number cv

such that all the usual properties hold: $u+v = v+u$, $v+0 = v$, ...

E1. $V = \mathbb{R}^n$: the basic examples
E2. $V = M_{m\times n}$ matrices
E3. $V = C^\infty[a,b]$ differentiable functions.

What do we do in a vector space: **linear combinations** $c_1v_1 + c_2v_2 + \dots + c_pv_p$ $v_1, v_2, \dots \in V$, $c_1, c_2, \dots \in \mathbb{R}$

Ⓑ A **subspace** U of a vector space V: a subset U which is closed under operations:
- $U \ni 0$
- if $u_1, u_2 \in U$ then $u_1 + u_2 \in U$
- if $u \in U$ & $c \in \mathbb{R}$ then $cu \in U$.

E1. $V = \mathbb{R}^n$, $A \in M_{m\times n}$ gives $\left.\begin{array}{l} U \overset{\text{def}}{=} \text{solutions of } Ax=0 \\ \quad x \in \mathbb{R}^n \end{array}\right\}$ $\mathrm{Nul}(A)$ $\overset{\shortparallel}{U}$

= solutions of a system of Linear equations. homogeneous

Pf. if $x, y \in \mathrm{Nul}(A)$, want $x+y \in \mathrm{Nul}(A)$

means $Ax = 0$, $Ay = 0$ so $A(x+y) = \overset{0}{Ax} + \overset{0}{Ay} = 0$

If $x \in \mathrm{Nul}(A)$ then $cx \in \mathrm{Nul}(A)$ (2)

$\underbrace{\phantom{x \in \mathrm{Nul}(A)}}_{Ax=0}$ $\underbrace{\phantom{cx \in \mathrm{Nul}(A)}}_{A(cx)=0}$

$$A(cx) = cAx = c\cdot 0 = 0. \quad \boxtimes$$

Need: $\underbrace{\mathrm{Nul}(A) \ni 0}$

means: $A0 = 0$, true $\square$

L1. Any subspace U of a vector space V, is itself a vector space!

Pf. Obvious: need operations $u_1 + u_2$, cu on U. ~~He~~ But U inherits operations from V: for $u_1, u_2 \in U$ to add them in U, just form the sum $u_1 + u_2$ in V and recall that by definition of subspace $u_1 + u_2$ is actually in U. $\square$

A subspace of a vector space: this is just a subset which itself is a vector space.

Ex. Let $v_1, v_2, \dots, v_p$ be p $\underbrace{\text{elements}}_{\text{vectors}}$ of a vector space V.

$V \supseteq \text{Span}(v_1, v_2, \dots, v_p) =$

= the set of all linear combinations

$c_1v_1 + c_2v_2 + \dots + c_pv_p$

of $v_1, \dots, v_p$.

L. $\underline{\text{Span}(v_1, \dots, v_p)}$ is always a $\underline{\text{subspace}}$ (so it is again a vector space!)

Pf. $S = \text{Span}(v_1, \dots, v_p)$.

If $s_1, s_2 \in S$ want $s_1 + s_2 \in S$.

$s_1 = a_1v_1 + \dots + a_pv_p$
$s_2 = b_1v_1 + \dots + b_pv_p$ $\Longrightarrow$ $s_1 + s_2 = c_1v_1 + \dots + c_pv_p$.

Add: $\underline{s_1 + s_2} = (a_1 + b_1)v_1 + \dots + (a_p + b_p)v_p$

$\underline{0 \in S}$: $0 = \underbrace{0}_{=0} v_1 + \dots + \underbrace{0 v_p}_{=0} = 0$ ✓ $\square$

R. If $A \in \mathbb{R}_{m,n}$ it gives two subspaces

① $\text{Nul}(A) \subseteq \mathbb{R}^n$ — $Ax = 0$ in $\mathbb{R}^n$

② $\text{Col}(A)$ = Span of columns of A

range of A

$A \in M_{m \times n}$ $\quad A = [A_1 \cdots A_n]$

$A_i \in \mathbb{R}^m$, ① $\underline{\mathrm{Col}}(A) = \mathrm{Span}(A_1, \dots, A_n) \subseteq \mathbb{R}^m$.

② $\mathrm{Nul}(A) \subseteq \mathbb{R}^n$.

For a vector space V

$v_1, \dots, v_p \in V$ is a spanning set

if $\mathrm{Span}(v_1, \dots, v_p) = V$.

$\boxed{\text{Idea:}}$ a spanning set gives a "concrete" description of V.

Vectors in V = linear combs. of $v_1, \dots, v_p$.

Ex. $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \in M_{3\times 5}$

$\mathrm{Nul}(A)$ = all with $Ax = 0$

(1) REF: $\left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

pivots 1, 3

free: 2, 4, 5

$$\left.\begin{aligned} x_1 - 2x_2 \quad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \end{aligned}\right\} \mathrm{Nul}(A)$$

$$x_1 = 2x_2 + x_4 - 3x_5$$
$$x_2 = 1x_2 \quad 0 \quad 0$$
$$x_3 = -2x_4 + 2x_5$$

$$x_4 = x_4$$
$$x_5 = x_5$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{v_2} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{v_3}$$

$$Nul(A) = Span(v_1, v_2, v_3)$$

$$Col(A) = Span(\underbrace{A_1, A_2, \cdots, A_5}_{\text{spanning set}})$$

$$=$$

1. A spanning set for Col(A) is given by pivotal columns of A.

Ex, pivotal columns for A: 1, 3

A spanning set for Ran(A): $\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}$.

Vectors $v_1, \dots, v_p$ in V are linearly independent if the only linear combination

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p$$

which is 0 is the trivial one:

$$c_1 = \dots = c_p = 0$$

(none of v_i's is a lin. com. of the others)

A $\boxed{\text{basis}}$ of a vector space V is any ordered sequence $v_1, v_2, \dots, v_d$ such that

① $v_1, \dots, v_d$ span (V) } "they contain all info on V"

② $v_1, \dots, v_d$ is linearly indep.

("in a most efficient way").

---

dimension of (V) = $\dim(V)$

= # of elements in a basis
any

It is a measure of "size" of V.

Later:

T2. Any two bases have the same size

Ex. $V = \mathbb{R}^n$ then $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \dots$

is a basis of $\mathbb{R}^n$.

("standard basis").

Pf. $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1 e_1 + x_2 e_2 + \dots + x_n e_n}_{\begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \dots}$

$\text{span}(e_1, \dots, e_n) = \mathbb{R}^n$

Lin. ind. if $0 = \underbrace{c_1 e_1 + \dots + c_n e_n}_{}$

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

so $c_1 = 0, \dots, c_n = 0$ $\square$

$$\boxed{\dim(\mathbb{R}^n) = n}$$

$\dim(\mathbb{R}) = 1$ line

$\dim(\mathbb{R}^2) = 2$ plane

$\dim(\mathbb{R}^3) = 3$ space

$\vdots$

examples agree with our experience in geometry