LINEAR ALGEBRA 235

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Notes on the distance learning part of the course This covers three chapters

- 4. Vector spaces.
- 5. Eigenvalues and eigenvectors
- 6. Inner products and orthogonal vectors

The notes are organized the same as sections in the book (I tried to supply more expressive titles of sections).

Current State of the Notes. At the moment only 4.1 is covered in detail.

For parts that are not covered here one should rely on the handwritten class notes. Hopefully I will be able to add some typed summaries here in time.

Contents

Date: ?

4. **Chapter 4: Vector spaces**

This chapter expands the methods we have developed from $\mathbb{R}^n$'s to the larger class of *vector spaces*.

We start with survey of the development of Linear Algebra in this course in 4.0

4.0. **Survey: The Development of Linear Algebra.** This part is not strictly needed. The goal is to help with understanding of the flow of subjects in our course.

Linear Algebra starts historically with the problem of solving systems of linear equations and progresses to notions of vector spaces and linear transforms on vector spaces.

The following sketch of the evolution of this material should become more and more clear through the semester:

4.0.1. *Vector spaces and linear transforms come from systems of linear equations.* This evolution can be described as

A. From unknowns to vector spaces.

- (i) n unknown numbers $x_1, ..., x_n$.
- (ii) An element $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ of $\mathbb{R}^n$.
- (iii) A vector v in a vector space V.

B. From systems of linear equations to linear transforms.

- (i) A system of m linear equations in n unknowns $x_1, ..., x_n$

$$a_{i1}x_1 + \cdots + a_{in}x_n \;=\; b_i \quad \text{for} \quad i = 1, 2, ..., m.$$

- (ii) A matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$.
- (iii) A linear transform T from a vector space U to a vector space V.

4.0.2. *Study of vector spaces and linear transforms.* These are the more advanced topics:

C. Things to do in a vector space. What one can do in a vector space is make *linear combinations*. This leads us to consider whether a given sequence $v_1, ..., v_p$ of elements of a vector space V is

(1) *linearly independent*;
(2) a *spanning set* for V;
(3) a *basis* for V.

Then the *dimension* of a vector space V is the size (i.e., the number of elements) of any basis of V.

Vector spaces also often posses *geometric notions* of lengths of vectors and angles between vectors.

D. How to analyze a linear transform. To a linear transform T from a vector space U to a vector space V, one associates two vector spaces

- The *null space* $Nul(T)$ is a subspace of U.
- The *range* $Ran(T)$ is a subspace of V.

A deep understanding of a linear transform T from a vector space V to itself is achieved by finding its *eigenvalues* and *eigenvectors*. This makes calculations with T “easy”.

4.1. **Vector spaces and subspaces.**

4.1.1. *Sets.* A *set* X means a collection of objects. Then the objects in the collection X are said to be *elements* of X. As a shorthand we write $a \in X$ or $X \ni a$ to mean "a is an object in the collection X", i.e., "a is an element of the set X".

Example. (0) $\mathbb{R}$ is the set of *real numbers*. So, $a \in \mathbb{R}$ means that a is a real number.

(1) $[a, b]$ denotes the closed interval with ends a and b, so $x \in [a, b]$ means that x is a real number and $a \leq x \leq b$.

4.1.2. A vector space V is an object that "behaves like some $\mathbb{R}^n$ in the sense that,

- V is a set with two operations:
  (1) the addition operation, usually denoted $u + v$, combines two elements u, v of V into a third one which is denoted $u + v$;
  (2) the *multiplication by numbers* operation, usually denoted cv, combines an element v of V and a number c in $\mathbb{R}$ into another element of V denoted cv;
- Moreover, these two operations have the same properties as in $\mathbb{R}^n$.

When V is a vector space we call its elements *vectors*.

Remarks. (0) The key thing one can do in any vector space V is that for elements $v_1, ..., v_p$ in V one can form *linear combinations* $c_1v_1 + c_2v_2 + \cdots + c_pv_p$ which are new elements of V.

(1) In every vector space V there is a particular element 0 with the property that $v+0 = v$ for $v \in V$. It is called *zero*.

It would be more precise to denote it something like 0_V in order to remember that this is an element of V with the property $v + 0_V = v$ for $v \in V$. Such more precise notation would avoid a confusion: not all zeros in all vector spaces are literally the same. For instance in $V = \mathbb{R}^3$ the zero is $(0, 0, 0)$ and in $\mathbb{R}^2$ the zero is $(0, 0)$. However, in practice we we use notation 0 for simplicity.

Example. (0) Set $\mathbb{R}^n$ with the usual operations of addition and multiplication by numbers.

() Set M_{mn} (a better notation would be $M_{m,n}$ of all $m \times n$ matrices with the usual operations of addition and multiplication by numbers.

(2) Set $C^\infty[a, b]$ of infinitely differentiable functions on an interval $[a, b]$. The operation of addition of functions f, g produces a function $f + g$ whose values are sums of values of f and g, i.e., $(f+g)(x) \stackrel{\text{def}}{=} f(x)+g(x)$ for any point $x \in [a, b]$. Similarly, the product of a function f and a number c is a new function cf whose values are obtained by multiplying values of f by c, i.e., $(cf)(x) =$ c□[1]□x) for $x \in [a, b]$.

[1] !(

(3) Set $\mathcal{P} = \mathbb{R}[X]$ of polynomials $P = a + a_1X + a_2X^2 + \cdots + a_nX^n$ of all polynomials in a variable X.

Usefulness 1. The notion of a vector space covers much beyond the $\mathbb{R}^n$'s that we have been considering but in all these new examples one can still do what we do in $\mathbb{R}^n$, i.e., *linear combinations.*

4.1.3. *Subspaces of vector spaces.* A *subspace* U of a vector space V means a subset U of V such that

- It is closed under addition and multiplication by numbers.

This means that if u, u' are elements of U and $c \in \mathbb{R}$ then the sum $u + u' \in V$ and the product $cu \in V$ (both calculated in the vector space V) happen to be elements of U.

Lemma. A. If A is an $m\times n$ matrix then the set $Nul(A)$ of all solutions $x \in \mathbb{R}^n$ of the homogeneous equation $Ax = 0$ is a subspace of $\mathbb{R}^n$.

Remark. $Nul(A)$ is therefore called the *null space of* A.

Proof. $\square$

Example. (0) The set U of all vectors $v = \left(v_1//v_2//v_3// \right)$ in $\mathbb{R}^3$ such that $v_1 = 2v_2$ and $v_1 + v_3 = 0$ is a subspace of the vector space $\mathbb{R}^3$. For instance it is the null space of the matrix $A = \left(\begin{smallmatrix} 1 & -2 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)$.

(1) For any number ω, the set S_ω of all functions $f \in C^\infty[0, 1]$ such that $f'' + \omega^2 f = 0$ is a subspace of $C^\infty[0, 1]$.

Lemma. B. If $v_1, ..., v_p$ are vectors in a vector space V then the set $Span(v_1, ..., v_p)$ of all linear combinations of these vectors is a subspace of the vector space V.

Proof. $\square$

Example. (2) In $\mathbb{R}^3$, the xy-plane is the subset $Span(e_1, e_2)$ so it is a subspace. Similarly, the z-axis is $Span(e_3)$ hence a subspace.

Summary of 4.1.

(1) We define the notion of a vector space to be a setting which "behaves like any $\mathbb{R}^n$" in the sense that one can form linear combinations with the usual properties.
(2) The notion of a "subspace U of a vector space V" is defined so that subspaces of V are just the subsets that themselves vector spaces. This produces many new vector spaces.
(3) A subset $\mathcal{S}$ of V gives a subspace $Span(\mathcal{S})$ of V.

(4) A matrix A of type $m{\times}n$ (it corresponds to a linear transform $T : \mathbb{R}^{\to}\mathbb{R}^m$ by $T(x) = Ax$), gives
   - The null-space of A is a subspace $Nul(A)$ of $\mathbb{R}^n$ (here R^n is the source of T);
   - The column space is a subspace $Col(A)$ of $\mathbb{R}^m$ (here $\mathbb{R}^m$ is the *target of* T.

(5) A subset $\mathcal{S} = \{v_1, ..., v_p\}$ of V is a
   - spanning set of V if ...;
   - linearly independent if ...
   - basis of V if both.

(6) A spanning subset $\mathcal{S}$ provides a concrete description of V (as all linear combinations of elements of $\mathcal{S}$).

(7) A basis $\mathcal{B}$ of V provides a most economical concrete description of V (as linear combinations of the smallest possible subset of V).

(8) Any two bases of V have the same number of elements. [This will be checked in 4.5.]

   This dimension of V is the size of any basis of V. It is denoted $\dim(V)$.

(9) Vectors $e_1, ..., e_n$ in $\mathbb{R}^n$ are a basis. We call it the *standard basis of* $\mathbb{R}^n$. Therefore $\dim(\mathbb{R}^n) = n$. (This agrees with the usual idea of dimension for a line $\mathbb{R} = \mathbb{R}^1$, plane $\mathbb{R}^2$ and space $\mathbb{R}^3$.)

(10) The *zero vector space* Is $V = \{0\}$ with only the vector zero. Its basis is the empty set, so its dimension is zero.

4.2. **4.2 Vector spaces** $Nul(A)$, $Col(A)$ **associated to a matrix** A **and** $\mathrm{Ker}(T)$, $Ran(T)$ **associated to a linear transform** T**.**

Summary of 4.2.

(1) For two vector spaces U, V we define *linear transforms* T from U to V.
   - A linear transform $T : U \to V$ defines a subspace $\mathrm{Ker}(T)$ of the source U (the "kernel" of T), and a subspace $Ran(T) = Im(T)$ of the target V (the "range" or "image" of T).
   - When $U = \mathbb{R}^n$ and $V = \mathbb{R}^m$ and $T(u) = Au$ for a matrix A then the new subspaces coincide with the old ones: $\mathrm{Ker}(T) = Nul(A)$ and $Ran(T) = Col(A)$.

4.3. **Bases of a vector space.**

Summary of 4.3.

(1) For a matrix A of type $m{\times}n$ (i.e., $A \in M_{mn}$),
   - A basis of $Col(U)$ is given by pivotal columns of A. Therefore, $\dim[Col(A)]$ is the number of pivots of A.
   - Recall that the general solution of the homogeneous system of equations $Ax = 0$ is of the form $y_1v_1 + \cdots + y_qv_q$ where $y_1, ..., y_q$ are the free variables and

v_i are some vectors we find from the REF of the system. Then a basis of $Nul(A)$ is given by the vectors $v_1, ..., v_q$ that appear with the free variables. Therefore, $\dim[Nul(A)$ is the number of free variables.

- $\dim[Nul(A))] + \dim[Col(A)]$ is n, the dimension of the source $\mathbb{R}^n$ of the transform $T(u) = Au$.

4.4. **Coordinate systems associated to bases.**

Summary of 4.4. Let us consider a basis $\mathcal{B} = \{b_1, ..., b_n\}$ of a vector space V.

(1) For any vector v in V there are unique numbers $c_1, ..., c_n$ such that $v = c_1v_1 + \cdots + c_nv_n$. The coefficients $c_1, ..., c_n$ that we use to write v as a linear combination of vectors in the basis $\mathcal{B}$ are called the <u>*coordinates of v in basis $\mathcal{B}$*</u> or just the <u>*$\mathcal{B}$-coordinates of v*</u>.

Vector $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is the <u>coordinate vector for v in basis $\mathcal{B}$</u> and denoted by $[v]_{\mathcal{B}}$.

(2) When the vector space V is $\mathbb{R}^n$, for any basis $\mathcal{B} = \{b_1, ..., b_n\}$ of $\mathbb{R}^n$ we get the matrix $P_{\mathcal{B}} = [b_1 \cdots b_n]$ whose columns are vectors $b_1, ..., b_n$. Matrix $P_{\mathcal{B}}$ is invertible.

(3) When the vector space V is $\mathbb{R}^n$ one can find the coordinate vector $[v]_{\mathcal{B}}$ of a vector $v \in \mathbb{R}^n$ as the solution of the equation $P_{\mathcal{B}}\ x = v$. So, one can write it as $[v]_{\mathcal{B}} = P_{\mathcal{B}}{}^{-1}v$.

(4) The *geometric meaning* of coordinates $c_1, ..., c_n$ of v in a basis $\mathcal{B}$ is that they describe how to decompose v into a sum of contributions $c_1b_1,\ ..., c_nb_n$ in the directions of basis vectors $b_1, ...b_n$.

For instance if $\mathcal{B} = \{e_1, ..., e_n\}$ is the standard basis of $\mathbb{R}^n$ then for a vector $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ we have $v = v_1e_1 + \cdots + v_ne_n$. So, the $\{e_1, ..., e_n\}$-coordinates of v are just the components $v_1, ..., v_n$ of v. In vector terms this says that $[v]_{\mathcal{B}} = v$.

(5) A key property of coordinates of vectors V in V for a basis $\mathcal{B}$ of V, is that they translate any computation in V into a computation in $\mathbb{R}^n$.

To make this clear I will consider the role of <u>invertible linear transforms</u>:

(1) A linear transform $T : U \to V$ is said to be *invertible* if there is a linear transform $S : V \to U$ in the opposite direction, such that each reverses the other, i.e., what one does the other one undoes. The precise meaning is that $S(T(u)) = u$ for $u \in U$ and $T(S(v)) = v$ for $v \in V$.

Then S is denoted by T^{-1}.

(2) If $T : \mathbb{R}^n \to \mathbb{T}^m$ is given by a matrix A, i.e., $T(x) = Ax$, then the linear transform T is invertible if and only iff ("iff") the matrix A is invertible. Then the inverse of T is the multiplication with A^{-1}, i.e., $T^{-1}(y)A^{-1}y$.

[Such T can only be invertible if $n = m$!.]

(3) If T is invertible it gives a 1-1 correspondence ("bijection") between vectors in U and V. Any vector $u\in U$ corresponds to a vector v in V which is $v=T(u)$. In the opposite direction one can recover u from v by $=T^{-1}(v)$.
(4) This allows us to translate any situation in of these vector spaces into the other –everything works the same in U and in V. For instance for vectors $u_1,...,u_p$ in U:
   (a) $u_1,...,u_p$ are linearly independent in U iff $T(u_1),...,T(u_p)$ are linearly independent in V;
   (b) $u_1,...,u_p$ span U iff $T(u_1),...,T(u_p)$ span V;
   (c) $u_1,...,u_p$ is a basis of U iff $T(u_1),...,T(u_p)$ is a basis of V.
(5) An invertible linear transform $T:U\to V$ is also called an *isomorphism of vector spaces*. The meaning is that T makes U and V have the "same shape" in the sense that "everything works the same in U and V'. (Here, "iso" means "the same" and "morphos" means "shape".)

Finally:

- For a basis $\mathcal{B}=\{b_1,...,b_n\}$ of V the coordinate map $[-]_{\mathcal{B}}:V\to\mathbb{R}^n$ is an isomorphism of vector spaces, i.e., a an invertible linear transform . So, all calculations for V can be translated into calculations for $\mathbb{R}^n$ by taking the $\mathcal{B}$-coordinates of vectors.
- An example. Let $\mathcal{P}$ be the vector space of polynomials in variable T. Let $\mathcal{P}_n = Span(1,T,T^2,...,T^n$ be all polynomials $P(T)=p_0+p_1T+\cdots+p_kT^n$, i.e., all polynomials with degree $\leq n$.

  Then $\mathcal{B}=\{1,T,...,T^n\}$ is a basis of $\mathcal{P}_n$ (so the dimension is $n+1$). For this basis the coordinate vector of $P(T)=p_0+p_1T+\cdots+p_kT^n$ is $[P(T)_{\mathbb{B}}=\begin{pmatrix}1\\T\\ \vdots\\p_n\end{pmatrix}$, i.e., the coordinates are just the coefficients in the polynomial.

4.5. **The dimension** $\dim(V)$ **of a vector space** V**.**

Summary of 4.5. This section deals with abstract properties of bases, linearly independent sets and spanning sets.

The first group of results is devoted to proving that two basis have the same size. This is the claim (4) below,and (1-3) are just preparation for (4).

(1) For subsets $\mathcal{S}\subseteq\mathcal{T}$ of V, $Span(\mathcal{S})$ is a subspace of $Span(\mathcal{T})$. Also,if $\mathcal{T}$ lies in $Span(\mathcal{S})$ then the inclusion $Span(\mathcal{S})\subseteq Span(\mathcal{T})$ is equality.
(2) [*Replacing a vector in a spanning set.*] Let $\mathcal{S}\subseteq V$ be a spanning set, so that any $v\in V$ is a linear combination of vectors of $\mathcal{S}$. If for a vector u in $\mathcal{S}$, the coefficient of vector u in this linear combination is not zero then one can replace u in $\mathcal{S}$ by v, in the sense that when $\mathcal{R}$ is obtained by removing u from $\mathcal{S}$ then $\mathcal{S}_1=\ \mathcal{R}\cup\{v\}$ is again a spanning set.

(3) [*Replacement in a spanning set by independent vectors.*] Let $\mathcal{S}\subseteq V$ be a spanning set and let the subset $\mathcal{I}\subseteq V$ be linearly independent. Then one can split $\mathcal{S}$ into union of subsets $\mathcal{S}'$ and $\mathcal{R}$, so that
- (i) $\mathcal{S}'$ and $\mathcal{I}$ have the same number of elements; and
- (ii) $\mathcal{R}\cup\mathcal{I}$ is still a spanning set.

So, some elements of $\mathcal{S}$ can be replaced by elements in $\mathcal{I}$, so that the resulting set is still a spanning set.

(4) [Spanning sets are larger then linearly independent subsets.] If $\mathcal{S}\subseteq V$ is a spanning set of q elements and $\mathcal{I}\subseteq V$ is a linearly independent subset of V with p elements then $q \geq p$.

(5) Any two basis of V have the same number of elements.
- This has been announced earlier. The number of elements in a basis of V is called $\dim(V)$. One can think of the dimension of V as a measure of how complicated vector space V is.

Next, we consider how to find a basis if one knows some spanning set or some linearly ind pendent set.

(1) Any spanning subset $\mathcal{S}$ of V contains a basis.
(2) We say that a vector space is *finite dimensional* if it has a finite spanning set. Then t actually has a finite basis $\mathcal{B}$, so $\dim(V)$ is a finite number. [We will really be interested only in finite dimensional vector spaces.]
(3) If V is a finite dimensional vector space then any linearly independent subset $\mathcal{I}$ of V can be completed (by adding more vectors) to a basis of V.
(4) [Basis theorem.] If $\mathcal{A}$ is a subset of V of size $\dim(V)$, then in order for $\mathcal{A}$ to be a basis it suffices that they satisfy one of the following two conditions: (i) $\mathcal{A}$ is linearly independent, or (ii) $\mathcal{A}$ is a spanning set.

    In other words, if we know that $\mathcal{A}$ has the correct size then one needs to check only one of the conditions (i) and (ii).
(5) If U is a subspace of V then $\dim(U) \leq \dim(V)$. One has equality of dimensions $\dim(U) \leq \dim(V)$ iff $U = V$.
(6) The dimension of the zero vector space $\{0\}$ is 0. This is the only vector space with dimension 0.

4.6. **Rank of a matrix.**

Summary of 4.6. The first topic is the *row space* $Row(A)$ of a matrix A :

(1) The*row space* $Row(A)$ is the span of rows of A. If $A \in M_{mn}$ then $Row(A)$ is a subspace of $\mathbb{R}^n$.
- If two matrices A, B are row equivalent then $Row(A) = Row(B)$.
- A basis of $Row(A)$ is given by the non-zero rows of the REF of A. [So the dimension of $Row(A)$ is again the number of pivots in A.]
- $Row(A)$ and $Col(A)$ are analogous – the spans of rows and of columns of A.

(2) $\dim[Row(A)] = \dim[Row(B)]$ and it is the number of pivots of A.
(3) This number is called the *rank of A* and denoted $rank(A)$.

The second topic is the Rank Theorem:

(1) [Rank Theorem.] For a matrix A of type $m\times n$: $rank(A) + \dim[Nul(A)] = n$.
   - This we have already noticed!
   - If one thinks of A in terms of the linear transform $T : \mathbb{R}^n \to \mathbb{R}^m$ then the intuitive meaning of the rank theorem is the following.
     Think of $\mathbb{R}^n$ and $\mathbb{R}^m$ as rooms containing n and m people respectively. Think of T as a process of some people going from the room $\mathbb{R}^n$ to the room $\mathbb{R}^m$. Then $rank(A)$ is the dimension of $Col(A) = Ran(T)$, it corresponds to those who have moved from $\mathbb{R}^n$ to $\mathbb{R}^m$. Also, $Nul(A)$ corresponds to people that T did not manage to displace, so $\dim[Nul(A)]$ corresponds to the number of people who have stayed in the room.
     Now, the Rank Theorem says hat the number of people who have left $\mathbb{R}^n$, plus the number of people who stayed in $\mathbb{R}^n$ is the total number of people in $\mathbb{R}^n$.

The last topic is the relation of invertibility of A and the rank of A.

(1) For a matrix A of type $n\times n$ the following is equivalent:
   (a) A is invertible;
   (b) $rank(A) = n$
   (c) $Nul(A) = \{0\}$ or $\dim[Nul(A)] = 0$.
(2) This is really just a reformulation of equivalences in the Invertible Matrix Theorem from chapter 2.

5. Chapter 5. Eigenvalues and eigenvectors of a square matrix

Summary.

5.1. What are eigenvalues and eigenvectors?

Summary.

5.2. The characteristic equation.

Summary.

5.3. Diagonalization of a matrix.

Summary.

5.4. Eigenvector and eigenvalues of linear transforms.

Summary.

5.5. Complex eigenvalues ?

6. Inner product on a vector space and orthogonal vectors

Summary.

6.1. Inner product, length, orthogonality.

Summary.

6.2. Orthogonal sets.

Summary.

6.3. Orthogonal projections.

Summary.

6.4. Orthonormal basis and the Gram-Schmidt process.