Math 235 Practice Final Exam Answers **Spring 2020**

Q1. (a) $det(A - \lambda I) = (1-\lambda)(\lambda^2 - 16)$.

(b) A is diagonalizable. $A = PDP^{-1}$, where $P = \begin{bmatrix} 15 & 0 & 0 \\ -7 & 1 & 7 \\ 2 & -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Q2. *Answer:*. $detA = 2(k-1)(k-2)(k-3)$.

Sketch of calculation: By row operations, A can be reduced to the following matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \\ 0 & k-1 & k^2-1 & k^3-1 \end{bmatrix}$$

Then use the identity $k^2 - 1 = (k-1)(k+1)$, $k^3 - 1 = (k-1)(k^2+k+1)$, we can factor out $2(k-1)$, so that by a cofactor expansion along the first column, we see that $detA$ equals $2(k-1)$ times the determinant of the following matrix

$$\begin{bmatrix} 1 & 3 & 7 \\ 1 & 4 & 13 \\ 1 & k+1 & k^2+k+1 \end{bmatrix}$$

Again by row operations, we reduce the above matrix to

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 6 \\ 0 & k-2 & k^2+k-6 \end{bmatrix}$$

It follows that

$$detA = 2(k-1) \times 1 \times (1 \times (k^2+k-6) - 6 \times (k-2)) = 2(k-1)(k-2)(k-3).$$

Q3.

(a) $\mathcal{B} = \{1, t^2, \cos t\}$.

(b) The $\mathcal{B}$-coordinates for $p(t) = 3\cos t + t^2$ is $[0, 1, 3]^T$.

Q4.

(a) The eigenvalues of A are $3 + 3i$ and $3 - 3i$.

(b) $A = PCP^{-1}$ where $C = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$.

Q5. C is invertible, because the equation $BCB = I_3$ implies $(detB)(detC)(detB) = 1$, which in turn implies $detC \neq 0$. Note that the equation $BCB = I_3$ also implies that $C = B^{-2}$, so $C^{-1} = B^2 = \begin{bmatrix} 1 & 0 & 1 \\ 7 & 2 & 10 \\ 4 & 1 & 6 \end{bmatrix}$.

Q6.

(a) Use the properties of transpose: $(A+B)^T = A^T + B^T$ and $(cA)^T = cA^T$, we obtain

$$S(A+B) = (A+B)^T - (A+B) = (A^T + B^T) - (A+B) = (A^T - A) + (B^T - B) = S(A) + S(B),$$

$$S(cA) = (cA)^T - cA = cA^T - cA = c(A^T - A) = cS(A).$$

Hence S is a linear transformation.

(b) The null space of S is the subspace of matrices A such that $A^T = A$. Hence a basis of the null space of S is given by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The range of S is the subspace of matrices A such that $A^T = -A$. So a basis of the range of S is given by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The sum of the dimensions of these subspaces equals the dimension of $\mathbb{R}^{2\times 2}$, which is 4 (*Note:* This is the Rank Theorem.)

Q7.

(a) A basis of $Nul(A)$ is $[-2, 3, 2]^T$.

(b) A basis of $Col(A)$ is $[1, 3, 5, -2]^T$, $[0, 2, 2, -2]^T$. (The first two columns of A.)

(c) A basis of $Row(A)$ is $[1, 0, 1]^T$, $[0, 2, -3]^T$.

(d) The rank of A is 2.

Hint: The reduced echelon form of A is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.