CHAPTER 2

Theories, Lagrangians and counterterms

1. Introduction

In this chapter, we will make precise the definition of quantum field theory we sketched in Chapter 1. Then, we will show the main theorem:

**Theorem A.** Let \( \mathcal{T}^{(n)}(M) \) denote the space of scalar field theories on a manifold \( M \), defined modulo \( \hbar^{n+1} \).

Then \( \mathcal{T}^{(n+1)}(M) \to \mathcal{T}^{(n)}(M) \) is (in a canonical way) a principal bundle for the space of local action functionals.

Further, \( \mathcal{T}^{(0)}(M) \) is canonically isomorphic to the space of local action functionals which are at least cubic.

This theorem has a less natural formulation, depending on an additional choice, that of a renormalization scheme. A renormalization scheme is an object of a “motivic” nature, defined in Section 9.

**Theorem B.** The choice of a renormalization scheme leads to a section of each principal bundle \( \mathcal{T}^{(n+1)}(M) \to \mathcal{T}^{(n)}(M) \), and thus to an isomorphism between the space of theories and the space of local action functionals of the form \( \sum \hbar^i S_i \), where \( S_0 \) is at least cubic.

1.1. Let me summarize the contents of this chapter.

The first few sections explain, in a leisurely fashion, the version of the renormalization group flow we use throughout this book. Sections 2 and 4 introduce the heat kernel version of high-energy cut-off we will use throughout the book. Section 3 contains a general discussion of Feynman graphs, and explains how certain finite dimensional integrals can be written as sums over graphs. Section 5 explains why infinities appear in the naive functional integral formulation of quantum field theory. Section
6 shows how the weights attached to Feynman graphs in functional integrals can be interpreted geometrically, as integrals over spaces of maps from graphs to a manifold.

In Section 7, we finally get to the precise definition of a quantum field theory and the statement of the main theorem. Section 8 gives a variant of this definition which doesn’t rely on the heat kernel, but instead works with an arbitrary parametrix for the Laplacian. This variant definition is equivalent to the one based on the heat kernel. Section 9 introduces the concept of renormalization scheme, and shows how the choice of renormalization scheme allows one to extract the singular part of the weights attached to Feynman graphs. Section 10 uses this to construct the local counterterms associated to a Lagrangian, which are needed to render the functional integral finite. Section 11 gives the proof of theorems A and B above.

Finally, we turn to generalizations of the main results. Section 13 shows how everything generalizes, mutatis mutandis, to the case when our fields are no longer just functions, but sections of some vector bundle. Section 14 shows how we can further generalize to deal with theories on non-compact manifolds, as long as an appropriate infrared cut-off is introduced.

2. The effective interaction and background field functional integrals

As in the introduction, a quantum field theory in our Wilsonian definition will be given by a collection of effective actions, related by the renormalization group flow. In this section we will write down a version of the renormalization group flow, based on the effective interaction, which we will use throughout the book.

2.1. Let us assume that our energy \( \Lambda \) effective action can be written as

\[
S[\Lambda](\phi) = -\frac{1}{2} \left\langle \phi, (D + m^2)\phi \right\rangle + I[\Lambda](\phi)
\]

where:

1. The function \( I[\Lambda] \) is a formal series in \( \hbar \), \( I[\Lambda] = I_0[\Lambda] + \hbar I_1[\Lambda] + \cdots \), where the leading term \( I_0 \) is at least cubic. Each \( I_i \) is a formal power series on the vector space \( C^\infty(M) \) of fields (later, I will explain what this means more precisely).

The function \( I[\Lambda] \) will be called the effective interaction.

2. \( \langle , \rangle \) denotes the \( L^2 \) inner product on \( C^\infty(M, \mathbb{R}) \) defined by \( \langle \phi, \psi \rangle = \int_M \phi \psi \).
(3) $D$ denotes\(^1\) the Laplacian on $M$, with signs chosen so that the eigenvalues of $D$ are non-negative; and $m \in \mathbb{R}_{>0}$.

Recall that the renormalization group equation relating $S[\Lambda]$ and $S[\Lambda']$ can be written

$$S[\Lambda'](\phi_L) = \hbar \log \left( \int_{\Phi_H \in C^\infty(M)} e^{S[\Lambda](\phi_L + \phi_H)/\hbar} \right).$$

We can rewrite this in terms of the effective interactions, as follows. The spaces $C^\infty(M)_{<\Lambda'}$ and $C^\infty(M)_{[\Lambda',\Lambda)}$ are orthogonal. It follows that

$$S[\Lambda](\phi_L + \phi_H) = -\frac{1}{2} \left( \phi_L, (D + m^2)\phi_L \right) - \phi_L, (D + m^2)\phi_H + I[\Lambda](\phi_L + \phi_H).$$

Therefore the effective interaction form of the renormalization group equation (RGE) is

$$I[\Lambda'](a) = \hbar \log \left( \int_{\Phi \in C^\infty(M)} \exp \left( -\frac{1}{2\hbar} \left( \phi, (D + m^2)\phi \right) + \frac{1}{\hbar} I[\Lambda](\phi + a) \right) \right).$$

Note that in this expression the field $a$ no longer has to be low-energy. We obtain a variant of the renormalization group equation by considering effective interactions $I[\Lambda]$ which are functionals of all fields, not just low-energy fields, and using the equation above. This equation is invertible; it is valid even if $\Lambda' > \Lambda$.

We will always deal with this invertible effective interaction form of the RGE. Henceforth, it will simply be referred to as the RGE.

2.2. We will often deal with integrals of the form

$$\int_{x \in U} \exp \left( \Phi(x)/\hbar + I(x + a)/\hbar \right)$$

over a vector space $U$, where $\Phi$ is a quadratic form (normally negative definite) on $U$. We will use the convention that the “measure” on $U$ will be the Lebesgue measure normalised so that

$$\int_{x \in U} \exp \left( \Phi(x)/\hbar \right) = 1.$$

Thus, the measure depends on $\hbar$.

\(^1\) The symbol $\Delta$ will be reserved for the BV Laplacian
2.3. Normally, in quantum field theory textbooks, one starts with an action functional

\[ S(\phi) = -\frac{1}{2} \left< \phi, (D + m^2)\phi \right> + I(\phi), \]

where the interacting term \( I(\phi) \) is a local action functional. This means that it can be written as a sum

\[ I(\phi) = \sum h^i I_{i,k}(\phi) \]

where

\[ I_{i,k}(a) = \sum_{j=1}^{s} \int_{M} D_{1,j}(a) \cdots D_{k,j}(a) \]

and the \( D_{i,j} \) are differential operators on \( M \). We also require that \( I(\phi) \) is at least cubic modulo \( \hbar \).

As I mentioned before, the local interaction \( I \) is supposed to be thought of as the scale \( \infty \) effective interaction. Then the effective interaction at scale \( \Lambda \) is obtained by applying the renormalization group flow from energy \( \infty \) down to energy \( \Lambda \). This is expressed in the functional integral

\[ I[\Lambda](a) = \hbar \log \left( \int_{\phi \in C^\infty(M,\Lambda,\infty)} \exp \left( -\frac{1}{2\hbar} \left< \phi, (D + m^2)\phi \right> + \frac{1}{\hbar} I(\phi + a) \right) \right). \]

This functional integral is ill-defined.

3. Generalities on Feynman graphs

In this section, we will describe the Feynman graph expansion for functional integrals of the form appearing in the renormalization group equation. This section will only deal with finite dimensional vector spaces, as a toy model for the infinite dimensional functional integrals we will be concerned with for most of this book. For another mathematical description of the Feynman diagram expansion in finite dimensions, one can consult, for example, [Man99].

3.1. Definition. A stable graph is a graph \( \gamma \), possibly with external edges (or tails); and for each vertex \( v \) of \( \gamma \) an element \( g(v) \in \mathbb{Z}_{\geq 0} \), called the genus of the vertex \( v \); with the property that every vertex of genus 0 is at least trivalent, and every vertex of genus 1 is at least 1-valent (0-valent vertices are allowed, provided they are of genus > 1).  

\(^2\)The term “stable” comes from algebraic geometry, where such graphs are used to label the strata of the Deligne-Mumford moduli space of stable curves.
If $\gamma$ is a stable graph, the genus $g(\gamma)$ of $\gamma$ is defined by
\[ g(\gamma) = b_1(\gamma) + \sum_{v \in V(\gamma)} g(v) \]
where $b_1(\gamma)$ is the first Betti number of $\gamma$.

More formally, a stable graph $\gamma$ is determined by the following data.

1. A finite set $H(\gamma)$ of half-edges of $\gamma$.
2. A finite set $V(\gamma)$ of vertices of $\gamma$.
3. An involution $\sigma : H(\gamma) \rightarrow H(\gamma)$. The set of fixed points of this involution is denoted $T(\gamma)$, and is called the set of tails of $\gamma$. The set of two-element orbits is denoted $E(\gamma)$, and is called the set of edges.
4. A map $\pi : H(\gamma) \rightarrow V(\gamma)$, which sends a half-edge to the vertex to which it is attached.
5. A map $g : V(\gamma) \rightarrow \mathbb{Z}_{\geq 0}$.

From this data we construct a topological space $|\gamma|$ which is the quotient of
\[ V(\gamma) \amalg (H(\gamma) \times [0, \frac{1}{2}]) \]
by the relation which identifies $(h, 0) \in H(\gamma) \times [0, \frac{1}{2}]$ with $\pi(h) \in V(\gamma)$; and identifies $(h, \frac{1}{2})$ with $(\sigma(h), \frac{1}{2})$. We say $\gamma$ is connected if $|\gamma|$ is. A graph $\gamma$ is stable, as above, if every vertex $v$ of genus 0 is at least trivalent, and every vertex of genus 1 is at least univalent.

We are also interested in automorphisms of stable graphs. It is helpful to give a formal definition. An element of $g \in \text{Aut}(\gamma)$ of the group $\text{Aut}(\gamma)$ is a pair of maps $H(g) : H(\gamma) \rightarrow H(\gamma), V(g) : V(\gamma) \rightarrow V(\gamma)$, such that $H(g)$ commutes with $\sigma$, and such that the diagram
\[
\begin{array}{ccc}
H(\gamma) & \xrightarrow{H(g)} & H(\gamma) \\
\downarrow & & \downarrow \\
V(\gamma) & \xrightarrow{V(g)} & V(\gamma)
\end{array}
\]
commutes.

3.2. Let $U$ be a finite-dimensional super vector space, over a ground field $\mathbb{K}$. Let $\mathcal{O}(U)$ denote the completed symmetric algebra on the dual vector space $U^\vee$. Thus, $\mathcal{O}(U)$ is the ring of formal power series in a variable $u \in U$. 
Let 
\[ \mathcal{O}^+ (U[[h]]) \subset \mathcal{O} (U[[h]]) \]
be the subspace of those functionals which are at least cubic modulo \( h \).

For an element \( I \in \mathcal{O} (U[[h]]) \), let us write
\[ I = \sum_{i,k \geq 0} h^i I_{i,k} \]
where \( I_{i,k} \in \mathcal{O} (U) \) is homogeneous of degree \( k \) as a function of \( u \in U \).

If \( f \in \mathcal{O} (U) \) is homogeneous of degree \( k \), then it defines an \( S_k \)-invariant linear map
\[ D^k f : U^\otimes k \to \mathbb{K} \]
\[ u_1 \otimes \cdots \otimes u_k \to \left( \frac{\partial}{\partial u_1} \cdots \frac{\partial}{\partial u_k} f \right) (0). \]

Thus, if we expand \( I \in \mathcal{O} (U)[[h]] \) as a sum \( I = \sum h^i I_{i,k} \) as above, then we have collection of \( S_k \) invariant elements \( D^k I_{i,k} \in (U^\vee)^\otimes k \).

Let \( \gamma \) be a stable graph, with \( n \) tails. Let \( \phi : \{1, \ldots, n\} \cong T(\gamma) \) be an ordering of the set of tails of \( \phi \). Let \( P \in \text{Sym}^2 U \subset U^\otimes 2 \), let \( I \in \mathcal{O}^+ (U[[h]]) \), and let \( a_1, \ldots, a_n \in U \) By contracting the tensors \( P \) and \( a_i \) with the dual tensor \( I \) according to a rule given by \( \gamma \), we will define
\[ w_{\gamma,\phi}(P,I)(a_1, \ldots, a_{T(\gamma)}) \in \mathbb{K}. \]

The rule is as follows. Let \( H(\gamma) \), \( T(\gamma) \), \( E(\gamma) \), and \( V(\gamma) \) refer to the sets of half-edges, tails, internal edges, and vertices of \( \gamma \), respectively. Recall that we have chosen an isomorphism \( \phi : T(\gamma) \cong \{1, \ldots, n\} \). Putting a propagator \( P \) at each internal edge of \( \gamma \), and putting \( a_i \) at the \( i \)th tail of \( \gamma \), gives an element of
\[ U^\otimes E(\gamma) \otimes U^\otimes E(\gamma) \otimes U^\otimes T(\gamma) \cong U^\otimes H(\gamma). \]

Putting \( D^k I_{i,k} \) at each vertex of valency \( k \) and genus \( i \) gives us an element of
\[ \text{Hom}(U^\otimes H(\gamma), \mathbb{K}). \]
Contracting these two elements yields the weight \( w_{\gamma,\phi}(P,I)(a_1, \ldots, a_n) \).

Define a function
\[ w_{\gamma}(P,I) \in \mathcal{O} (U) \]
by
\[ w_{\gamma}(P,I)(a) = w_{\gamma,\phi}(P,I)(a, \ldots, a) \]
The first few graphs in the expansion of $W(P, I)$. The variable $a \in U$ is placed at each external edge.

where $\phi$ is any ordering of the set of tails of $\gamma$. Note that $w_\gamma(P, I)$ is homogeneous of degree $n$, and has the property that, for all $a_1, \ldots, a_n \in U$,

$$\frac{\partial}{\partial a_1} \cdots \frac{\partial}{\partial a_n} w_\gamma(P, I) = \sum_{\phi \in \{1, \ldots, n\} \equiv T(\gamma)} w_{\gamma, \phi}(P, I)(a_1, \ldots, a_n)$$

where the sum is over ways of ordering the set of tails of $\gamma$.

Let $v_{i,k}$ denote the graph with one vertex of genus $i$ and valency $k$, and with no internal edges. Then our definition implies that

$$w_{v_{i,k}}(P, I) = k! I_{i,k}.$$  

3.3. Now that we have defined the function $w_\gamma(P, I) \in \mathcal{O}(U)$, we will arrange them into a formal power series.
Let us define
\[ W(P, I) = \sum_{\gamma} \frac{1}{|\text{Aut}(\gamma)|} \hbar^{g(\gamma)} w_\gamma(P, I) \in \mathcal{O}^+(U)[[\hbar]] \]
where the sum is over connected stable graphs \( \gamma \), and \( g(\gamma) \) is the genus of the graph \( \gamma \). The condition that all genus 0 vertices are at least trivalent implies that this sum converges. Figure 1 illustrates the first few terms of the graphical expansion of \( W(P, I) \).

Our combinatorial conventions are such that, for all \( a_1, \ldots, a_k \in U \),
\[ \left( \frac{\partial}{\partial a_1} \cdots \frac{\partial}{\partial a_k} W(P, I) \right)(0) = \sum_{\gamma, \phi} \frac{\hbar^{g(\gamma)}}{|\text{Aut}(\gamma, \phi)|} w_{\gamma, \phi}(P, I)(a_1, \ldots, a_k) \]
where the sum is over graphs \( \gamma \) with \( k \) tails and an isomorphism \( \phi : \{1, \ldots, k\} \cong T(\gamma) \), and the automorphism group \( \text{Aut}(\gamma, \phi) \) preserves the ordering \( \phi \) of the set of tails.

**3.3.1 Lemma.**
\[ W(0, I) = I. \]

**Proof.** Indeed, when \( P = 0 \) only graphs with no edges can contribute, so that
\[ W(0, I) = \sum_{i,k} \frac{\hbar^i}{|\text{Aut}(v_{i,k})|} w_{v_{i,k}}(0, I). \]
Here, as before, \( v_{i,k} \) is the graph with a single vertex of genus \( i \) and valency \( k \), and with no internal edges. The automorphism group of \( v_{i,k} \) is the symmetric group \( S_k \), and we have seen that \( w_{v_{i,k}}(0, I) = k! I_{i,k} \). Thus, \( W(0, I) = \sum_{i,k} \hbar^i I_{i,k} \), which (by definition) is equal to \( I \).

\[ \square \]

**3.4.** As before, let \( P \in \text{Sym}^2 U \), and let us write \( P = \sum P' \otimes P'' \). Define an order two differential operator \( \partial_P : \mathcal{O}(U) \to \mathcal{O}(U) \) by
\[ \partial_P = \frac{1}{2} \sum \frac{\partial}{\partial P'} \frac{\partial}{\partial P''} \]
A convenient way to summarize the Feynman graph expansion \( W(P, I) \) is the following.

**3.4.1 Lemma.**
\[ W(P, I)(a) = \hbar \log \{ \exp(\hbar \partial_P) \exp(I/\hbar) \} (a) \in \mathcal{O}^+(U)[[\hbar]]. \]
The expression \( \exp(h\partial P) \exp(I/h) \) is the exponential of a differential operator on \( U \) applied to a function on \( U \); thus, it is a function on \( U \).

**Proof.** We will prove this by first verifying the result for \( P = 0 \), and then checking that both sides satisfy the same differential equation as a function of \( P \). When \( P = 0 \), we have seen that

\[
W(0, I) = I
\]

which of course is the same as \( h \log \exp(h\partial P) \exp(I/h) \).

Now let us turn to proving the general case. It is easier to consider the exponentiated version: so we will actually verify that

\[
\exp \left( h^{-1}W(P, I) \right) = \exp(h\partial P) \exp(I/h).
\]

We will do this by verifying that, if \( \varepsilon \) is a parameter of square zero, and \( P' \in \text{Sym}^2 U \),

\[
\exp \left( h^{-1}W(P + \varepsilon P', I) \right) = (1 + \varepsilon \partial P) \exp \left( h^{-1}W(P, I) \right).
\]

Let \( a_1, \ldots, a_k \in U \), and let us consider

\[
\left( \frac{\partial}{\partial a_1} \cdots \frac{\partial}{\partial a_k} \exp \left( h^{-1}W(P, I) \right) \right)(0).
\]

It will suffice to prove similar differential equations for this expression.

It follows immediately from the definition of the weight function \( w_\gamma(P, I) \) of a graph that

\[
\left( \frac{\partial}{\partial a_1} \cdots \frac{\partial}{\partial a_k} \exp \left( h^{-1}W(P, I) \right) \right)(0) = \sum_{\gamma, \phi} \frac{h^{\gamma}(\gamma)}{|\text{Aut}(\gamma, \phi)|} w_{\gamma, \phi}(P, I)(a_1, \ldots, a_k)
\]

where the sum is over all possibly disconnected stable graphs \( \gamma \) with an isomorphism \( \phi: \{1, \ldots, k\} \cong T(\gamma) \). The automorphism group \( \text{Aut}(\gamma, \phi) \) consists of those automorphisms preserving the ordering \( \phi \) of the set of tails of \( \gamma \).

Let \( \varepsilon \) be a parameter of square zero, and let \( P' \in \text{Sym}^2 U \). Let us consider varying \( P \) to \( P + \varepsilon P' \). We find that

\[
\frac{d}{d\varepsilon} \left( \frac{\partial}{\partial a_1} \cdots \frac{\partial}{\partial a_k} \exp \left( h^{-1}W(P + \varepsilon P', I) \right) \right)(0) = \sum_{\gamma, \phi} \frac{h^{\gamma}(\gamma)}{|\text{Aut}(\gamma, \varepsilon, \phi)|} w_{\gamma, \varepsilon, \phi}(P, I)(a_1, \ldots, a_k).
\]

Here, the sum is over possibly disconnected stable graphs \( \gamma \) with a distinguished edge \( e \in E(\gamma) \). The weight \( w_{\gamma, \varepsilon, \phi} \) is defined in the same way as \( w_{\gamma, \phi} \) except that the distinguished edge \( e \) is labelled by \( P' \), whereas all other edges are labelled by \( P \). The automorphism group considered must preserve the edge \( e \) as well as \( \phi \).
Given any graph $\gamma$ with a distinguished edge $e$, we can cut along this edge to get another graph $\gamma'$ with two more tails. These tails can be ordered in two different ways. If we write $P' = \sum u' \otimes u''$ where $u', u'' \in U$, these extra tails are labelled by $u'$ and $u''$. Thus we find that
\[
\frac{d}{d\varepsilon} \left( \frac{\partial}{\partial a_1} \cdots \frac{\partial}{\partial a_k} \exp \left( h^{-1} W \left( P + \varepsilon P', I \right) \right) \right) (0)
= \frac{1}{2} \left( \frac{h^8(\gamma)}{\left| Aut(\gamma, \Phi) \right|} \right) \sum_{\gamma, \phi} w_{\gamma, \phi} (P, I) (a_1, \ldots, a_k, u', u'').
\]
Here the sum is over graphs $\gamma$ with $k + 2$ tails, and an ordering $\phi$ of these $k + 2$ tails. The factor of $\frac{1}{2}$ arises because of the two different ways to order the new tails. Comparing this to the previous expression, we find that
\[
\frac{d}{d\varepsilon} \left( \frac{\partial}{\partial a_1} \cdots \frac{\partial}{\partial a_k} \exp \left( h^{-1} W \left( P + \varepsilon P', I \right) \right) \right) (0)
= \frac{1}{2} \sum_{\gamma, \phi} \left( \frac{h^8(\gamma)}{\left| Aut(\gamma, \Phi) \right|} \right) \sum_{\gamma, \phi} w_{\gamma, \phi} (P, I) (a_1, \ldots, a_k, u', u'').
\]
Since
\[
\partial_{\varepsilon} = \frac{1}{2} \sum_{\gamma, \phi} \frac{\partial}{\partial u'} \cdot \frac{\partial}{\partial u''}
\]
this completes the proof. \qed

This expression makes it clear that, for all $P_1, P_2 \in Sym^2 U$,
\[
W (P_1, W (P_2, I)) = W (P_1 + P_2, I).
\]

3.5. Now suppose that $U$ is a finite dimensional vector space over $\mathbb{R}$, equipped with a non-degenerate negative definite quadratic form $\Phi$. Let $P \in Sym^2 U$ be the inverse to $-\Phi$. Thus, if $e_i$ is an orthonormal basis for $-\Phi$, $P = \sum e_i \otimes e_i$. (When we return to considering scalar field theories, $U$ will be replaced by the space $C^\infty (M)$, the quadratic form $\Phi$ will be replaced by the quadratic form $-\langle \phi, (D + m^2)\phi \rangle$, and $P$ will be the propagator for the theory).

The Feynman diagram expansion $W (P, I)$ described above can also be interpreted as an asymptotic expansion for an integral on $U$.

3.5.1 Lemma.
\[
W (P, I) (a) = h \log \int_{x \in U} \exp \left( \frac{1}{2h} \Phi (x, x) + \frac{1}{h} I (x + a) \right).
\]
The integral is understood as an asymptotic series in $\hbar$, and so makes sense whatever the signature of $\Phi$. As I mentioned before, we use the convention that the measure on $U$ is normalized so that

$$
\int_{x \in U} \exp \left( \frac{1}{2\hbar} \Phi(x,x) \right) = 1.
$$

This normalization accounts for the lack of a graph with one loop and zero external edges in the expansion.

If $U$ is a complex vector space and $\Phi$ is a non-degenerate complex linear inner product on $U$, then the same formula holds, where we integrate over any real slice $U_\mathbb{R}$ of $U$.

**Proof.** It suffices to show that, for all functions $f \in \mathcal{C}(U)$,

$$
\int_{x \in U} e^{(2\hbar)^{-1}\Phi(x,x)} f(x + a) = e^{\hbar \partial_{l} f}
$$

(where both sides are regarded as functions of $a \in U$).

The result is clear when $f = 1$. Let $l \in U^\vee$. Note that

$$
e^{\hbar \partial_{l}}(lf) - le^{\hbar \partial_{l}}(lf) = \hbar[\partial_{l}, l]e^{\hbar \partial_{l}}(lf).
$$

In this expression, $\hbar[\partial_{l}, l]$ is viewed as an order 1 differential operator on $\mathcal{C}(U)$.

The quadratic form $\Phi$ on $U$ provides an isomorphism $U \rightarrow U^\vee$. If $u \in U$, let $u^\vee \in U^\vee$ be the corresponding element; and, dually, if $l \in U^\vee$, let $l^\vee \in U$ be the corresponding element.

Note that

$$
[\partial_{l}, l] = -\frac{\partial}{\partial l^\vee}.
$$

It suffices to verify a similar formula for the integral. Thus, we need to check that

$$
\int_{x \in U} e^{(2\hbar)^{-1}\Phi(x,x)} l(x) f(x + a) = -\hbar \frac{\partial}{\partial l^\vee} \int_{x \in U} e^{(2\hbar)^{-1}\Phi(x,x)} f(x + a).
$$

(The subscript in $l^\vee_a$ indicates we are applying this differential operator to the $a$ variable).

Note that

$$
\frac{\partial}{\partial l^\vee} e^{(2\hbar)^{-1}\Phi(x,x)} = \hbar^{-1} l(x) e^{\Phi(x,x)}/\hbar.
$$
Thus,
\[
\int_{x \in U} e^{(2\hbar)^{-1}\Phi(x,x)} l(x) f(x + a) = \hbar \int_{x \in U} \left( \frac{\partial}{\partial l_x^a} e^{(2\hbar)^{-1}\Phi(x,x)} \right) f(x + a)
\]
\[
= -\hbar \int_{x \in U} e^{(2\hbar)^{-1}\Phi(x,x)} \frac{\partial}{\partial l_x^a} f(x + a)
\]
\[
= -\hbar \int_{x \in U} e^{(2\hbar)^{-1}\Phi(x,x)} l(x) \frac{\partial}{\partial l_x^a} f(x + a)
\]
\[
= -\hbar \left( \frac{\partial}{\partial l_x^a} \right) \int_{x \in U} e^{(2\hbar)^{-1}\Phi(x,x)} f(x + a)
\]
as desired.

\[\square\]

3.6. One can ask, how much of this picture holds if \(U\) is replaced by an infinite dimensional vector space? We can’t define the Lebesgue measure in this situation, thus we can’t define the integral directly. However, one can still contract tensors using Feynman graphs, and one can still define the expression \(W(P, I)\), as long as one is careful with tensor products and dual spaces. (As we will see later, the singularities in Feynman graphs arise because the inverse to the quadratic forms we will consider on infinite dimensional vector spaces do not lie in the correct completed tensor product.)

Let us work over a ground field \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\). Let \(M\) be a manifold and \(E\) be a super vector bundle on \(M\) over \(\mathbb{K}\). Let \(\mathcal{E} = \Gamma(M, E)\) be the super nuclear Fréchet space of global sections of \(E\). Let \(\otimes\) denote the completed projective tensor product, so that \(\mathcal{E} \otimes \mathcal{E} = \Gamma(M \times M, E \boxtimes E)\). (Some details of the symmetric monoidal category of nuclear spaces, equipped with the completed projective tensor product, are presented in Appendix 2).

Let \(\mathcal{O}(\mathcal{E})\) denote the algebra of formal power series on \(\mathcal{E}\),
\[
\mathcal{O}(\mathcal{E}) = \prod_{n \geq 0} \text{Hom}(\mathcal{E} \otimes^n, \mathbb{K})_{S_n}
\]
where \(\text{Hom}\) denotes continuous linear maps and the subscript \(S_n\) denotes coinvariants. Note that \(\mathcal{O}(\mathcal{E})\) is an algebra: direct product of distributions defines a map
\[
\text{Hom}(\mathcal{E} \otimes^n, \mathbb{K}) \times \text{Hom}(\mathcal{E} \otimes^m, \mathbb{K}) \to \text{Hom}(\mathcal{E} \otimes^{n+m}, \mathbb{K}).
\]
These maps induce an algebra structure on \(\mathcal{O}(\mathcal{E})\).
We can also regard \( \mathcal{O}(\mathcal{E}) \) as simply the completed symmetric algebra of the dual space \( \mathcal{E}^\vee \), that is,

\[
\mathcal{O}(\mathcal{E}) = \hat{\text{Sym}}^* (\mathcal{E}^\vee).
\]

Here, \( \mathcal{E}^\vee \) is the strong dual of \( \mathcal{E} \), and is again a nuclear space. The completed symmetric algebra is taken in the symmetric monoidal category of nuclear spaces, as detailed in Appendix 2.

As before, let

\[
\mathcal{O}^+(\mathcal{E})[[\hbar]] \subset \mathcal{O}(\mathcal{E})[[\hbar]]
\]

be the subspace of those functionals \( I \) which are at least cubic modulo \( \hbar \).

Let \( \text{Sym}^n \mathcal{E} \) denote the \( S_n \)-invariants in \( \mathcal{E} \otimes^n \). If \( P \in \text{Sym}^2 \mathcal{E} \) and \( I \in \mathcal{O}^+(\mathcal{E})[[\hbar]] \) then, for any stable graph \( \gamma \), one can define

\[
w_\gamma(P, I) \in \mathcal{O}(\mathcal{E}).
\]

The definition is exactly the same as in the finite dimensional situation. Let \( T(\gamma) \) be the set of tails of \( \gamma \), \( H(\gamma) \) the set of half-edges of \( \gamma \), \( V(\gamma) \) the set of vertices of \( \gamma \), and \( E(\gamma) \) the set of internal edges of \( \gamma \). The tensor products of interactions at the vertices of \( \gamma \) define an element of

\[
\text{Hom}(\mathcal{E} \otimes H(\gamma), \mathbb{R}).
\]

We can contract this tensor with the element of \( \mathcal{E} \otimes^{2E(\gamma)} \) given by the tensor product of the propagators; the result of this contraction is

\[
w_\gamma(P, I) \in \text{Hom}(\mathcal{E} \otimes T(\gamma), \mathbb{R}).
\]

Thus, one can define

\[
W(P, I) = \sum_{\gamma} \frac{1}{|\text{Aut}(\gamma)|} w_\gamma(P, I) \in \mathcal{O}^+(\mathcal{E})[[\hbar]]
\]

exactly as before.

The interpretation in terms of differential operators works in this situation too. As in the finite dimensional situation, we can define an order two differential operator

\[
\partial_P : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E}).
\]

On the direct factor

\[
\text{Hom}(\mathcal{E} \otimes^n, \mathbb{K})_{S_n} = \text{Sym}^n \mathcal{E}^\vee
\]

of \( \mathcal{O}(\mathcal{E}) \), the operator \( \partial_P \) comes from the map

\[
\text{Hom}(\mathcal{E} \otimes^n, \mathbb{K}) \to \text{Hom}(\mathcal{E} \otimes^{n-2}, \mathbb{K})
\]
given by contracting with the tensor \( P \in \mathcal{E}^2 \).

Then,

\[
W(P, I) = \hbar \log \{ \exp(\hbar \partial P) \exp(I/\hbar) \}
\]
as before.

4. Sharp and smooth cut-offs

4.1. Let us return to our scalar field theory, whose action is of the form

\[
S(\phi) = -\frac{1}{2} \left< \phi, (D + m^2)\phi \right> + I(\phi).
\]
The propagator \( P \) is the kernel for the operator \( (D + m^2)^{-1} \). There are several natural ways to write this propagator. Let us pick a basis \( \{e_i\} \) of \( C^\infty(M) \) consisting of orthonormal eigenvectors of \( D \), with eigenvalues \( \lambda_i \in \mathbb{R}_{\geq 0} \). Then,

\[
P = \sum_i \frac{1}{\lambda_i^2 + m^2} e_i \otimes e_i.
\]

There are natural cut-off propagators, where we only sum over some of the eigenvalues. For a subset \( U \subset \mathbb{R}_{\geq 0} \), let

\[
P_U = \sum_{i \text{ such that } \lambda_i \in U} \frac{1}{\lambda_i^2 + m^2} e_i \otimes e_i.
\]

Note that, unlike the full propagator \( P \), the cut-off propagator \( P_U \) is a smooth function on \( M \times M \) as long as \( U \) is a bounded subset of \( \mathbb{R}_{\geq 0} \).

By analogy with the case of finite-dimensional integrals, we have the formal identity

\[
W(P, I) (a) = \hbar \log \int_{\phi \in C^\infty(M)} \exp \left( -\frac{1}{2\hbar} \left< \phi, (D + m^2)\phi \right> + \frac{1}{\hbar} I(\phi + a) \right)
\]

Both sides of this equation are ill-defined. The propagator \( P \) is not a smooth function on \( C^\infty(M \times M) \), but has singularities along the diagonal; this means that \( W(P, I) \) is not well defined. And, of course, the integral on the right hand side is infinite dimensional.

In a similar way, we have the following (actual) identity, for any functional \( I \in \mathcal{E}^+(C^\infty(M))[[\hbar]] \):

\[
W(P_{[\Lambda, \Lambda]}, I) (a) = \hbar \log \int_{\phi \in C^\infty(M)_{[\Lambda, \Lambda]}} \exp \left( -\frac{1}{2\hbar} \left< \phi, (D + m^2)\phi \right> + \frac{1}{\hbar} I(\phi + a) \right).
\]
Both sides of this identity are well-defined; the propagator $P_{[\Lambda', \Lambda]}$ is a smooth function on $M \times M$, so that $W \left( P_{[\Lambda', \Lambda]}, I \right)$ is well-defined. The right hand side is a finite dimensional integral.

The equation (1) says that the map
\[
\partial^+ (C^\infty(M))[[\hbar]] \rightarrow \partial^+ (C^\infty(M))[[\hbar]]
\]
\[I \mapsto W \left( P_{[\Lambda', \Lambda]}, I \right)
\]
is the renormalization group flow from energy $\Lambda$ to energy $\Lambda'$.

4.2. In this book we will use a cut-off based on the heat kernel, rather than the cut-off based on eigenvalues of the Laplacian described above.

For $l \in \mathbb{R}_{>0}$, let $K^0_l \in C^\infty(M \times M)$ denote the heat kernel for $D$; thus,
\[
\int_{y \in M} K^0_l(x, y) \phi(y) = \left( e^{-lD} \phi \right)(x)
\]
for all $\phi \in C^\infty(M)$.

We can write $K^0_l$ in terms of a basis of eigenvalues for $D$ as
\[
K^0_l = \sum_i e^{-\lambda_i l} e_i \otimes e_i.
\]
Let
\[
K_l = e^{-lm^2} K^0_l
\]
be the kernel for the operator $e^{-l(D + m^2)}$. Then, the propagator $P$ can be written as
\[
P = \int_{l=0}^{\infty} K_l dl.
\]

For $\varepsilon, L \in [0, \infty]$, let
\[
P(\varepsilon, L) = \int_{l=\varepsilon}^{L} K_l dl.
\]
This is the propagator with an infrared cut-off $L$ and an ultraviolet cut-off $\varepsilon$. Here $\varepsilon$ and $L$ are length scales rather than energy scales; length behaves as the inverse to energy. Thus, $\varepsilon$ is the high-energy cut-off and $L$ is the low-energy cut-off.

The propagator $P(\varepsilon, L)$ damps down the high energy modes in the propagator $P$. Indeed,
\[
P(\varepsilon, L) = \sum_i \frac{e^{-\varepsilon \lambda_i} - e^{-L \lambda_i}}{\lambda_i^2 + m^2} e_i \otimes e_i
\]
so that the coefficient of $e_i \otimes e_i$ decays as $\lambda_i^{-2} e^{-\varepsilon \lambda_i}$ for $\lambda_i$ large.
The first few expressions in the renormalization group flow from scale \( \varepsilon \) to scale \( L \).

Because \( P(\varepsilon, L) \) is a smooth function on \( M \times M \), as long as \( \varepsilon > 0 \), the expression \( W(P(\varepsilon, L), I) \) is well-defined for all \( I \in \mathcal{O}^+(C^\infty(M))[[h]] \). (Recall the superscript + means that \( I \) must be at least cubic modulo \( h \).

**4.2.1 Definition.** The map

\[
\mathcal{O}^+(C^\infty(M))[[h]] \rightarrow \mathcal{O}^+(C^\infty(M))[[h]]
\]

\[
I \mapsto W(P(\varepsilon, L), I)
\]

is defined to be the renormalization group flow from length scale \( \varepsilon \) to length scale \( L \).

From now on, we will be using this length-scale version of the renormalization group flow.

Figure 2 illustrates the first few terms of the renormalization group flow from scale \( \varepsilon \) to scale \( L \).
5. SINGULARITIES IN FEYNMAN GRAPHS

As the effective interaction $I[L]$ varies smoothly with $L$, there is an infinitesimal form of the renormalization group equation, which is a differential equation in $I[L]$. This is illustrated in figure 3.

The expression for the propagator in terms of the heat kernel has a very natural geometric/physical interpretation, which will be explained in Section 6.

5. Singularities in Feynman graphs

5.1. In this section, we will consider explicitly some of the simple Feynman graphs appearing in $W(P(\varepsilon, L), I)$ where $I(\phi) = \frac{1}{3!} \int_M \phi^3$, and try to take the limit as $\varepsilon \to 0$. We will see that, for graphs which are not trees, the limit in general won’t exist. The $\frac{1}{3!}$ present in the interaction term simplifies the combinatorics of the Feynman diagram expansion. The Feynman diagrams we will consider are all trivalent, and as explained
in Section 3, each vertex is labelled by the linear map
\[ C^\infty(M)^{\otimes 3} \to \mathbb{R} \]
\[ \phi_1 \otimes \phi_2 \otimes \phi_3 \mapsto \frac{\partial}{\partial \phi_1} \frac{\partial}{\partial \phi_2} \frac{\partial}{\partial \phi_3} I \]
\[ = \int_M \phi_1(x) \phi_2(x) \phi_3(x) \]
Consider the graph \( \gamma_1 \), given by

We see that
\[ w_{\gamma_1}(P(\varepsilon, L), I)(a) = \int_{l \in [\varepsilon, L]} \int_{x \in M} a(x) K_l(x, x) d\text{Vol}_M dl \]
where \( d\text{Vol}_M \) is the volume form associated to the metric on \( M \), and \( a \in C^\infty(M) \) is the field we place at the external edge of the graph. Since there are no external edges there is no dependence on \( a \in C^\infty(M) \), and we find
\[ w_{\gamma_2}(P(\varepsilon, L), I) = \int_{l_1, l_2, l_3 \in [\varepsilon, L]} \int_{x, y \in M} K_{l_1}(x, y) K_{l_2}(x, y) K_{l_3}(x, y) d\text{Vol}_M x d\text{Vol}_M y dl_1 dl_2 dl_3. \]
Using the fact that
\[ K_l(x, y) \simeq I^{-\dim M/2} e^{-d(x-y)^2/l} + \text{higher order terms} \]
for \( l \) small, we can see that the limit of \( w_{\gamma_2}(P(\varepsilon, L), I) \) as \( \varepsilon \to 0 \) is singular.

Let \( \gamma_3 \) be the graph

We see that
\[ w_{\gamma_3}(P(\varepsilon, L), I)(a) = \int_{l \in [\varepsilon, L]} \int_{x \in M} a(x) K_l(x, x) d\text{Vol}_M dl \]
for \( l \) small, the limit \( \lim_{\varepsilon \to 0} w_{\gamma_1}(P(\varepsilon, L), I)(a) \) doesn’t exist.
This graph is a tree; the integrals associated to graphs which are trees always admit \( \varepsilon \to 0 \) limits. Indeed,

\[
\omega_{\gamma_3}(P(\varepsilon, L), I)(a) = \int_{L \in [\varepsilon, L]} \int_{x, y \in M} a(x)^2 K_l(x, y) a(y)^2 \\
= \left< a^2, \int_{L}^{L} \left( e^{-\varepsilon D_a^2} \right) \, dl \right>
\]

and the second expression clearly admits an \( \varepsilon \to 0 \) limit.

All of the calculations above are for the interaction \( I(\phi) = \frac{1}{4} \int \phi^4 \). For more general interactions \( I \), one would have to apply a differential operator to both \( a \) and \( K_l(x, y) \) in the integrands.

6. The geometric interpretation of Feynman graphs

From the functional integral point of view, Feynman graphs are just graphical tools which help in the perturbative calculation of certain functional integrals. In the introduction, we gave a brief account of the world-line interpretation of quantum field theory. In this picture, Feynman graphs describe the trajectories taken by some interacting particles. The length scale version of the renormalization group flow becomes very natural from this point of view.

As before, we will work in Euclidean signature; Lorentzian signature presents significant additional analytical difficulties. We will occasionally comment on the formal picture in Lorentzian signature.

6.1. Let us consider a massless scalar field theory on a compact Riemannian manifold \( M \). Thus, the fields are \( C^\infty(M) \) and the action is

\[
S(\phi) = -\frac{1}{2} \int_M \phi \, D \phi.
\]
The propagator $P(x, y)$ is a distribution on $M^2$, which can be expressed as a functional integral

$$P(x, y) = \int_{\phi \in C^\infty(M)} e^{S(\phi)} \phi(x) \phi(y).$$

Thus, the propagator encodes the correlation between the values of the fields $\phi$ at the points $x$ and $y$.

We will derive an alternative expression of the propagator as a \textit{one-dimensional} functional integral.

Recall that we can write the propagator of a massless scalar field theory on a Riemannian manifold $M$ as an integral of the heat kernel. If $x, y \in M$ are distinct points, then

$$P(x, y) = \int_0^\infty K_t(x, y) \, dt$$

where $K_t \in C^\infty(M^2)$ is the heat kernel. This expression of the propagator is sometimes known as the Schwinger representation, and the parameter $t$ as the Schwinger parameter. One can also interpret the parameter $t$ as proper time, as we will see shortly.

The heat kernel $K_t(x, y)$ is the probability density that a particle in Brownian motion on $M$, which starts at $x$ at time zero, lands at $y$ at time $t$. Thus, we can rewrite the heat kernel as

$$K_t(x, y) = \int_{f: [0, t] \to M} \mathcal{D}_{\text{Wiener}} f \delta_{\text{Wiener}} f$$

where $\mathcal{D}_{\text{Wiener}} f$ is the Wiener measure on the path space.

We can think of the Wiener measure as the measure for a quantum field theory of maps

$$f : [0, t] \to M$$

with action given by

$$E(f) = \int_0^t \langle df, df \rangle.$$ 

Thus, we will somewhat loosely write

$$K_t(x, y) = \int_{f: [0, t] \to M} e^{-E(f)} \delta_{\text{Wiener}} f$$

where we understand that the integral can be given rigorous meaning using the Wiener measure.
Combining these expressions, we find the desired expression for the propagator as a one-dimensional functional integral:

$$P(x, y) = \int_{\phi \in C^\infty(M)} e^{S(\phi)} \phi(x) \phi(y) = \int_{t=0}^{\infty} \int_{f:[0,t] \rightarrow M} \frac{e^{-E(f)}}{f(0)=x, f(t)=y}.$$  

This expression is the core of the world-line formulation of quantum field theory. This expression tells us that the correlation between the values of the fields at points $x$ and $y$ can be expressed in terms of an integral over paths in $M$ which start at $x$ and end at $y$.

If we work in Lorentzian signature, we find the (formal) identity

$$\int_{\phi \in C^\infty(M)} e^{S(\phi)} \phi(x) \phi(y) = \int_{t=0}^{\infty} \int_{f:[0,t] \rightarrow M} \frac{e^{iE(f)}}{f(0)=x, f(t)=y}.$$  

This expression is difficult to make rigorous sense of; I don’t know of a rigorous treatment of the Wiener measure when the target manifold has Lorentzian signature.

6.2. We should interpret these identities as follows. We should think of particles moving through space-time as equipped with an “internal clock”; as the particle moves, this clock ticks at a rate independent of the time parameter on space-time. The world-line of such a particle is a parameterized path in space-time, that is, a map $f : \mathbb{R} \rightarrow M$. This path is completely arbitrary: it can go backwards or forwards in time. Two world-lines which differ by a translation on the source $\mathbb{R}$ should be regarded as the same. In other words, the internal clock of a particle doesn’t have an absolute starting point.

If $I = [0, \tau]$ is a closed interval, and if $f : I \rightarrow M$ is a path describing part of the world-line of a particle, then the energy of $f$ is, as before,

$$E(f) = \int_{[0,\tau]} \langle df, df \rangle.$$  

In quantum field theory, everything that can happen will happen, but with a probability amplitude of $e^{iE}$ where $E$ is the energy. Thus, to calculate the probability that a particle starts at the point $x$ in space-time and ends at the point $y$, we must integrate over all paths $f : [0,\tau] \rightarrow M$, starting at $x$ and ending at $y$. We must also integrate over the parameter $\tau$, which is interpreted as the time taken on the internal clock of the particle as it moves from $x$ to $y$. This leads to the expression (in Lorentzian signature) we discussed earlier,

$$P(x, y) = \int_{t=0}^{\infty} \int_{f:[0,t] \rightarrow M} \frac{e^{iE(f)}}{f(0)=x, f(t)=y}.$$
6.3. We would like to have a similar picture for interacting theories. We will consider, for simplicity, the \( \phi^3 \) theory, where the fields are \( \phi \in C^\infty(M) \), and the action is

\[
S(\phi) = \int_M -\frac{1}{2} \phi \mathcal{D} \phi + \frac{1}{6} \phi^3.
\]

We will discuss the heuristic picture first, ignoring the difficulties of renormalization. At the end, we will explain how the renormalization group flow and the idea of effective interactions can be explained in the world-line point of view.

The fundamental quantities one is interested in are the correlation functions, defined by the heuristic functional integral formula

\[
\mathbb{E}(x_1, \ldots, x_n) = \int_{\phi \in C^\infty(M)} e^{S(\phi)/\hbar} \phi(x_1) \cdots \phi(x_n).
\]

We would like to express these correlation functions in the world-line point of view.

6.4. The \( \phi^3 \) theory corresponds, in the world-line point of view, to a theory where three particles can fuse at a point in \( M \). Thus, world-lines in the \( \phi^3 \) theory become world-graphs; further, just as the world-lines for the free theory are parameterized, the world-graphs arising in the \( \phi^3 \) theory have a metric, that is, a length along each edge. This length on the edge of the graph corresponds to time traversed by the particle on this edge in its internal clock.

Measuring the value of a field \( \phi \) at a point \( x \in M \) corresponds, in the world-line point of view, to observing a particle at a point \( x \). We would like to find an expression for the correlation function \( \mathbb{E}(x_1, \ldots, x_n) \) in the world-line point of view. As always in quantum field theory, one should calculate this expectation value by summing over all events that could possibly happen. Such an event is described by a world-graph with end points at \( x_1, \ldots, x_n \). Since only three particles can interact at a given point in space-time, such world-graphs are trivalent. Thus, the relevant world-graphs are trivalent, have \( n \) external edges, and the end points of these external edges maps to the points \( x_1, \ldots, x_n \).

Thus, we find that

\[
\mathbb{E}(x_1, \ldots, x_n) = \sum_{\gamma} \frac{1}{|\text{Aut} \gamma|} \hbar^{-|\gamma|} \int_{g \in \text{Met} \gamma} \int_{f: \gamma \to M} e^{-E(f)}.
\]

Here, the sum runs over all world-graphs \( \gamma \), and the integral is over those maps \( f: \gamma \to M \) which take the endpoints of the \( n \) external edges of \( \gamma \) to the points \( x_1, \ldots, x_n \).
The symbol $\text{Met}(\gamma)$ refers to the space of metrics on $\gamma$, in other words, to the space $\mathbb{R}_{>0}^{E(\gamma) \cap T(\gamma)}$ where $E(\gamma)$ is the set of internal edges of $\gamma$, and $T(\gamma)$ is the set of tails.

If $\gamma$ is a metrized graph, and $f : \gamma \to M$ is a map, then $E(f)$ is the sum of the energies of $f$ restricted to the edges of $\gamma$, that is,

$$E(f) = \sum_{e \in E(\gamma)} \int_0^{l(e)} \langle df, df \rangle.$$ 

The space of maps $f : \gamma \to M$ is given a Wiener measure, constructed from the usual Wiener measure on path space.

This graphical expansion for the correlation functions is only a formal expression: if $\gamma$ has a non-zero first Betti number, then the integral over $\text{Met}(\gamma)$ will diverge, as we will see shortly. However, this graphical expansion is precisely the expansion one finds when formally applying Wick’s lemma to the functional integral expression for $E(x_1, \ldots, x_n)$. The point is that we recover the propagator when we consider the integral over all possible maps from a given edge.

In Lorentzian signature, of course, one should use $e^{iE(f)}$ instead of $e^{-E(f)}$.

6.5. As an example, we will consider the path integral $f : \gamma \to M$ where $\gamma$ is the metrized graph

Then $\gamma$ has two vertices, labelled by interactions $I_{0,3}$.

The integral

$$\int_{f : \gamma \to M} e^{-E(f)}$$

is obtained by putting the heat kernel $K_l$ on each edge of $\gamma$ of length $l$, and integrating over the position of the two vertices. Thus, we find

$$\int_{f : \gamma \to M} e^{-E(f)} = \int_{x,y \in M} K_{l_1}(x, x)K_{l_2}(x, y)K_{l_3}(y, y).$$
However, the second integral, over the space of metrized graphs, does not make sense. Indeed, the heat kernel $K_I(x, y)$ has a small $l$ asymptotic expansion of the form

$$K_I(x, y) = l^{-n/2} e^{-\|x-y\|^2/4l} \sum_l l^i f_i(x, y).$$

This implies that the integral

$$\int_{l_1, l_2, l_3} \int_{f: \gamma \to M} e^{-E(f)} = \int_{x, y, \in M} K_{l_1}(x, x) K_{l_2}(x, y) K_{l_3}(y, y)$$

does not converge.

This second integral is the weight attached to the graph $\gamma$ in the Feynman diagram expansion of the functional integral for the $\frac{1}{3!} \phi^3$ interaction.

6.6. Next, we will explain (briefly and informally) how to construct the correlation functions from a general scale $L$ effective interaction $I[L]$. We will not need this construction of the correlation functions elsewhere in this book. A full treatment of observables and correlation functions will appear in [CG10].

The correlation functions will allow us to give a world-line formulation for the renormalization group equation on a collection $\{I[L]\}$ of effective interactions: the renormalization group equation is equivalent to the statement that the correlation functions computed using $I[L]$ are independent of $L$.

The correlations $E^n_{I[L]}(f_1, \ldots, f_n)$ we will define will take, as their input, functions $f_1, \ldots, f_n \in C^\infty(M)$. Thus, the correlation functions will give a collection of distributions on $M^n$:

$$E^n_{I[L]} : C^\infty(M^n) \to \mathbb{R}[\hbar].$$

These correlation functions will be defined as a sum over graphs.

Let $\Gamma_n$ denote the set of graphs $\gamma$ with $n$ univalent vertices, which are labelled as $v_1, \ldots, v_n$. These vertices will be referred to as the external vertices. The remaining vertices will be called the internal vertices. The internal vertices of a graph $\gamma \in \Gamma_n$ can be of any valency, and are labelled by a genus $g(v) \in \mathbb{Z}_{\geq 0}$. The internal vertices of genus 0 must be at least trivalent.

For a graph $\gamma \in \Gamma_n$, and smooth functions $f_1, \ldots, f_n \in C^\infty(M)$, we will define

$$C_\gamma(I[L])(f_1, \ldots, f_n) \in \mathbb{R}$$

by contracting certain tensors attached to the edges and the vertices.
We will label each internal vertex \( v \) of genus \( i \) and valency \( k \) by
\[
I_{i,k}[L] : C^\infty(M)^{\otimes H(v)} \to \mathbb{R}.
\]

Let \( f_1, \ldots, f_n \in C^\infty(M) \) be smooth functions on \( M \). The external vertex \( v_i \) of \( \gamma \) will be labelled by the distribution
\[
C^\infty(M) \to \mathbb{R}
\]
\[
\phi \mapsto \int_M f \phi.
\]

Any edge \( e \) joining two internal vertices will be labelled by
\[
P(L, \infty) \in C^\infty(M^2).
\]

The remaining edges, which join two external vertices or join an external and an internal vertex, will be labelled by \( P(0, \infty) \), which is a distribution on \( M^2 \).

As usual, we can contract all these tensors to define an element
\[
C_\gamma(I[L])(f_1, \ldots, f_n) \in \mathbb{R}.
\]

One may worry that because some of the edge are labelled by the distribution \( P(0, \infty) \), this expression is not well defined. However, because the external edges are labelled by smooth functions \( f_i \), there are no problems. Figure 4 describes \( C_\gamma(I[L], f_1, f_2, f_3) \) for a particular graph \( \gamma \).

Then, the correlation function for the effective interaction \( I[L] \) is defined by
\[
\mathbb{E}_{I[L]}(f_1, \ldots, f_n) = \sum_{\gamma \in \Gamma_n} \frac{1}{\text{Aut}(\gamma)} h^{n-g(\gamma)-1} C_\gamma(I[L])(f_1, \ldots, f_n).
\]

Unlike the heuristic graphical expansion we gave for the correlation functions of the \( \phi^3 \) theory, this expression is well-defined.

We should interpret this expansion as saying that we can compute the correlation functions from the effective interaction \( I[L] \) by allowing particles to propagate in the usual way, and to interact by \( I[L] \); except that in between any two interactions, particles must travel for a proper time of at least \( L \). This accounts for the fact that edges which join to internal vertices are labelled by \( P(L, \infty) \).

If we have a collection \( \{ I[L] \mid L \in (0, \infty) \} \) of effective interactions, then the renormalization group equation is equivalent to the statement that all the correlation functions constructed from \( I[L] \) using the prescription given above are independent of \( L \).
7. A definition of a quantum field theory

7.1. Now we have some preliminary definitions and an understanding of why the terms in the graphical expansion of a functional integral diverge. This book will describe a method for renormalizing these functional integrals to yield a finite answer.

This section will give a formal definition of a quantum field theory, based on Wilson’s philosophy of the effective action; and a precise statement of the main theorem, which says roughly that there’s a bijection between theories and Lagrangians.

7.1.1 Definition. A local action functional \( I \in \mathcal{O}(\mathcal{C}^\infty(M)) \) is a functional which arises as an integral of some Lagrangian. More precisely, if we Taylor expand \( I \) as \( I = \sum_k I_k \) where

\[
I_k(\lambda a) = \lambda^k I_k(a)
\]

(so that \( I_k \) is homogeneous of degree \( k \) of the variable \( a \in \mathcal{C}^\infty(M) \)), then \( I_k \) must be of the form

\[
I_k(a) = \sum_{j=1}^s \int_M D_{1,j}(a) \cdots D_{k,j}(a)
\]

where \( D_{i,j} \) are arbitrary differential operators on \( M \).

Let \( \mathcal{O}_{loc}(\mathcal{C}^\infty(M)) \subset \mathcal{O}(\mathcal{C}^\infty(M)) \) be the subspace of local action functionals.
As before, let
\[ \mathcal{O}_{\text{loc}}^+(C^\infty(M))[\hbar] \subset \mathcal{O}_{\text{loc}}(C^\infty(M))[\hbar] \]
be the subspace of those local action functionals which are at least cubic modulo \( \hbar \).

Thus, local action functionals are the same as Lagrangians modulo those Lagrangians which are a total derivative.

**7.1.2 Definition.** A perturbative quantum field theory, with space of fields \( C^\infty(M) \) and kinetic action \(-\frac{1}{2} \langle \phi, (D + m^2) \phi \rangle\), is given by a set of effective interactions \( I[L] \in \mathcal{O}^+(C^\infty(M))[\hbar] \) for all \( L \in (0, \infty) \), such that

1. The renormalization group equation
   \[ I[L] = W(P(\epsilon, L), I[\epsilon]) \]
   is satisfied, for all \( \epsilon, L \in (0, \infty) \).

2. For each \( i, k \), there is a small \( L \) asymptotic expansion
   \[ I_{i,k}[L] \simeq \sum_{r \in \mathbb{Z}_{\geq 0}} g_r(L) \Phi_r \]
   where \( g_r(L) \in C^\infty((0, \infty)_L) \) and \( \Phi_r \in \mathcal{O}_{\text{loc}}(C^\infty(M)) \).

Let \( \mathcal{F}^{(\infty)} \) denote the set of perturbative quantum field theories, and let \( \mathcal{F}^{(n)} \) denote the set of theories defined modulo \( \hbar^{n+1} \). Thus, \( \mathcal{F}^{(\infty)} = \lim_{n \to \infty} \mathcal{F}^{(n)} \).

Let me explain more precisely what I mean by saying there is a small \( L \) asymptotic expansion
\[ I_{i,k}[L] \simeq \sum_{r \in \mathbb{Z}_{\geq 0}} g_r(L) \Phi_r. \]

Without loss of generality, we can require that the local action functionals \( \Phi_r \) appearing here are homogeneous of degree \( k \) in the field \( a \).

Then, the statement that there is such an asymptotic expansion means that there is a non-decreasing sequence \( d_R \in \mathbb{Z} \), tending to infinity, such that for all \( R \), and for all fields \( a \in C^\infty(M) \),
\[ \lim_{L \to 0} L^{-d_R} \left( I_{i,k}[L](a) - \sum_{r=0}^{R} g_r(L) \Phi_r(a) \right) = 0. \]

In other words, we are asking that the asymptotic expansion exists in the weak topology on \( \text{Hom}(C^\infty(M)^{\otimes k}, \mathbb{R})_{S_k} \).
The main theorem of this chapter (in the case of scalar field theories) is the following.

**Theorem A.** Let \( \mathcal{T}^{(n)} \) denote the set of perturbative quantum field theories defined modulo \( \hbar^{n+1} \). Then \( \mathcal{T}^{(n+1)} \) is, in a canonical way, a principal bundle over \( \mathcal{T}^{(n)} \) for the abelian group \( \mathcal{O}_{\text{loc}}(C^\infty(M)) \) of local action functionals on \( M \).

Further, \( \mathcal{T}^{(0)} \) is canonically isomorphic to the space \( \mathcal{O}_{\text{loc}}^+(C^\infty(M)) \) of local action functionals which are at least cubic.

There is a variant of this theorem, which states that there is a bijection between theories and Lagrangians once we choose a renormalization scheme, which is a way to extract the singular part of certain functions of one variable. The concept of renormalization scheme will be discussed in Section 9; this is a choice that only has to be made once, and then it applies to all theories on all manifolds.

**Theorem B.** Let us fix a renormalization scheme.

Then, we find a section of each torsor \( \mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)} \), and so a bijection between the set of perturbative quantum field theories and the set of local action functionals \( I \in \mathcal{O}_{\text{loc}}^+(C^\infty(M))[\hbar] \). (Recall the superscript + means that \( I \) must be at least cubic modulo \( \hbar \).)

7.2. We will first prove theorem B, and deduce theorem A (which is the more canonical formulation) as a corollary.

In one direction, the bijection in theorem B is constructed as follows. If \( I \in \mathcal{O}_{\text{loc}}^+(C^\infty(M))[\hbar] \) is a local action functional, then we will construct a canonical series of counterterms \( I^{\text{CT}}(\varepsilon) \). These are local action functionals, depending on a parameter \( \varepsilon \in (0, \infty) \) as well as on \( \hbar \). The counterterms are zero modulo \( \hbar \), as the tree-level Feynman graphs all converge. Thus, \( I^{\text{CT}}(\varepsilon) \in \hbar \mathcal{O}_{\text{loc}}(C^\infty(M))[\hbar] \otimes C^\infty((0, \infty)) \) where \( \otimes \) denotes the completed projective tensor product.

These counterterms are constructed so that the limit

\[
\lim_{\varepsilon \to 0} W \left( P(\varepsilon, L), I - I^{\text{CT}}(\varepsilon) \right)
\]

exists. This limit defines the scale \( L \) effective interaction \( I[L] \).

Conversely, if we have a perturbative QFT given by a collection of effective interactions \( I[l] \), the local action functional \( I \) is obtained as a certain renormalized limit of
8. An alternative definition

In the previous section I presented a definition of quantum field theory based on the heat-kernel cut-off. In this section, I will describe an alternative, but equivalent, definition, which allows a much more general class of cut-offs. This alternative definition is a little more complicated, but is conceptually more satisfying. One advantage of this alternative definition is that it does not rely on the heat kernel.

As before, we will consider a scalar field theory where the quadratic term of the action is \( \frac{1}{2} \int \phi (D + m^2) \phi \).

8.0.1 Definition. A parametrix for the operator \( D + m^2 \) a distribution \( P \) on \( M \times M \), which is symmetric, smooth away from the diagonal, and is such that

\[
((D + m^2) \otimes 1) P - \delta_M \in C^\infty(M \times M)
\]

is smooth; where \( \delta_M \) refers to the delta distribution along the diagonal in \( M \times M \).

For any \( L > 0 \), the propagator \( P(0, L) \) is a parametrix. In the alternative definition of a quantum field theory presented in this section, we can use any parametrix as the propagator.

Note that if \( P, P' \) are two parametrices, the difference \( P - P' \) between them is a smooth function. We will give the set of parametrices a partial order, by saying that

\[
P \leq P'
\]

if \( \text{Supp}(P) \subset \text{Supp}(P') \). For any two parametrices \( P, P' \), we can find some \( P'' \) with \( P'' < P \) and \( P'' < P' \).

8.1. Before we introduce the alternative definition of quantum field theory, we need to introduce a technical notation. Given any functional \( f \in \mathcal{G}(C^\infty(M)) \), we get a
continuous linear map

\[ C^\infty(M) \rightarrow \mathcal{O}(C^\infty(M)) \]

\[ \phi \mapsto \frac{dJ}{d\phi}. \]

**8.1.1 Definition.** A function \( J \) has smooth first derivative if this map extends to a continuous linear map

\[ \mathcal{D}(M) \rightarrow \mathcal{O}(C^\infty(M)), \]

where \( \mathcal{D}(M) \) is the space of distributions on \( M \).

**8.1.2 Lemma.** Let \( \Phi \in C^\infty(M)^{\otimes 2} \) and suppose that \( J \in \mathcal{O}^+(C^\infty(M))[[h]] \) has smooth first derivative. Then so does \( W(\Phi,J) \in \mathcal{O}^+(C^\infty(M))[[h]] \).

**Proof.** Recall that

\[ W(\Phi,J) = h \log \left( e^{b\partial_\Phi e^{i/h}} \right). \]

Thus, it suffices to verify two things. Firstly, that the subspace \( \mathcal{O}(C^\infty(M)) \) consisting of functionals with smooth first derivative is a subalgebra; this is clear. Secondly, we need to check that \( \partial_\Phi \) preserves this subalgebra. This is also clear, because \( \partial_\Phi \) commutes with \( \frac{d}{d\phi} \) for any \( \phi \in C^\infty(M) \).

\[ \square \]

**8.2.** The alternative definition of quantum field theory is as follows.

**8.2.1 Definition.** A quantum field theory is a collection of functionals

\[ I[P] \in \mathcal{O}^+(C^\infty(M))[[h]], \]

one for each parametrix \( P \), such that the following properties hold.

1. If \( P, P' \) are parametrices, then

\[ W(P - P', I[P']) = I[P]. \]

This expression makes sense, because \( P - P' \) is a smooth function on \( M \times M \).

2. The functionals \( I[P] \) satisfy the following locality axiom. For any \( (i, k) \), the support of

\[ \text{Supp } I_{i,k}[P] \subset M^k \]

can be made as close as we like to the diagonal by making the parametrix \( P \) small. More precisely, we require that, for any open neighbourhood \( U \) of the small diagonal

\[ M \subset M^k, \]

we can find some \( P \) such that

\[ \text{Supp } I_{i,k}[P'] \subset U \]
9. Extracting the singular part of the weights of Feynman graphs

9.1. In order to construct the local counterterms needed for theorem A, we need a method for extracting the singular part of the finite-dimensional integrals $w(\varepsilon, L, I)$ attached to Feynman graphs. This section will describe such a method, which relies on an understanding of the behaviour of the functions $w(\varepsilon, L, I)$ as $\varepsilon \to 0$. We will see that $w(\varepsilon, L, I)$ has a small $\varepsilon$ asymptotic expansion

$$w(\varepsilon, L, I)(a) \simeq \sum g_i(\varepsilon)\Phi_i(L, a),$$

where the $\Phi_i(L, a)$ are well-behaved functions of the field $a$ and of $L$. Further, the $\Phi_i(L, a)$ have a small $L$ asymptotic expansion in terms of local action functionals.

The functionals $g_i(\varepsilon)$ appearing in this expansion are of a very special form: they are periods of algebraic varieties. For the purposes of this book, the fact that these functions are periods is not essential. Thus, the reader may skip the definition of periods without any loss. However, given the interest in the relationship between periods and quantum field theory in the mathematics literature (see [KZ01], for example) I felt that this point is worth mentioning.

Before we state the theorem precisely, we need to explain what makes a function of $\varepsilon$ a period.
9.2. According to Kontsevich and Zagier [KZ01], most or all constants appearing in mathematics should be periods.

9.2.1 Definition. A number \( \alpha \in \mathbb{C} \) is a period if there exists an algebraic variety \( X \) of dimension \( d \), a normal crossings divisor \( D \subset X \), and a form \( \omega \in \Omega^d(X) \) vanishing on \( D \), all defined over \( \mathbb{Q} \); and a homology class

\[
\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C})) \otimes \mathbb{Q}
\]

such that

\[
\alpha = \int_{\gamma} \omega.
\]

We are interested in periods which depend on a variable \( \varepsilon \in (0, \infty) \). Such families of periods arise from families of algebraic varieties over the affine line.

Suppose we have the following data.

1. an algebraic variety \( X \) over \( \mathbb{Q} \);
2. a normal crossings divisor \( D \subset X \);
3. a Zariski open subset \( U \subset \mathbb{A}^1_\mathbb{Q} \) such that \( U(\mathbb{R}) \) contains \( (0, \infty) \);
4. a smooth map \( X \to U \), of relative dimension \( d \), whose restriction to \( D \) is a also a smooth map to \( U \);
5. a relative \( d \)-form \( \omega \in \Omega^d(X/U) \), defined over \( \mathbb{Q} \), and vanishing along \( D \);
6. a homology class \( \gamma \in H_d((X_1(\mathbb{C}), D_1(\mathbb{C})), \mathbb{Q}) \).

Let us assume that the map

\[
(X(\mathbb{C}), D(\mathbb{C})) \to U(\mathbb{C})
\]

is a locally trivial fibration of pairs of smooth manifolds. For \( t \in (0, \infty) \subset U(\mathbb{R}) \), we will let \( X_t(\mathbb{C}) \) and \( D_t(\mathbb{C}) \) denote the fibre over \( t \).

We can transfer the homology class \( \gamma \in H_*(X_1(\mathbb{C}), D_1(\mathbb{C})) \) to any fibre \( (X_t(\mathbb{C}), D_t(\mathbb{C})) \) for \( t \in (0, \infty) \). This allows us to define a function \( f \) on \( (0, \infty) \) by

\[
f(t) = \int_{\gamma_t} \omega_t.
\]

The function \( f \) is real analytic.

9.2.2 Definition. Let \( \mathcal{P}_\mathbb{Q}((0, \infty)) \subset C^\infty((0, \infty)) \) be the subalgebra of functions of this form. Elements of this subalgebra we be called rational periods.
Note that, if \( f \) is a rational period, then, for every rational number \( t \in \mathbb{Q} \cap (0, \infty) \), \( f(t) \) is a period in the sense of Kontsevich and Zagier.

**9.3 Definition.** Let
\[
\mathcal{P}((0, \infty)) = \mathcal{P}_\mathbb{Q}((0, \infty)) \otimes \mathbb{R} \subset C^\infty((0, \infty))
\]
be the real vector space spanned by the space of rational periods. Elements of \( \mathcal{P}((0, \infty)) \) will be called periods.

### 9.3. Now we are ready to state the theorem on the small \( \varepsilon \) asymptotic expansions of the functions \( w_\gamma(P(\varepsilon, L), I)(a) \).

We will regard the functional \( w_\gamma(P(\varepsilon, L), I)(a) \) as a function of the three variables \( \varepsilon, L \) and \( a \in C^\infty(M) \). It is an element of the space of functionals
\[
\mathcal{O}(C^\infty(M), C^\infty((0, \infty)_\varepsilon) \otimes C^\infty((0, \infty)_L)).
\]
The subscripts \( \varepsilon \) and \( L \) indicate the coordinates on the copies of \( (0, \infty) \). If we fix \( \varepsilon \) but allow \( L \) and \( a \) to vary, we get a functional
\[
w_\gamma(P(\varepsilon, L), I) \in \mathcal{O}(C^\infty(M), C^\infty((0, \infty)_L)).
\]
This is a topological vector space; we are interested in the behaviour of \( w_\gamma(P(\varepsilon, L), I) \) as \( \varepsilon \to 0 \).

The following theorem describes the small \( \varepsilon \) behaviour of \( w_\gamma(P(\varepsilon, L), I) \).

**9.3.1 Theorem.** Let \( I \in \mathcal{O}_{\text{loc}}(C^\infty(M))[[h]] \) be a local functional, and let \( \gamma \) be a connected stable graph.

1. There exists a small \( \varepsilon \) asymptotic expansion
\[
w_\gamma(P(\varepsilon, L), I) \sim \sum_{i=0}^{\infty} g_i(\varepsilon) \Psi_i
\]
where the
\[
g_i \in \mathcal{P}((0, \infty)_\varepsilon)
\]
are periods, and \( \Psi_i \in \mathcal{O}(C^\infty(M), C^\infty((0, \infty)_L)) \).

The precise meaning of “asymptotic expansion” is as follows: there is a non-decreasing sequence \( d_R \in \mathbb{Z} \), indexed by \( R \in \mathbb{Z}_{>0} \), such that \( d_R \to \infty \) as \( R \to \infty \), and such that for all \( R \),
\[
\lim_{\varepsilon \to 0} \varepsilon^{-d_R} \left( w_\gamma(P(\varepsilon, L), I) - \sum_{i=0}^{R} g_i(\varepsilon) \Psi_i \right) = 0.
\]
where the limit is taken in the topological vector space \( \mathcal{O}(C^\infty(M), C^\infty((0, \infty)_L)) \).

(2) The \( g_i(\varepsilon) \) appearing in this asymptotic expansion have a finite order pole at zero: for each \( i \) there is a \( k \) such that \( \lim_{\varepsilon \to 0} \varepsilon^k g_i(\varepsilon) = 0 \).

(3) Each \( \Psi_i \) appearing in the asymptotic expansion above has a small \( L \) asymptotic expansion of the form

\[
\Psi_i \simeq \sum_{j=0}^{\infty} f_{i,j}(L) \Phi_{i,j}
\]

where the \( \Phi_{i,j} \) are local action functionals, that is, \( \Phi_{i,j} \in \mathcal{O}_i(C^\infty(M)) \); and each \( f_{i,k}(L) \) is a smooth function of \( L \in (0, \infty) \).

This theorem is proved in Appendix 1. All of the hard work required to construct counterterms is encoded in this theorem. The theorem is proved by using the small \( t \) asymptotic expansion for the heat kernel to approximate each \( w_\gamma(P(\varepsilon, L), I) \) for small \( \varepsilon \).

9.4. These results allow us to extract the singular part of the finite-dimensional integral \( w_\gamma(P(\varepsilon, L), I) \). Of course, the notion of singular part is not canonical, but depends on a choice.

9.4.1 Definition. Let \( \mathcal{P}((0, \infty))_{\geq 0} \subset \mathcal{P}((0, \infty)) \) be the subspace of those functions \( f \) of \( \varepsilon \) which are periods and which admit a limit as \( \varepsilon \to 0 \).

A renormalization scheme is a complementary subspace

\[
\mathcal{P}((0, \infty))_{< 0} \subset \mathcal{P}((0, \infty))
\]

to \( \mathcal{P}((0, \infty))_{\geq 0} \).

Thus, once we have chosen a renormalization scheme we have a direct sum decomposition

\[
\mathcal{P}((0, \infty)) = \mathcal{P}((0, \infty))_{\geq 0} \oplus \mathcal{P}((0, \infty))_{< 0}.
\]

A renormalization scheme is the data one needs to define the singular part of a function in \( \mathcal{P}((0, \infty)) \).

9.4.2 Definition. If \( f \in \mathcal{P}((0, \infty)) \), define the singular \( \text{Sing}(f) \) of \( f \) to be the projection of \( f \) onto \( \mathcal{P}((0, \infty))_{< 0} \).
9.5. We can now use this definition to extract the singular part of the functions \( w_\gamma(P(\varepsilon, L), I) \). As before, let us think of \( w_\gamma(P(\varepsilon, L), I) \) as a distribution on \( M^k \). Then, Theorem 9.3.1 shows that \( w_\gamma(P(\varepsilon, L), I) \) has a small \( \varepsilon \) asymptotic expansion of the form

\[
w_\gamma(P(\varepsilon, L), I) \simeq \sum_{i=0}^{\infty} g_i(\varepsilon) \Phi_i
\]

where the \( g_i(\varepsilon) \) are periods and the \( \Phi_i \in \text{Hom}(C^\infty(M^k), C^\infty((0, \infty)_L)) \). Theorem 9.3.1 also implies that there exists an \( N \in \mathbb{Z}_{\geq 0} \) such that, for all \( n > N \), \( g_n(\varepsilon) \) admits an \( \varepsilon \to 0 \) limit.

Denote the \( N^\text{th} \) partial sum of the asymptotic expansion by

\[
\Psi_N(\varepsilon) = \sum_{i=0}^{N} g_i(\varepsilon) \Phi_i.
\]

Then, we can define the singular part of \( w_\gamma(P(\varepsilon, L), I) \) simply by

\[
\text{Sing}_\varepsilon w_\gamma(P(\varepsilon, L), I) = \text{Sing}_\varepsilon \Psi_N(\varepsilon) = \sum_{i=0}^{N} (\text{Sing}_\varepsilon g_i(\varepsilon)) \Phi_i.
\]

This singular part is independent of \( N \), because if \( N \) is increased the function \( \Psi_N(\varepsilon) \) is modified only by the addition of functions of \( \varepsilon \) which are periods and which tend to zero as \( \varepsilon \to 0 \).

Theorem 9.3.1 implies that \( \text{Sing}_\varepsilon w_\gamma(P(\varepsilon, L), I) \) has the following properties.

9.5.1 Theorem. Let \( I \in \mathcal{O}_{loc}(C^\infty(M))[h] \) be a local functional, and let \( \gamma \) be a connected stable graph.

1. \( \text{Sing}_\varepsilon w_\gamma(P(\varepsilon, L), I) \) is a finite sum of the form

\[
\text{Sing}_\varepsilon w_\gamma(P(\varepsilon, L), I) = \sum f_i(\varepsilon) \Phi_i
\]

where

\[
\Phi_i \in \mathcal{O}_{loc}(C^\infty(M), C^\infty((0, \infty)_L)),
\]

and

\[
f_i \in \mathcal{P}((0, \infty))_{<0}
\]

are purely singular periods.

2. The limit

\[
\lim_{\varepsilon \to 0} (w_\gamma(P(\varepsilon, L), I) - \text{Sing}_\varepsilon w_\gamma(P(\varepsilon, L), I))
\]

exists in the topological vector space \( \mathcal{O}_{loc}(C^\infty(M), C^\infty((0, \infty)_L)) \).
Each \( \Phi_i \) appearing in the finite sum above has a small \( L \) asymptotic expansion

\[
\Phi_i \simeq \sum_{j=0}^{\infty} f_{i,j}(L) \Psi_{i,j}
\]

where \( \Psi_{i,j} \in \mathcal{O}_{\text{loc}}(C^\infty(M)) \) is local, and \( f_{i,j}(L) \) is a smooth function of \( L \in (0, \infty) \).

## 10. Constructing local counterterms

### 10.1. The heart of the proof of theorem A is the construction of local counterterms for a local interaction \( I \in \mathcal{O}_{\text{loc}}(C^\infty(M)) \). This construction is simple and inductive, without the complicated graph combinatorics of the BPHZ algorithm.

The theorem on the existence of local counterterms is the following.

#### 10.1.1 Theorem. There exists a unique series of local counterterms

\[
I_{i,k}^{\text{CT}}(\varepsilon) \in \mathcal{O}_{\text{loc}}(C^\infty(M)) \otimes_{\text{alg}} \mathcal{P}((0, \infty))_{<0},
\]

for all \( i > 0, k \geq 0 \), with \( I_{i,k}^{\text{CT}} \) homogeneous of degree \( k \) as a function of \( \alpha \in C^\infty(M) \), such that, for all \( L \in (0, \infty] \), the limit

\[
\lim_{\varepsilon \to 0} W \left( P(\varepsilon, L), I - \sum_{i,k} h^i I_{i,k}^{\text{CT}}(\varepsilon) \right)
\]

exists.

Here the symbol \( \otimes_{\text{alg}} \) denotes the algebraic tensor product, so only finite sums are allowed.

### 10.2. We will construct our counterterms using induction on the genus and number of external edges of the Feynman graphs. Later, we will see a very short (though unilluminating) construction of the counterterms, which does not use Feynman graphs. For reasons of exposition, we will introduce the Feynman graph picture first.

Let \( \Gamma_{i,k} \) denote the set of all stable graphs of genus \( i \) with \( k \) external edges. Let

\[
W_{i,k}(P,I) = \sum_{\gamma \in \Gamma_{i,k}} w_{\gamma}(P(\varepsilon,L), I).
\]

Thus,

\[
W(P,I) = \sum h^i W_{i,k}(P,I).
\]
If the graph $\gamma$ is of genus zero, and so is a tree, then $\lim_{\varepsilon \to 0} w_\gamma(P(\varepsilon, L), I)$ converges. Thus, the first counterterms we need to construct are those from graphs with one loop and one external edge. Let us define

$$I_{1,1}^{CT}(\varepsilon, L) = \text{Sing}_\varepsilon W_{1,1}(P(\varepsilon, L), I).$$

Section 9 explains the meaning of the singular part $\text{Sing}_\varepsilon$ of $W_{1,1}(P(\varepsilon, L), I)$.

We need to check that this has the desired properties. It is immediate from the definition that

$$W_{1,1}\left(P(\varepsilon, L), I - hI_{1,1}^{CT}(\varepsilon, L)\right) = W_{1,1}(P(\varepsilon, L), I) - I_{1,1}^{CT}(\varepsilon, L),$$

and so the limit $\lim_{\varepsilon \to 0} W_{1,1}\left(P(\varepsilon, L), I - hI_{1,1}^{CT}(\varepsilon, L)\right)$ exists.

Next, we need to check that

10.2.1. $I_{1,1}^{CT}(\varepsilon, L)$ is local.

First we will show that

10.2.2. $I_{1,1}^{CT}(\varepsilon, L)$ is independent of $L$.

Figure 5 illustrates $\frac{d}{d\varepsilon} W_{1,1}(P(\varepsilon, L), I)$. This expression is non-singular, as it is obtained by contracting the distribution $l_{0,3}$ on $C^\infty(M^3)$ with the smooth function $K_L \in C^\infty(M^2)$. Therefore $I_{1,1}^{CT}(\varepsilon, L)$, which we defined to be the singular part of $W_{1,1}(P(\varepsilon, L), I)$, is independent of $L$.

Since $I_{1,1}^{CT}(\varepsilon, L)$ is independent of $L$, to verify that it is local we only need to examine the behaviour of $W_{1,1}(P(\varepsilon, L), I)$ at small $L$. Theorem 9.5.1 implies that $\text{Sing}_\varepsilon W_{1,1}(P(\varepsilon, L), I)$ has a small $L$ asymptotic expansion in terms of local action functionals. Therefore, since we know $I_{1,1}^{CT}(\varepsilon, L)$ is independent of $L$, it follows that it is local.

Now that we know $I_{1,1}^{CT}(\varepsilon, L)$ is independent of $L$, we will normally drop $L$ from the notation.

10.3. The next step is to construct $I_{1,2}^{CT}(\varepsilon, L)$. However, it is just as simple to construct directly the general counterterm $I_{i,k}^{CT}(\varepsilon, L)$. Let us lexicographically order the set $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, so that $(i, k) < (j, l)$ if $i < j$ or if $i = j$ and $k < l$. Let us write

$$W_{<(i,k)}(P, I) = \sum_{(j,l)<(i,k)} h^j W_{j,l}(P, I).$$
The expression in terms of stable graphs, as follows. Let $\Gamma_{<(i,k)}$ denote the set of stable graphs with genus smaller than $i$, or with genus equal to $i$ and less than $k$ external edges. Then,

$$W_{<(i,k)}(P, I) = \sum_{\gamma \in \Gamma_{<(i,k)}} \frac{h^g(\gamma)}{|\text{Aut} \gamma|} w_\gamma(P, I).$$

Let us assume, by induction, we have constructed counterterms $I^{CT}_{j,l}(\varepsilon)$ for all $(j,l) < (i,k)$, with the following properties.

1. For all $L$, the limit

$$\lim_{\varepsilon \to 0} W_{<(i,k)} \left( P(\varepsilon, L), I - \sum_{(j,l) < (i,k)} h^j I^{CT}_{j,l}(\varepsilon) \right)$$

exists.

2. The counterterms $I^{CT}_{j,l}(\varepsilon)$ are local for $(j,k) < (i,k)$.

Then, we define the counterterm

$$I^{CT}_{i,k}(\varepsilon, L) = \text{Sing}_\varepsilon W_{i,k} \left( P(\varepsilon, L), I - \sum_{(j,l) < (i,k)} h^j I^{CT}_{j,l}(\varepsilon) \right).$$
This diagram, which is just the infinitesimal version of the renormalization group equation, explains why $I_{i,k}^{CT}(\varepsilon, L)$ is independent of $L$. In this diagram, $W_{s,s}$ is shorthand for

$$W_{s,s} \left( P(\varepsilon, L), I - \sum_{a,b<\langle i,k \rangle} I_{a,b}^{CT}(\varepsilon) \right).$$

As before, it is immediate that

$$W_{i,k} \left( P(\varepsilon, L), I - \sum_{(j,l)<(i,k)} h^i I_{j,l}^{CT}(\varepsilon) - h^i I_{i,k}^{CT}(\varepsilon, L) \right)$$

$$= W_{i,k} \left( P(\varepsilon, L), I - \sum_{(j,l)<(i,k)} h^i I_{j,l}^{CT}(\varepsilon) \right) - I_{i,k}^{CT}(\varepsilon, L)$$

is non-singular.

What we need to show is that

1. $I_{i,k}^{CT}(\varepsilon, L)$ is independent of $L$.
2. $I_{i,k}^{CT}(\varepsilon, L)$ is local.
As before, the second statement follows from the first one. To show independence of \( L \) it suffices to show that

\[
\left( \text{i} \right) \quad \frac{d}{dL} W_{i,k} \left( P(\xi, L), I - \sum_{(j,l) < (i,k)} \hbar^i I_{j,l}^{CT} (\xi) \right) \tag*{}
\]

is non-singular, that is, the limit of this expression as \( \xi \to 0 \) exists. A proof of this is illustrated in figure 6. In this diagram, an expression for \( \left( \text{i} \right) \) is given as a sum over graphs whose vertices are labeled by \( W_{j,l} \left( P(\xi, L), I - \sum_{(r,s) < (j,l)} \hbar^j I_{r,s}^{CT} (\xi) \right) \) for various \( (j,l) < (i,k) \). We know by induction that these are non-singular. On the unique edge of these graphs we put the heat kernel \( K_L \), which is smooth. Thus, the expression resulting from each graph is non-singular.

10.4. In fact, the use of Feynman graphs is not at all necessary in this proof; I first came up with the argument by thinking of \( W(P, I) \) as

\[
W(P, I) = \hbar \log \left( \exp \left( \hbar \partial_p \right) \exp(I/\hbar) \right).
\]

From this expression, one can see that

\[
W \left( P(L, L'), W \left( P(\xi, L), I \right) \right) = W \left( P(\xi, L'), I \right)
\]

\[
W_{i,k} \left( P(\xi, L), I \right) = W_{i,k} \left( P(\xi, L), I_{\leq (i,k)} \right) + I_{i,k}.
\]

The first identity is obvious, and the second identity can be seen (for instance) using the expression of \( W(P, I) \) in terms of Feynman graphs.

These two identities are that is really needed for the argument. Indeed, suppose we have constructed counterterms \( I_{r,s}^{CT} (\xi) \) for all \( (r,s) < (i,k) \), such that, for all \( L \), the limit

\[
\lim_{\xi \to 0} W_{\leq (i,k)} \left( P(\xi, L), I - \sum_{(r,s) < (i,k)} \hbar^r I_{r,s}^{CT} (\xi) \right)
\]

exists. Let us suppose, by induction, that these counterterms are local and independent of \( L \).

Then, we define the next counterterm by

\[
I_{i,k}^{CT} (L, \xi) = \text{Sing}_\xi W_{i,k} \left( P(\xi, L), I - \sum_{(r,s) < (i,k)} \hbar^r I_{r,s}^{CT} (\xi) \right).
\]
The identity

\[ W_{i,k} \left( P(\varepsilon, L), I - \sum_{(r,s) < (i,k)} h^r I^\text{CT}_{r,s} (\varepsilon) - h^i I^\text{CT}_{i,k} (L, \varepsilon) \right) = \\
W_{i,k} \left( P(\varepsilon, L), I - \sum_{(r,s) < (i,k)} h^r I^\text{CT}_{r,s} (\varepsilon) \right) - I^\text{CT}_{i,k} (L, \varepsilon) \]

shows that the limit

\[ \lim_{\varepsilon \to 0} W_{,i,k} \left( P(\varepsilon, L), I - \sum_{(r,s) < (i,k)} h^r I^\text{CT}_{r,s} (\varepsilon) - h^i I^\text{CT}_{i,k} (L, \varepsilon) \right) \]

exists.

To show locality of the counterterm \( I^\text{CT}_{i,k} (L, \varepsilon) \), it suffices, as before, to show that it is independent of \( L \). If \( L' > L \), we have

\[ I^\text{CT}_{i,k} (L', \varepsilon) = \text{Sing}_\varepsilon W_{i,k} \left( P(\varepsilon, L'), I - \sum_{(r,s) < (i,k)} h^r I^\text{CT}_{r,s} (\varepsilon) \right) = \\
= \text{Sing}_\varepsilon W_{i,k} \left( P(L, L'), W \left( P(\varepsilon, L), I - \sum_{(r,s) < (i,k)} h^r I^\text{CT}_{r,s} (\varepsilon) \right) \right) = \\
= \text{Sing}_\varepsilon W_{i,k} \left( P(L, L'), W_{< (i,k)} \left( P(\varepsilon, L), I - \sum_{(r,s) < (i,k)} h^r I^\text{CT}_{r,s} (\varepsilon) \right) \right) = \\
+ h^i W_{i,k} \left( P(\varepsilon, L), I - \sum_{(r,s) < (i,k)} h^r I^\text{CT}_{r,s} (\varepsilon) \right) = \\
= \text{Sing}_\varepsilon W_{i,k} \left( P(L, L'), W_{< (i,k)} \left( P(\varepsilon, L), I - \sum_{(r,s) < (i,k)} h^r I^\text{CT}_{r,s} (\varepsilon) \right) \right) + \\
+ \text{Sing}_\varepsilon W_{i,k} \left( P(\varepsilon, L), I - \sum_{(r,s) < (i,k)} h^r I^\text{CT}_{r,s} (\varepsilon) \right) \]

Since

\[ W_{i,k} \left( P(L, L'), W_{< (i,k)} \left( P(\varepsilon, L), I - \sum_{(r,s) < (i,k)} h^r I^\text{CT}_{r,s} (\varepsilon) \right) \right) \]
is non-singular, the last equation reduces to
\[
I_{i,k}^{\text{CT}}(L', \varepsilon) = \text{Sing}_{\varepsilon} W_{i,k} \left( P(\varepsilon, L), I - \sum_{(r,s) < (i,k)} h^r I_{r,s}^{\text{CT}}(\varepsilon) \right)
\]
\[
= I_{i,k}^{\text{CT}}(L, \varepsilon)
\]
as desired.

Thus, \( I_{i,k}^{\text{CT}}(L, \varepsilon) \) is independent of \( L \), and can be written as \( I_{i,k}^{\text{CT}}(\varepsilon) \); and we can continue the induction.

11. Proof of the main theorem.

Let \( I \in \mathcal{O}_{\text{loc}}^+ (C^\infty(M))[[\hbar]] \) be a local action functional. What we have shown so far allows us to construct the corresponding theory. Let
\[
W^R (P(0, L), I) = \lim_{\varepsilon \to 0} W \left( P(\varepsilon, L), I - I_{i,k}^{\text{CT}}(\varepsilon) \right)
\]
be the renormalized version of the renormalization group flow from scale 0 to scale \( L \) applied to \( I \). The theory associated to \( I \) has scale \( L \) effective interaction \( W^R (P(0, L), I) \). It is easy to see that this satisfies all the axioms of a perturbative quantum field theory, as given in Definition 7. The locality axiom of the definition of perturbative quantum field theory follows from Theorem 9.5.1.

We need to show the converse: that to each theory there is a corresponding local action functional. This is a simple induction. Let \( \{ I[L] \} \) denote a collection of effective interactions defining a theory. Let us assume, by induction on the lexicographic ordering as before, that we have constructed local action functionals \( I_{r,s} \) for \( (r, s) < (i, k) \) such that
\[
W_{a,b}^R \left( P(0, L), \sum_{(r,s) < (i,k)} h^r I_{r,s} \right) = I_{a,b}[L]
\]
for all \((a, b) < (i, k)\).

Then, the infinitesimal renormalization group equation implies that
\[
W_{i,k}^R \left( P(0, L), \sum_{(r,s) < (i,k)} h^r I_{r,s} \right) = I_{i,k}[L]
\]
is independent of \( L \). The locality axiom for the theory \( \{ I[L] \} \) – which says that each \( I_{i,k}[L] \) has a small \( L \) asymptotic expansion in terms of local action functionals – implies
that this quantity is local. Thus, let
\[ I_{i,k} = I_{i,k}[L] - W_{i,k}^R \left( P(0, L), \sum_{(r,s) \leq (i,k)} h^r I_{r,s} \right). \]
Then,
\[ W_{a,b}^R \left( P(0, L), \sum_{(r,s) \leq (i,k)} h^r I_{r,s} \right) = I_{a,b}[L] \]
for all \((a, b) \leq (i, k)\), and we can continue the induction.

11.1. Recall that we defined \( \mathcal{T}^{(n)} \) to be the space of theories defined modulo \( h^{n+1} \), and \( \mathcal{T}^{(\infty)} \) to be the space of theories. Thus, we have shown that the choice of renormalization scheme sets up a bijection between \( \mathcal{T}^{(\infty)} \) and functionals \( I \in \mathcal{O}^{+}_{\text{loc}}(C^{\infty}(M))[h] \), and between \( \mathcal{T}^{(n)} \) and functionals \( I \in \mathcal{O}^{+}_{\text{loc}}(C^{\infty}(M))[h]/h^{n+1} \).

The more fundamental statement of the theorem on the bijection between theories and local action functionals is that the map \( \mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)} \) makes \( \mathcal{T}^{(n+1)} \) into a principal bundle for the group \( \mathcal{O}^{+}_{\text{loc}}(C^{\infty}(M)) \), in a canonical way (independent of any arbitrary choices, such as that of a renormalization scheme). In this subsection we will use the bijection constructed above to prove this statement.

The bijection between theories and Lagrangians shows that the map \( \mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)} \) is surjective; this is the only place the bijection is used.

To show that \( \mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)} \) is a principal bundle, suppose that \( \{I[L]\}, \{J[L]\} \) are two theories which are defined modulo \( h^{n+2} \) and which agree modulo \( h^{n+1} \).

Let \( I_0[L] \in \mathcal{T}^{(0)} \) be the classical theory corresponding to both \( I[L] \) and \( J[L] \). Let us consider the tangent space to \( \mathcal{T}^{(0)} \) at \( I_0[L] \), which includes infinitesimal deformations of classical theories which do not have to be at least cubic. More precisely, let \( T_{I_0[L]} \mathcal{T}^{(0)} \) be the set of \( H[L] \in \mathcal{O}(C^{\infty}(M)) \) such that
\[ I_0[L] + \delta H[L] \]
satisfies the classical renormalization group equation modulo \( \delta^2 \),
\[ I_{0,i}[L] + \delta H_i[L] = W_{0,i} \left( P(0, L), \sum \delta H_j[L] \right) \]
and which satisfy the usual locality axiom, that \( H[L] \) has a small \( L \) asymptotic expansion in terms of local action functionals.

The bijection between classical theories and local action functionals is canonical, independent of the choice of renormalization scheme. This is true even if we include
non-cubic terms in our effective interaction, as long as these non-cubic terms are accompanied by nilpotent parameters.

Thus, we have a canonical isomorphism of vector spaces

\[ T_{I_0[L]}(\mathcal{F}^{(0)}) \cong \mathcal{O}_{\text{loc}}(C^\infty(M)). \]

The following lemma now shows that \( \mathcal{F}^{(n+1)} \to \mathcal{F}^{(n)} \) is a torsor for \( \mathcal{O}_{\text{loc}}(C^\infty(M)). \)

11.1.1 Lemma. Let \( I, J \in \mathcal{F}^{(n+1)} \) be theories which agree in \( \mathcal{F}^{(n)} \). Then, the functional

\[ I_0[L] + \delta h^{-(n+1)}(I[L] - J[L]) \in \mathcal{O}(C^\infty(M)) \]

satisfies the classical renormalization group equation modulo \( \delta^2 \), and so defines an element of

\[ T_{I_0[L]}(\mathcal{F}^{(0)}) \cong \mathcal{O}_{\text{loc}}(C^\infty(M)). \]

Note that \( h^{-(n+1)}(I[L] - J[L]) \) is well-defined as \( I[L] \) and \( J[L] \) agree modulo \( h^{n+1} \).

PROOF. This is a simple calculation. \( \square \)

12. Proof of the parametrix formulation of the main theorem

In this page we will prove the equivalence of the definition of theory based on arbitrary parametrices, explained in Section 8 with the definition based on the heat kernel. Since this result is not used elsewhere in the book, I will not give all the details.

Thus, suppose we have a theory in the heat kernel sense, given by a family \( I[L] \) of effective interactions satisfying the renormalization group equation and the locality axiom. If \( P \) is a parametrix, let us define a functional \( I[P] \) by

\[ I[P] = W(P - P(0,L), I[L]) \in \mathcal{O}(C^\infty(M))[[h]]. \]

Since \( P(0,L) \) and \( P \) are both parametrices for the operator \( D + m^2 \), the difference between them is smooth. Thus, \( W(P - P(0,L), I[L]) \) is well-defined.

12.0.2 Lemma. The collection of effective interactions \( \{I[P]\} \), defined for each parametrix \( P \), defines a theory using the parametrix definition of theory.

PROOF. To prove this, we need to verify the following.
(1) If \( P' \) is another parametrix, then
\[
I[P'] = W \left( P' - P, I[P] \right)
\]
(this is the version of the renormalization group equation for the definition of theory based on parametrices.

(2) By choosing a parametrix \( P \) with support close to the diagonal, we can make the distribution
\[
I_i,k[P] \in \mathcal{D}(M^k)_{s_k}
\]
on \( M^k \) supported as close as we like to the small diagonal.

(3) The functional \( I[P] \) has smooth first derivative. Recall, as explained in Section 8, that this means the following. There is a continuous linear map
\[
C^\infty(M) \to \mathcal{E}(C^\infty(M))[[\hbar]]
\]
\[
\phi \mapsto \frac{dI[P]}{df}.
\]
Saying that \( I[\Phi] \) has smooth first derivative means that this map extends to a continuous linear map
\[
\mathcal{D}(M) \to \mathcal{E}(C^\infty(M))[[\hbar]]
\]
where \( \mathcal{D}(M) \) is the space of distributions on \( M \).

In order to verify these properties, it is convenient to choose a renormalization scheme, so that we can write
\[
I[L] = \lim_{\epsilon \to 0} W \left( P(\epsilon, L), I - I^{cl}(\epsilon) \right).
\]
Now let us choose a cut-off function \( \Psi \in C^\infty(M \times M) \) which is 1 in a neighbourhood of the diagonal, and 0 outside a small neighbourhood of the diagonal. Then, \( \Psi P(0, L) \) is a parametrix, which agrees with \( P(0, L) \) near the diagonal. Thus, we have
\[
I[\Psi P(0, L)] = \lim_{\epsilon \to 0} W \left( \Psi P(\epsilon, L), I - I^{cl}(\epsilon) \right).
\]
Since \( I \) and \( I^{cl}(\epsilon) \) are local, we can, by choosing the function \( \Psi \) to be supported in a very small neighbourhood of the diagonal, ensure that \( I_i,k[\Psi P(0, L)] \) is supported within an arbitrarily small neighbourhood of the small diagonal in \( M^k \).

From this and from the identity
\[
I[P] = W \left( P - \Psi P(0, L), I[\Psi P(0, L)] \right)
\]
we can check that, by choosing the parametrix \( P \) to have support very close to the diagonal, we can ensure that \( I_i,k[P] \) has support arbitrarily close to the small diagonal in \( M^k \). The point is that the combinatorial formulae for \( W(\Phi, J) \) (where \( \Phi \in C^\infty(M^2) \))
and \( J \in \mathcal{E}(C^\infty(M^2))^+[[h]] \) allows one to control the support of \( W_{i,k}(\Phi, J) \) in terms of the support of \( J \) and that of \( \Phi \).

This shows that \( I[P] \) satisfies axioms the first two properties we want to verify. It remains to check that \( I[P] \) has smooth first derivative. This follows from the fact that all local functionals have smooth first derivative, and that the property of having smooth first derivative is preserved under the renormalization group flow.

\[
\square
\]

Now, we need to prove the converse. This follows from the following lemma.

**12.0.3 Lemma.** Let \( \{I_{(r,s)}[P]\} \) and \( \{I'_{(r,s)}[P]\} \) be two parametrix theories, defined for all \((r, s) \leq (I, K)\). Suppose that \( I_{(r,s)}[P] = I'_{(r,s)}[P] \) if \((r, s) < (I, K)\). Then,

\[
I_{(I,K)} = I_{(I,K)}[P] - I'_{(I,K)}[P]
\]

is a local functional, that is, an element of \( \mathcal{E}_{\text{loc}}(C^\infty(M)) \).

**Proof.** Note that the renormalization group equation implies that \( I_{(I,K)} \) is independent of \( P \). The locality axiom implies that \( I_{(I,K)} \) is supported on the small diagonal of \( M^K \). Further, \( I_{(I,K)} \) has smooth first derivative. Any distribution \( J \in \mathcal{D}(M^K)_{\bar{s}_K} \) which is supported on the small diagonal and which has smooth first derivative is a local functional.

\[
\square
\]

**13. Vector-bundle valued field theories**

We would like to have a bijection between theories and Lagrangians for a more general class of field theories. The most general set-up we will need is when the fields are sections of some vector bundle on a manifold; and the interactions depend smoothly on some additional supermanifold. In this section we will explain how to do this on a compact manifold.

**13.0.4 Definition.** A nilpotent graded manifold is the following data:

(1) A smooth manifold with corners \( X \),

(2) A sheaf \( A \) of commutative superalgebras over the sheaf of algebras \( C^\infty_X \),

satisfying the following properties:
(1) \( A \) is locally free of finite rank as a \( C^\infty_X \)-module. In other words, \( A \) is the sheaf of sections of some super vector bundle on \( X \).

(2) \( A \) is equipped with an ideal \( I \) such that \( A/I = C^\infty_X \), and \( I^k = 0 \) for some \( k > 0 \). The ideal \( I \), its powers \( I^1 \), and the quotient sheaves \( A/I^1 \), are all required to be locally free sheaves of \( C^\infty_X \)-modules.

The algebra \( \Gamma(X, A) \) of \( C^\infty \) global sections of \( A \) will be denoted by \( \mathcal{A} \).

Everything in this section will come in families, parameterized by a nilpotent graded manifold \((X, A)\).

13.1. We are interested in vector-valued theories on a compact manifold \( M \). As in the case of scalar field theories, we will fix the data of the free theory, which gives us our propagator; and then consider possible interacting theories which deform this.

The following definition aims to be broad enough to capture all of the free field theories used in this book, and in future applications. Unfortunately, it is not particularly transparent.

13.1.1 Definition. A free theory on a manifold \( M \) consists of the following data.

(1) A super vector bundle \( E \) over the field \( \mathbb{R} \) or \( \mathbb{C} \) on \( M \), equipped with a direct sum decomposition \( E = E_1 \oplus E_2 \) into the spaces of propagating and non-propagating fields, respectively. We will denote the space of smooth global sections of \( E \) or \( E_i \) by \( \mathcal{E}, \mathcal{E}_i \) respectively.

We will let

\[
\mathcal{E}_1^i = \Gamma(M, E_i^\vee \otimes \text{Dens}(M)).
\]

There is an inclusion

\[
\mathcal{E}_1^i \subset \mathcal{E}_1^\vee.
\]

(2) An even, \( \mathcal{A} \)-linear, order two differential operator

\[
D_{\mathcal{E}_1} : \mathcal{E}_1 \otimes \mathcal{A} \to \mathcal{E}_1 \otimes \mathcal{A}
\]

(where the tensor product is the completed projective tensor product).

\( D_{\mathcal{E}_1} \) must be a generalized Laplacian, which means that the symbol

\[
\sigma(D_{\mathcal{E}_1}) \in \Gamma(T^*M, \text{Hom}(E, E)) \otimes \mathcal{A}
\]

must be the identity on \( E \) times a smooth family of Riemannian metrics

\[
g \in C^\infty(T^*M) \otimes C^\infty(X).
\]
(Recall that \( \mathcal{C}^\infty(X) \subset \mathcal{A} \) is a subalgebra, as \( \mathcal{A} \) is the global sections of a bundle of algebras on \( X \)).

(3) A differential operator

\[
D' : \mathcal{E}^1_1 \rightarrow \mathcal{E}_1.
\]

This operator is required to be symmetric: the formal adjoint

\[
(D')^* : \mathcal{E}^1_1 \rightarrow \mathcal{E}_1
\]

is required to be equal to \( D' \).

(4) Let

\[
D_{\mathcal{E}_1} : \mathcal{E}^1_1 \rightarrow \mathcal{E}_1
\]

be the formal adjoint of \( D_{\mathcal{E}_1} \). We require that

\[
D' D_{\mathcal{E}_1} = D_{\mathcal{E}_1} D'.
\]

We will abuse notation and refer to the entirely of the data of a free theory on \( M \) as \( \mathcal{E} \).

The most basic example of this definition is the free scalar field theory, as considered in chapter 2. There, the space \( \mathcal{E}_1 \) of fields is \( \mathcal{C}^\infty(M) \). The space \( \mathcal{E}_2 \) of non-interacting fields is 0. The operator \( D_{\mathcal{E}_1} \) is the usual positive-definite Laplacian operator \( \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M) \). The operator \( D' \) is the identity, where we have used the Riemannian volume element to trivialize the bundle of densities on \( M \), and so to identify \( \mathcal{E}_1 \) with \( \mathcal{E}^1_1 \).

More interesting examples will be presented in chapter 5, when we consider the Batalin-Vilkovisky formalism. The use of graded vector bundles will be essential in the BV formalism.

**13.2.** The space \( \mathcal{E}_2 \) of non-propagating fields is introduced into this definition with an eye to future applications: none of the examples treated in this book will have non-propagating fields. Thus, the reader will lose nothing by ignoring the space of non-propagating fields.

For those who are interested, however, let me briefly explain the reason for considering non-propagating fields. Let us consider a free scalar field theory on a Riemannian manifold \( M \) with metric \( g_0 \). Let us consider perturbing the metric to \( g_0 + h \). The action of the scalar field theory is given by \( S(\phi) = \int \phi \triangle_{g_0 + h} \phi \), as usual. Notice that the action is not just local as a function of the field \( \phi \), but also local as a function
of the perturbation \( h \) of the metric. However, \( h \) is not treated as a quantum field, only as a classical field: we do not consider integration over the space of metrics.

In this situation, the tensor \( h \) is said to be a background field, or (in the terminology adopted here) a non-propagating field.

In this example, the space of fields is
\[
\mathcal{E} = C^\infty(M) \oplus \Gamma(M, \text{Sym}^2 TM)
\]
where \( \mathcal{E}_1 = C^\infty(M) \) is the space of propagating fields and \( \mathcal{E}_2 = \Gamma(M, \text{Sym}^2 TM) \) is the space of non-propagating fields.

The action of the theory is
\[
S(\phi) = \int_M \phi \Delta_{g_0} \phi + \int_M \phi \left( \Delta_{g_0 + h} - \Delta_{g_0} \right) \phi.
\]
The quadratic part of the action is the only part relevant to the definition of a free field theory as presented above. As we see, the quadratic part only depends on the propagating field \( \phi \), and not on \( h \). However, the interaction term depends both on \( \phi \) and \( h \). It is a general feature of the interaction terms that they must have some dependence on \( \phi \); they cannot be functions just of the non-propagating field \( h \). We will build this into our definition of interactions in the presence of non-propagating fields shortly.

If we take the free theory associated to this example – given by discarding the interacting terms in the action \( S \) – we fit it into the general definition as follows. The operator \( D_{\mathcal{E}_1} \) is the Laplacian \( \Delta_{g_0} : C^\infty(M) \to C^\infty(M) \). The operator \( D' \) is the identity map \( C^\infty(M) \to C^\infty(M) \).

13.3. Now that we have the general definition of a free field theory, we can start to define the concept of effective interaction in this context. First, we have to define the heat kernel.

If the manifold \( M \) is compact, there is a unique heat kernel
\[
K_t \in \mathcal{E}_1 \otimes \mathcal{E}_1 \otimes C^\infty(\mathbb{R}_{>0}) \otimes \mathcal{A}
\]
for the operator \( D_{\mathcal{E}_1} \).

Composing with the operator \( D' \) gives an element
\[
D'K_t \in \mathcal{E}_1 \otimes \mathcal{E}_1 \otimes C^\infty(\mathbb{R}_{>0}) \otimes \mathcal{A}.
\]
We will view this as an element of $\mathcal{E} \otimes \mathcal{E} \otimes C^\infty(\mathbb{R}_{>0})$.

The adjointness properties of the differential operators $D'$, $D_{\mathcal{E}_1}$ imply that $D'K_t$ is symmetric.

The propagator for the theory is

$$P(\mathcal{E}, L) = \int_L D'K_t \in \mathcal{E}^{\otimes 2} \otimes \mathcal{A}$$

which is again symmetric.

Note that unless we impose additional positivity conditions on the operator $D_{E_1}$, the heat kernel $K_t$ may not exist at $t = \infty$; thus, the propagator $P(\mathcal{E}, \infty)$ may not exist. In almost all examples, however, the operator $D_{\mathcal{E}_1}$ is positive, and so the heat kernel $K_\infty$ does exist.

If we specialize the case of the free scalar field theory on a Riemannian manifold $(M, g_0)$, then, as we have seen, $\mathcal{E}_1 = C^\infty(M)$, $D_{\mathcal{E}_1} = \Delta_{g_0}$ is the non-negative Laplacian for the metric $g_0$. In this example the operator $D'$ is the identity. Thus, the propagator prescribed by this general definition coincides with propagator presented in our earlier analysis of the free field theory.

13.4. As before, we can define the algebra

$$\mathcal{O}(\mathcal{E}, \mathcal{A}) = \prod \text{Hom}(\mathcal{E}^{\otimes n}, \mathcal{A})_{S_n}$$

of all functionals on $\mathcal{E}$ with values in $\mathcal{A}$. Here Hom denotes the space of continuous linear maps. The properties of the symmetric monoidal category of nuclear spaces, as detailed in Appendix 2, show that

$$\mathcal{O}(\mathcal{E}, \mathcal{A}) = \left( \prod_n \text{Sym}^n \mathcal{E}^{\vee} \right) \otimes \mathcal{A}.$$  

There is a subspace

$$\mathcal{O}_1(\mathcal{E}, \mathcal{A}) \subset \mathcal{O}(\mathcal{E}, \mathcal{A})$$

of $\mathcal{A}$-valued local action functionals, defined as follows.

13.4.1 Definition. A functional $\Phi \in \mathcal{O}(\mathcal{E}, \mathcal{A})$ is a local action functional if, when we expand $\Phi$ as a sum $\Phi = \sum \Phi_n$ of its homogeneous components, each

$$\Phi_n : \mathcal{E}^{\otimes n} \to \mathcal{A}.$$
can be written in the form
\[ \Phi_n(e_1, \ldots, e_n) = \sum_{j=1}^{k} \int_M (D_{1,j}e_1) \cdots (D_{n,j}e_n) d\mu \]
where
\[ d\mu \in \text{Densities}(M) \]
is some volume element on \( M \), and each
\[ D_{i,j} : \mathcal{E} \otimes \mathcal{A} \to C^\infty(M) \otimes \mathcal{A} \]
is an \( \mathcal{A} \)-linear differential operator.

Note that \( \mathcal{O}_1(\mathcal{E}, \mathcal{A}) \) is not a closed subspace. However, as we will see in Appendix 2, \( \mathcal{O}_{\text{loc}}(\mathcal{E}, \mathcal{A}) \) has a natural topology making it into a complete nuclear space, and a module over \( \mathcal{A} \) in the symmetric monoidal category of nuclear spaces.

Our interactions will be elements of
\[ \mathcal{O}_1(\mathcal{E}, \mathcal{A})[[\hbar]]. \]

We would like to allow our interactions to have quadratic and linear terms modulo \( \hbar \). However, we require that these quadratic terms are accompanied by elements of the nilpotent ideal
\[ \mathcal{I} = \Gamma(X, I) \subset \mathcal{A} \]
(recall that \( \mathcal{A}/\mathcal{I} = C^\infty(X) \)). If we don’t impose this condition, we will encounter infinite sums.

Thus, let us denote by
\[ \mathcal{O}^+(\mathcal{E}, \mathcal{A})[[\hbar]] \subset \mathcal{O}(\mathcal{E}, \mathcal{A})[[\hbar]] \]
the subset of those functionals which are at least cubic modulo the ideal generated by \( \mathcal{I} \) and \( \hbar \).

Then, the renormalization group operator
\[ W(P(\mathcal{E}, L), I) = \hbar \log \left( \exp(\hbar \partial_{P(\mathcal{E}, L)}) \exp(I/\hbar) \right) : \mathcal{O}^+(\mathcal{E}, \mathcal{A})[[\hbar]] \to \mathcal{O}^+(\mathcal{E}, \mathcal{A})[[\hbar]] \]
is well-defined.

Because we now allow quadratic and linear interaction terms modulo \( \hbar \), the Feynman graph expansion of this expression involves one- and two-valent genus 0 vertices. However, each such vertex is accompanied by an element of the ideal \( \mathcal{I} \) of \( \mathcal{A} \). Since
this ideal is nilpotent, there is a uniform bound on the number of such vertices that can occur, so there are no infinite sums.

**13.4.2 Definition.** A theory is given by a collection of even elements

\[ I[L] \in \mathcal{O}^+(\mathcal{E}, C^\infty((0, \infty)_L) \otimes \mathcal{A})[[h]], \]

such that

1. The renormalization group equation

\[ I[L'] = W \left( P(L, L'), I[L] \right) \]

holds.

2. Each \( I_{i,(k)}[L] \) has a small \( L \) asymptotic expansion

\[ I_{i,(k)}[L](e) \simeq \sum \Psi_r(e) f_r(L) \]

where \( \Psi_r \in \mathcal{O}_i(\mathcal{E}) \) are local action functionals.

Let \( \mathcal{F}^{(\infty)}(\mathcal{E}) \) denote the space of such theories, and let \( \mathcal{F}^{(n)}(\mathcal{E}) \) denote the space of theories defined modulo \( \hbar^{n+1} \), so that \( \mathcal{F}^{(\infty)}(\mathcal{E}) = \lim_{n \to \infty} \mathcal{F}^{(n)}(\mathcal{E}) \).

Let me explain more precisely what I mean by saying there is a small \( L \) asymptotic expansion

\[ I_{i,k}[L] \simeq \sum_{j \in \mathbb{Z}_{\geq 0}} g_j(L) \Phi_j. \]

Without loss of generality, we can require that the local action functionals \( \Phi_r \) appearing here are homogeneous of degree \( k \) in the field \( e \in \mathcal{E} \).

Recall that \( \mathcal{A} \) is the global sections of some bundle of algebras \( A \) on a manifold with corners \( X \). Let \( A_x \) denote the fibre of \( A \) at \( x \in X \). For every element \( \alpha \in \mathcal{A} \), let \( \alpha_x \in A_x \) denote the value of \( \alpha \) at \( x \).

The statement that there is such an asymptotic expansion means that there is a non-decreasing sequence \( d_R \in \mathbb{Z} \), tending to infinity, such that for all \( R \), for all fields \( e \in \mathcal{E} \), for all \( x \in X \),

\[ \lim_{L \to 0} L^{-d_R} \alpha_x \left( I_{i,k}[L](e) - \sum_{r=0}^{R} g_r(L) \Phi_r(e) \right) = 0 \]

in the finite dimensional vector space \( A_x \).

Then, as before, the theorem is:
13.4.3 Theorem. The space \( \mathcal{F}^{(n+1)}(E) \) has the structure of a principal \( \mathcal{O}_{\text{loc}}(E, \mathcal{A}) \) bundle over \( \mathcal{F}^{(n)}(E) \), in a canonical way. Further, \( \mathcal{F}^{(0)}(E) \) is canonically isomorphic to the space \( \mathcal{O}_{\text{loc}}^+(E, \mathcal{A}) \) of \( \mathcal{A} \)-valued local action functionals on \( E \) which are at least cubic modulo the ideal \( \mathcal{I} \subset \mathcal{A} \).

Further, the choice of renormalization scheme gives rise to a section \( \mathcal{F}^{(n)}(E) \rightarrow \mathcal{F}^{(n+1)}(E) \) of each torsor, and so a bijection between \( \mathcal{F}^{(\infty)}(E) \) and the space

\[
\mathcal{O}_{\text{loc}}^+(E, \mathcal{A})[[h]]
\]

of local action functionals with values in \( \mathcal{A} \), which are at least cubic modulo \( h \) and modulo the ideal \( \mathcal{I} \subset \mathcal{A} \).

Proof. The proof is essentially the same as before. The extra difficulties are of two kinds: working with an auxiliary parameter space \( X \) introduces extra analytical difficulties, and working with quadratic terms in our interaction forces us to use Artinian induction with respect to the powers of the ideal \( \mathcal{I} \subset \mathcal{A} \).

For simplicity, I will only give the proof when the effective interactions \( I[L] \) are all at least cubic modulo \( h \). The argument in the general case is the same, except that we also must perform Artinian induction with respect to the powers of the ideal \( \mathcal{I} \subset \mathcal{A} \).

As before, we will prove the renormalization scheme dependent version of the theorem, saying that there is a bijection between \( \mathcal{F}^{(\infty)}(E) \) and \( \mathcal{O}_{\text{loc}}^+(E, \mathcal{A})[[h]] \). The renormalization scheme independent formulation is an easy corollary.

Let us start by showing how to construct a theory associated to a local interaction

\[
I = \sum h^i I_{(i,k)} \in \mathcal{O}_{\text{loc}}(E, \mathcal{A})[[h]].
\]

We will assume that \( I_{(0,k)} = 0 \) if \( k < 3 \).

The argument is essentially the same as the argument we gave earlier. We will perform induction on the set \( \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \) with the lexicographical order: \((i,k) < (r,s)\) if \( i < r \) or if \( i = r \) and \( k < s \).

Suppose, by induction, we have constructed counterterms

\[
I_{(i,k)}^{CT} \in \mathcal{O}_{\text{loc}}(E, C^\infty((0,\infty), \mathcal{A}))
\]

for all \((i,k) < (I,K)\). The \( I_{(i,k)}^{CT} \) are supposed, by induction, to have the following properties:
(1) Each $I^{CT}_{(i,k)}(\varepsilon)$ is homogeneous of degree $k$ as a function of the field $e \in \mathcal{E}$.

(2) Each $I^{CT}_{(i,k)}(\varepsilon)$ is required to be a finite sum

$$I^{CT}_{(i,k)}(\varepsilon) = \sum g_r(\varepsilon) \Phi_r$$

where $g_r(\varepsilon) \in C^\infty((0, \infty)_{\varepsilon})$ and $\Phi_r \in \mathcal{O}_{loc}(\mathcal{E}, \mathcal{A})$. Each $g_r(\varepsilon)$ is required to have a finite order pole at $0$; that is, $\lim_{\varepsilon \to 0} \varepsilon^k g_r(\varepsilon) = 0$ for some $k > 0$.

(3) Recall that $\mathcal{A}$ is the space of global sections of a vector bundle $A$ on $X$. For any element $\alpha \in A$, let $\alpha_x \in A_x$ denote its value at $x \in X$.

We require that, for all $L \in (0, \infty)$ and all $x \in X$, the limit

$$\lim_{\varepsilon \to 0} W_{(r,s)} \left( P(\varepsilon, L), I - \sum_{(i,k) \leq (r,s)} h^i L^{CT}_{(i,k)}(\varepsilon) \right)$$

exists in the topological vector space $\text{Hom}(\mathcal{E}^\otimes r, \mathbb{R}) \otimes A_x$. Here, $\text{Hom}(\mathcal{E}^\otimes r, \mathbb{R})$ is given the strong topology (i.e. the topology of uniform convergence on bounded subsets).

Now we need to construct the next counterterm $I^{CT}_{(l,k)}(\varepsilon)$. We would like to define

$$I^{CT}_{(l,k)}(\varepsilon) = \text{Sing}_\varepsilon W_{(l,k)} \left( P(\varepsilon, L), I - \sum_{(r,s) < (l,k)} h^r L^{CT}_{(r,s)}(\varepsilon) \right).$$

In order to be able to define the singular part like this, we need to know that

$$W_{(l,k)} \left( P(\varepsilon, L), I - \sum_{(r,s) < (l,k)} h^r L^{CT}_{(r,s)}(\varepsilon) \right)$$

has a nice small $\varepsilon$ asymptotic expansion. The required asymptotic expansion is provided by the following theorem, proved in Appendix 1.

13.4.4 Theorem. For all graphs $\gamma$, and all $I \in \mathcal{O}_{loc}(\mathcal{E}, \mathcal{A})[[h]]$, there exist local action functionals $\Phi_r \in \mathcal{O}_{loc}(\mathcal{E}, \mathcal{A}) \otimes C^\infty((0, \infty)_L)$ and functions $g_r$ in the space $P((0, \infty)_{\varepsilon}) \subset C^\infty((0, \infty)_{\varepsilon})$ of periods, such that for all $l \in \mathbb{Z}_{\geq 0}$, there is a small $\varepsilon$ asymptotic expansion

$$\frac{\partial^l}{\partial L^l} w_{\gamma}(P(\varepsilon, L), I) \simeq \frac{\partial^l}{\partial L^l} \sum_{r \geq 0} g_r(\varepsilon) \Phi_r.$$ 

This means that for each $l$ and each $x \in X$, there exists a non-decreasing sequence $d_R \in \mathbb{Z}$, tending to $\infty$, such that

$$\lim_{\varepsilon \to 0} \varepsilon^{-d_R} \frac{\partial^l}{\partial L^l} \left( w_{\gamma}(P(\varepsilon, L), I) - \sum_{r \geq 0} g_r(\varepsilon) \Phi_r \right) x = 0.$$
in the topological vector space $\text{Hom}(\mathcal{E} \otimes T(\gamma), \mathbb{R}) \otimes A_x$. Here, $\text{Hom}(\mathcal{E} \otimes T(\gamma), \mathbb{R})$ is given the strong topology (i.e. the topology of uniform convergence on bounded subsets).

Also, each $g_r$ has a finite order pole at $\varepsilon = 0$, meaning that $\lim_{\varepsilon \to 0} \varepsilon^k g_r(\varepsilon) = 0$ for some $k > 0$.

Further, each

$$\Phi_r(L, e) \in \text{Hom}(\mathcal{E} \otimes T(\gamma), C^\infty((0, \infty)_L))$$

has a small $L$ asymptotic expansion

$$\Phi_r \simeq \sum h_s(L) \Psi_{r,s}$$

where each $\Psi_{r,s} \in \mathcal{O}_{\text{loc}}(\mathcal{E}, \mathcal{A})$. The definition of small $L$ asymptotic expansion is in the same sense as before: there exists a non-decreasing sequence $d_s \in \mathbb{Z}_{\geq 0}$, tending to $\infty$, such that, for all $x \in X$, and all $S \in \mathbb{Z}_{\geq 0},$

$$\lim_{L \to 0} L^{-d_s} \left( \Phi_r(e) - \sum_{s=0}^{S} h_s(L) \Psi_{r,s}(e) \right) = 0$$

in the topological vector space $\text{Hom}(\mathcal{E} \otimes T(\gamma), \mathbb{R}) \otimes A_x$.

It follows from this theorem that it makes sense to define the next counterterm $I_{(i,K)}^{CT}$ simply by

$$I_{(i,K)}^{CT}(\varepsilon, L) = \text{Sing}_\varepsilon \left( W_{(i,K)} \left( P(\varepsilon, L), I - \sum_{(r,s) < (i,K)} h_r I_{(r,s)}^{CT}(\varepsilon) \right) \right).$$

We would like to show the following properties of $I_{(i,K)}^{CT}(\varepsilon, L)$.

1. $I_{(i,K)}^{CT}(\varepsilon, L)$ is independent of $L$.
2. $I_{(i,K)}^{CT}(\varepsilon)$ is local, that is, it is an element of

$$\mathcal{O}_{\text{loc}}(\mathcal{E}, \mathcal{A}) \otimes_{\text{alg}} C^\infty((0, \infty)_{\varepsilon}).$$

3. For all $L \in (0, \infty)$ and all $x \in X$, the limit

$$\lim_{\varepsilon \to 0} W_{(i,K)} \left( P(\varepsilon, L), I - \sum_{(i,k) \leq (i,K)} h_i I_{(i,k)}^{CT}(\varepsilon) \right)$$

exists in the topological vector space $\text{Hom}(\mathcal{E} \otimes K, \mathbb{R}) \otimes A_x$. 
If we can prove these three properties, we can continue the induction.

The third property is immediate: it follows from the small $\varepsilon$ asymptotic expansion of Theorem 13.4.4.

For the first property, observe that

$$\frac{\partial}{\partial L} \text{Sing}_\varepsilon W_{(I,K)}\left(P(\varepsilon, L), I - \sum_{(r,s) < (I,K)} h^r I_{(r,s)}^{CT}(\varepsilon)\right)$$

$$= \text{Sing}_\varepsilon \frac{\partial}{\partial L} W_{(I,K)}\left(P(\varepsilon, L), I - \sum_{(r,s) < (I,K)} h^r I_{(r,s)}^{CT}(\varepsilon)\right).$$

This follows from the fact that the small $\varepsilon$ asymptotic expansion proved in Theorem 13.4.4 commutes with taking $L$ derivatives.

Thus, to show that $I_{(I,K)}^{CT}$ is independent of $L$, it suffices to show that, for all $L$, all $x \in X$, and all $e \in \mathcal{E}$,

$$\frac{\partial}{\partial L} W_{(I,K)}\left(P(\varepsilon, L), I - \sum_{(r,s) < (I,K)} h^r I_{(r,s)}^{CT}(\varepsilon)\right)_{x}$$

has an $\varepsilon \to 0$ limit. This is immediate by induction, using the renormalization group equation.

The small $L$ asymptotic expansion in Theorem 13.4.4 now implies that $I_{(I,K)}^{CT}(\varepsilon)$ is local.

Thus, $I_{(I,K)}^{CT}(\varepsilon)$ satisfies all the required properties, and we can continue our induction, to construct all the counterterms.

So far, we have shown how to construct the effective interactions

$$I[L] = \lim_{\varepsilon \to 0} W\left(P(\varepsilon, L), I - \sum_{(r,s)} h^r I_{(r,s)}^{CT}(\varepsilon)\right).$$

These effective interactions satisfy the renormalization group equation. It is immediate from Theorem 13.4.4 that the $I[L]$ satisfy the locality axiom, so that they define a theory.

Now, we need to prove the converse. This is again an inductive argument. Suppose we have a theory, given by effective interactions

$$I[L] \in \mathcal{O}^+(\mathcal{E}, \mathcal{A})[[h]].$$
Suppose that we have a local action functional
\[
J = \sum_{(i,k) < (I,K)} \hbar^i J_{(i,k)} \in \mathcal{O}_\text{loc}^+(\mathcal{E}, \mathcal{A})[[\hbar]]
\]
with associated effective interactions \(J[L]\).

Suppose, by induction, that
\[
J_{(i,k)}[L] = I_{(i,k)}[L]
\]
for all \((i, k) < (I, K)\).

We need to find some \(J'_{(I,K)}\) such that, if we set \(J' = J + \hbar^i J'_{(I,K)}\), then
\[
J'_{(I,K)}[L] = I_{(I,K)}[L].
\]

We simply let
\[
\]

The renormalization group equation implies that \(J'_{(I,K)}\) is independent of \(L\). It is automatic that
\[
J'_{(I,K)}[L] = I_{(I,K)}[L].
\]

Finally, the fact that both \(I_{(I,K)}[L]\) and \(J_{(I,K)}[L]\) satisfy the small \(L\) asymptotics axiom of a theory implies that \(J'_{(I,K)}\) is local.

\[\square\]

14. Field theories on non-compact manifolds

A second generalization is to non-compact manifolds. On non-compact manifolds, we don’t just have ultraviolet divergences (arising from small scales) but infrared divergences, which arise when we try to integrate over the non-compact manifold.

However, by imposing a suitable infrared cut-off, we will find a notion of theory on a non-compact manifold; and a bijection between theories and Lagrangians.

The infrared cut-off we impose is rather brutal: we multiply the propagator by a cut-off function so that it becomes supported on a small neighbourhood of the diagonal in \(M^2\). However, the notion of theory is independent of the cut-off chosen.

We will also show that there are restriction maps, allowing one to restrict a theory on a non-compact manifold \(M\) to any open subset \(U\). This allows one to define a sheaf
of theories on any manifold. If we are on a compact manifold, global sections of this sheaf are theories in the sense we defined before.

The sheaf-theoretic statement of our main theorem asserts that this sheaf is isomorphic to the sheaf of local action functionals. As always, this isomorphism depends on the choice of a renormalization scheme. The renormalization scheme independent statement of the theorem is that the sheaf of theories defined modulo $\hbar^{n+1}$ is a torsor over the sheaf of theories defined modulo $\hbar^n$, for the sheaf of local action functionals on $M$.

14.1. Now let us start defining the notion of theory on a possibly non-compact manifold $M$.

As in Section 13, we will fix a nilpotent graded manifold $(X, A)$, and a family of free field theories on $M$ parameterized by $(X, A)$. The free field theory is given by a super vector bundle $E$ on $M$, whose space of global sections will be denoted by $\mathcal{E}$; together with various auxiliary data detailed in Definition 13.1.1.

We will use the following notation. We will let $\mathcal{E}$ denote the space of all smooth sections of $E$, $\mathcal{E}_c$ the space of compactly supported sections, $\mathcal{E}'$ the space of distributional sections and $\mathcal{E}'_c$ the space of compactly supported distributional sections. The bundle $E \otimes \text{Dens}_M$ will be denoted $E'$. We will use the notation $\mathcal{E}'$, $\mathcal{E}'_c$, $\mathcal{E}'_s$ and $\mathcal{E}'_c$ to denote spaces of smooth, compactly supported, distributional and compactly supported distributional sections of the bundle $E'$. Sections of $\mathcal{E}$ will let $\mathcal{E}'$ denote the space of distributional sections of $E$, and $\mathcal{E}'_c$ denote the compactly supported distributional sections. With this notation, $\mathcal{E}' = \mathcal{E}'_c$, $\mathcal{E}'_c$ and so on.

14.2.

14.2.1 Definition. Let $M, X$ be topological spaces. A subset $C \subset M^n \times X$ is called proper if each of the projection maps $\pi_i : M^n \times X \to M \times X$ is proper when restricted to $C$.

Thus, we can talk about sections of various bundles on $M$ which have proper support.

Recall that we can identify the space $\mathcal{O}(\mathcal{E}_c, \mathcal{A})$ of $\mathcal{A}$-valued functions on $\mathcal{E}_c$ (modulo constants) with the completed symmetric algebra

$$\mathcal{O}(\mathcal{E}_c, \mathcal{A}) = \prod_{n > 0} \text{Sym}^n(\mathcal{E}'_c) \otimes \mathcal{A},$$
where we have identified $\mathcal{E}^!$ – the space of distributional sections of the bundle $E^!$ on $M$ – with $(\mathcal{E}_c)^\vee$. We will let

$$\mathcal{O}_p(\mathcal{E}_c, \mathcal{A}) \subset \mathcal{O}(\mathcal{E}_c, \mathcal{A})$$

be the subset consisting of those functionals $\Phi$ each of whose Taylor components

$$\Phi_n \in \text{Sym}^n \mathcal{E}^!$$

have proper support. We are only interested in functions on $\mathcal{E}_c$ modulo constants.

Note that $\mathcal{O}_p(\mathcal{E}, \mathcal{A})$ is not an algebra; the direct product of two properly supported distributions does not necessarily have proper support.

Note also that every $\mathcal{A}$-valued local action functional $I \in \mathcal{O}_1(\mathcal{E}, \mathcal{A})$ is an element of $\mathcal{O}_p(\mathcal{E}, \mathcal{A})$.

**14.3.** Recall that the super vector bundle $E$ on $M$ has additional structure, as described in Definition 13.1.1. This data includes a decomposition $E = E_1 \oplus E_2$ and a generalized Laplacian $D_{\mathcal{E}_1} : \mathcal{E}_1 \to \mathcal{E}_1$.

We are interested in the heat kernel for the Laplacian $\Delta_{E_1}$. On a compact manifold $M$, this is unique, and is an element of

$$K_t \in \mathcal{E}^!_1 \otimes \mathcal{E}_1 \otimes C^\infty(\mathbb{R}_{>0}) \otimes \mathcal{A}.$$ 

On a non-compact manifold, there are many heat kernels, corresponding to various boundary conditions. In addition, such heat kernels may grow on the boundary of the non-compact manifold in ways which are difficult to control.

To remedy this, we will introduce the concept of fake heat kernel. A fake heat kernel is something which solves the heat equation but only up to the addition of a smooth kernel.

**14.3.1 Definition.** A fake heat kernel is a smooth section

$$K_t \in \mathcal{E}^!_1 \otimes \mathcal{E}_1 \otimes C^\infty(\mathbb{R}_{>0}) \otimes \mathcal{A}$$

with the following properties.

1. $K_t$ extends, at $t = 0$ to a distribution. Thus, $K_t$ extends to an element of

$$\mathcal{E}^!_1 \otimes \mathcal{E}_1 \otimes C^\infty(\mathbb{R}_{\geq 0}) \otimes \mathcal{A}$$

Further, $K_0$ is the kernel for the identity map $\mathcal{E}_1 \to \mathcal{E}_1$. 
(2) The support

$$\text{Supp } K_t \subset M \times M \times \mathbb{R}_{>0} \times X$$

is proper. Recall that this means that both projection maps $\text{Supp } K_t \to M \times \mathbb{R}_{>0} \times X$ are proper.

(3) The heat kernel $K_t$ satisfies the heat equation up to exponentially small terms in $t$. More precisely, $\frac{d}{dt} K_t + D_{\partial_t} K_t$ extends to a smooth section

$$\frac{d}{dt} K_t + D_{\partial_t} K_t \in \mathcal{E}^1_t \otimes \mathcal{E}^1_t \otimes \mathcal{C}^\infty(\mathbb{R}_{>0}) \otimes \mathcal{A}$$

which vanishes at $t = 0$, with all derivatives in $t$ and on $M$, faster than any power of $t$.

(4) The heat kernel $K_t$ admits a small $t$ asymptotic expansion which can be written, in normal coordinates $x, y$ near the diagonal of $M$, in the form

$$K_t \simeq t^{-\dim M/2} e^{-\|x-y\|^2/t} \sum_{i \geq 0} t^i \Phi_i(x, y).$$

Let me explain more carefully about what I mean by a small $t$ asymptotic expansion. We will let

$$K^N_t = \psi(x, y) t^{-\dim M/2} e^{-\|x-y\|^2/t} \sum_{i=0}^N t^i \Phi_i(x, y)$$

be the $N^{th}$ partial sum of this asymptotic expansion (where we have introduced a cut-off $\psi(x, y)$ so that $K^N_t$ is zero outside of a small neighbourhood of the diagonal).

Then, we require that for all compact subsets $C \subset M \times M \times X$,

$$\left\| \partial^k_{l,m} \left( K_t - K^N_t \right) \right\|_{C,l,m} = O(t^{N - \dim M/2 - l/2 - k}).$$

where $\| - \|_{l,m}$ refers to the norm on $\Gamma(M, E^\mathcal{V} \otimes \text{Dens}(M)) \otimes \mathcal{E} \otimes \mathcal{A}$ where we differentiate $l$ times on $M \times M$, $m$ times on $X$, and take the supremum over the compact subset $C$.

These estimates are the same as the ones satisfied by the actual heat kernel on a compact manifold $M$, as detailed in [BGV92].

The fake heat kernel $K_t$ satisfies the heat equation up to a function which vanishes faster than any power of $t$. This implies that the asymptotic expansion

$$t^{-\dim M/2} e^{-\|x-y\|^2/t} \sum_{i \geq 0} t^i \Phi_i(x, y).$$
of $K_t$ must be a formal solution to the heat equation (in the sense described in [BGV92], Section 2.5. This characterizes the functions $\Phi_i(x, y)$ (defined in a neighbourhood of the diagonal) uniquely.

**14.3.2 Lemma.** Let $K_t, \tilde{K}_t$ be two fake heat kernels. Then $K_t - \tilde{K}_t$ extends across $t = 0$ to a smooth kernel, that is,

$$K_t - \tilde{K}_t \in \mathcal{E}^1 \otimes \mathcal{E} \otimes \mathcal{A} \otimes C^\infty(\mathbb{R}_{\geq 0}).$$

Further, $K_t - \tilde{K}_t$ vanishes to all orders at $t = 0$.

**Proof.** This is clear from the existence and uniqueness of the small $t$ asymptotic expansion. □

**14.3.3 Lemma.** A fake heat kernel always exists.

**Proof.** The techniques of [BGV92] allow one to construct a fake heat kernel by approximating it with the partial sums of the asymptotic expansion. □

### 14.4

Let us suppose that $M$ is an open subset of a compact manifold $N$, and that the free field theory $\mathcal{E}$ on $M$ is restricted from one, say $\mathcal{F}$, on $N$.

Then, the restriction of the heat kernel for $\mathcal{F}$ to $M$ is an element

$$\tilde{K}_t \in \mathcal{E}^1 \otimes \mathcal{E}_1 \otimes C^\infty(\mathbb{R}_{\geq 0}) \otimes \mathcal{A},$$

which satisfies all the axioms of a fake heat kernel except that of requiring proper support.

If $\Psi \in C^\infty(M \times M)$ is a smooth function, with proper support, which takes value 1 on a neighbourhood of the diagonal, then

$$\Psi \tilde{K}_t \in \mathcal{E}^1 \otimes \mathcal{E}_1 \otimes C^\infty(\mathbb{R}_{\geq 0}) \otimes \mathcal{A}$$

is a fake heat kernel.

### 14.5

The bundle $E_1$ of propagating fields is a direct summand of the bundle $E$ of all fields. Thus, the fake heat kernel $K_t$ can be viewed as an element of the space

$$\mathcal{E}^1 \otimes \mathcal{E} \otimes C^\infty(\mathbb{R}_{\geq 0}) \otimes \mathcal{A}.$$ 

The operator $D' : \mathcal{E}_1^1 \rightarrow \mathcal{E}_1$ extends to an operator $\mathcal{E}^1 \rightarrow \mathcal{E}$. For $0 < \varepsilon < L < \infty$, let us define the fake propagator by

$$P(\varepsilon, L) = \int_{\varepsilon}^{L} \sigma((D' \otimes 1)K_t) \, dt \in \text{Sym}^2 \mathcal{E} \otimes \mathcal{A},$$
where \( \sigma : \mathcal{E} \otimes \mathcal{E} \rightarrow \text{Sym}^2 \mathcal{E} \) is the symmetrization map.

We would like to define a theory, for the fake heat kernel \( K_t \), to be a collection of effective interactions

\[
I[L] \in \mathcal{O}_p^+(\mathcal{E}_c, \mathcal{A})[[\hbar]]
\]
satisfying the renormalization group equation defined using the fake propagator \( P(\varepsilon, L) \).

Recall that the subscript \( p \) in the expression \( \mathcal{O}_p(\mathcal{E}_c, \mathcal{A}) \) indicates that we are looking at functionals which are distributions with proper support. A function in \( \mathcal{O}(\mathcal{E}_c, \mathcal{A}) \) is in \( \mathcal{O}_p(\mathcal{E}_c, \mathcal{A}) \) if its Taylor components, which are \( \mathcal{A} \)-valued distributions on \( M^n \), have support which is a proper subset of \( M^n \times X \).

In order to do this, we need to know that the renormalization group flow is well defined. Thus, we need to check that if we construct the weights attached to graphs using the propagators \( P(\varepsilon, L) \) and interactions \( I \in \mathcal{O}_p(\mathcal{E}_c, \mathcal{A})^+[[\hbar]] \).

**14.5.1 Lemma.** Let \( \gamma \) be a connected graph with at least one tail. Let \( P \in \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{A} \) have proper support. Suppose for each vertex \( v \) of \( \gamma \) we have a continuous linear map

\[
I_v : \mathcal{E}_c \otimes H(\gamma) \rightarrow \mathcal{A}
\]

which has proper support.

Then,

\[
\omega(\gamma, P, \{ I_v \}) : \mathcal{E}_c \otimes T(\gamma) \rightarrow \mathcal{A}
\]

is well defined, and is a continuous linear map with proper support.

**Proof.** Let \( f \in \mathcal{E}_c \otimes T(\gamma) \). The expression \( \omega(\gamma, P, \{ I_v \})(f) \) is defined by contracting the tensor

\[
f \otimes_{e \in E(\gamma)} P_e \in \mathcal{E}_c \otimes H(\gamma)
\]
given by putting a propagator \( P \) on each edge of \( \gamma \) and \( f \) at the tails of \( \gamma \), with the distribution

\[
\otimes_{v \in V(\gamma)} I_v : \mathcal{E}_c \otimes H(\gamma) \rightarrow \mathcal{A}.
\]

Neither quantity has compact support. However, the restrictions we placed on the supports of \( f \), \( P \) and each \( I_v \) means that the intersection of the support of \( \otimes I_v \) with that of \( f \otimes P_e \) is a compact subset of \( M^H(\gamma) \). Thus, we can contract \( \otimes I_v \) with \( f \otimes P_e \) to get an element of \( \mathcal{A} \).
The resulting linear map
\[
E \otimes T(\gamma) \rightarrow \mathcal{A}
\]
\[
f \mapsto w_\gamma(P, \{I_v\})(f)
\]
is easily seen to have proper support.

Note that this lemma is false if the graph $\gamma$ has no tails. This is the reason why we only consider functionals on $E$ modulo constants, or equivalently, without a constant term.

**14.5.2 Corollary.** The renormalization group operator
\[
\mathcal{O}_p^+(\mathcal{E}_c, \mathcal{A})[[\hbar]] \rightarrow \mathcal{O}_p^+(\mathcal{E}_c, \mathcal{A})[[\hbar]]
\]
\[
I \rightarrow W(P(\varepsilon, L), I)
\]
is well-defined.

As always, $\mathcal{O}_p^+(\mathcal{E}_c, \mathcal{A})[[\hbar]]$ refers to the space of elements of $\mathcal{O}_p^+(\mathcal{E}_c, \mathcal{A})[[\hbar]]$ which are at least cubic modulo $\hbar$ and the ideal $\Gamma(X, m) \subset \mathcal{A}$.

**PROOF.** The renormalization group operator is defined by
\[
W(P(\varepsilon, L), I) = \sum_\gamma \frac{1}{\|\text{Aut}(\gamma)\|} h^{\gamma(\gamma)} w_\gamma(P(\varepsilon, L), I).
\]
The sum is over connected stable graphs; and, as we are working with functionals on $E$ modulo constants, we only consider graphs with at least one tail. Lemma 14.5.1 shows that each $w_\gamma(P(\varepsilon, L), I)$ is well-defined.

**14.6.** Now we can define the notion of theory on the manifold $M$, using the fake propagator $P(\varepsilon, L)$.

**14.6.1 Definition.** A theory is a collection $\{I[L] \mid L \in \mathbb{R}_{>0}\}$ of elements of $\mathcal{O}_p^+(\mathcal{E}_c, \mathcal{A})[[\hbar]]$ which satisfy

1. The renormalization group equation,
\[
I[L] = W(P(\varepsilon, L), I[\varepsilon])
\]
2. The asymptotic locality axiom: there a small $L$ asymptotic expansion
\[
I[L] \simeq \sum f_i(L)\Psi_i
\]
in terms of local action functionals \( \Psi_i \in \mathcal{O}^+_\text{loc}(\mathcal{E}, \mathcal{A})[[\hbar]] \). We will assume that the functions \( f_i(L) \) appearing in this expansion have at most a finite order pole at \( L = 0 \); that is, we can find some \( n \) such that \( \lim_{L \to 0} L^n f_i(L) = 0 \).

Let \( \mathcal{F}^{(n)}(\mathcal{E}, \mathcal{A}) \) denote the set of theories defined modulo \( \hbar^{n+1} \), and let \( \mathcal{F}^{(\infty)}(\mathcal{E}, \mathcal{A}) \) denote the set of theories defined to all orders in \( \hbar \).

14.7. One can ask how the definition of theory depends on the choice of fake heat kernel. It turns out that there is no dependence. Let \( \mathcal{K}_t, \mathcal{K}_t' \) be two heat fake heat kernels, with associated propagators \( P(\mathcal{E}, L), \tilde{P}(\mathcal{E}, L) \). We have seen that \( \mathcal{K}_t - \mathcal{K}_t' \) vanishes to all orders at \( t = 0 \). It follows that \( P(0, L) - \tilde{P}(0, L) \) is smooth, that is, an element of \( \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{A} \). Also, \( P(0, L) - \tilde{P}(0, L) \) has proper support. Further, as \( L \to 0 \), \( P(0, L) \) and all of its derivatives vanish faster than any power of \( L \).

Thus, the renormalization group operator

\[
W \left( P(0, L) - \tilde{P}(0, L), - \right) : \mathcal{O}^+_p(\mathcal{E}, \mathcal{A})[[\hbar]] \to \mathcal{O}^+_p(\mathcal{E}, \mathcal{A})[[\hbar]]
\]

is well-defined.

14.7.1 Lemma. Let \( \{\tilde{I}[L]\} \) be a collection of effective interactions defining a theory for the propagator \( \tilde{P}(\mathcal{E}, L) \). Then,

\[
I[L] = W \left( P(0, L) - \tilde{P}(0, L), \tilde{I}[L] \right)
\]

defines a theory for the propagator \( P(\mathcal{E}, L) \).

Further, the small \( L \) asymptotic expansion of \( I[L] \) is the same as that of \( \tilde{I}[L] \).

Proof. The renormalization group equation

\[
W \left( P(\mathcal{E}, L), I[\mathcal{E}] \right) = I[L]
\]

is a corollary of the general identity,

\[
W \left( P, W \left( P', I \right) \right) = W \left( P + P', I \right)
\]

for any \( P, P' \in \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{A} \) of proper support, and any \( I \in \mathcal{O}^+_p(\mathcal{E}, \mathcal{A})[[\hbar]] \). The statement about the small \( L \) asymptotics of \( I[L] \) - and hence the locality axiom which says that \( I[L] \) defines a theory - follows from the locality axiom for \( \tilde{I}[L] \) and the fact that \( P(0, L) \) and all its derivatives tend to zero faster than any power of \( L \), as \( L \to 0 \). \( \square \)
14.8. Now let $U \subset M$ be any open subset. The free field theory on $M$ – defined by the vector bundle $E$, together with certain differential operators on it – restrict to define a free field theory on $U \subset M$. One can ask if a theory on $M$ will restrict to one on $U$ as well.

The following proposition, which will be proved later, shows that one can do this.

14.8.1 Proposition. Let $U \subset M$ be an open subset.

Given any theory

$$\{ I[L] \in \mathcal{O}^+_p(\mathcal{E}, \mathcal{A})[[\hbar]] \}$$

on $M$ (defined using any fake heat kernel on $M$), there is a unique theory

$$\{ I_U[L] \in \mathcal{O}^+_p(\Gamma(U, E|_U), \mathcal{A})[[\hbar]] \}$$

again defined using any fake heat kernel on $U$, with the property that the small $L$ asymptotic expansion of $I_U[L]$ is the restriction to $U$ of the small $L$ asymptotic expansion of $I[L]$.

The existence of this restriction map shows that there is a presheaf on $M$ which assigns to an open subset $U \subset M$ the set $\mathcal{F}^{(n)}(\Gamma(U, E|_U), \mathcal{A})$ of theories on $U$. We will denote this presheaf by $\mathcal{F}^{(n)}(\mathcal{E}, \mathcal{A})$.

14.9. Now we are ready to state the main theorem.

14.9.1 Theorem. (1) The presheaves $\mathcal{F}^{(n)}(\mathcal{E}, \mathcal{A})$ and $\mathcal{F}^{(\infty)}(\mathcal{E}, \mathcal{A})$ of theories on $M$ are sheaves.

(2) There is a canonical isomorphism between the sheaf $\mathcal{F}^{(0)}(\mathcal{E}, \mathcal{A})$ and the sheaf of local action functionals $I \in \mathcal{O}_{\text{loc}}(\mathcal{E}_c, \mathcal{A})$ which are at least cubic modulo the ideal $\Gamma(X, m) \subset \mathcal{A}$.

(3) For $n > 0$, $\mathcal{F}^{(n)}(\mathcal{E}, \mathcal{A})$ is, in a canonical way, a torsor over $\mathcal{F}^{(n-1)}(\mathcal{E}, \mathcal{A})$ for the sheaf of abelian groups $\mathcal{O}_{\text{loc}}(\mathcal{E}_c, \mathcal{A})$, in a canonical way.

(4) Choosing a renormalization scheme leads to a map

$$\mathcal{F}^{(n)}(\mathcal{E}, \mathcal{A}) \rightarrow \mathcal{F}^{(n+1)}(\mathcal{E}, \mathcal{A})$$

of sheaves for each $n$, which is a section. Thus, the choice of a renormalization scheme leads to an isomorphism of sheaves

$$\mathcal{F}^{(\infty)}(\mathcal{E}, \mathcal{A}) \cong \mathcal{O}^+_c(\mathcal{E}_c, \mathcal{A})[[\hbar]]$$

on $M$. 

14.10. The proof of the theorem, and of Proposition 14.8.1, is along the same lines as before, using counterterms.

14.10.1 Proposition. Let us fix a renormalization scheme, and a fake heat kernel $K_t$ on $M$.

Let

$$I = \sum \hbar^i I_{i,k} \in \mathcal{O}^+_\text{loc}(\mathcal{E}, \mathcal{A})[[\hbar]].$$

Then:

1. There is a unique series of counterterms

$$I^{CT}(\epsilon) = \sum \hbar^i I^{CT}_{i,k}(\epsilon)$$

where

$$I^{CT}_{i,k}(\epsilon) \in \mathcal{O}^+_\text{loc}(\mathcal{E}, \mathcal{A}) \otimes_{\text{alg}} \mathcal{P}((0, \infty))_{<0}$$

is purely singular as a function of $\epsilon$, with the property that the limit

$$\lim_{\epsilon \to 0} W \left(P(\epsilon, L), I - I^{CT}(\epsilon)\right)$$

exists, for all $L$.

2. This limit defines a collection of elements $I[L] \in \mathcal{O}^+_\text{p}(\mathcal{E}, \mathcal{A})[[\hbar]]$, satisfying the renormalization group equation and locality axiom, and so defines a theory.

3. The counterterms $I^{CT}_{i,k}(\epsilon)$ do not depend on the choice of a fake heat kernel.

4. If $U \subset M$ is an open subset, then the counterterms for $I$ restricted to $U$ are the restrictions to $U$ of the counterterms $I^{CT}_{i,k}(\epsilon)$ for $I$. Thus, counterterms define a map of sheaves

$$\mathcal{O}^+_\text{loc}(\mathcal{E}, \mathcal{A})[[\hbar]] \to \mathcal{O}^+_\text{loc}(\mathcal{E}, \mathcal{A}) \otimes_{\text{alg}} \mathcal{P}((0, \infty)_{<0})[[\hbar]].$$

Proof. The proof of the first two statements is identical to the proof of the corresponding statement on a compact manifold, and so is mostly omitted. One point is worth mentioning briefly, though: the counterterms are defined to be the singular parts of the small $\epsilon$ asymptotic expansion of the weight $w_\gamma(P(\epsilon, L), I)$ attached to a graph $\gamma$. One can ask whether such asymptotic expansions exist when we use a fake heat kernel rather than an actual heat kernel.

The asymptotic expansion, as constructed in Appendix 1, only relies on the small $t$ expansion of the heat kernel $K_t$, and thus exists whenever the propagator is constructed from a kernel with such a small $t$ expansion.

The third clause is proved using the uniqueness of the counterterms, as follows. Suppose that $K_t, \tilde{K}_t$ are two fake heat kernels, with associated fake propagators $P(\epsilon, L)$,
\[ \bar{P}(\varepsilon, L). \text{ Let } I^{CT}(\varepsilon), \bar{I}^{CT}(\varepsilon) \text{ denote the counterterms associated to the two different fake} \\
\text{heat kernels. We need to show that they are the same.} \]

Note that
\[ \lim_{\varepsilon \to 0} W \left( \bar{P}(0, L) - P(0, L), W \left( P(\varepsilon, L), I - I^{CT}(\varepsilon) \right) \right) \]
exists. We can write the expression inside the limit as
\[ W \left( P(0, \varepsilon) - \bar{P}(0, \varepsilon), W \left( \bar{P}(\varepsilon, L), I - I^{CT}(\varepsilon) \right) \right). \]

Note that \( P(0, \varepsilon) - \bar{P}(0, \varepsilon) \) tends to zero, with all derivatives, faster than any power
of \( \varepsilon \). Also, \( W \left( \bar{P}(\varepsilon, L), I - I^{CT}(\varepsilon) \right) \) has a small \( \varepsilon \) asymptotic expansion in terms of
functions of \( \varepsilon \) of polynomial growth at the origin.

From these two facts it follows that
\[ \lim_{\varepsilon \to 0} W \left( \bar{P}(\varepsilon, L), I - I^{CT}(\varepsilon) \right) \]
exists.

Uniqueness of the counterterms implies that \( I^{CT}(\varepsilon) = \bar{I}^{CT}(\varepsilon) \).

The fourth clause can be proved easily using independence of the counterterms of
the fake heat kernel. \( \square \)

14.11. Now we can prove Proposition 14.8.1 and Theorem 14.9.1. The proof is
easy once we know that counterterms exist.

Let us start by regarding the set of theories \( \mathcal{T}^{(\infty)}(\mathcal{E}, \mathcal{A}) \) as just a set, and not as
arising from a sheaf on \( M \). (After all, we have not yet proved Proposition 14.8.1, so we
do not know that we have a presheaf of theories).

14.11.1 Lemma. Let us choose a renormalization scheme, and a fake heat kernel. Then the
map of sets
\[ \mathcal{O}^{+}_{\text{loc}}(\mathcal{E}, \mathcal{A})[[h]] \to \mathcal{T}^{(\infty)}(\mathcal{E}, \mathcal{A}) \]
\[ I \mapsto \{I[L] = \lim_{\varepsilon \to 0} W \left( P(\varepsilon, L), I - I^{CT}(\varepsilon) \right) \} \]
is a bijection.

PROOF. This is proved by the usual inductive argument. \( \square \)
Now we can prove Proposition 14.8.1, which we restate here for convenience.

14.11.2 Proposition. Let $U \subset M$ be an open subset.

Given any theory
\[
\{ I[L] \in \mathcal{O}_p^+ (\mathcal{E}, \mathcal{A})[[\hbar]] \}
\]
on $M$, for any fake heat kernel, $U$, there is a unique theory
\[
\{ I_U[L] \in \mathcal{O}_p^+ (\Gamma(U, \mathcal{E} | U), \mathcal{A})[[\hbar]] \}
\]
with the property that the small $L$ asymptotics of $I_U[L]$ is the restriction to $U$ of the small $L$ asymptotics of $I[L]$.

Proof. Uniqueness is obvious, as any two theories on $U$ with the same small $L$ asymptotic expansions must coincide.

For existence, we will use the bijection between theories and Lagrangians which arises from the choice of a renormalization scheme. We can assume that the theory \( \{ I[L] \} \) on $M$ arises from a local action functional $I \in \mathcal{O}_\text{loc}^+ (\mathcal{E}_r, \mathcal{A})[[\hbar]]$. Then, we define $I_U[L]$ to be the theory on $U$ associated to the restriction of $I$ to $U$.

It is straightforward to check that, with this definition, $I[L]$ and $I_U[L]$ have the same small $L$ asymptotics. \( \square \)

It follows from the proof of this lemma that the map
\[
\mathcal{O}_\text{loc}^+ (\mathcal{E}_r, \mathcal{A})[[\hbar]] \to \mathcal{F}^{(\infty)} (\mathcal{E}, \mathcal{A})
\]
of sets actually arises from a map of presheaves on $M$. Since $\mathcal{O}_\text{loc}^+ (\mathcal{E}_r, \mathcal{A})[[\hbar]]$ is actually a sheaf on $M$, it follows that $\mathcal{F}^{(\infty)} (\mathcal{E}, \mathcal{A})$ is also a sheaf, and similarly for $\mathcal{F}^{(n)} (\mathcal{E}, \mathcal{A})$.

The remaining statements of Theorem 14.9.1 are proved in the same way as the corresponding statements on a compact manifold.