ELECTRO-MAGNETIC DUALITY AND GEOMETRIC LANGLANDS DUALITY
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Contents

1. Witten-Kapustin approach to Geometric Langlands ("GL") conjecture

We will be interested in what physicists *do* for the GL program and how they *think* about it. The former part is substantial but the latter may be much more important.

The key ingredients:

- Constructions of relevant families of Quantum Field Theories from 4d gauge theory and from 2d sigma models.
- Comparison of parameterizations of these two overlapping families of theories.
- Action of S-duality on these theories
- Branes.
- Wilson operators and t'Hooft operators.
- Wilson operators are the tensoring operators on $\mathcal{C}oh(\mathcal{LS}_{\check{G}}(C))$, t'Hooft operators are the Hecke operators in $Fukaya(T^*\mathrm{Bun}_G(C))$.
- $Fukaya(T^*\mathrm{Bun}_G(C))$ is identified with the microlocalization of D-modules on $\mathrm{Bun}_G(C)$.

Remarks. (1) There is also a Gukov-Witten extension of this work to *Tamely Ramified Geometric Langlands*, and Witten has announced further extension to the general (possibly wildly ramified) global Geometric Langlands.

(2) *Local Geometric Langlands conjectures* have recently been formulated and explored by Gaitsgory and Frenkel.

(3) A *Quantum version of Langlands conjectures* has been formulated by Feigin and Stoyanowski. Recently this formulation has been essentially improved and extended by Gaitsgory based on suggestions of Bezrukavnikov and Lurie.

Question. A priori, Geometric Langlands conjectures concern curves defined over any closed field $\Bbbk$. So, it is not quite clear while in the case $\Bbbk = \mathbb{C}$ we get a particularly nice point of view on these (the hyperkähler structure on the Hitchin moduli), using a subfield $\mathbb{R}\subseteq\mathbb{C}$.

1.1. New features of the KW-approach.
What does the KW-approach do towards proving GL? At the moment it offers no new results, however, there are several new and important perspectives.

1.1.1. *Context.* From the point of view of physics Complex Geometric Langlands conjecture is interpreted in terms of QFT. This makes GL a very special part of the S-duality conjectures in 4d super Young-Mills, however this special case has exceptional amount of structure. In particular, GL is seen as a close relative of the Donaldson theory.

From the mathematical point of view, the main achievement is the retelling of the complex Geometric Langlands program in the language of Differential-Geometry.

1.1.2. *New perspectives.*

(1) Prediction of elements of the *geometric Langlands program for surfaces.*

This is the most exciting aspect as mathematicians have not been making much progress on this question, at least until the recent work of Braverman and Finkelberg.

(2) Relevance of the *hyperkähler structure in representation theory.*

This is also a crucial achievement since this has been a persistent mystery for 20 years since Kronheimer constructed hyperkähler structures on nilpotent orbits. Various Kähler structures become relevant through (4), i.e., through the categories of supersymmetric branes that they define. From this point of view mathematicians did not have a chance to see the relevance of hyperkähler structures before the development and of the Fukaya category. The hyperkähler structure relevant in this work is the one on the Hitchin moduli.

(3) The machinery of *branes.*

A striking application of the point of view of branes in the KW approach is that it promises to provide a more advanced and useful theory of microlocalization given by Fukaya category, and it gives microlocalization a precise role as a bridge between local systems and D-modules.

(4) Relation to *S-duality* conjectures in 4d.

S-duality conjectures in 4d currently have no mathematical formulation. However the relevance of this bigger picture is illustrated by its role in a new treatment of the Beilinson-Drinfeld quantization of the Hitchin system.

(5) *Ramification phenomena.*

Once physicists found out what geometric Langlands is about, it took them no time at all to extend the formulations to the tamely ramified case (regular singularities) and maybe to the general case of wild ramification (irregular singularities). The tame case is in a paper of Gukov-Witten and the wild case is announced as a work of Witten. The new element is the understanding of Hecke operators at a singular point of a connection. This is interesting since the linear differential equations with irregular singularities are still something of a mystery (at least beyond curves).

There are other charming aspects of the KW approach such as

- "Additional dimensions" that appear when one studies Hitchin system as a target of maps from a surface.
- A new differential geometric view on Hecke modifications through monopoles with singularities and also as G-bundles on $\mathbb{P}^1$.
- Explicit Langlands correspondence, i.e., an explicit construction of a D-module from a local system.

1.2. **The setting and strategy of the KW-approach.** The first observation is that the two basic geometric objects in the GL-story, $\mathrm{Bun}_G(C)$ and $\mathcal{LS}_G(C)$ appear as faces of the *Hitchin moduli* $\mathcal{H}_C^U$ – two Kähler structures on this real manifold attached to a compact form U of G. Second, the geometric mechanism that allows us to pass between G and $\check{G}$ is based on the structure of a completely integrable system on $\mathcal{H}_C^U$. Then the somewhat mysterious relation between G and $\check{G}$ takes a precise form in terms of Hitchin moduli – the duality of completely integrable systems $\mathcal{H}_C^U$ and $\mathcal{H}_C^{\check{U}}$. The third step is to lift this duality of completely integrable systems to a level of equivalence of categories. Here, a new ingredient appears. Mathematicians knew such equivalence in a special case when one considers categories of coherent sheaves on $T^*\mathrm{Bun}_G(C)$ its $\check{G}$ counterpart. However these categories are just tiny specializations of the categories of *branes* that are natural for physicists; and in this larger setting the duality of completely integrable systems gives (conjecturally) an equivalence of categories of branes on $\mathcal{H}_C^U$ and $\mathcal{H}_C^{\check{U}}$ which is an example of T-duality equivalences of categories of branes. We are interested in one of specializations of this T-duality equivalence, but not the one known to mathematicians – equivalence of coherent sheaves on $\mathcal{LS}_{\check{G}}(C)$ and the *Fukaya category* on $\mathcal{LS}_{\mathbb{G}}(C)$. The fourth ingredient is the *microlocalization – Fukaya category* on $\mathcal{LS}_G(C)$. is found to consist of microlocalizations of D-modules on $BunG$.

1.2.1. *Hitchin moduli* $\mathcal{H}_C^U$. Let U be a compact Lie group and G its complexification, so G is a reductive algebraic group over $\mathbb{C}$. To a compact group U Hitchin associated for each complex curve C the *Hitchin moduli* (of *Higgs bundles*)

$$\mathcal{H}_C^U.$$

Its standard incarnation is as a real manifold with a hyperkähler structure. This in particular means a family of Kähler structures indexed by the 2-sphere

$$S \stackrel{\text{def}}{=} \{u \in \mathbb{H};\ u^2 = -1\}$$

consisting of all square roots of -1 in quaternions. We will concentrate on three basic Kähler structures corresponding to $I, J, K \in S$. However, Hitchin moduli also has a stack version (stack in differential manifolds), where one simply dispenses with bothersome stability conditions.

Both heroes in the GL story, $\mathrm{Bun}_G(C)$ and $\mathcal{LS}_G(C)$ appear as faces of $\mathcal{H}_C^U$ – from the point of view of the Kähler structure I, $\mathcal{H}_C^U$ is $\mathcal{H}_C^U(I) \cong T^*\mathrm{Bun}_G(C)$, while $\mathcal{H}_C^U(J) \cong \mathcal{LS}_G(C) \cong \mathcal{H}_C^U(K)$.

1.2.2. *Duality of completely integrable systems.* However, we really need to move between $\mathrm{Bun}_G(C)$ and $\mathcal{LS}_{\check{G}}(C)$. The framework for the passage from G to $\check{G}$, i.e., U and $\check{U}$ is now given by Hutchins's observation that $\mathcal{H}_C^U$ is a completely integrable system over the "*Hitchin base*" $\mathcal{S}_U(C)$. Then the relation between U and $\check{U}$ takes form of the duality of completely integrable systems $\mathcal{H}_C^U$ and $\mathcal{H}_C^{\check{U}}$.

The Hitchin map $\mathcal{H}_C^U \to \mathcal{S}_U(C)$ is only holomorphic for the complex structure I and from this point of view it is the affinization of $T^*\mathrm{Bun}_G(C)$, i.e., $\mathcal{S}_U(C)$ is the affine variety associated to the algebra of global functions on $T^*\mathrm{Bun}_G(C)$. Actually the algebra of global functions is explicitly known and we find that

$$\mathcal{S}_U(C) \ = \ \Gamma[C,\ (\Omega^1_C \otimes \mathfrak{g}^*)//G]$$

where // denotes the invariant theory quotient, i.e., categorical quotient.[(1)] This explicit description gives an (almost canonical) identification of bases of Hitchin systems for dual groups

$$\mathcal{S}_U(C) \ \cong \ \mathcal{S}_{\check{U}}(C), \quad b \leftrightarrow \check{b}.$$

Next, the fibers $(\mathcal{H}^U_C)_b,\ b \in \mathcal{S}_U(C)$, of the Hitchin map $\mathcal{H}^U_C \to \mathcal{S}_U(C)$ are "tori" $(S^1)^{2N}$ for a generic b, and in the complex structure I they are abelian varieties (Jacobians of certain "*spectral curves*" C_b parameterized by the Hitchin base elements b). Now the duality of completely integrable systems means that the abelian varieties $(\mathcal{H}^U_C)_b$ and $(\mathcal{H}^{\check{U}}_C)_{\check{b}}$ are canonically dual abelian varieties.

Remark. The duality of completely integrable systems is only a generic property. When $b \in \mathcal{S}_U(C)$ is not sufficiently generic, the fibers $(\mathcal{H}^U_C)b$ and $(\mathcal{H}^{\check{U}}_C)_{\check{b}}$ are singular and at present no one knows what the meaning of duality should be.

1.2.3. *Categories of branes.* Mathematicians tend to concentrate on one complex structure on $\mathcal{H}^U_C$ at a time, and they encode this complex geometry algebraically through the triangulated category of coherent sheaves. Physicists however have a larger setting, the *category of branes* $Br[\mathcal{H}^U_C]$ on the real manifold $\mathcal{H}^U_C$; which so far has not been defined mathematically.

For each $u \in S$ the corresponding Kaḧler structures on $\mathcal{H}^U_C$ defines two subcategories of branes, the B-model category uses the complex structure part of the Kaḧler structure u, and the A-model subcategory the symplectic part of the Kaḧler structure u:

- The B-model subcategory is the category of coherent sheaves

$$u_B = \ Br^B_u[\mathcal{H}^U_C] = \ Br[\mathcal{H}^U_C, J_u] \stackrel{\text{def}}{=} D^b[\mathcal{C}oh(\mathcal{H}^U_C(J_u))]$$

  for the complex structure J_u associated to u.
- The A-model subcategory is the Fukaya category

$$u_A = \ Br^A_u[\mathcal{H}^U_C] = \ Br_{\omega_u}[\mathcal{H}^U_C] \stackrel{\text{def}}{=} \mathrm{Fu}[\mathcal{H}^U_C, \omega_u]$$

  for the symplectic structure J_u associated to u.

These are subcategories of branes which are invariant under certain symmetry associated to J_u or ω_u. This symmetry is odd so its called a supersymmetry.

[1] $\mathfrak{g}^*//G$ can be viewed as a vector space but maybe not completely canonically.

Actually there is more, for each pair $(u,v)\in S^2$ we get one Generalized Kähler structure on $\mathcal{H}_C^U$ and one subcategory $Br[\mathcal{H}_C^U;u,v]$ of (u,v)-supersymmetric branes. The above subcategories are the just the special (anti)diagonal cases :

$$J_u \leftrightarrow (u,u) \quad \text{and} \quad \omega_u \leftrightarrow (u,-u).$$

1.2.4. *Fourier transforms in mathematics and physics.* Mathematically, the generic duality of completely integrable systems (in the holomorphic view of dual disconnected abelian varieties), yields the Fourier-Mukai equivalence of derived categories of coherent sheaves on $T^*\mathrm{Bun}_G(C)$ and $T^*\mathrm{Bun}_{\check{G}}(C)$ over the "regular" parts $\mathcal{S}_U(C)_{reg} \cong \mathcal{S}_{\check{U}}(C)_{reg}$. This has been interpreted by Pantev as (generic part of) the classical limit of the Langlands correspondence.

The (generic) Fourier-Mukai equivalence of coherent sheaves $D^b[\mathcal{C}oh(T^*\mathrm{Bun}_G(C))] \cong D^b[\mathcal{C}oh(T^*\mathrm{Bun}_{\check{G}}(C))]$ can be viewed as an equivalence of certain categories of branes

$$Br_I^B[\mathcal{H}_C^U] \;\cong\; Br_I^B[\mathcal{H}_C^{\check{U}}].$$

Physicists see this as a very special case of a general equivalence

$$Br[\mathcal{H}_C^U] \;\cong\; Br[\mathcal{H}_C^{\check{U}},$$

which is a special case of several conjectural kinds of *dualities*, so it can be viewed as Mirror Symmetry, T-duality or S-duality.

We will most often refer to it as S-duality. The standard meaning of this expression is a very important and mysterious duality (i.e., equivalence) of two supersymmetric gauge theories in four real dimensions predicted by the Montonen-Olive conjecture. The relation to Langlands duality has been a prominent question since these two gauge theories correspond to dual groups U and $\check{U}$. Now, the above conjectural S-duality equivalence $Br_I^B[\mathcal{H}_C^U] \;\cong\; Br_I^B[\mathcal{H}_C^{\check{U}}]$ is the brane formulation of a certain 2-dimensional limit ("*compactification*") of a certain simplification ("the GL-*topological twist*") of the original 4-dimensional S-duality.

The standard properties expected from S-duality are then used to show that it exchanges subcategories $I_B(U)$ and $I_B(\check{U})$ as well as $J_B(U)$ and $K_A(\check{U})$. The first case is the Fourier-Mukai transform which is a classical limit of the Langlands duality. However, the second case will turn out to be a key step in Langlands duality itself.

1.2.5. *Kapustin-Witten strategy in terms of branes for three holomorphic structures.* We start with the category $D^b[\mathcal{C}oh(\mathcal{LS}_{\check{G}}(C))]$ of coherent sheaves on the moduli of $\check{G}$ local systems. We view this category as B-branes for the complex structure J. There are two steps

1. S-duality exchanges J_B and K_A so it takes a coherent sheaf $\mathcal{F}$ to an A-brane $\mathbb{S}(\mathcal{F})$ for the Kähler structure K.
2. Now, we use the observation that

(∗) *global A-branes on $\mathcal{H}_C^{\check{U}}(K)$ can be identified with modules for microdifferential operators on $Bun_{\check{G}}(C)$.*

Then the A-brane $\mathbb{S}(\mathcal{F})$ for $K.$ is a microlocalization of a D-module on $Bun_G(C)$ which we call the Langlands transform $\mathcal{L}(\mathcal{F})$ of $\mathcal{F}$.

Remark. Notice that in the first two steps we use Kähler structures J, K which are both isomorphic to the the moduli of local systems. In the third step the complex structure I, i.e., the moduli of Higgs bundles, appears as an ingredient of the procedure (∗). This procedure is based on a choice of a certain "large A-brane", and that choice involves the complex structure I. So, the third step is a very nontrivial form of a hyperkähler rotation. (The naive one is obviously not compatible with categories.)

1.3. **Construction of relevant Quantum Field Theories via Gauge Theory.** A priori, the Geometric Langlands duality appears in this approach as a relation between certain 2-dimensional Quantum Field Theories called *sigma models*, attached to compact forms of dual groups $G, \check{G}$ and to a curve C. However, these theories turn out to be essentially related to certain gauge theories in dimensions 10 and 4. While these sigma models can be introduced directly, their most interesting aspect (at least for physicists), is that they are specializations of gauge theories. The gauge theory background turns out to have various consequences for GL, which are so far physical proofs of known mathematical results such as commutativity of Hecke operators or quantization of Hitchin's completely integrable system (a deep result of Beilinson and Drinfeld).

So, the setup of the KW treatment of GL is the construction of certain Quantum Field Theories in dimensions 10,4 and 2. Eventually, different constructions will yield different but overlapping families of theories in 2 dimensions so we will have to compare the natural parameterizations of families coming from sigma models and from gauge theory.

From the point of view of gauge theory it is natural to starts with a 10 dimensional gauge theory. Then one deduces the 2d sigma models through a sequence of "simplifications". We first list the steps and then supply the details bellow.

(1) In dimension 10 there is an essentially unique supersymmetric gauge theory $\mathcal{T}$.
(2) The straightforward dimensional reduction gives a gauge theory $\mathcal{T}'$ in 4d, but this theory has no supersymmetry.
(3) However, a *twisted* dimensional reduction gives a 4d theory $\mathcal{T}''$ with two supersymmetries, the $N = 4$ super Young-Mills theory.
    - This twisted dimensional reduction has a well known analogue which produces Donaldson theory
(4) Two supersymmetries give a $\mathbb{P}^1$-family of topological twists of this $N = 4$ super Young-Mills , hence a $\mathbb{P}^1$-family $\mathcal{T}_t''$, $t \in \mathbb{P}^1$ of topological gauge theories in 4d.

(5) The fields for the theory $\mathcal{T}_t''$ are solutions of certain system of PDEs, the equations of invariance under the supersymmetry corresponding to $t \in \mathbb{P}^1$. So the nature of these theories is glimpsed through various results on these PDEs.(2)
(6) "Compactification" of 4d topological Quantum Field Theories to a compact Riemann surface C, gives a $\mathbb{P}^1$-family of sigma models on C, with target the Hitchin moduli .

1.3.1. *Super gauge theory in 10d.* Here are some clues for the role of dimension 10:

- *Nahm's theorem.* 10d is the maximal dimension for a supersymmetric gauge theory. The theory in $10d$ is in some reasonable sense unique.
- The simplest construction of the maximally supersymmetric 4d super Young-Mills is by dimensional reduction from $10d$.
- We will see later that the existence of interesting theories parallels the chain $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ of division algebras.

Now we go through some of the features of the theory in 10d:

(1) The Lagrangian in 10d is kinetic and contains one coupling parameter denoted e.
(2) The 10d super gauge theory lives on a superspacetime $\mathcal{H}^{1,9}$. The spin bundle $\mathcal{S}$ on $\mathbb{R}^{1,9}$ is chosen to be $\mathbb{R}^{1,9}$-equivariant, hence in particular trivial. Then the superspacetime is a Heisenberg type supergroup, an extension of the odd abelian Lie group $\mathcal{S}^-$ of *odd spinors* on $\mathbb{R}^{1,9}$, by the group $\mathbb{R}^{1,9}$ of even spacetime translations. On the level of Lie algebras
$$0 \to \mathbb{R}^{1,9} \to \mathfrak{h}^{1,9} \to \mathcal{S}^- \to 0.$$
The extension is defined by the *canonical* (hence $Spin(1,9)$ invariant), *self-pairing* of odd spinors $\mathcal{S}^-$ into vectors $\mathbb{R}^{1,9}$.
(3) The fields in a super gauge theory are pairs (A, λ) where fermion λ ("superpartner"), is an even spinor with "values in $\mathfrak{g}$", i.e.,
$$\lambda \in \Gamma(M,^E \mathfrak{g} \otimes \mathcal{S}^+).$$
(4) The group of symmetries of the theory is the *super Poincare group* $s\mathcal{P}^{1,9}$ which is the semidirect product
$$s\mathcal{P}^{1,9} \stackrel{\text{def}}{=} \mathcal{H}^{1,9} \rtimes \; Spin(1,9)$$
of the *superspacetime translations* group $\mathcal{H}^{1,9}$ and the linear rotations $Spin(1,9)$. Here, $Spin(1,9)$ appears rather then $SO(1,9)$ because fields include spinors.
(5) *Supersymmetries* of $\mathcal{H}^{1,9}$ (also called *odd* symmetries or *fermionic* symmetries). They are given by translations of the superspace $\mathcal{H}^{1,9}$, i.e., by the *odd spinors* $\mathcal{S}^-$ ("odd chirality spinors"). It will be important that they are organized as the odd spinorial representation of $Spin(1,9)$.

[2]These are 4d PDEs. However, one also studies their reductions to 3d and 2d which will become important later.

They form an odd vector bundle of odd symmetries on $\mathbb{R}^{1,9}$ of the rank $2^{\lfloor\frac{9+1}{2}}/2 = 2^{10/2-1} = 2^4 = 16$.

(6) *Action of supersymmetries on fields.* Supersymmetries, i.e., $\mathcal{S}^-_{1,9}$ act on fields via ...

1.3.2. *The straightforward dimensional reduction to 4d.* The straightforward dimensional reduction gives a super gauge theory in 4d valid on any spin manifold M. We will see that the problem with this theory is that in the full generality of arbitrary M it is not supersymmetric, i.e., it has no supersymmetry.

To start with, this reduction of the supersymmetric theory $\mathcal{T}$ on $\mathbb{R}^{1,9}$ consists of restricting the fields on the 10 dimensional Minkowski space $\mathbb{R}^{1,9}$ (i.e., on $\mathcal{H}^{1,9}$), to those which are constant in the direction of $\mathbb{R}^{1,5}$, i.e., equivariant under the group of translations $\mathbb{R}^{1,5\cdot}$. We think of the restricted fields as fields on the quotient $\mathbb{R}^4 = \mathbb{R}^{1,9}/\mathbb{R}^{1,9}$. This gives a 4d theory $\mathcal{T}'$. The use of the factorization reduces the rotational symmetry from $Spin(1,9$ to $Spin(1,5)\times Spin(4)$. So, $\mathcal{T}'$ on $\mathbb{R}^4$ carries a sophisticated symmetry group because of its 10d origin.

(1) The effect on bosonic fields is that from one gauge field $\underline{A}$ in 10d we get a pair $(A,\underline{\phi})$ of one gauge field A in 4d and six scalar fields $\underline{A}_i$, $3 < I \leq 9$, on $\mathbb{R}^4$.(3)
(2) The first effect of the reduction on the Lagrangian (=action) is to make it look more complicated. The kinetic term in 10d Lagrangian turns into a sum of three terms. One is the analogous kinetic term of the 4d gauge field, but there is also the bracket of scalar fields in $ad(E)$ and a mixed term – the covariant derivatives of scalar fields.
(3) There is also another source of complexity of the Lagrangian of the 4d theory. In 4d we can (and we do) add to the action a *topological term* which is specific for gauge theories in dimension four. The new coupling θ combines with the old coupling e into a complexified coupling $\tau \in \mathbb{H}$.
(4) The key information for the reduced theory $\mathcal{T}'$ is the organization of supersymmetries $\mathcal{S}^-$ according to the rotational symmetry group. This is just elementary representation theory or linear algebra. the restriction of the representation $\mathcal{S}^-$ of $Spin(1,9)$ to the subgroup $Spin(4)\times Spin(1,5)$.(4)

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[3] The 4d interpretation of a component of a 10d field is based on how it transforms under the rotations of the 4d space. For instance, a *scalar field* or *zero spin field*, will mean a *functions* on the space (here with values in $\mathfrak{g}$), and it is characterized by the trivial action of rotations of the 4d space.

Explicitly, $\underline{A} = \sum_{I=0}^{9} \underline{A}_I dx^I$ and $A = \sum_{I=0}^{3} \underline{A}_I dx^I$. We view the six scalars as a single field

$$\underline{\phi} = \sum_{I=4}^{9} \underline{A}_I dx^I$$

which is a map from $\mathbb{R}^4$ to translation invariant one 1-forms on $\mathbb{R}^{1,5} = \mathbb{R}^6$ with values in $\mathfrak{g}$.

[4] Using special isomorphisms of low rank semisimple groups, we view $Spin(4)$ as a product $SU(2)_L\times SU(2)_R$ of the left and right copy of $SU(2)$. To some extent in QFT one can pass between

The important role of the rotational symmetry $Spin(4)$ is that it allows to extend the theory from $\mathbb{R}^4$ to any 4d Riemannian manifold (M,g). The supersymmetries that extend to (M,g) are those that are fixed under the holonomy subgroup $Hol(M,g)\subseteq Spin(4)$. So, the ones that carry over to any (M,g) are those invariant under $Spin(4)$. However, the RT shows that $(\mathcal{S}_-)^{Spin(4)}=0$, so no supersymmetries survive if we try to extend the theory to a general 4d Riemannian manifold.[5] However, there is a way around this:

1.3.3. *Twisted dimensional reduction from 10d to 4d.* Here we adapt the straightforward reduction $\mathcal{T}'$ this will allow us us to keep half of the $4\times 4=16$ SUSY charges, ending up with 4 left movers and 4 right movers. This is achieved by twisting the action of rotational symmetries $Spin(4)$ on the fields. I will call this a *holonomy twist*. For this, we use a nonstandard embedding of 4d rotations into the group of symmetries of the theory $\mathcal{T}'$

$$\kappa:\ Spin(4)\hookrightarrow Spin(4)\times Spin(6).$$

One can think of it as a new copy $Spin(4)'\subseteq Spin(4)\times Spin(6)$.

(1) A particular choice of κ used here (called the *GL twist*), gives a theory $\mathcal{T}''$ valid on arbitrary 4d Riemannian manifold, but this time with two supersymmetries as $\mathbf{ß}\overset{\text{def}}{=}\mathcal{S}_-^{Spin(4)'})$ is two dimensional.
(2) The first component of the holonomy twist κ is identity. The second is the canonical embedding of $Spin(4)\cong SU(2)\times SU(2)$ as the semisimple part of a Levi factor in $Spin(6)\cong SU(4)$.
(3) An important symmetry of the theory is given by the center of the this Levi subgroup. This is a copy of $U(1)$ which acts on fields and centralizes $Spin(4)'$.
(4) Once we have changed the action of $Spin(4)$ on the fields, the interpretation of the fields also changes! The effect is that in the holonomy twisted theory $\mathcal{T}''$ the bosonic fields are triples (A,ϕ,σ)[6] of

the Minkowski space and the Euclidean space, or real and complex representations of groups, so we replace $Spin(1,5)$ with $Spin(6)\cong SU(4)$.

Now the restriction of $\mathcal{S}_-$ to rotational symmetries is

$$\mathcal{S}^-|_{SL(2)\times SL(2)\times SL(4)}\ \cong\ L(1)\otimes L(0)\otimes L(\omega_3)\,\oplus\, L(0)\otimes L(1)\otimes L(\omega_1),$$

which we denote $(\mathbf{2},\mathbf{1},\overline{\mathbf{4}})\oplus(\mathbf{1},\mathbf{2},\mathbf{4})$ or just $(\mathbf{2}_L,\overline{\mathbf{4}})\oplus(\mathbf{2}_R,\mathbf{4})$ (**??**).

Notice that this decomposes supersymmetries into *left-handed* and *right-handed* ones, according to which copy of $SU(2)$ acts nontrivially. Say, left-handed ones are invariant under $SU(2)_R$.

This at the same time explains the effect of the reduction to fermions since the restrictions of $\mathcal{S}^{\pm}$ are the same.

[5]Some supersymmetry may survive on manifolds with sufficiently small holonomy.

[6]The bosonic fields for $\mathcal{T}'$ are pairs $(A,\underline{\phi})$. The new component A is the same as before. The field $\underline{\phi}$ can be written as $\sum_4^9\ \underline{A}_I dx^I$ for any coordinates x^I on $\mathbb{R}^6$. For certain choice of coordinates x^I it splits into ϕ formed from four components $\underline{A_4},...,\underline{A_7}$ and

$$\sigma\overset{\text{def}}{=}\frac{\underline{A}_8-i\underline{A}_9}{\sqrt{2}},\qquad \overline{\sigma}=\frac{\underline{A}_8+i\underline{A}_9}{\sqrt{2}}.$$

- (i) a gauge field $A = \sum_0^3 \ A_\mu dx^\mu$ (a $\mathfrak{g}$-valued one form),
- (ii) a $\mathfrak{g}$-valued one form $\phi = \sum_0^3 \ \phi_\mu dx^\mu$, and
- (iii) two conjugate *complex scalars* $\sigma, \overline{\sigma}$ which are functions on $\mathbb{R}^4$ with values in the complexification $\mathfrak{g} \stackrel{\text{def}}{=} \mathfrak{u}_{\mathbb{C}}$.

This is one of three possible holonomy twists of $\mathcal{T}'$. Before KW work it was neglected as it does not produce invariants of 4 manifolds. The other two twists give the Donaldson theory and its relatives.

1.3.4. *Topological twists give a $\mathbb{P}^1$-family of topological Field Theories in 4d.* We have a QFT $\mathcal{T}''$ with two fermionic symmetries which we call Q_L, Q_R (certain basis of invariants $\boldsymbol{\mathcal{S}} = \mathcal{S}_-^{Spin(4)'}$). We will chose $0 \neq Q = uQ_L + vQ_R \in \boldsymbol{\mathcal{S}}$ and we will pass from fields to their Q-cohomology. This gives a theory $\mathcal{T}''_Q$ which will turn out to be topological, i.e., independent of the metric on a 4-manifold M. The symmetry Q will then be called the *topological symmetry.* The theory, i.e., the cohomology, depends only on the $\mathbb{C}$-line through Q. We denote the parameter for these lines by $t = \frac{u}{v}$. Then the two supersymmetries of the super gauge theory $\mathcal{T}''$ give a $\mathbb{P}^1{}_t$-family of topological twists $\mathcal{T}''_t$, i.e., a $\mathbb{P}^1{}_t$-family of topological gauge theories in 4d. (Here $\mathbb{P}^1$ means $\mathbb{CP}^1$.)

Now the key question is to identify the fields of each of these theories $\mathcal{T}''_t$. So, we need to identify (i) the fields which are killed by Q (called *supersymmetric* fields or *topological* fields), and (ii) the effect of killing the Q-exact fields. Since Q is a vector field supersymmetric fields are solutions of a system of PDEs. So the nature of these theories is studied through

1.3.5. *Study of supersymmetry PDEs.* These PDE's originally make sense in 10d as the supersymmetries in 4d come from 10d and one takes the point of view of reducing them to 4d, and later further to 3d or 2d.[(7)]

A. PDEs. The dimensional reduction of equations means that we fix values of some components of the field in a higher dimensional equation and then consider *consistency equations* in these fixed components. These equations are the necessary equations for the higher dimensional system to have a solution in the remaining components of the field. Often these will be special kinds of a solution in which the remaining components behave in some simple way.

(1) These *consistency equations* in bosonic fields (A, ϕ, σ) take form of (here $D = \nabla$ and $D^* \stackrel{\text{def}}{=} \star D \star$)
   - **Bosonic equations**:

$$(\star) \quad [F - [\phi, \phi] \ \pm t^{\mp 1} \ (D\phi)]^{\pm} \ = \ 0 \quad \text{and} \quad D^*\phi = 0,$$

[7]The relevant moduli of solutions in 2d will be the Hitchin moduli of Higgs fields (moduli of solutions of Hitchin equations), and in and 3d the moduli will give t'Hooft operators, a new expression for Hecke operators (moduli of solutions of extended Bogomolny equation).

where for a 2-form Ω one denotes by $\Omega^{\pm}$ the (anti)self-dual parts of Ω.

- **Fermionic equations.**

$$D\sigma \ \pm \ t^{\pm 1}[\phi,\sigma]=0 \quad \text{and} \quad [\sigma,\overline{\sigma}]=0.$$

So, A appears only through curvature F and there are three bosonic equations
(a) $[F-[\phi,\phi]+t\ (D\phi)]^{+}=0$,
(b) $[F-[\phi,\phi]-t^{-1}(D\phi)]^{-}=0$
(c) $D^{*}\phi=0$;
and three fermionic equations
(a) $D\sigma+t[\phi,\sigma]=0$,
(b) $D\sigma-t^{-1}[\phi,\sigma]=0$
(c) $[\sigma,\overline{\sigma}]=0$.

(2) Fermionic equations imply that when the solution has a finite automorphism group (these we call "*irreducible solutions*"), then $\sigma=0$. This is really the case we are interested in. When we pass to $d=2$ this condition will mean that we are away from singularities of the Hitchin moduli.

B. Analogies with 2d sigma models. Roughly, the result is that the theories behave somewhat like 2-dimensional sigma models.$^{(8)}$

- When $t\notin\mathbb{R}$ the moduli of solutions is $\mathcal{LS}_{U_{\mathbb{C}}}(C)$.
- This is most straightforward at $t=\pm i$, here bosonic equations amount to flatness of the complexified connection $\mathcal{A}\overset{\text{def}}{=}A+i\phi$. The analogy with the B-model with target $\mathcal{LS}_{U_{\mathbb{C}}}(C)$: in B-model the supersymmetric fields are constant maps.
- For $t\in\mathbb{R}$ equations are elliptic. This is analogous to A-model which counts solutions of elliptic equations.
- For generic t, the analogy is with the 2d sigma model based on a generalized complex geometry.

C. Vanishing theorems for solutions of SUSY PDEs. Besides the information on the nature of the theories, PDEs give another relevant family of results, the *vanishing theorems*. PDE's derived from SUSY often have unusual vanishing properties. On a closed 4 manifold M the solutions of the consistency equations depend on the first Pontryagin class $\boldsymbol{p}$:

Theorem. (a) If $\boldsymbol{p}\neq 0$ there is a solution at just one point of $\mathbb{CP}^1$, either 0 or ∞ depending on the sign of $\boldsymbol{p}$.

(b) If $\boldsymbol{p}=0$ then any solution (A,ϕ) for some t is also a solution for all t's, and can be viewed as a flat connection $\mathcal{A}=A+i\phi$ with values in $G_{\mathbb{C}}$.

D. Reduction of consistency equations to 2d and 3d. Later we will find out that we are also interested in reduction of SUSY PDEs to 2d and 3d.

[8] When we reduce theories to 2d, we will in fact get 2d sigma models.

(1) Reduction of consistency equations to 2d are the *Hitchin equations*. Here the four manifold is a product $M = \mathbb{R}^2 \times \Sigma$ for a 2d surface Σ and $\mathbb{R}^2$-invariant solutions of consistency equations on M are pull-backs of fields on Σ. This reduces equations to Σ and there we get Hitchin's equations for Higgs fields.
(2) Reduction to 3d gives an extended ("complexified") version of the Bogomolny equation for monopoles. These equations appear to be new and they provide another incarnation of Hecke operators.

E. Topological action and the canonical parameter Ψ. To complete the passage to topological theories on arbitrary 4d Riemann manifolds M, we need to find the corresponding action, i.e., the *topological Lagrangian.* It should satisfy two properties:

- (i) when the spacetime M is flat the new Lagrangian should be the same as the old one, i.e., the one for $\mathcal{N} = 4$ super Young-Mills .
- (ii) For each M, new Lagrangian should be invariant under *each* of the topological supersymmetries Q_t corresponding to $t \in \mathbb{P}^1{}_t$.

The solution is a Lagrangian $\boldsymbol{I} = \boldsymbol{I}^{A,\phi} + \boldsymbol{I}^{\sigma} + \boldsymbol{I}^{top}$ where

$$\boldsymbol{I}^{A,\phi} = \frac{-1}{e^2} \int_M d^4x \,\sqrt{g}\, Tr[\frac{1}{2}|\mathcal{F}|^2 + (D^*\phi)^2]$$

$$\boldsymbol{I}^{\sigma} = \frac{21}{e^2} \int_M d^4x \,\sqrt{g} \quad Tr[\frac{1}{2}[\sigma, \overline{\sigma}]^2 - D_\mu \overline{\sigma} D^\mu \sigma - [\phi_\mu, \sigma][\phi_\mu, \overline{\sigma}]]$$

$$\boldsymbol{I}^{top} = i\frac{\theta}{8\pi^2} \int_M Tr[F \wedge F].$$

This structure of the action leads to the following seemingly remarkable coincidence between the action and the SUSY equations:

Lemma. (a) The action is minimized iff

$$\mathcal{F} = 0 = D^*\phi \quad \text{and} \quad D\sigma = 0,\ [\sigma, \overline{\sigma}] = 0,\ [\phi, \sigma] = 0.$$

This is equivalent to requiring that the field satisfies SUSY equations for *all* $t \in \mathbb{P}^1{}_t$.

(b) For each $t \in \mathbb{P}^1{}_t$ the action $\boldsymbol{I}$ can be written as a sum of a Q_t-exact term ($Q_t V(t)$ for some expression $V(t)$), and a topological term (the first Pontryagin class) :

$$\boldsymbol{I} = Q_t V(t) + \frac{i\Psi}{4\pi} \int_M Tr(F \wedge F)$$

for the so called *canonical parameter*

$$\Psi \overset{\text{def}}{=} Re(\tau) + i\ Im(\tau) \cdot \frac{t - t^{-1}}{t + t^{-1}} = \frac{\theta}{2\pi} + e^2 \cdot \frac{t - t^{-1}}{t + t^{-1}}.$$

This form implies that $\boldsymbol{I}$ is killed by each of the supersymmetries Q_t.

Proof. (a) follows from the above formula – apart from the topological term the action is a sum of squares of the above supersymmetry equations.

Remark. We see that the parameters τ and t combine into one relevant parameter Ψ. So we end up with a family of topological theories parameterized by $\Pi \in \mathbb{P}^1{}_\Psi$.

1.3.6. *Compactification of 4d topological Quantum Field Theories to 2d sigma models.* In this step, we pass from our $\mathbb{P}^1$-family of 4d topological Quantum Field Theories to a $\mathbb{P}^1$-family of sigma models on a surface Σ, with target the Hitchin moduli for C. For this we consider a product $M = \Sigma \times C$ of Riemann surfaces Σ, C with C compact, and we rescale the metric on C (this does not affect the complex structure!), so that C is much smaller then Σ.

The limiting process makes action *large* for fields that change much along C. According to the stationary phase principle. the path integral localizes to fields which minimize the action in the C-direction and change freely in the Σ-direction. However, the lemma in 1.3.5.**E** says that the minimum of action appears for fields which satisfy SUSY equations for *all* $t \in \mathbb{P}^1{}_t$, i.e.,

$$\mathcal{F} = 0 = D^*\phi \quad \text{and} \quad D\sigma = 0,\ [\sigma, \overline{\sigma}] = 0,\ [\phi, \sigma] = 0.$$

This relation between the action and supersymmetry equations now gives

A. The limit of a 4d theory when C gets small is a 2d sigma model. The logic is approximately the following.

(1) The simplest solutions are those (E, A, ϕ, σ) which are pull-backs from C and have $\sigma = 0$.
(2) So these are the solutions (E, A, ϕ) on C of *Hitchin equations*

$$\mathcal{F} \ = \ 0 \ = \ D^*\phi.$$

(3) The *Hitchin moduli* $\mathcal{H} = \mathcal{H}_C^U$ is the moduli of solutions (E, A, ϕ) of Hitchin's equations on C. In terms of the real fields A, ϕ the equations are

$$F = \phi \wedge \phi \quad \text{and} \quad D\phi = 0 = D^*\phi.$$

Here *moduli* means *modulo gauge transforms*, i.e., automorphisms of E.
(4) The general solutions vary freely in Σ but the restriction to each copy $q \times C$, $q \in \Sigma$, of C, is a minimizer. So, the fields in the effective theory on Σ are maps from Σ to $\mathcal{H}$.

B. Singularities of $\mathcal{H}$ and a partial translation between fields on M and Σ, when C is much smaller then Σ. This translation is a *low energy transition* in the sends that it works the best *on the shell*, i.e., for fields of minimal energy, and also for the nearby fields of "*almost minimal energy*".

(1) The moduli of solutions $\mathcal{H}$ has singularities at reducible solutions. A solution is said to be *reducible* if it has a continuous symmetry group, i.e., its stabilizer in the group of gauge transformations is infinite. We will assume that $g > 1$, then the generic solutions are irreducible.

(2) The above partial translation breaks down at reducible solutions or equivalently at the maps $\Sigma \to \mathcal{H}$ which visit the singularities of $\mathcal{H}$.[(9)]
(3) We will consider only the case of a semisimple group U. If $G = U(1)$ then all solutions are reducible since they have $U(1)$ symmetry. This causes the above low energy description of the theory to have besides a sigma model a factor which is a supersymmetric gauge theory.
(4) KW stay away from singularities of $\mathcal{H}$. (In particular G is semisimple and $g > 1$.) This should suffice for Langlands correspondence for irreducible local systems.
(5) A precise form of the translation is

Lemma. (a) The singularity avoiding fields with zero action (energy) are precisely the constant maps into $\mathcal{H}_{reg}$, i.e., elements of $\mathcal{H}_{reg}$.

(b) We say that a map $\Sigma \to \mathcal{H}$ *varies slowly* if The value changes significantly only
over the distances in Σ which are
$>>$ size of C. Then the slowly varying maps $\Sigma \to \mathcal{H}$ are the same as the gauge theory fields of *almost zero action.*

Question. Can one repair the above inadequacies (by considering $\mathcal{H}$ as a stack ...) and "localize" the whole 4d gauge theory on the Hitchin moduli?

1.4. **Appendix. 10d super Young-Mills as an "octonic theory".** These are some general comments by Witten. The upshot is a "justification" of the existence of a maximally supersymmetric gauge theory in 10d.

A. Extra dimensions in physics. When physics and geometry look at the same problem, physics has at least one extra direction. If we interpret the meaning of physics treating geometric problems as the method of sigma models, the extra dimensions are given by the dimension of the source manifold, i.e., the dimensions that are used to probe the target manifold. In particular, the number of extra directions is here the dimension of the QFT.

For instance in Quantum Mechanics there is one extra dimension and as Geometric Langlands is related to 2d sigma models, here physics has two extra dimensions.

B. Super Quantum Mechanics on a manifold M. Quantum Mechanics is the 1-dimensional QFT. In the Lagrangian approach, Quantum Mechanics on a manifold M consists of studying the maps from an interval I into M (the possible time evolutions in a configuration space M). So it is a sigma model of dimension one. The most obvious observables are the evaluations at $t \in I$ of functions on M. In the standard Super Quantum Mechanics on a manifold M the super ingredient is the extension from functions

[9] A solution in 4d may contain more information then a map in 2d. The additional degrees of freedom at singularities include (i) non-vanishing of the component σ, (ii) solution need not be a pull-back from C. This loss of information should be compensated by refining the map (say by going to a resolution), so that it reflects the extra information in the solution.

on M to differential forms. There is one supersymmetry- the De Rham differential acting on forms. From the Hamiltonian point of view, QM is a synonym for quantization of a Poisson structure. In the Hamiltonian approach to super QM on a manifold M, the cohomology $H^*(M,\mathbb{R})$ is a Hilbert space of states that the theory produces (more precisely this is the *chiral ring* of the theory).

C. Lifting to a higher dimension. This is the idea that one may be able to trade extra symmetry for for extra dimensions.

- First, any problem $\mathcal{T}'$ should involve time (i.e., quantization), so to start with our problem is on $\mathbb{R}^{1,0}$, hence in dimension one.
- Next, for a problem $\mathcal{T}'$ with extra symmetry one may look for a related problem $\mathcal{T}$ in a higher dimension, i.e., on $R^{1,n-1}$, such that when we restrict to fields constant in the direction of the factor $\mathbb{R}^{n-1}$, then we get the 1d problem $\mathcal{T}'$. Then the part of the natural symmetry in higher dimension that survives the reduction to 1d, may provide an "explanation" for the extra symmetry in the original problem.

This is a standard unification method for differential equations of Mathematical Physics as reductions of simpler equations in higher dimension (equations for monopoles, Hitchin equations,...).

C. 10d super Young-Mills as an "octonic theory". Now we want to apply the idea of trading symmetry and dimensions to explain why is there a gauge theory in 4d which has supersymmetry $\mathcal{N}=4$, i.e., a maximally supersymmetric gauge theory in 4d, and why do we encounter it in GL? The partial (intuitive) answer is that it comes from an *octonic* theory – the super Young-Mills on $\mathbb{R}^{1,9}$. Being an octonic theory it is very exceptional in that it posses the largest supersymmetry.

The conjectural relation to octonions can be motivated through the analogy with Hodge theory. The claim is that for each of the division algebras $D=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$, there exist

(1) a maximally supersymmetric geometry $\mathcal{G}$ based on D (D acts on tangent spaces), and
(2) a QFT $\mathcal{T}$ on a Minkowski space of dimension $(1,d+1)$ where d is the dimension of the algebra D,

which are related by:

- The reduction $\mathcal{T}'$ of $\mathcal{T}$ to dimension one (i.e,, to a Quantum Mechanics theory), gives rise to a geometry $\mathcal{G}$ – the data on M needed to construct the theory $\mathcal{T}'$ on M amounts precisely to to a $\mathcal{G}$-manifolds structure. So, the reduced theory $\mathcal{T}'$ "is" the QM view on the geometry $\mathcal{G}$.

This claim has an interesting and well known consequence.

Notice that the theory $\mathcal{T}$ on $\mathbb{R}^{1,d+1}$ being "natural", has a $Spin(1,d+1)$ symmetry. When we reduce to 1d, we use a factorization $\mathbb{R}^{1,d+1} \cong \mathbb{R}^{1,d} \times \mathbb{R}^{0,1}$ and this reduces the symmetry, so $\mathcal{T}'$carries symmetry group $Spin(1,d) \times Spin(0,1) = Spin(1,d)$.

Now consider the Hamiltonian view on $\mathcal{T}'$. Because $\mathcal{T}'$ is a super QM theory on a $\mathcal{G}$-manifold M, the Hilbert space of states for $\mathcal{T}'$ is something like the cohomology $H^*(M)$ of the manifold M. So we get a well known

Theorem. The cohomology of any compact $\mathcal{G}$-manifold carries an action of $Spin(1,d)$.

The meaning of this for division algebras $D = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ is :

(1) Kahler geometry is the maximally symmetric geometry based on $D = \mathbb{C}$. According to the above scheme it should be a reduction of a QFT $\mathcal{T}$ on $\mathbb{R}^{1,\dim(\mathbb{C})+1} = \mathbb{R}^{1,3}$, and this explains why for a compact Kahler manifold M there is an action of $Spin(1,\dim(\mathbb{C})) = Spin(1,2) \cong SL_2(\mathbb{R})$ on $H^*(M,\mathbb{R})$ ("Weak Lefschetz").
(2) hyperkähler geometry is the maximally symmetric geometry based on $D = \mathbb{H}$ and should be a reduction of a theory $\mathcal{T}$ on $\mathbb{R}^{1,\dim(\mathbb{H})+1} = \mathbb{R}^{1,5}$. This explains the action of $Spin(1,\dim(\mathbb{H})) = Spin(1,4) \cong Sp_2(\mathbb{R})$ on $H^*(M,\mathbb{R})$ for a compact hyperkähler manifold M.
(3) The theorem does not give anything for $D = \mathbb{R}$ since $Spin(1,1) = \mathbb{R}^*$.
(4) If there would exist "octonic geometry", i.e., a maximally symmetric geometry based on $D = \mathbb{O}$, it should be a reduction of a QFT $\mathcal{T}$ on $\mathbb{R}^{1,\dim(\mathbb{O})+1} = \mathbb{R}^{1,9}$. Then $Spin(1,\dim(\mathbb{O})) = Spin(1,8) \cong$? would act on cohomology of compact octonic manifolds.

While there are real, Kahler and hyperkähler manifolds, we do not know of octonic manifolds. If there were such manifolds they would give a QM theory with symmetry $Spin(1,8)$. However, for each compact U there does exist a Quantum Mechanical Problem with a $Spin(1,8)$ symmetry – the reduced to dimension one of the super Young-Mills theory on $\mathbb{R}^{(1,9)}$. So one can regard this QM theory as nature's substitute for octonic manifolds.

Remark. On the quantum level, "octonic geometry" in 10d involves incorporating (super)strings.

1.5. **Construction of relevant Quantum Field Theories via 2d sigma models.** So far, we have constructed a $\mathbb{P}^1{}_\Psi$ of 2d sigma models with target the Hitchin moduli $\mathcal{H}_C^U$. The existence of these theories is based on the hyperkähler structure on $\mathcal{H}$.[10] We will see

[10]One way to see this is to count the supersymmetries. Recall from 1.3.3 that the theory $\mathcal{T}''$ (the GL-twist) on a flat 4d manifold M has a $(4,4)$ supersymmetry (4 "left movers" and 4 "right movers".) Actually this is true for all manifolds of the form $\Sigma \times C$, Then the $(4,4)$-symmetry is inherited by the passage to 2d. However it is known that 2d sigma models with $4+4$ SUSY have hyperkähler targets ("hypermultiplets parameterize a hyperkähler manifold").

Question. The analogous hyperkähler structure on the cotangent bundle to a flag variety belongs to a family of "similar" hyperkähler structures. Is this true for $\mathcal{H}$, and if so is this useful at all?

that the hyperkähler structure actually gives a larger family of $\mathbb{P}^1\times\mathbb{P}^1$ supersymmetric sigma models through the mechanism of *generalized Kähler structures.*

1.5.1. *Hyperkähler structure on the Hitchin moduli $\mathcal{H}_C^U$.* **Hyperkähler manifolds.** Recall that on any hyperkähler manifold X each element u of the two sphere $S=\{u\in\mathbb{H};\ u^2=-1\}$ defines a Kähler structure $X(u)$ on X, it consists of a complex structure given by u and a symplectic structure ω_u. We parameterize $S=\{u\in\mathbb{H};\ u^2=-1\}$ by ${\mathbb{P}^1}_w$ by

$$I_w \overset{\text{def}}{=} \frac{1-w\overline{w}}{1+w\overline{w}}I \ + \ i\frac{w-\overline{w}}{1+w\overline{w}}J \ + \ \frac{w+\overline{w}}{1+w\overline{w}}K.$$

Say, $I_{\pm i}=\mp J,\ I_{\pm 1}=\pm K$ while $I_0=\ I$ and $I_\infty=\ -I$. One has $X(-1/\overline{w})\cong\overline{X(w)}$.

A geometric way to think of a hyperkähler manifold X is as a complex manifold $\widetilde{X}$ that fibers over ${\mathbb{P}^1}_w$ and the fiber at w is $X(I_w)$ (so this *twistor map* has a smooth trivialization). The fibers carry a holomorphic symplectic structure which is actually twisted by a line bundle on ${\mathbb{P}^1}_w$ but we can untwist it at I,J,K, then the holomorphic symplectic form takes form $\Omega_I=\omega_J+\omega_K$ (plus cyclic permutations of this).

B. Quaternionic vector space $\mathcal{W}$. One way to see the hyperkähler structure on $\mathcal{H}$ is to reconstruct $\mathcal{H}$ as a hyperkähler quotient of a very simple hyperkähler manifold – a quaternionic vector space. Let us fix a U-bundle E over C, and consider the vector space $\mathcal{W}$ of smooth one-forms on C with values in $\mathfrak{u}_{\mathbb{C}}={}^E\ \mathfrak{g}$, i.e., $\mathcal{W}\overset{\text{def}}{=}\mathrm{Hom}[T(C_{\mathbb{R}}),\ {}^E\mathfrak{u}_{\mathbb{C}}]=\Gamma[C,\Omega^1_{C_{\mathbb{R}}}\otimes{}^E\mathfrak{g}]$, where $C_{\mathbb{R}}$ means that we consider C as a real manifold. As above, we denote the real and imaginary part of $\mathcal{A}\in\mathcal{W}$ by $A=Re(\mathcal{A})$ and $\phi=Im(\mathcal{A})$.

One can combine the complex structures on C and on $\mathfrak{u}_{\mathbb{C}}=\mathfrak{g}$ in a certain way, to give $\mathcal{W}$ a structure of a hyperkähler manifold – a quaternionic vector space with an invariant metric $ds^2\overset{\text{def}}{=}\int_C\ Tr[A\wedge\overline{A}+\phi\wedge\overline{\phi}]$. The actions of $I,J,K\in\mathbb{H}$ on $\mathcal{W}$ are chosen so that the action of J does not depend on the complex structure on C. $\omega_I,\omega_K,\Omega_J$ do not depend on the complex structure of C.

C. Hyperkähler quotient construction of $\mathcal{H}$. The hyperkähler reduction of a hyperkähler manifold X with respect to a group K of symmetries is the hyperkähler manifold $({\mu_K}^{-1}0)/K$ where μ_K is the hyperkähler moment map. It has three components $\mu_I,\mu_J,\mu_K:X\to\mathfrak{k}^*$ which are moment maps for the action of K on $X(I),X(J),X(K)$, i.e., they are defined using the symplectic structures $(\omega_I,\omega_J,\omega_K)$ on X. We can combine two of these into $\nu_I\overset{\text{def}}{=}\mu_J+i\mu_K$ a complex moment map into $\mathfrak{k}^*_{\mathbb{C}}$.

Now, a hyperkähler structure on $\mathcal{H}$ is a consequence of[(11)]

[11]This seems to be a general construction of hyperkähler structure on a target. In a 2d super Young-Mills with $(4,4)$ symmetry and a gauge transform group $\mathcal{G}$:

- (i) Hypermultiplets parameterize a hyperkähler manifold $\mathcal{W}$,
- If $\mathcal{G}$ acts freely on $\mathcal{W}$ then the *low energy physics* is a sigma model with target the hyperkähler reduction of $\mathcal{W}$ by K. Actually, this logic is the historical origin of hyperkähler reduction.

Theorem. The hyperkähler reduction of $\mathcal{W}$ with respect to the action of the gauge group $\mathcal{G} = \mathrm{Aut}(E) = \Gamma(C,\ {}^{E}U)$ is isomorphic to $\mathcal{H}$.

Remark. The *twistor space* $\widetilde{\mathcal{W}}$ has a symmetry group $\mathbb{C}^*_w$ which acts on $\mathbb{P}^1{}_w$ and permutes the fibers of $\widetilde{\mathcal{W}}$ by $\lambda(I_w) = I_{\lambda^{-1}w}$. In particular it acts holomorphically on the fiber at 0. Its compact part $U(1)_w$ is a symmetry of the hyperkähler structure on $\mathcal{W}$. $\mathbb{C}^*_w$ acts on the component ϕ and uses the complex structure on C.[12] All of this descends to $\mathcal{H}$. So the hyperkähler structure on $\mathcal{H}$ is particularly simple – it has a $\mathbb{C}^*_w$-symmetry so there are only three different complex structures $\mathcal{H}(J_0) = \mathcal{H}(I)$, $\mathcal{H}(J_1) = \mathcal{H}(J)$ and $\mathcal{H}(J_\infty) = \mathcal{H}(-I) = \overline{\mathcal{H}(I)}$.

D. Comparison of metrics and B-fields from gauge theory and sigma models. We are really comparing the actions for 2d sigma models and our topological theories in 4d. The B-field in a sigma model is a closed 2-form B on the target. The B-field term in 2d action is $\int_\Sigma\ \Phi^* B$ for a map Φ on Σ. The kinetic energy term $\int_\Sigma\ |d\Phi|^2$ depends on the metric on the target, which is here a part of the hyperkähler structure on $\mathcal{H}$

Lemma. (a) The two metrics on $\mathcal{H}$, metric ds^2_{gau} given by the 4d gauge theory and the metric $ds^2_{hk} = ds^2$ given by the hyperkähler structure, are related by:

$$ds^2_{gau}\ =\ Im(\tau)\ ds^2_{hk}\ =\ \frac{4\pi}{e^2}\ ds^2_{hk}.$$

(b) The B-field of the sigma model is proportional to the theta angle coupling of gauge theory:

$$B\ =\ -Re(\tau)\ \omega_I\ =\ -\frac{\theta}{2\pi}\ \omega_I.$$

Proof. (a) The kinetic energy term for a gauge field in 4d is $\boldsymbol{I}_{kin} = -\frac{1}{2e^2}\ \int_{M=\Sigma\times C}\ |d^2y|\ |d^2z|\ |F|^2$. As Ω^1_M is the sum of pull-backs of Ω^1_Σ and Ω^1_C, we can write our gauge field as $A = A_\Sigma + A_C$ and decompose $|F|^2$ according to (Σ, C)-types. $(1,1)$-summand gives the *kinetic energy* of the corresponding map Φ in the sigma model. (The $(2,0)$ and $(0,2)$ summands are the kinetic and potential energy of A_Σ.) Comparison of kinetic energy formulas from 4d and 2d gives the relation of two metrics on $\mathcal{H}$.

(b) From the point of view of gauge theory the B-field comes from the topological term $\frac{i\theta}{8\pi^2}\ \int_{\Sigma\times C}\ Tr(F\wedge F)$.

1.5.2. $\mathbb{P}^1{}_w$ *of complex structures on* $\mathcal{H}$. We now consider $\mathcal{H}$ as a complex manifold with one of the complex structures I, J, K. The understanding of $\mathcal{H}$ in a complex structure I_w is based on the "coincidence" of the hyperkähler-reduction of $\mathcal{W}$ (the quotient of $\mu^{-1}0$

[12] $U(1)_w$ acts on the physical theory in 4d but does not preserve the $\mathbb{P}^1{}_\Psi$-family of topological twists.

by U-gauge transforms) and the $\mathcal{I}$-holomorphic reduction of $\mathcal{W}$ (for $\mathcal{I} = I$ this is the quotient of the subspace $\nu_0{}^{-1}0$ by a larger group of $U_{\mathbb{C}}$-gauge transforms).[13]

Lemma. **A.** The I-holomorphic reduction $\mathcal{H}(I)$ of $\mathcal{W}$ is the moduli of Higgs bundles, i.e., pairs $(\mathcal{E}, \varphi)$ of a holomorphic $\mathbb{G}$-bundle $\mathcal{E}$ on C and a holomorphic section of $\omega_C \otimes {}^{\mathcal{E}}\mathfrak{g}$.[14] Notice that this is the cotangent bundle to the moduli $\mathrm{Bun}_G(C) = \mathcal{M}_C^G$ of G-bundles on C.

B. The J-holomorphic reduction of $\mathcal{W}$ is the moduli of all $G_{\mathbb{C}}$-local systems on C.[15]

Proof. **B.** comes from the observation above that Hitchin equations impose flatness on $\mathcal{A}$. Part **A.** is based on

Sublemma. (a) Any connection $\nabla = d + A$ on a U-bundle E over a complex curve C defines a complexification $E_{\mathbb{C}}$ which is a a holomorphic $G_{\mathbb{C}}$-bundle.

(b) If the holomorphic moment map vanishes at (E, A, ϕ) then the $(1,0)$-part φ of $\phi = \varphi + \overline{\varphi}$ is holomorphic (as a section of $ad(E_{\mathbb{C}}) \otimes \omega_C$), for the complex structure given by the connection A.

Proof. (a) One defines $\overline{\partial}$-operator on $\mathbb{E}_{\mathbb{C}}$ by $d\overline{z} \stackrel{\text{def}}{=} \overline{dz} \nabla_{\bar{\partial}_C}$.

Remarks. (0) Recall that these are all interesting complex structures – for instance for $w \neq 0, \infty$, complex manifold $\mathcal{H}(I_w)$ is independent of w, so it is isomorphic to $\mathcal{H}(J) = \mathcal{LS}_{\check{G}}(C)$.

(1) $\mathbb{C}_w^*$ acts on on $\mathcal{H}(I)$ by $\lambda \cdot (\mathcal{E}, \varphi) \stackrel{\text{def}}{=} (\mathcal{E}, \lambda\varphi)$, the natural action on $T^*\mathrm{Bun}_G(C)$. So on (A, ϕ) the action involves only ϕ and can be reconstructed from $\phi = \varphi + \overline{\varphi}$.

1.5.3. *Hitchin fibration: $\mathcal{H}$ as a completely integrable system.* This has been covered above.

1.6. **Parameterization of theories.** We are interested in

[13] The results are generally close, with some "high codimension" differences that appear as differences in stability notions. The holomorphic reduction is easier to understand and gives an approximation of $\mathcal{H}$ in a complex structure $\mathcal{I}$.

[14] I will ignore the questions of stability as these are not germane for GL. For instance, with stabilities switched on $\mathcal{H}(I)$ is only the moduli of semistable Higgs bundles. Smooth points of $\mathcal{H}(I)$ are stable points, and the remaining stable points are orbifold singularities. The strictly semistable points give singularities worse then orbifolds. If genus is > 1 then the Higgs bundle moduli differs from $\mathcal{H}(I)$ in high codimension.

[15] [Corlette, Donaldson] $\mathcal{H}$ is the moduli of semistable maps $\pi_1 q(C) \to G_{\mathbb{C}}$, i.e., it consists of stable local systems and equivalence classes of semistable local systems.

1.6.1. *Parameters for 4d theories.* The physical theory in 4d depends on the choice of coupling constants. A topologically twisted theory then depends on both the coupling and the choice of a topological twist Very roughly one may try to imagine the choice of the twist as the choice of a "nature" of the theory, and the choice of a coupling as a choice of the strength in the theory. However, the parameters of the family of topological theories that we get, is not quite a product of coupling and topological parameters but a quotient of this product since the two ingredients interact.

(1) Coupling parameter $\tau \in \mathbb{H}$.
    The action (Lagrangian) of any gauge theory contains two basic summands – the kinetic energy term and the topological term. Each come with a choice of a coupling (i.e., a coefficient), and this gives two coupling parameters, the kinetic parameter g and the theta-angle θ[16] It turns out that the symmetries of the the dependence of the theory on the coupling are most obvious when one combines g, θ into one complex coupling parameter τ.
(2) Topological twisting parameter t.
    We will denote the parameter in the $\mathbb{P}^1$ of supersymmetry lines by t. So, the topologically twisted gauge theories in 4d are parameterized by $\mathbb{P}^1{}_t$.
(3) Topological theories parameter ("canonical parameter") Ψ
    It turns out that the topological theory depends on τ, t only through their combination Ψ. So, topologically twisted theories are parameterized by $\mathbb{P}^1{}_\Psi$.

1.6.2. *Parameterization of relevant geometries on a hyperkähler manifold.* The standard point of view on a hyperkähler manifold is that they have a $\mathbb{P}^1$ family of Kähler geometries, however there is a larger relevant class of geometries – the $\mathbb{P}^1 \times \mathbb{P}^1$ family of Generalized Kähler geometries.

(1) Kähler parameter w.
    The natural hyperkähler structure on the Hitchin moduli gives a family of Kähler structures on the Hitchin moduli, parameterized by the set S of square roots of -1 in $\mathbb{H}$. Topologically $S \cong S^2 \cong \mathbb{P}^1$ and we call this projective line $\mathbb{P}^1{}_w$.
    - $I \in S$, i.e., $w = 0$ gives Kähler structure of Higgs fields, i.e., $T^*Bun_G(C)$.
    - $J \in S$, i.e., $w = -i$ gives Kähler structure of local systems, i.e., $\mathcal{LS}_{G_\mathbb{C}}(C)G(C)$.
    - For $w \neq 0, \infty$ (including $K \in S$ corresponding to $w = 1$), all Kähler structures are identified through a certain G_m-action, in particular they are all isomorphic to $\mathcal{LS}_{G_\mathbb{C}}(C)G(C)$.
(2) Generalized Kähler parameter (w_+, w_-).
    However, the $\mathbb{P}^1{}_w$ of Kähler structures is a part of a larger family of Generalized Kähler structures associated to the hyperkähler structure. These

[16]One calls the coefficients of the topological term the *theta angle* because integrality of the topological term implies that quantum mechanically the parameter is only relevant modulo 2π.

are parameterized by ${\mathbb P^1}_w\times{\mathbb P^1}_w$ and Kähler structures appear as the diagonal ${\mathbb P^1}_w$. We denote the parameters (w_+,w_-).

1.6.3. *Parametrization of* $(2,2)$*-supersymmetric sigma models with a hyperkähler target.*

(1) Each of these Kähler structures produces two supersymmetric sigma models called A and B, that use only half of the Kähler structure – the symplectic and the complex structure.
(2) Actually, each of the above Generalized Kähler structures produces one supersymmetric sigma model. So we get a ${\mathbb P^1}_{w_+}\times{\mathbb P^1}_{w_-}$ of sigma models. It contains the $\mathbb P^1$ of complex sigma models (i.e., type B) as the diagonal $\Delta_{{\mathbb P^1}_w}$ and the $\mathbb P^1$ of symplectic sigma models (i.e., type A) as the antidiagonal $\Delta^-_{{\mathbb P^1}_w}$.
(3) Moduli of sigma models parameter q.

    Different geometries may give equivalent theories and here the repetitions are counted by the diagonal action of $G_{m,w}$ on ${\mathbb P^1}_{w_+}\times{\mathbb P^1}_{w_-}$. Therefore, the moduli of theories is parameterized by ${\mathbb P^1}_q$ for

$$q \stackrel{\text{def}}{=} \frac{w_+}{w_-}.$$

    (This is actually just a geometric invariant theory quotient so there may exist a finer moduli?)

1.6.4. *The relation between two families of theories.*

(1) The topological twist determines the Generalized Kähler structure by

$${\mathbb P^1}_t \ni t\mapsto(-t,t^{-1}) \ \in\ {\mathbb P^1}_{w_+}\times{\mathbb P^1}_{w_-}.$$

(2) Therefore, the topological twist determines the equivalence class of theories by

$$q \stackrel{\text{def}}{=} \frac{w_+}{w_-} = \frac{-t}{t^{-1}} = -t^2.$$

    So, the theories for $\pm t$ are equivalent.
(3) The role of the canonical parameter Ψ is that
    - When $\Psi\neq\infty$, i.e., $t\neq\pm i$, we get a theory which is (equivalent to) an A-model and
    - Ψ determines its complexified Kähler class

$$[B+i\omega] = -\Psi\cdot[\omega_I].$$

1.7. **Langlands correspondence.**

1.7.1. *Interesting categories of branes on a hyperkähler manifold.* On $\mathcal{H}$ we have complex structures IJK and for each a category of A-branes and B-branes. However, they all lie in one large category of branes for the sigma model where elementary objects are just connections on submanifolds. The subcategories consist of branes which are fixed by the supersymmetries corresponding to the complex structure and the choice of a topological sigma model (*supersymmetric branes*). In particular, we can take intersections of any of these six categories, i.e., look at branes with several supersymmetries.

1.7.2. *S-duality.* This is a conjectural claim that – in our case – categories of branes on $\mathcal{H}(G,C)$ and $\mathcal{H}(\check{G},C)$ are equivalent. Then one checks that, according to standard properties expected from S-duality, it exchanges some complex structures and models:

Lemma. (a) S-duality exchanges

(1) $I_B \cong I_B$
(2) $I_A \cong I_A$
(3) $J_B \cong K_A$
(4) $J_A \cong K_B$
(5) $K_B \cong J_A$
(6) $K_A \cong J_B$

(b) The restriction of S-duality to the case (1) is[(17)] the Fourier-Mukai transform.[(18)]

(c) Restriction to (2) is its (less known) analogue for A-branes.

1.8. **Hecke operators as singular monopoles (*t'Hooft operators*).**

1.8.1. *The content of Langlands correspondence.* Recall that the traditional content of a Langlands conjecture is that $G_{\mathbb{C}}$-local systems on a curve correspond to D-modules on $Bun_{\check{G}_{\mathbb{C}}}(C)$ which are eigenvectors for Hecke operators. These Hecke operators are parameterized by $Irr(\check{G})$.

In the categorical formulation of Beilinson-Drinfeld, a local system is viewed as a coherent sheaf on the moduli of local systems (the structure sheaf of a point), and then it is clearly an eigenvector for the operation of tensoring of coherent sheaves with vector bundles. In particular it is an eigenobject for the tautological family of vector bundles associated to irreducible representations of $\check{G}$. From this point of view the content of the Langlands conjecture becomes the matching of two families of operators and of their eigenobjects.

17 In the "generic case", i.e., where Hitchin fibers are dual tori.

18 Hitchin fibrations $\mathcal{H}^G \to B \cong \check{B} \leftarrow \mathcal{H}^{\check{G}}$ are holomorphic and (generically) dual for ye complex structure I.

1.8.2. *L-functions.* Actually the Langlands correspondence should have an additional key property that it *preserves the L-functions.* However this is implicitly included in the KW-approach, i.e., in lifting GL duality into a duality of two Quantum Field Theories. The notion of equivalence of Quantum Field Theories means a correspondence that *preserves the correlators* and this is what L-functions are. Now our difficulties in defining L-functions may "disappear" by turning into the problem of making sense of Feynman integrals.

1.8.3. *S-duality is more symmetric then Langlands duality.* In the formulation of S-duality groups G and $\check{G}$ are in symmetric roles – one has equivalence of categories of branes on the Hitchin moduli of U and $\check{U}$. The asymetry in GL is caused by specializing the category of branes on one of the sides to branes with a certain supersymmetry, this leads to a specialization on the other side which is no longer of the same nature.

1.8.4. *Wilson and t'Hooft operators.* An aspect of this symmetry is that in the theory associated to U there are both Wilson operators parameterized by $Irr(G)$ and t'Hooft operators parameterized by $Irr(\check{G})$. Now as S-duality exchanges the theories for G and $\check{G}$, it also exchanges these two kinds of operators (this is Kapustin's formulation of S-duality).

When we specialize the sigma model branes to topological theories parameterized by Ψ, for a given Ψ at most one kind of operators survives, and not all operators of this kind either.

There are two values of Ψ that play role in GL, $\Psi = 0, \infty$. At $\Psi = 0$ the branes of the theory for U specialize to coherent sheaves in $\mathcal{LS}_{\check{G}}(C)$. Here Wilson operators survive and they take the form of tensoring with canonical vector bundles on $\mathcal{LS}_G(C)$. This is the more obvious part of the picture.

At $\Psi = \infty$ the theory for U specializes to Fukaya category on $\mathrm{Bun}_G(C)$ Here t'Hooft operators survive and they get interpreted as cetain singularities of monopoles which are given by G-bundles on $\mathbb{P}^1$. Conjecturally, these are microlocalizations of Hecke operators. This is the deeper part.

1.8.5. *KW-construction and operators.* Here we go once again through the the KW-construction of the Langlands correspondence and we follow what happens with operators. We concentrate on what happens to a $\check{G}$-local system, i.e., to a ponit of $\mathcal{LS}_{\check{G}}(C)$, which we view as a point brane. There is a funny feature of this brane that it has some supersymmetry for all three of the standard Kaḧler structures I, J, K on the Hitchin moduli – for all three Kaḧler structures this is a B-brane. As a consequence the branes we produce from it will again have SUSY for I, J, K but for each of these the type of SUSY may change and we indicate these changes bellow.

(1) **BBB**. We start with a G-local system $\mathcal{E}$ on C. We think of it as a point p in $\mathcal{LS}_G(C)$, hence in the Hitchin moduli. Then it can be viewed as a brane $(p, \mathcal{O}_p, d)$ (point with the trivial line bundle with a trivial connection on it), and

it is clearly holomorphic in all complex structures, so it is of type BBB for the complex structures IJK.

(2) **BAA**. Applying the S-duality to the brane $(p, \mathcal{O}_p, d)$ we get a brane $\mathbb{S}(p, \mathcal{O}_p, d)$ which we know is of type BAA. However, we know what it is since $\mathbb{S}(p, \mathcal{O}_p, d)$ can be thought of as the Fourier-Mukai transform $\mathcal{F}\mathcal{O}_p$ of the coherent sheaf $\mathcal{O}_p$ in the complex structure I. If p lies above a point b in the Hitchin base B, and we identify b with $\check{b} \in \check{B}$, then the transform of $\mathcal{O}_\mathcal{E}$ is an I-holomorphic line bundle $L_\mathcal{E}$ on the fiber $(\mathcal{H}^{\check{G}})_{\check{b}}$ of the Hitchin fibration for $\check{G}$.

Moreover, the property of $\mathcal{O}\mathcal{E}$ of being an eigenobject for tensoring with vector bundles translates in the property of the corresponding A-brane of being an eigenobject for t'Hooft operators.

(3) **ABA**. Now, we pass to D-modules using observation $(*)$.It says that the A-brane $L_\mathcal{E}$ for K. is a microlocalization of a D-module $\mathcal{L}(\mathcal{E})$, the Langlands transform of $\mathcal{E}$.

Again, the property of the A-brane $L_\mathcal{E}$ of being an eigenobject for t'Hooft operators. translates into the property of the corresponding D-module of being an eigenobject for Hecke operators.

Appendix A. **Electro-magnetic Duality and S-duality**

This is the origin and the baby case of S-duality. On the classical level electromagnetism is decribed by Maxwell's equations. On the quantum level, electromagnetism will be described in terms of the $U(1)$ gauge theory. The S-duality is then the extension of the EM duality to the gauge theory for the general group G.

A.1. **Classical theory of electromagnetism: fields and Maxwell's equations.**

A.1.1. *Electro-magnetic fields.*

- Electric field E is a vectorfield on $\mathbb{R}^3$ or $\mathbb{R}^{3,1}$.
- Magnetic field B is a 1-form on $\mathbb{R}^3$ or $\mathbb{R}^{3,1}$.
- Relativistically, E and B combine into a 2-form on $\mathbb{R}^3$

$$F \stackrel{\text{def}}{=} dt \wedge E \ + \ \star B,$$

for the Hodge star operator $\star$ on $\mathbb{R}^3$.

A.1.2. *Maxwell's equations in vacuum.* The equations are

$$dF = 0 \quad \text{and} \quad d(\star F) = 0.$$

A.1.3. *Classical duality.* The original 19th century version of Electro-magnetic Duality is the obvious symmetry of Maxwell's equations

$$F \mapsto \ \star F \quad \text{and} \quad \star F \mapsto \ -F.$$

The signs are chosen so that the symmetry is $\star$-equivariant (use $\star^2 = -1$).

A.2. **Quantum mechanical setting for electromagnetism.** In the QM formulation the above obvious symmetry between F and $\star F$ is broken:

- Equation $dF = 0$ is interpreted as

  F is the curvature ∇^2 of a connection $\nabla = d + A$ on a line bundle L over the spacetime M_4.

  This interpretations is the solution of $dF = 0$ and this equation now disappears (it becomes trivial: "Bianchi identity").
- In this geometric setting physics consists of one differential equation $d(\star F) = 0$, so in the Lagrangian approach this has to be the criticality equation for a Lagrangian.

A.2.1. *Kinetic action.* In order to interpret Maxwell 's second equation $d(\star F) = 0$ as the criticality equation of an action, we integrate it to a "kinetic" action

$$I_{kin} \stackrel{\text{def}}{=} \int_{M_4} ||F||^2 \ = \int_{M_4} F \wedge \star F.$$

This is analogous to harmonic maps, the difference is only that one uses the "kinetic energy" of a connection rather then a "kinetic energy" of a map.

A.2.2. *Topological action.* However, in 4d there is a possibility to enrich the action with a topological term

$$I_{top} \stackrel{\text{def}}{=} \int_{M_4} F \wedge F = \ i\theta \int_{M_4} c_1(L)^2.$$

Remark. The topological term does not influence the criticality equations (it is locally constant in F !), hence it does not influence the classical solutions.

A.2.3. *Total action.* The total action is a combination

$$I \stackrel{\text{def}}{=} \frac{1}{4e^2} I_{kin} + i \ \frac{\theta}{(2\pi)^2} I_{top} = \ \frac{1}{4e^2} \int_{M_4} F \wedge \star F \ + \ i \ \frac{\theta}{(2\pi)^2} \int_{M_4} F \wedge F.$$

A.2.4. *Couplings e, θ.* Physically, e is the charge of an electron. θ is an angle variable. We combine them into one complex coupling

$$\tau \stackrel{\text{def}}{=} \frac{\theta}{2\pi} + i \frac{4\pi}{e^2}.$$

Remark. Couplings are a purely *quantum phenomena*. Notice that they do not influence the classical solutions, i.e., the criticality equations (remark A.2.1). However, they certainly influence the action and hence the quantum theory.

A.2.5. *Path integral.* The partition function is an integral over the space $\mathcal{A}$ of connections modulo the group $\mathcal{G}$ of gauge transformations

$$Z = \langle 1 \rangle \stackrel{\text{def}}{=} \int_{\mathcal{A}/\mathcal{G}} dA \quad e^{-I(A)}.$$

Notice that the integral depends on connectionns not only on curvatre – the integrand depends only on the curvature F, however the measure does depend on A.

A.3. **Quantum Electro-Magnetic Duality.** The obvious symmetry of the classical picture of EM force, i.e., symmetry of Maxwell's equations, does not disappear in the in the quantum mechanical setting but here it becomes a more complicated symmetry denoted S.

(1) For one thing the new symmetry S is non-clasical since it notices the coupling. It acts on the complex coupling by $S(\tau) = -1/\tau$.
(2) Still, S exchanges F and $\star F$, so can be viewed as the quantum version of the classical EM duality.
(3) S is given by the *Fourier transform* in the space of all connections. The action on τ is roughly a version of the Fourier transform formula

$$\int \frac{dx}{\sqrt{2\pi}} e^{ixy} \, e^{-\lambda x^2/2} = e^{-\lambda^{-1} y^2/2}$$

which inverts λ.
(4) S exchanges F and $\star F$, so it can be viewed as the quantum version of the classical EM duality.
(5) There is an additional classical synmmetry T which shifts the θ-angle $\theta \mapsto \theta + 2\pi$. Equivalently, $T(\tau) = \tau + 1$. This affects the action but not the path integral.[19]
(6) Quantum symmetry S combines with a classical synmmetry T to generate a symmetry group $\Gamma \cong SL_2(\mathbb{Z})$ called the S-duality group (or Hecke group).

A.4. **S-duality.** A conjectural quantum symmetry S is a generalization of the quantum EM duality. It arises when one considers gauge theory for arbitrary compact groups U. Then the electric charges are irreducible represntations of U and the magnetic charges are irreducible represntations of $\check{U}$.[20]

[19] At least not when M is closed, then the integral produces an integer.
[20] In the case $U \cong U(1) \cong \check{U}$ both kinds of charges can be thought of simply as integers, i.e., integer multiples of one basic charge.

Conjecture. There is a quantum symmetry S which

(1) "inverts" τ by $S(\tau) = -1/l_U\tau$,[(21)]
(2) exchanges U and $\check{U}$ and therefore also the electric and magnetic charges.
(3) combines with a classical synmmetry T (shift of the θ-angle), to give a finite index subgroup Γ of $SL_2(\mathbb{Z})$ – the S-duality group of U.

One can try to make this more precise by introducing two copies $\mathbb{H}$ and $\mathbb{H}'$ of the upper half plane, related to U and $\check{U}$. Then T acts the same on both by $\tau\mapsto\tau+1$ and $\tau'\mapsto\tau'+1$, However, S exchanges the two copies by

$$S = \begin{pmatrix} 0 & 1/\sqrt{n} \\ -\sqrt{n} & 0 \end{pmatrix}, \quad \text{i.e.,} \quad S(\tau) = \frac{-1}{n\tau'}, \quad S(\tau') = \frac{-1}{n\tau}.$$

If $\check{\mathfrak{g}} \cong \mathfrak{g}$ one can pretend that $\mathbb{H}' = \mathbb{H}$.

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[21] l_U is the lacing for both $\mathfrak{u}$ and $\check{\mathfrak{u}}$.