# NOTES ON SUPER MATH
MOSTLY FOLLOWING
# BERNSTEIN–DELIGNE–MORGAN

CONTENTS



*Date*: Long Long Time Ago in a Land Far Away .

*The main source.* This is the text [BDM] :

- Deligne, Pierre; Morgan, John W. *Notes on supersymmetry* (following Joseph Bernstein).

## References

[BDM] Deligne, Pierre; Morgan, John W. *Notes on supersymmetry* (following Joseph Bernstein). Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), 41–97, Amer. Math. Soc., Providence, RI, 1999.

## 1. **Linear algebra**

The super math is the mathematics that obeys certain sign rule. One can introduce super versions of standard objects by writing formulas enriched with some signs and then claiming that these formulas work well. A more systematic approach is the description of the sign rule as an additional structure – a *braiding* – on the tensor category of $\mathbb{Z}_2$-graded vector spaces. A braiding on a tensor category $\mathcal{T}$ provides a notion of *commutative algebras* in the setting of $\mathcal{T}$, as a consequence one obtains notions of $\mathcal{T}$-versions of geometric objects, Lie groups etc., i.e., the standard bag of mathematical ideas. The above braiding (*the super braiding*) gives the *super math.* We will survey the effect of the *super braiding* on linear algebra, geometry (super manifolds) and analysis (integration on super manifolds).

*Some unusual aspects.* Some concepts develop unexpected subtleties. For instance on super manifolds there are three objects that generalize various aspects of differential forms: (super) differential forms, densities, integral forms.

The odd part contributes in the direction opposite from what one expects. This is familiar in the case of super dimension which is just the Euler characteristic: $even - odd$. However as this principle propagates through more complicated objects it gets more surprising. We will see this when we study integration on super manifolds.

*Applications.*

(1) Some non-commutative situations are commutative from the super point of view.
(2) Some standard constructions have a more "set-theoretic" interpretation in the super setting :
   (a) The differential forms on a manifold $M$ can be viewed as functions on a super manifold which is the moduli of maps from the super point $\mathbb{A}^{0|1}$ to $M$.
   (b) The differential forms on the loop space $\Lambda(M)$ are functions on the super manifold which is the moduli of maps from the super circle $S^{1|1}$ to $M$. This explains the non-trivial structure of a vertex algebra on these differential forms, for instance the vector fields on $S^{1|1}$ give the (a priori sophisticated) structure of $N = 2$ topological vertex algebra.

(c) Complexes in homological algebra are representations of a certain super group with the underlying manifold $S^{1|1}$.

(3) Supersymmetry: this is a symmetry of a mathematical object which mixes even and odd components. These are more difficult to spot without the super point of view. For instance integrals with supersymmetry of will be easier to calculate. (Our example will be the baby case of Witten's approach to Morse theory.)

(4) Fermions: elementary particles break into bosons and fermions depending on whether they obey usual mathematics or require super-mathematics.

*Development.* The underlying structure of this theory is the category $s\mathcal{V}ect_{\Bbbk}$ of super vector spaces over a basic field$^{(1)}$ (or ring) $\Bbbk$, with a structure of a tensor category with a super braiding. The next level is the linear algebra in $s\mathcal{V}ect_{\Bbbk}$, it has two notions of liner operators: (i) the *inner* Hom, i.e., $\underline{\mathrm{Hom}}(U,V)$ is a super vector space, it consists of all $\Bbbk$-liner operators, (i) the *categorical* Hom, i.e., $\mathrm{Hom}(U,V) = \mathrm{Hom}_{s\mathcal{V}ect_{\Bbbk}}(U,V)$ is an ordinary vector space, it consists of all $\Bbbk$-liner operators that preserve parity,

1.1. **Super-math as the math in the braided tensor category of super vector spaces.** A *super vector space* is simply a vector space graded by $\mathbb{Z}_2 = \{0,1\}$: $V = V_0 \oplus V_1$, i.e., a representation of the group$^{(2)}$ $\{\pm 1\}$. Therefore, $\mu_2$ acts on any category of super objects.[3]

1.1.1. *Parity.* We will say that vectors $v \in V_p$ are *homogeneous of parity p* and we will denote the parity of $v$ by $p_v$ or $\overline{v}$. Another way to keep track of parity is the "fermionic sign" $(-1)^F$. On each super vector space this is the linear operator which is $+1$ on $V_0$ and $-1$ on $V_1$. Here, "F" for fermionic, will sometimes be used to indicate the super versions of standard constructions.

1.1.2. *Sign Rule and super braiding.* The meaning of "super" is that all calculations with super vector spaces have to obey the

"*Sign Rule*: when $a$ passes $b$, the sign $(-1)^{p_a p_b}$ appears.

More precisely (and more formally) the calculations are done in the tensor category $\mathcal{V}ect^s_{\Bbbk}$ of super vector spaces over $\Bbbk$, enriched by a certain structure called "braiding". The braiding on a tensor category is a (consistent) prescription of what we mean by a natural identification of $V \otimes W$ and $W \otimes V$, i.e., a commutativity isomorphism ("commutativity constraint) $c_{VW}: V \otimes W \to W \otimes V$, functorial in $V$ and $W$.

[1]Here $\Bbbk$ is even, i.e., there is no parity grading in "numbers".

[2]To cover the case of the arbitrary ground ring $\Bbbk$, the correct group is the group scheme $\mu_2$ of second roots of unity, defined over integers. The difference matters only when 2 is not invertible in $\Bbbk$.

[3]Moreover, $\mu_2$ acts identically on objects, so it lies in the center of that category.

The braiding in the "ordinary" math is $c_{V,W}(v\otimes w) = w\otimes v$ on $VVect_{\Bbbk}$. The super math is based on the braiding in $(\mathcal{V}ect^s_{\Bbbk}, \underset{\Bbbk}{\otimes})$ given by the sign rule

$$c_{VW} : V\otimes W\to W\otimes V, \quad v\otimes w\mapsto (-1)^{p_v p_w} w\otimes v,$$

which is the formalization of the above *Sign Rule.*

1.1.3. *Braiding gives geometry.* While one can define associative algebras in any tensor category, in order to have a notion of commutative algebras the tensor category needs a braiding. In this way, each braiding gives one version of the notion of commutative algebras, hence one version of standard mathematics.

1.1.4. *General arguments and calculations in coordinates.* While the calculations in a braided tensor category $\mathcal{V}ect^s_{\Bbbk}$ are "natural", and of a general nature (arguments valid for any braiding), working in coordinates will involve applications of this specific commutativity constraint and also careful sign conventions.

1.1.5. *Unordered tensor products in braided tensor categories.* (i) In any tensor category a tensor product of a finite ordered family $\otimes_1^n V_{i_k} \stackrel{\text{def}}{=} V_{i_1}\otimes\cdots\otimes V_{i_n}$ is defined canonically, and the associativity constraint can be viewed as identity.

(ii) In a braided tensor category, the tensor product is also defined for unordered families, here $\otimes_{i\in I} V_i$ is defined as the projective limit of all tensor products given by a choice of order (the consistency property of commutativity constraints ensures that this is a projective system).

1.1.6. *Special property of the super braiding.* The super braiding is very special – it is self-inverse, i.e., $c_{VW} = {c_{WV}}^{-1}$. In particular, $c^2_{VV} = 1$.

1.2. **The effect of the sign rule on linear algebra over the base ring $\Bbbk$.** Some mathematical constructions extend to any tensor category, for instance the notion of an algebra. In our case it gives the following notion: a super $\Bbbk$-algebra is a $\Bbbk$-algebra $A$ with a compatible super structure, i.e., $A_p{\cdot}A_q\subseteq A_{p+q}$.

1.2.1. *Commutativity.* Mathematical constructions related to commutativity require the tensor category to have a braiding.

The *commutator* in an algebra $A$ in a braided category is obtained by applying the multiplication to $a\otimes b - c_{A,A}(b\otimes a)$. So, the (super)commutator in a super-algebra $A$ is

$$[a,b]_F \stackrel{\text{def}}{=} ab - (-1)^{p_a p_b}\, ba.$$

So we say that elements $a$ and $b$ of a super-algebra $A$ super-commute if

$$ab = (-1)^{p_a p_b}\, ba.$$

An abstract reason for usefulness of the notion of super-commutativity is that it allows one to think of some non-commutative situations as if they were commutative, and this in particular gives notions of a super-commutative algebra $A$, i.e., of a super-space $X$ with the super-commutative algebra of functions $\mathcal{O}(X) = A$.

1.2.2. *Functors between vector spaces, super vector spaces and graded vector spaces.*

- *Inclusion* $\mathcal{V}ect_{\Bbbk} \subseteq \mathcal{V}ect^s_{\Bbbk}$. This is an inclusion of a full braided tensor subcategory.
- *Forgetful functor* $\mathcal{V}ect^s_{\Bbbk} \xrightarrow{\mathcal{F}} \mathcal{V}ect_{\Bbbk}$. It forgets the super structure, i.e., the $\mathbb{Z}_2$-grading. It is a functor between tensor categories but not the braided tensor categories.
- *Projection to the even part* $\mathcal{V}ect^s_{\Bbbk} \xrightarrow{-_0} \mathcal{V}ect_{\Bbbk}$. This is an exact functor but it does not preserve the tensor category structure.
- *Forgetful functors* $\mathcal{V}ect^{\bullet}_{\Bbbk} \xrightarrow{\boldsymbol{s}} \mathcal{V}ect^s_{\Bbbk} \xrightarrow{\mathcal{F}} \mathcal{V}ect_{\Bbbk}$ Let $\mathcal{V}ect^{\bullet}_{\Bbbk}$ be the graded vector spaces, i.e., vector spaces $V$ with a $\mathbb{Z}$-decomposition $V = \oplus_{n\in\mathbb{Z}} V_n$. Any graded vector space $V = \oplus_{n\in\mathbb{Z}} V^n$ defines a super vector space
$$\boldsymbol{s}(V) \stackrel{\text{def}}{=} V \text{ with the decomposition } V = V_0 \oplus V_1 \text{ for } V_0 \stackrel{\text{def}}{=} \oplus_{n\ even} V^n \text{ and } V_1 \stackrel{\text{def}}{=} \oplus_{n\ odd} V^n.$$
The standard braiding on $\mathcal{V}ect^{\bullet}_{\Bbbk}$ is the super grading!

We denote by $\underline{1}$ the unit object $\Bbbk$ in $\mathcal{V}ect^s_{\Bbbk}$.

1.2.3. *Some notions in a braided tensor category.* In a braided tensor category we automatically have the notions of

- standard classes of algebras: (associative, commutative, unital, Lie),
- standard operations on algebras (tensor product of algebras, opposite algebra),
- modules over algebras,
- linear algebra of such modules,
- symmetric and exterior algebras of modules over commutative algebras
- etc.

1.2.4. *Parity change of super vector spaces.* Operation $\Pi : \mathcal{V}ect^s_{\Bbbk} \to \mathcal{V}ect^s_{\Bbbk}$ is defined by $(\Pi V)_p \stackrel{\text{def}}{=} V_{1-p}$. So, $\Pi(V)$ can be canonically identified with $V$ as a vector space but the parities have changed.

We will also denote by $\Pi$ the one dimensional *odd* vector space $\Bbbk\pi$ with a chosen basis $\pi$, then the functor $\Pi$ is canonically identified with the left tensoring functor

$$\Pi V \cong \Pi \otimes V.$$

Observe that we have made a *choice* of tensoring with $\Pi$ on the *left.*

1.2.5. *The inner* Hom *and duality in super vector spaces.* There are two related and easily confused concepts.

(1) Hom *for the category of super vector spaces.* For two super vector spaces $V$ and $W$,
$$\mathrm{Hom}_{\Bbbk}(V,W) \overset{\text{def}}{=} \mathrm{Hom}_{super\ \Bbbk-modules}(V,V)$$
denotes all maps of super vector spaces, i.e., $\Bbbk$-linear maps which preserve the super structure (i.e., the parity). So this is an *ordinary*vector space, i.e., an *even* vector space.

(2) *Inner* Hom *in the category of super vector spaces.* The vector space of all $\Bbbk$-linear maps $\mathrm{Hom}_{\Bbbk}[\mathcal{F}(V),\mathcal{F}(W)]$ has a canonical structure of a super vector space which we denote $\underline{\mathrm{Hom}}_{\Bbbk}(V,W)$.

The relation is given by composing with the projection to the even part: $\mathrm{Hom} = -_0 \circ \underline{\mathrm{Hom}}$, i.e., the even part of the inner Hom consists of maps that preserve parity
$$\underline{\mathrm{Hom}}_{\Bbbk}(V,W)_0 = \mathrm{Hom}_{\Bbbk}[V,W],$$
and the odd part is the maps that reverse the parity.

The *dual super vector space* is defined in terms of inner Hom
$$\check{V} \overset{\text{def}}{=} \underline{\mathrm{Hom}}_{\Bbbk}(V,\Bbbk).$$

1.2.6. *Some canonical maps.* The *convention* we use is that *linear operators act on the left* , i.e.,

*there is a canonical evaluation map of super vector spaces*
$$\underline{\mathrm{Hom}}(U,V)\otimes U \to V, \quad A\otimes u \mapsto Au.$$

(1) The pairing with linear functionals. Applying this to linear functionals yields for each super vector space $V$, its *evaluation map* (or its *canonical pairing*), which is a map of super vector spaces
$$ev_V:\ \check{V}\otimes V \to \underline{1}, \quad \langle\omega,v\rangle \overset{\text{def}}{=} ev_V(\omega\otimes v) \overset{\text{def}}{=} \omega(v).$$

(2) The map $(V\otimes\check{V})\otimes V = V\otimes(\check{V}\otimes V) \to V\otimes\underline{1} \cong V$ gives a map
$$\check{V}\otimes V \to \underline{\mathrm{End}}(V).$$

(3) The map $V\otimes\check{V} \xrightarrow{c_{V,\check{V}}} \check{V}\otimes V \xrightarrow{ev_V} V\otimes\underline{1} \cong V$ gives the *biduality map* for $V$
$$\iota:\ V \to \check{\check{V}}, \quad \langle\iota_v,\omega\rangle \overset{\text{def}}{=} \langle v,\omega\rangle = (-1)^{p_\omega p_v}\ \langle\omega,v\rangle, \quad v\in V,\ \omega\in\check{V}.$$

*Remarks.* (1) If $V\otimes\check{V}\to \underline{\mathrm{Hom}}_{\Bbbk}(V,V)$ is an isomorphism the coevaluation map can be interpreted as a diagonal $\delta_V:\ \underline{1}\to V\otimes\check{V}$.

(2) We can define the canonical *wrong way maps* such as $\check{V}\otimes_{\Bbbk}V\to \underline{\mathrm{Hom}}_{\Bbbk}(V,V)$, by inserting braiding in appropriate places.

1.2.7. *The trace and the dimension.* We will see in 1.6.1 that the super-trace of a linear operator $T : V{\to}V$ can be calculated defined using its block decomposition

$$s\mathrm{tr}_F(T) = \ \mathrm{tr}(T_{00}) - \mathrm{tr}(T_{11}).$$

In particular, the super-dimension (fermionic dimension) is $s\dim_F(V) = \ \dim(V_0) - \dim(V_1)$.

1.2.8. *Tensor category of graded super vector spaces.* A graded super vector space is a graded vector space $V = \ \oplus_{\mathbb{Z}} V_p$ with a super structure on each $V_p$. There seem to be two (equivalent) ways to choose the braiding on the tensor category of super graded vector spaces.

- Bernstein's convention uses the sign given by the *total parity* $\deg(v) + p_v$

  $$c_{VW}(v{\otimes}w) = \ (-1)^{(deg(v)+p_v)(deg(w)+p_w)} w{\otimes}v.$$

  Then, taking the total parity $(deg(v) + p_v$ is a functor $\boldsymbol{s}$ into the tensor category of super-vector spaces.
- Deligne's convention is that commutativity constraint given by the sign which is the *product of the signs for the degree and for the parity*

  $$c_{VW}(v{\otimes}w) \overset{\text{def}}{=} (-1)^{deg(v)deg(w)}{\cdot}(-1)^{p_v p_w} \ w{\otimes}v = \ (-1)^{deg(v)deg(w)+p_v p_w} \ w{\otimes}v,$$

  this means that we combine (multiply) the commutativity constraints due to the $\mathbb{Z}$-grading and the $\mathbb{Z}_2$-grading.

*Remark.* The above two choices of braidings on the same tensor category are equivalent by an involution $\iota$ on the tensor category of graded super vector spaces. $\iota$ is given by changing the $\mathbb{Z}_2$-degree by adding the $\mathbb{Z}$-degree. (The tensoring constraint for $\iota$ is $\iota_{V,W} = (-1)^{\deg(v)\cdot p_q} : \ \iota(V{\otimes}W)\xrightarrow{\cong} \iota(V){\otimes}\iota(W)$.)

1.2.9. *Symmetric and exterior algebras $S(V)$ and $\wedge^*V$.* For a super vector space $V$, $S(V)$ is defined as the super commutative algebra freely generated by $V$.

- If $V$ is even we are imposing the ordinary commutativity $uv = vu$ and $S(V) = \Bbbk[v_1, ..., v_n]$ for any basis of $V$.
- If $V$ is odd, we are imposing the anti-commutativity $uv = -vu$ and therefore $\mathcal{F}[S(V)] = \ \overset{\bullet}{\wedge}\mathcal{F}(V)$, i.e., if we forget parity this is an ordinary exterior algebra. A more precise formulation in the odd case is

  $$S(V) \ = \ \boldsymbol{s}[\overset{\bullet}{\wedge}\mathcal{F}(V)].$$

$\wedge^*(V)$ is defined as the algebra generated by $V$ and by the anticommutativity relations for elements of $V$. On the level of ordinary algebras

$$\mathcal{F}[\wedge^*(V)] \ \cong \ \mathcal{F}[S^*(\Pi V)].$$

1.3. **Super algebras.** Let $A$ be a super algebra. The constructions bellow are not ad hoc, there is no smart choice. On one hand these are special case of definitions in general braided categories, and on the other hand they are also forced on us by desire of compatibility of super vector spaces with ordinary vector spaces (see 1.7).

1.3.1. *The opposite algebra $A^o$.* $A^o$ is given by the following multiplication structure on the super vector spaces $A$

$$a \underset{A^o}{\cdot} b \stackrel{\text{def}}{=} (-1)^{p_a p_b}\, b{\cdot}a, \quad a,b\in A^o = A.$$

There is an equivalence of categories $M\mapsto M^o$ of left $A$-modules and right $A^o$-modules by $M^0 = m$ as a super vector space and

$$m{\cdot}a \stackrel{\text{def}}{=} (-1)^{p_a p_m}\, a{\cdot}m.$$

$A$ is super commutative iff $A^o = A$. In particular, for super-commutative $A$ left and right modules are the same in the sense that one has an equivalence as above $\mathfrak{m}^l(A) \ni M\mapsto M^o \in \mathfrak{m}^r(A^o) = \mathfrak{m}^r(A)$.

1.3.2. *Tensor product of algebras.* The algebra structure on $A\otimes B$ is

$$(a'\otimes b)(a''\otimes b'')\stackrel{\text{def}}{=} (-1)^{p_{b'}p_{a''}}\, a'a''\otimes b'b''.$$

Algebra structure on a tensor product of algebra requires braiding so that multiplication can be defined by

$$(A\otimes B)\otimes(A\otimes B) = A\otimes B\otimes A\otimes B \xrightarrow[\cong]{1\otimes c_{B,A}\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow[\cong]{m_A\otimes m_B} A\otimes B.$$

1.3.3. *Derivatives.* A linear map $\partial:\ A\to A$ is said to be a *(left) derivative of $A$ of parity $p$* if

$$\partial(ab) = (\partial a)b + (-1)^{p\cdot p_a}\, a(\partial b).$$

There is also an (equivalent) notion of right derivatives, but we follow the convention that operators act on the left of vectors.

A consequence of this convention, we will (later) write the pairing of a vector field $\xi$ and a 1-form $\omega$ (a differential), in the form $\langle\xi,\omega\rangle$, so that it agrees with the left action of vector fields on functions: $\langle\xi, df\rangle = \xi(f)$.

1.3.4. *Parity change on modules for a super algebra $A$.* For a left $A$-module $M$, super vector space $\Pi M$ has a canonical $A$-action

$$a \underset{\Pi M}{\cdot} m = (-1)^{p_a}\, am.$$

For a right $A$-module $M$, the actions on $M$ and $\Pi M$ are the same. The reason seems to be that the parity change is viewed as a left tensoring $\Pi(M) = \Pi\underset{\Bbbk}{\otimes} M$.

## 1.4. **Lie algebras and their enveloping algebras.**

### 1.4.1. *Lie algebras.*

*Remark. Non-triviality of* $[x,x]=0$ In a super Lie algebra $\mathfrak{g}$ let us specialize the relation $[x,y]+(-1)^{p_x p_y}[y,x]=0$ to $x=y$. If $x$ is even, it says that $[x,x]=0$, however if $x$ is odd it does not say anything. So, condition $[x,x]=0$ (i.e., $\Bbbk{\cdot}x$ is an abelian subalgebra), is non-trivial for odd $x$.

### 1.4.2. *Enveloping algebras of Lie algebras.*

*Theorem.* [Poincare-Birkhoff-Witt]

*Proof.* The Poincare-Birkhoff-Witt theorem is proved in any tensor category with a $\mathbb{Q}$-structure, by constructing explicitly the enveloping algebra multiplication $*$ on the symmetric algebra $S^\bullet(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. Its relation to the standard product in $S^\bullet(\mathfrak{g})$ is

$$x_1\cdots x_n = \int_{S_n} d\sigma\ x_{\sigma 1} * \cdots * x_{\sigma n} = \frac{1}{|S_n|}\sum_{\sigma\in S_n}\ x_{\sigma 1} * \cdots * x_{\sigma n}.$$

One defines multiplication $*$ inductively, for $x_i, y_j \in \mathfrak{g}$

$$(x_1\cdots x_p) * (y_1\cdots y_q) \stackrel{\text{def}}{=} x_1 * (x_2 * (\cdots(x_p * (y_1\cdots y_q))\cdots)$$

and

$$x * (y_1\cdots y_q) \stackrel{\text{def}}{=} xy_1\cdots y_q + \int_{S_{q+1}} \sum_{i=1}^{q}\ (q-i+1)\ y_{\sigma 1} * \cdots * [x, y_{\sigma i}] * \cdot * y_{\sigma q}.$$

## 1.5. **Linear algebra on free modules over super-algebras (inner Hom, free modules and matrices).** Here $A$ is a super-algebra. The subtle parts are only done when $A$ is super-commutative.

### 1.5.1. *The inner* Hom *for $A$-modules.* 
Let $\mathcal{F}$ denote forgetting the super structure. For $A$-modules $M,N$, the vector space of $\mathcal{F}(A)$-linear maps $\mathrm{Hom}_{\mathcal{F}(A)}[\mathcal{F}M,\mathcal{F}N]$ has a canonical super-structure $\underline{\mathrm{Hom}}_A(M,N)$, such that the even part $\underline{\mathrm{Hom}}_A(M,N)_0$ is the space of maps of $A$-modules $\mathrm{Hom}_A(M,N)$ (i.e., the even maps in $\underline{\mathrm{Hom}}_A(M,N)$ are those that preserve the super structure). In particular, one has the duality operation on $A$-modules

$$\check{M} \stackrel{\text{def}}{=} \underline{\mathrm{Hom}}_A(M,A) \quad \text{hence} \quad \mathcal{F}(\check{M}) \ = \ \mathrm{Hom}_{\mathcal{F}(A)}[\mathcal{F}M,\mathcal{F}A].$$

1.5.2. *Free modules.* By a basis of a module over a super-algebra $A$ one means a *homogeneous* basis. Then a free module $M$ over a super-algebra $A$ means a module that has a basis. The *standard free left* $A$-modules are

$$A^{p|q} \stackrel{\text{def}}{=} \oplus_1^{p+q} Ae_i,$$

with $e_i$ even precisely for $i \le p$.

One has $A^{p|q} \cong A^p \oplus \Pi(A^q)$, etc.

1.5.3. *Linear operators.* We will be only interested in supercommutative $A$, but we take a moment for the general case. In the general case we view $A$ as a right $A$-module so that $A^{p|q} \stackrel{\text{def}}{=} \oplus_1^{p+q} Ae_i$ is a right $A$-module (and therefore a left $A^o$-module). This is convenient because the *inner* Hom *super algebra* $\underline{\mathrm{End}}_A(A^{p|q})$, acts on $A^{p|q}$ on the left.

If $A$ happens to be super-commutative, then the above left action of $A^o$ on $A^{p|q}$ can be viewed as a n action of $A = A^o$, and this is the original action of $A$ on $A^{p|q}$ viewed as a left $A$-module. So, the above convention using right action gives in this case the standard construction of $\underline{\mathrm{End}}_A(A^{p|q})$ for commutative algebras $A$.

1.5.4. *Coordinatization and matrices.* In general the Coordinatization of vectors in $A^{p|q}$ is by

$$x = x^i e_i \quad \text{with } x^i \in A^o.$$

If $A$ is super commutative this is the ordinary $A$-Coordinatization.

The Coordinatization of operators uses the right $A$-action

$$Te_j = e_i T^i_j, \quad T^i_j \in A.$$

*Bloc form* of matrices. Since each row and column in a matrix has a parity, the positions in a matrix come with a pair of signs[4]

$$(T^i_j) = \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix}.$$

1.6. **Berezinian (super determinant) of free modules and automorphisms of free modules.** Let $A$ be a super-commutative algebra. We consider the notions of trace and its nonlinear analogue, the determinant.

Super trace The notion of trace in the super setting is given by general principles (the braiding of the tensor structure). It applies to the *inner endomorphisms* of a free $A$-module $M$ of finite rank, and yields an even map of $A$-modules

$$Tr : \underline{\mathrm{Hom}}_A(M, M) \to A.$$

[4]This should not be confused with parity of matrix coefficients – all of them can be arbitrary elements of $A$.

In particular it applies to endomorphisms $\mathrm{Hom}_A(M,M) = \underline{\mathrm{Hom}}_A(M,M)_0$ and gives a map of $A_0$-modules

$$\mathrm{Hom}_A(M,M) = \underline{\mathrm{Hom}}_A(M,M)_0 \to A_0.$$

Super determinant General principles also determine what the notion of super determinant (called Berezinian) should be. One requires that

- ($\star 1$) $\det(e^T) = e^{Tr(T)}$ when $e^T$ makes sense,
- ($\star 2$) det is in some sense algebraic (map of algebraic groups).

We will construct super determinant using its linear algebra characterization as the action of the operator on the top exterior power. The requisite notion of the "super top exterior power" will be constructed in a somewhat ad hoc way. Among all formulas that yield the top exterior power in classical mathematics we observe that one of these produces a rank one module even in the super case.[5]

One should notice that super-determinant is defined in a very restrictive situation – on automorphisms of free modules. So, there are two step restrictions (i) to $\mathrm{Hom}_A(M,M)$ ($= \underline{\mathrm{Hom}}_A(M,M)$ rather then all of $\underline{\mathrm{Hom}}_A(M,M)$, and (ii) to only the invertible part $\mathrm{Aut}_A(M,M)$ of $\mathrm{Hom}_A(M,M)$. For instance if $A$ is even then a matrix of $T \in \mathrm{Aut}_A(M,M)$ has $T_{+-} = 0 = T_{-+}$ while $T_{++}, T_{--}$ are some ordinary matrices with coefficients in $A$. Then the Berezian is given by

$$Ber(T) = \det(T_{++}) \cdot \det(T_{--})^{-1}.$$

Here, $M$ will be a free $A$-module, hence isomorphic to one of $A^{p|q}$.

1.6.1. *Super trace.* If $\check{M} \underset{A}{\otimes} M \to \underline{\mathrm{Hom}}_A(M,M)$ is an isomorphism, one has a categorical notion of the trace

$$Tr \overset{\text{def}}{=} [\underline{\mathrm{Hom}}_A(M,M) \overset{\cong}{\leftarrow} \check{M} \underset{A}{\otimes} M \xrightarrow{ev_M} \underline{1} = A].$$

*Lemma.* (a) In terms of dual bases $e_i, e^i$ of $M$ and $\check{M}$, this reduces to $Tr(T) = (-1)^{p_i} \langle e^i, Te_i \rangle$.

(b) In terms of the matrix $(T^i_j) \overset{\overset{?}{\text{def}}}{=} \langle e^i, Te_i \rangle$ that we defined above using the *right* action of $A$, this is

$$Tr(T) = (-1)^{p_i} T^i_i = Tr(T_{++}) - Tr(T_{--}).$$

(c) The trace of a commutator is still zero: $Tr[A,B] = 0$, i.e., $Tr(AB) = (-1)^{p_A p_B}\, Tr(BA)$.

*Proof.* (a) and (b) follow from

$$Tr[ev_{M,M}(m\check{\omega})] = (-1)^{\overline{\omega} \cdot \overline{m}} \cdot \langle \omega, m \rangle.$$

[5] In the end we get a computable formula and we can check the characterizing property ($\star$).

For this one recalls that the map $\check{M}\otimes M \to \underline{\mathrm{End}}_A(M)$ is a composition $\check{M}\otimes M \xrightarrow{c_{\check{M},M}} M\otimes\check{M} \xrightarrow{ev_{M,M}} \underline{\mathrm{End}}_A(M)$ (remark 1.2.6). So, the operator corresponding to $\omega\otimes m \in \check{M}\otimes M$ is $ev_{M,M}[c_{\check{M},M}\ \omega\otimes m] = (-1)^{\overline{\omega}\cdot\overline{m}}\cdot ev_{M,M}(\omega\otimes m)$, and according to the above definition, its trace is $\langle\omega, m\rangle$.

1.6.2. *Berezinian of a free module (= "top exterior power").* Classically, the determinant of a linear operator is its action on the top exterior power. However, for an odd line $L = \Bbbk\theta$, exterior algebra $\overset{\bullet}{\wedge}L = \oplus_0^\infty \Bbbk\theta^n$ has no top power. Instead we will use another classical formula for the top exterior power of a free $A$-module $L$. Let $V$ be a vector space over $\Bbbk$, then[6]

$$(0\hookrightarrow V)^! \mathcal{O}_V \overset{\mathrm{def}}{=} \mathrm{Ext}^\bullet_{\mathcal{O}(V)}[\mathcal{O}_0,\ \mathcal{O}(V)] = \overset{top}{\wedge} V\ [-\dim(V)].$$

LHS is In the super setting this will be the definition of the RHS. So the LHS is the correct super analogue of the top exterior power, that gives the super version of the determinant.

*Lemma.* Let $A$ be a commutative super-algebra and let $L = A^{p|q}$ be a free $A$-module.

(a) The graded object

$$Ber(L) \overset{\mathrm{def}}{=} \mathrm{Ext}^\bullet_{S^\bullet_A(\check{L})}[A,\ S^\bullet_A(\check{L})] = \mathrm{Ext}^\bullet_{\mathcal{O}(L)}[\mathcal{O}(0),\ \mathcal{O}(L)]$$

is concentrated in the degree $p$ where it is a free $A$-module of rank 1, and of parity the same as $q$.

(b) Berezian is canonically isomorphic to a line bundle

$$Ber(L) \cong \Lambda^{\mathrm{top}} L_0 \otimes S^{\mathrm{top}}(L_1)^*.$$

(c) To any ordered basis $e_1, ..., e_n$ one canonically associates a basis $[e_1, ..., e_n]$ of $Ber(L)$.[7]

*Proof.* (a) **(0)** Any ordered decomposition $L = L_1 \oplus L_2$ induces $S(L_1^*)\otimes_A S(L_2^*) \xrightarrow{\cong} S(L^*)$, and then $Ber(L_1)\overset{L}{\otimes}_A Ber(L_2) \xrightarrow{\cong} Ber(L)$.

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[6]From the point of view of algebraic geometry, Berezian is by definition the "relative dualizing sheaf" for $0\hookrightarrow V$. Its computation above is essentially the computation of the dualizing sheaf on $V$ since $\omega_V \overset{\mathrm{def}}{=} (V \to pt)^! \Bbbk$ has form $\omega_V = \mathcal{O}_V \otimes_\Bbbk \Omega$ and $\Bbbk = \omega_0 = (0\hookrightarrow V)^! \omega_V = (0\hookrightarrow V)^!(\mathcal{O}_V\otimes_\Bbbk \Omega) = \Omega\otimes_\Bbbk \overset{top}{\wedge} V\ [-\dim(V)]$ gives $\Omega = \overset{top}{\wedge} V^*\ [\dim(V)]$.

[7]The coming calculation of the Berezian determinant can be bi viewed as a description of the functoriality of $[e_1, ..., e_n]$ in $e_1, ..., e_n$. Roughly, even ones are covariant and odd ones contravariant, so we may write $[e_1, ..., e_p; e_{p+1}, ..., e_{p+q}]$

**(1)** *The case* $p|q = 1|0$. If $L = Ae$ then the $S_A^\bullet(\check{L})$-module $A$ (via the augmentation), has a free Koszul resolution[8] $S_A(\check{L})\underset{A}{\otimes}L^* \xrightarrow{s\otimes e^*\mapsto\ se^*} S_A(\check{L}) \to A\to 0$, hence

$$\mathrm{Ext}^\bullet_{S_A^\bullet(\check{L})}[A,\ S_A^\bullet(\check{L})] \ =\ H^\bullet\ \mathrm{Hom}[S_A(\check{L})\underset{A}{\otimes}Ae^* \xrightarrow{s\otimes e^*\mapsto\ se^*} S_A(\check{L}),\ S_A(\check{L})]$$

$$=\ H^\bullet[\ S_A^\bullet(\check{L}) \xrightarrow{s\mapsto\ se^*\otimes e} S_A^\bullet(\check{L})\underset{A}{\otimes}Ae] =\ Ae\ [-1] =\ \overset{top}{\underset{A}{\wedge}}L\ [-1].$$

**(2)** *The case* $p|q = 0|q$. If $L = A^{0|q} = \oplus_1^q A\theta_i$ for odd $\theta_i$'s, then $S_A^\bullet(\check{L}) = A[\theta_1^*, ..., \theta_q^*]$ is a Frobenius algebra, hence it is an injective module over itself. So,

$$\mathrm{Ext}^\bullet_{S_A^\bullet(\check{L})}[A,\ S_A^\bullet(\check{L})] =\ \mathrm{Hom}_{S_A^\bullet(\check{L})}[A,\ S_A^\bullet(\check{L})] =\ \mathrm{Hom}_{A[\theta_1^*,...,\theta_q^*]}(A,\ A[\theta_1^*, ..., \theta_q^*])$$

$$=\ \cap\ Ker(\theta_i^*:\ A[\theta_1^*, ..., \theta_q^*] \to A[\theta_1^*, ..., \theta_q^*])\ =\ A\theta_1^*\cdots\theta_q^*.$$

□

Now (b) and (c) follow. Let $L = \oplus\ Ae_i \oplus \oplus A\theta_j$ with $e_i$ even and $\theta_j$ odd. The factorization from **(0)**,

$$Ber(L) \cong\ \otimes_i\ \ Ber(Ae_i) \otimes Ber(\oplus\ A\theta_j)$$

is canonical since $Ae_i$'s are even. According to **(1)** and **(2)** it provides a basis of $Ber(L)$ of the form $e_1\otimes\cdots\otimes e_p\otimes\theta_1\cdots\theta_q^*$ which depends on the choice of order of $e_i$ and $\theta_j$'s.

More precisely, it is a tensor product of basis $e_1\wedge\cdots\wedge e_p$ of $\Lambda^{\mathrm{top}}\ L_0$ (if one calculates $Ber(L_0)$ in one step, using the Koszul resolution of the $S_A(L_0^*)$-module $A$), and $\theta_1^*\cdots\theta_q^*$ of $A\theta_1\cdots\theta_q^* \subseteq S^{\mathrm{top}}(A\theta_1^*\oplus\cdots\oplus A\theta_q^*)$.

1.6.3. *Remarks.* (1) A basis $e_1, ..., e_p, \theta_1, ..., \theta_q$ of $L$ gives a basis $[e_1\cdots e_p\theta_1^*\cdots\theta_q^*]$ of $Ber(L)$. This gives a more elementary approach to Berezinians – a free module with a basis $[e_1\cdots e_p\theta_1^*\cdots\theta_q^*]$, with a given rule on how this basis element transforms under a change of basis of $L$.

(2) We will remember that $Ber(L)$ is in degree $p$, i.e., we will consider it as an object of the category of graded $A$-modules.

*Corollary.* A short exact sequence of free modules $0 \to L' \to L \to L'' \to 0$, gives a canonical isomorphism

$$Ber(L) \cong Ber(L')\otimes Ber(L'').$$

[8] More generally, in the *case* $p|q = p|0$, the resolution is

$$S_A^\bullet(\check{L})\underset{A}{\otimes}\overset{p}{\underset{A}{\wedge}}\check{L} \to S_A^\bullet(\check{L})\underset{A}{\otimes}\overset{p-1}{\underset{A}{\wedge}}\check{L} \to \cdots \to S_A^\bullet(\check{L})\underset{A}{\otimes}\overset{1}{\underset{A}{\wedge}}\check{L} \to\ A\to 0.$$

1.6.4. *Berezinian of a map (= "determinant").* For an isomorphism $T : L \to M$ of free $A$-modules,

$$Ber(T) \in \operatorname{Hom}[Ber(L), Ber(M)] \stackrel{\text{def}}{=} \text{ the induced isomorphism of Berezinians.}$$

In particular, for an automorphism $T$ of a free $A$-module $L$,

$$Ber(T) \in (A_0)^* \text{ is the action of } T \text{ on } Ber(L).$$

Observe that in order for $T : L \to L$ to act on $Ber(L) = \operatorname{Ext}^{\bullet}_{S^{\bullet}_A(\check{L})}[A,\ S^{\bullet}_A(\check{L})]$,

- $T$ needs to be even and
- $T$ needs to be invertible.

*Lemma.* (a) (Berezinian in matrix terms.) If $T : L \to M$ is invertible, then so are the diagonal components$T_{++}$ and $T_{--}$ of $T = \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix}$. Then

$$T = \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix} = \begin{pmatrix} 1 & T_{+-}T_{--}{}^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} T_{++} - T_{+-}\cdot(T_{--})^{-1}\cdot T_{-+} & 0 \\ T_{-+} & T_{--} \end{pmatrix},$$

and

$$Ber(T) = \det(T_{++} - T_{+-}\cdot(T_{--})^{-1})\cdot \det(T_{--})^{-1}.$$

(b) (Berezinian in terms of algebraic super groups.) Berezinian is characterized as a map of groups $GL(p|q, A) \to GL(1|0, A)$ which satisfies

$$Ber(1 + \varepsilon T) = 1 + \varepsilon T$$

when $\varepsilon$ is even of square zero.

*Remarks.* (0) Clearly the appearance of inversion in $Ber\begin{pmatrix} T_{++} & 0 \\ 0 & T_{--} \end{pmatrix} = \det(T_{++})\cdot \det(T_{--})^{-1}$ corresponds to subtraction in the trace formula.

(1) If $A = \Bbbk$ is even then the automorphism group of $\Bbbk^{p|q}$ is $GL_p(\Bbbk)\times GL_q(\Bbbk)$, and $Ber = det_p/det_q : \operatorname{Aut}(\Bbbk^{p|q}) \to \Bbbk^*$.

(2) As usual, one important application of super determinants (over non-trivial super commutative algebras) comes from (local) isomorphisms of super manifolds $F : M \to N$. Then $dF : \mathcal{T}_M \to \mathbb{F}^*\mathcal{T}_N$ is an isomorphism of locally free $\mathcal{O}_M$-modules, hence one has $Ber(dF) : Ber(\mathcal{T}_M) \to F^*Ber(\mathcal{T}_N)$. We will use this for change of variables in integrals.

1.7. **The automatic extension of algebraic concepts to the super setting ("Even rules").** This is Bernstein's? idea to reduce the sign calculations to super-commutative algebras. A super vector space $V$ defines a functor from super-commutative algebras to ordinary vector spaces, $B \mapsto V(B) \stackrel{\text{def}}{=} (V_B)_0$ for $V_B \stackrel{\text{def}}{=} B\otimes V$.

For instance, a super *Lie algebra* structure on $V$ is the same as an ordinary *Lie algebra* structure on the functor $B \mapsto V(B)$. What this means is that for any super commutative algebra $B$, on $V(B)$ one is given a Lie algebra structure over $B_0$, which is functorial (natural) in $B$. This principle allows one to calculate the defining relations for super Lie algebras (instead of inventing the signs in these relations).

The same works if we replace the *Lie algebra* by any other algebraic structure.

## 2. **Manifolds**

2.0.1. *Definition of super manifolds as ringed spaces.* By definition a super manifold is a topological space $|M|$ with the sheaf $\mathcal{O}_M$ of supercommutative algebras which is locally the same as some $(\mathbb{R}^{p,q}, C^\infty_{\mathbb{R}^p}\otimes\wedge^*\mathbb{R}^q)$. This is analogous to one of characterizations of smooth manifolds $N$ as a topological space $|N|$ with the sheaf $\mathcal{O}_N$ of commutative algebras which is locally the same as some $(\mathbb{R}^p, C^\infty_{\mathbb{R}^p})$. A way to make this *less* abstract is provided by

2.0.2. *The set theoretical view on a supermanifold $M$ as the functor* $\mathrm{Hom}(-,M)$ *of* **points** *of $M$.* In differential geometry we visualize a manifold $N$ as a set of points and we would like to do the same in super geometry. The categorical way of thinking of a set of points of $N$ is $\mathrm{Hom}_{manifolds}(\mathbb{R}^0, N)$. The same construction in super manifolds does not notice enough information as $\mathrm{Hom}_{super\ manifolds}(\mathbb{R}^0, M) = |M|$ is the same as the set of points of the underlying ordinary manifold. The problem is that it is not at all sufficient to probe a super manifold with an even object $\mathbb{R}^0$. It works better if one probes with all super points, the collection of sets $\mathrm{Hom}_{super\ manifolds}(\mathbb{R}^{0|q}, M)$, $q \geq 0$, contains more information. In the end, as emphasized by Grothendieck, to restore the set theoretic point of view on $M$ one should look not at a single set but at the functor

$$\mathrm{Hom}(-,M): \ SuperManifolds^o \to Sets.$$

One says that $\mathrm{Hom}(X.M)$ is the *set of $X$-points of $M$.*

2.0.3. *Example.* As an example, the super manifold $GL(p|q)$ has underlying ordinary manifold $GL_p\times GL_q$, and the two are the same on the level of ordinary points. However for a super manifold $X$ with a supercommutative algebra of functions $A = \mathcal{O}(X)$, the set of $X$-points of $GL(p|q)$ is more interesting – this is the set of automorphisms of the free $A$-module $A^{p|q}$.

2.1. **Super manifolds – definitions.** A super manifold $M$ is a ringed topological space $(|M|, \mathcal{O}_M)$ locally isomorphic to some

$$\mathbb{R}^{p|q} \stackrel{\text{def}}{=} (\mathbb{R}^p, \mathcal{C}^\infty_{\mathbb{R}^p}[\psi^1, ..., \psi^q]),$$

where

$$\mathcal{C}^\infty_{\mathbb{R}^p}[\psi^1, ..., \psi^q] \ \stackrel{\text{def}}{=} \ \oplus_{I=\{I_1<\cdot<I_k\}} \ \mathcal{C}^\infty_{\mathbb{R}^p}\cdot\psi^{I_1}\cdot\cdot\cdot\psi^{I_k}$$

is a super-commutative algebra freely generated over the smooth functions $\mathcal{C}^\infty_{\mathbb{R}^p}$, by odd generators $\psi^1, ..., \psi^q$.

So, the functions on $\mathbb{R}^{p|q}$ are $\mathcal{C}^\infty_{\mathbb{R}^p} \underset{\Bbbk}{\otimes} S(W)$ for a $q$-dimensional odd vector space $W$.

2.1.1. *Maps.* A map of ringed spaces $f : (|M|, \mathcal{O}_M) \to (|N|, \mathcal{O}_N)$ is by definition a pair of a map $|f| : |M| \to |N|$ of topological spaces and a map of sheaves of rings $f^! : |f|^*\mathcal{O}_N \to \mathcal{O}_M$.

*Remarks.* (1) Intuitively, $f^!$ is the pull-back of functions under the map $f$. The compatibility of $f^!$ and $|f|$ is contained in the fact that $f^!$ is defined on $|f|^*\mathcal{O}_N$.

(2) In a "more set theoretic" situation like that of ordinary smooth manifolds we find that this compatibility implies that $|f|$ is a $C^\infty$ map and that $f^!$ is the pull back under $|f|$. In other words maps of

*Example.* For instance a map $f : \mathbb{R}^{p|q} \to \mathbb{R}^{r|s}$ consists of a continuous map $|f| = (f_1, ..., f_r) : \mathbb{R}^p \to \mathbb{R}^r$ with component functions $f_i : \mathbb{R}^p \to \mathbb{R}$, and a map of sheaves of rings

$$(|f|^*(\mathcal{C}^\infty_{\mathbb{R}^r}))[\phi^1, ..., \phi^s]) = |f|^*(\mathcal{C}^\infty_{\mathbb{R}^r}[\phi^1, ..., \phi^s]) \xrightarrow{f^!} \mathcal{C}^\infty_{\mathbb{R}^p}[\psi^1, ..., \psi^q].$$

Now, if $x^i$ and $y^k$ are coordinates on $\mathbb{R}^p$ and $\mathbb{R}^r$, then $F^*y^k \in \mathcal{C}^\infty_{\mathbb{R}^p}[\psi^1, ..., \psi^q]$ can be written as $\sum_I \ f^k_I \psi^I$ with $f^k_I \in \mathcal{C}^\infty(\mathbb{R}^p)$.

In particular, $f^k = f^k_\emptyset \in \mathcal{C}^\infty_{\mathbb{R}^p}$ is obtaine quite naturally by using the quotient map

$$\mathcal{C}^\infty_{\mathbb{R}^p}[\psi^1, ..., \psi^q] \twoheadrightarrow \mathcal{C}^\infty_{\mathbb{R}^p}[\psi^1, ..., \psi^q]/\sum \ \psi^i \cdot \mathcal{C}^\infty_{\mathbb{R}^p}[\psi^1, ..., \psi^q] = \mathcal{C}^\infty_{\mathbb{R}^p}$$

which kills the idea generated by all odd functions.

Now one checks that

(1) The collection of all $f^! y^k$ and $f^! \phi^l$, determines $f^!$.[9]
(2) $|f| = (f^1, ..., f^k)$, so in particular $f : \mathbb{R}^p \to \mathbb{R}^r$ is smooth.

So, all together

- A map $f$ is freely determined by a collection of $r$ even functions $f^! y^k$ and $s$ odd functions $f^! \phi^l$.
- The reduction of $f^! y^k$'s are the component functions of the corresponding smooth map $|f| : \mathbb{R}^p \to \mathbb{R}^r$.

2.1.2. *The associated ordinary manifold $M_{red}$.* The *reduced manifold of $M$* is the ringed space $M_{red} \overset{\text{def}}{=} (|M|, \mathcal{O}_M/J_M)$ where $J_M$ is the ideal generated by odd functions. This is a $\mathcal{C}^\infty$-manifold since $\mathbb{R}^{p|q}_{red}$ is clearly $\mathbb{R}^p$. Notice that

The reduced version $M_{red}$ is a submanifold of $M$.[10]

(1) $\mathcal{O}_M \twoheadrightarrow \mathcal{O}_{M_{red}}$ corresponds to the canonical *closed inclusion* of super manifolds $M_{red} \hookrightarrow M$.
(2) $M$ is not a fiber bundle over $M_{red}$ since there is *no canonical map $M \to M_{red}$*.

[9] $f^k$'s do not determine all $f^k_I$'s, i.e., $f^! y^k$. For instance a map $M \to \mathbb{R}$ is the same as an even function on $M$.

[10] It is also a quotient but non-canonically, see 2.1.5.

Let me mention another (less useful) passage to ordinary world – where one simply keeps the even functions. This gives a (usually nonreduced) scheme $M_{even} \overset{\text{def}}{=} \text{Spec}(\mathcal{O}_{M,0})$, which fits into

$$\mathcal{O}_{M_{even}} = (\mathcal{O}_M)_0 \subseteq \mathcal{O}_M \twoheadrightarrow \mathcal{O}_M/J_M, \quad \text{hence} \quad M_{even} \twoheadleftarrow M \hookleftarrow M_{red}.$$

The composition $M_{red} \to M_{even}$ makes $M_{even}$ into an infinitesimal extension of $M_{red}$.

2.1.3. *The value of a function $f \in \mathcal{O}_M$ at a point $x \in |M|$.* One can define the value $f(x)$ as the unique number $c$ such that $f - c$ is not invertible in any neighborhood of $x$.

A more global way to think about it is that $f$ gets restricted to a smooth function on the smooth manifold $M_{red}$ which on the level of a topological space is just is the same as $|M|$, and this restriction can be evealuated at points $x$ of $|M|$.

Implicite in the above definition of $f(x)$ is the observation that $\mathcal{O}_M$ is a sheaf of local rings on the topological space $|M|$ - the maximal ideal in the stalk at $x$ is $\mathfrak{m}_x = \{f \in \mathcal{O}_{M,x};\ f(x) = 0\}$, and the residue field is $\mathbb{R}$.

2.1.4. *Coordinates.* On $\mathbb{R}^{p|q}$ one has $x^\mu$'s and $\psi^i$'s.

A map $M \to \mathbb{R}^{p|q}$ is the same as $p$ even and $q$ odd functions on $M$.

2.1.5. *Classification of super manifolds.* On the level of isomorphism classes, for any manifold $N$, supermanifolds $M$ with $M_{red} \cong M$ are the same as vector bundles over $N$. However, the morphisms in two categories are quite different.

2.1.6. *Lemma.* (a) Any super manifold $M$ is locally a product of the corresponding even manifold $M_{red}$ and a super point $\mathbb{R}^{0|q}$.

(b) All super manifolds are of the form $\mathcal{O}_M = \boldsymbol{s}(\overset{\bullet}{\wedge}\mathcal{V}^*)$ for a vector bundle $\mathcal{V}$ over a smooth manifold $N$ $(= M_{red})$, however non-canonically.

(c) For any manifold $N$, $\mathcal{V} \mapsto \text{Spec}[\boldsymbol{s}(\overset{\bullet}{\wedge}\mathcal{V}^*)]$, is a bijection pf isomorphism classes of vector bundles over $N$ and supermanifolds $M$ with $M_{red} \cong N$.

*Proof.* (a) is just the definition of super-manifolds. (b) Part (a) shows that locally $\mathcal{O}_M = \boldsymbol{s}(\overset{\bullet}{\wedge}\mathcal{V})$ for a trivial vector bundle $\mathcal{V} = |M| \times \mathbb{R}^q$ of rank $q$.

2.1.7. *Remarks.* (a) Operation $\mathcal{V} \mapsto \text{Spec}[\boldsymbol{s}(\overset{\bullet}{\wedge}\mathcal{V}^*)]$ will later be called the parity change of a vector bundle $\mathcal{V}$ and denoted $\Pi(\mathcal{V})$.

(b) For a super manifold $M$, the choice $\mathcal{V}$ is an additional rigidification of $M$, observe that it lifts a super-manifold $M$ to a graded manifold. The functor from "manifolds with a vector bundle" to super manifolds: $(N, \mathcal{V}) \mapsto \text{Spec}[\boldsymbol{s}(\overset{\bullet}{\wedge}\mathcal{V}^*)] = \Pi\mathcal{V}$, is surjective on objects but not an equivalence.

2.1.8. *Remark.* Super $C^r$-manifolds do not make sense.

## 2.2. **Versions: $\infty$ dimensional super manifolds and super-schemes.**

2.2.1. $\infty$ *dimensional super manifolds.* Douady noticed that in $\infty$ dimension one needs local charts – the approach through the sheaf of functions is not enough to define manifolds.

$\infty$-dimensional super manifolds appear as various spaces of fields $\mathcal{F}$ in QFT, for instance the map spaces such as $Map(S^{1|1}, M)$. Deligne-Freed treat such spaces only as functors. So they do not spell the structure of a super manifold on $\mathcal{F}$, but only the functor of points $Map(-, \mathcal{F})$ defined on finite dimensional super manifolds.

2.2.2. *Super-schemes.* A super space $M = (|M|, \mathcal{O}_M)$ is a ringed space (topological space $|M|$ with a sheaf of super-rings $\mathcal{O}_M$), such that

- the structure sheaf $\mathcal{O}_M$ is super-commutative and
- the stalks are local rings.

A super scheme is a super space $M$ such that the even part $M_0 \stackrel{\text{def}}{=} (|M|, \mathcal{O}_{M,0})$ is a scheme, and that the odd part $\mathcal{O}_{M,1}$ is a coherent module for the even part $\mathcal{O}_{M,0}$. Some examples:

(1) The algebraic versions of $\mathbb{R}^{p|q}$'s are affine (super) schemes $\mathbb{A}^{n|m}$ over a ground ring $\Bbbk$, given by the super-commutative algebras of polynomial functions on these spaces

$$\mathcal{O}(\mathbb{A}^{n|m}) = \Bbbk[x^1, ..., x^n, \psi^1, ..., \psi^m] = S^\bullet(\Bbbk x^1 \oplus \cdots \oplus \mathbb{R}x^n) \otimes \overset{*}{\wedge} \mathcal{F}(\Bbbk\psi^1 \oplus \cdots \oplus \Bbbk\psi^m),$$

with $x^\mu$'s even and $\psi^k$'s odd. It is a product of the affine space $\mathbb{A}^n = \mathbb{A}^{n|0}$ and a super point $\mathbb{A}^{0|m}$. The functions on a super point $\mathcal{O}(\mathbb{A}^{0|m})$ have a finite basis of monomials $\psi^{i_1 < \cdots < i_k} \stackrel{\text{def}}{=} \psi^{i_1} \cdots \psi^{i_k}$.

(2) A vector bundle $\mathcal{V}$ over an ordinary scheme $X$. defines a super scheme $\Pi(\mathcal{V}) \stackrel{\text{def}}{=} (|X|, \overset{\bullet}{\wedge}_{\mathcal{O}_X} \mathcal{V}^*)$. The first infinitesimal neighborhood of $X$ in $\Pi(\mathcal{V})$ is $\mathcal{N} \stackrel{\text{def}}{=} (|X|, \mathcal{O}_X \oplus \mathcal{V}^*)$, given by imposing $\mathcal{V}^* \wedge \mathcal{V}^* = 0$. If $X$ is smooth so is $\Pi(\mathcal{V})$, but $\mathcal{N}$ is not smooth.

2.3. **Super-manifold as a functor.** When one wants to think of a super manifold $M$ in set theoretic terms, one associates to each super manifold $S$ the set of *$S$-points of $M$*

$$M(S) \stackrel{\text{def}}{=} Map(S, M) \stackrel{\text{def}}{=} \mathrm{Hom}_{SuperManifolds}(S, M).$$

One can think of this as all ways to probe $M$ with $S$. In this way we consider a super-manifold $M$ as a functor $S \mapsto M(S)$ from super manifolds to sets. For instance,

- (0) $M(\mathbb{R}^{0|0}) \stackrel{\text{def}}{=} Map(\mathbb{R}^{0|0}, M) = |M|$.

- (1) $M(\mathbb{R}^{0|1}) \stackrel{\text{def}}{=} Map(\mathbb{R}^{0|1}, M)$ is the moduli of pairs of a point $x \in M$ and an odd derivation of the local ring at $x$, i.e., these are the odd tangent vectors.
- (2) $\mathbb{R}^{1|0}(S) \stackrel{\text{def}}{=} Map(S, \mathbb{R}^{1|0}) = \mathcal{O}_S(S)_0$ and $Map(S, \mathbb{R}^{0|1}) = \mathcal{O}_S(S)_1$.
  More generally,
$$\mathbb{R}^{p|q}(S) \stackrel{\text{def}}{=} Map(S, \mathbb{R}^{p|q}) = \mathcal{O}_S(S)_0{}^{\oplus p} \oplus \mathcal{O}_S(S)_1{}^{\oplus q}.$$
- We say that the *universal point of* $M$ is the $M$-point $M \xrightarrow{1_M} M$ in $M(M) = \mathrm{Hom}(M, M)$ (it does have a universal property).

The idea is that $\mathbb{A}^{0|1}$ has only one $\Bbbk$-point so in this respect it looks like $\mathbb{A}^{0|0}$; however, $S$-points of $\mathbb{A}^{0|1}$ are numerous – the same as odd functions on $S$.

2.3.1. *The analogy with non-reduced schemes.* Odd functions on a super manifold $M$ are nilpotent, so in some respect it will behave like non-reduced schemes, and one can think of $M$ as $|M|$ (i.e., $M_{red}$), plus "fuzz", a little cloud around $M_{red}$. On the other hand we will have interesting and unusual analysis on $M$, in analogy with smooth manifolds.

2.3.2. *Fiber products.* Fiber products $S' \underset{S}{\times} M$ exist for maps $S' \to S$ that are locally projections of the form $S' = \mathbb{R}^{p|q} \times S \to S$.

2.3.3. *The use of a base $S$.* A super space $M/S$ (also called an $S$-super space $M$, or a super space $M$ with a base $S$), means simply a map of super spaces $M \to S$. The basic idea is that one studies such relative super space $M/S$ using all super spaces $T/$ with base $S$ – one studies the the set of $T/S$-points of $M/S$, defined by $(M/S)(T/S) \stackrel{\text{def}}{=} Hom_S(T, M)$.

In order to be able to construct maps $M \xrightarrow{\phi} N$ in terms of the corresponding maps of functors, one has to systematically use families of manifolds $M \to S$ (?). In particular, it means that one should with each family $M \to S$ also consider all families obtained by the base changes $M' = M \underset{S}{\times} S'$ (under projection-like maps $S' \to S$).

2.3.4. *"Functions are determined by their values on $S$-points".* A function $f$ on $M$ gives for any $S$-point $S \xrightarrow{\sigma} M$ a function $f\sigma$ on $S$, which we can think of as the value of $f$ on the $S$-point $\sigma$. So,tautologically, $f$ is the same as its value on the $M$-point $M \xrightarrow{id} M$.

2.4. **The functor of maps between two super spaces.** For two super spaces $M, N$ there are as usual *two notions of maps* from $M$ to $N$ (see the special case ). The *categorical one* is clear, this is the set $\mathrm{Hom}_{SuperManifolds}(M, N) = Map(M, N)$.

The other notion is suppossed to be an inner Hom in super spaces :

$$\underline{\mathrm{Hom}}(M, N) \stackrel{\text{def}}{=} \text{ moduli of all maps from } M \text{ to } N.$$

Here *all* means that we allow all maps of ringed topological spaces, without requiring that the map preserves parity. Then $\underline{\mathrm{Hom}}(M,N)$ itself should be a super space and $\underline{\mathrm{Hom}}(M,N)$ should be the underlying even space $\underline{\mathrm{Hom}}(M,N)_{red}$.

The problem is of course in how to organize the totality of all maps into a super manifold (usually infinite dimensional!) (or even, whether it can be done). We avoid this question by only defining a *functor* $\underline{\mathrm{Hom}}(M,N)$ on super manifolds, it is defined by

$$\underline{\mathrm{Hom}}(M,N)\ (S) \stackrel{\mathrm{def}}{=} Maps_S(M\times S, N\times S) = \ Map(M\times S, N),$$

i.e., the $S$-points of this functor are simply the $S$-families of maps form $M$ to $N$.

We leave out the question of what kind of a space would represent this functor (in nice cases it is given by a super manifold, possibly infinite dimensional).

## 2.5. **Lie groups and algebraic groups.**

2.5.1. *Super Lie groups.* $\mathbb{R}^{p|q}$ is a group (contrary to the intuition from commutative schemes where: group $\Rightarrow$ smooth $\Rightarrow$ reduced).

2.6. **Sheaves.** The sheaves on $M$ are by definition sheaves on the topological space $|M|$ (this is the only topological space around). The sheaves on $|M|$ that are related to its structure of a super manifold are the sheaves of $\mathcal{O}_M$-modules. Two notions of Hom give two relevant notions of global sections of an $\mathcal{O}_M$-module $\mathcal{A}$:

- The inner notion $\underline{\Gamma}$ gives a super module $\underline{\Gamma}(\mathcal{A})$ over the super-commutative algebra $\underline{\Gamma}(\mathcal{O}_M)$, by

$$\underline{\Gamma}(M,\mathcal{A}) \stackrel{\mathrm{def}}{=} \underline{\mathcal{H}\mathrm{om}}_{\mathcal{O}_M}(\mathcal{O}_M,\mathcal{A}) = \ \Gamma(|M|,\mathcal{A}) = \ \Gamma(|M|,\mathcal{A}_0) \oplus \Gamma(|M|,\mathcal{A}_1)\ .$$

- The notion given by the category of $\mathcal{O}_M$-modules is $\Gamma(\mathcal{A}) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{O}_M,\mathcal{A})$,

$$\Gamma(M,\mathcal{A}) \stackrel{\mathrm{def}}{=} \mathcal{H}\mathrm{om}_{\mathcal{O}_M}(\mathcal{O}_M,\mathcal{A}) = \ \Gamma(|M|,\mathcal{A}_0).$$

So, $\Gamma(\mathcal{A}) = \ [\underline{\Gamma}(\mathcal{A})]_0 = \ \Gamma(|M|,\mathcal{A}_0)$ and one can recover the odd part of the functor $\Gamma$ from the even part by

$$\underline{\Gamma}(M,\mathcal{A})_1 = \ \Gamma(|M|,\mathcal{A}_1) = \ \Pi\ \Gamma(M,\Pi\mathcal{A}).$$

## 3. **Differential Geometry**

As usual, on a super manifold $M$ one can identify vector bundles with locally free sheaves. However, the operation of change of parity which is obvious for locally free sheaves, is quite unexpected for vector bundles.

### 3.1. **Vector bundles.** We consider two kinds of objects over a super-manifold $M$

- Vector bundles over $M$. A vector bundle $V$ over $M$ of rank $p|q$, is a fiber bundle $V \to M$ which is (i) locally $\mathbb{A}^{p|q} \times M \to M$, (ii) has a structure group $GL(p|q)$ (i.e., the transition functions lie in that group).
- Locally free $\mathcal{O}_M$-modules $\mathcal{V}$.

3.1.1. *Super-vector space $\underline{V}$ in the category of super-manifolds, are the same as super-vector spaces.* This is the way we have obtained our basic manifolds $\mathbb{A}^{p|q}$.

*Lemma. Proof.* A super vector space $V$ gives a super manifold $\boldsymbol{V}$ with $|\boldsymbol{V}| = V_0$ and $\mathcal{O}_{\boldsymbol{V}} = \mathcal{C}^\infty(V_0) \otimes S^\bullet(V_1^*)$. (In algebraic geometry $\mathcal{O}(\boldsymbol{V}) = S^\bullet(V^*)$.) The structure of a "super-vector space in the category of super-manifolds" is clear.

In the opposite direction, a "super-vector space in the category of super-manifolds" $\boldsymbol{V}$, defines a super-vector space $V = [\mathcal{O}_{linear}(\boldsymbol{V})]^*$, the dual of linear functions. In the infinitesimal language, $V = T_0(\boldsymbol{V})$. Similarly, $V = Map(\mathbb{A}^{0|1}, \boldsymbol{V})$ with $V_0 = Map(\mathbb{A}^{0|0}, \boldsymbol{V})$ (the maps that factor thru the point $\mathbb{A}^{0|0}$) and $V_1 = Map[(\mathbb{A}^{0|1}, 0), (\boldsymbol{V}, 0)]$ (maps that send the point $\mathbb{A}^{0|0} = \mathbb{A}^{0|1}_{red} \subseteq \mathbb{A}^{0|1}$ to $0 \in \boldsymbol{V}$).

3.1.2. *Sheaf of sections.* For the equivalence of the two notions, a vector bundle $V$ gives a a sheaf

$$\mathcal{V} \stackrel{\text{def}}{=} \mathcal{O}^*_{V,lin} = \mathcal{H}\text{om}_{\mathcal{O}_M}(\mathcal{O}_{V,lin},\ \mathcal{O}_M) = \mathcal{M}ap_M(M \times \mathbb{A}^{0|1}, V) = \mathcal{T}_{V \to M}.$$

The last interpretation is as vertical vector fields, the one with $\mathbb{A}^{0|1}$ is the closest to the idea of a "sheaf of sections".

3.1.3. *The underlying vector bundle.* In the opposite direction, $\mathcal{V}$ defines a functor

$$S \mapsto \{(f,v),\ f: S \to M \text{ is an } S\text{-point of } M \text{ and } v \in \Gamma(S,[f^*\mathcal{V}]_0) \text{ is an even section of } \mathcal{V} \text{ over } f\ \},$$

which is represented by $V \stackrel{\text{def}}{=} \text{Spec}[S^\bullet_{\mathcal{O}_M}(\mathcal{V}^*)]$.

3.1.4. *The underlying topological space* $|V|$. The restriction of $\mathcal{V}$ to $M_{red}$, $\mathcal{V}|M_{red} = \mathcal{O}_{M_{red}} \underset{\mathcal{O}_M}{\otimes} \mathcal{V} = \mathcal{V}/J_M{\cdot}\mathcal{V}$, is a super vector bundle over the even manifold $M_{red}$. Say, if $\mathcal{V} = \mathcal{O}_M^{p|q} = \oplus_i \mathcal{O}_M e_i \oplus \oplus_j \mathcal{O}_M \theta_j$, then $\mathcal{V}|M_{red} = \mathcal{O}_{M_{red}}^{p|q} = \oplus_i \mathcal{O}_{M_{red}} e_i \oplus \oplus_j \mathcal{O}_{M_{red}} \theta_j$.

The underlying topological space $|V|$ is the vector bundle $|V| \to |M|$, corresponding to the sheaf $(\mathcal{V}|M_{red})_0$. For instance if $V = M \times \mathbb{A}^{p|q}$, then $|V| = |M| \times \mathbb{A}^p$ , while $\mathcal{V} = \mathcal{O}_M^{p|q}$ gives $(\mathcal{V}|M_{red})_0 = \oplus_i \mathcal{O}_{M_{red}} e_i = \mathcal{O}_{M_{red}}^p$.

3.1.5. *Change of parity of a vector bundle.* It is defined on locally free sheaves, hence also on vector bundles. From the point of view of locally free sheaves the change seems simple and formal, but on vector bundles it is drastic. Locally, $V \cong M \times \mathbb{A}^{p|q}$ and $\Pi(V) \cong M \times \Pi(\mathbb{A}^{p|q}) \cong M \times \mathbb{A}^{q|p}$. Observe, that the parity change on vector bundles changes the underlying topological space: while locally $|V| \cong |M| \times \mathbb{A}^p$ corresponds to the sheaf $(\mathcal{V}|M_{red})_0 \cong \oplus_i \mathcal{O}_{M_{red}} e_i = \mathcal{O}_{M_{red}}^p$, $|\Pi(V)| \cong |M| \times \mathbb{A}^q$ corresponds to $(\mathcal{V}|M_{red})_1 \cong \oplus_i \mathcal{O}_{M_{red}} \pi\theta_i \cong \mathcal{O}_{M_{red}}^q$.

However, this operation is still elementary. For instance, suppose that $V$ is an ordinary vector bundle over an ordinary manifold $M$. One can describe the super manifold $\Pi(V)$ as a pair $(M, \mathcal{O}_{\Pi(V)})$, i.e., the underlying manifold is the base $M$ of the vector bundle $V$ and the algebra of functions is $\mathcal{O}_{\Pi(V)} \overset{\text{def}}{=} \overset{*}{\wedge} \mathcal{V}^*$.

3.2. **(Co)tangent bundles.** A vector field means a derivative of the algebra of functions, so the vector fields on $\mathbb{A}^{n|m}$ are all $\xi = \xi^\mu \frac{\partial}{\partial x^\mu} + \xi^k \frac{\partial}{\partial \psi^k}$, where $\frac{\partial}{\partial \psi^k}$ has the usual properties that it kills $x^\mu$'s and $\frac{\partial}{\partial \psi^k} p^j = \delta_{jk}$, but

$$\frac{\partial}{\partial \psi^k}\ (fg) = (\frac{\partial}{\partial \psi^k} f)\ g + (-1)^{p_f}\ f(\frac{\partial}{\partial \psi^k} g).$$

A possible confusion regarding the $\mathbb{Z}_2$-grading: one could say that $\mathcal{T}_M^{(0)} = \oplus\ \mathcal{O}_M \partial_{x^\mu}$ are "even" vector fields and $\mathcal{T}_M^{(1)} = \oplus\ \mathcal{O}_M \partial_{\psi^i}$ are "odd", while the correct parity is $(\mathcal{T}_M)_0 = \oplus\ \mathcal{O}_{M,0} \partial_{x^\mu} \oplus \oplus\ \oplus\ \mathcal{O}_{M,1} \partial_{\psi^i}$.

The differential $df = \frac{\partial f}{\partial x^\mu} dx^\mu + \frac{\partial f}{\partial \psi^k} d\psi^k$ is of parity 0, so it satisfies $d(fg) = df{\cdot}g + f{\cdot}dg$.

3.2.1. *Differential forms.* Define $\Omega_M^1$ as the dual of $\mathcal{T}_M$ and $\Omega_M^\bullet \overset{\text{def}}{=} \overset{\bullet}{\wedge}\ \Omega_M^1$. Then $\Omega_M^\bullet$ is a graded object in the category of (sheaves of) super vector spaces, the associated super vector space $\boldsymbol{s}\Omega_M^\bullet$ combines the parity of the grading and the parity of $\Omega_M^1$. Say, if $M$ were even then $\Omega_M^1$ would be even, but $(\boldsymbol{s}\Omega_M^\bullet)^1$ would be odd.

Observe that if $M$ is not even, there are no highest degree forms: $\Omega_{\mathbb{A}^{p|q}}^n = \oplus_{r+s=n}\ \Omega_{\mathbb{A}^p}^r \otimes \overset{s}{\wedge}(\oplus\ \Bbbk\psi^i)$.

*Lemma.* (a) Differential $d: \mathcal{O}_M \to \Omega^1_M$ extends to the De Rham differential on $\Omega^\bullet_M$.

(b) (Poincare lemma) $(\Omega^\bullet_M, d)$ is a resolution of $\mathbb{R}_{|M|}$.

(c) $H^\bullet(\Omega^\bullet_M, d) = H^\bullet(|M|, \mathbb{R})$.

*Proof.* (b) reduces to the local setting $M = \mathbb{R}^{p|q} \times \mathbb{R}^p \times \mathbb{R}^{0|q}$ and then to factors $\mathbb{R}^p$ (standard Poincare lemma) and $\mathbb{R}^{0|q}$, or even $\mathbb{R}^{0|1}$ (Koszul complex).

3.3. **Parity change of the tangent bundle.** The most natural appearance of super commutative algebras are the algebras of differential forms $\Omega^*_M$ on an ordinary manifold $M$. This is the algebra of functions on a super manifold which has two natural interpretations as either (1) the moduli of maps of the super point $\mathbb{A}^{0|1}$ into $M$, or (2) the super manifold obtained from the tangent vector bundle $TM$ by parity change.

3.3.1. *Lemma.* (a) $\mathrm{Spec}(\boldsymbol{s}\Omega^\bullet_M) = \Pi(TM)$.

(b) $\Pi(TM)$ represents the functor $\underline{\mathrm{Hom}}(\mathbb{A}^{0|1}, M)$ defined by

$$\underline{\mathrm{Hom}}(\mathbb{A}^{0|1}, M)\ (S) \stackrel{\text{def}}{=} Maps_S(\mathbb{A}^{0|1} \times S,\ M \times S) = Map(\mathbb{A}^{0|1} \times S,\ M),$$

i.e., $S$-points are "$S$-families of odd tangent vectors on $M$".

*Proof.* In (a)

$$\mathcal{O}_{\Pi\ TM/M} = S^\bullet_{\mathcal{O}_M}([\Pi \otimes \mathcal{T}_M]^*) = S^\bullet_{\mathcal{O}_M}(\Pi \otimes \mathcal{T}^*M) = \oplus_k\ \Pi^{\otimes k} \otimes \overset{k}{\underset{\mathcal{O}_M}{\wedge}} (\ \Omega^1_M) = \boldsymbol{s}(\Omega^\bullet_M).$$

In (b), let $S$ be the spectrum of a commutative super algebra $A$. Then

$$\underline{\mathrm{Hom}}(\mathbb{A}^{0|1}, M)\ (S) = Map(\mathbb{A}^{0|1} \times S,\ M) = \mathrm{Hom}_{\Bbbk-alg}[\mathcal{O}(M),\ (\Bbbk \oplus \psi\Bbbk) \underset{\Bbbk}{\otimes} A] = \mathrm{Hom}_{\Bbbk-alg}(\mathcal{O}(M),\ A \oplus \psi A)$$

An element is a map $\phi = \alpha + \psi\beta : \mathcal{O}(M) \to A \oplus \psi A$ with

$$\alpha(fg) + \psi\beta(fg) = (\alpha(f) + \psi\beta(f)) \cdot (\alpha(g) + \psi\beta(g)) = \alpha(f)\alpha(g) + \psi\beta(f)\alpha(g) + \alpha(f)\psi\beta(g)) =$$

$$\alpha(f)\alpha(g) + \psi[\beta(f)\alpha(g) + (-1)^{p_f}\alpha(f)\beta(g))].$$

So $\alpha : \mathcal{O}(M) \to A$ is a morphism of algebras and $\beta : \mathcal{O}(M) \to A$ is an odd $\alpha$-derivative. So $\phi$ consists of a map $\alpha : S \to M$ and $\beta \in \Gamma[S,\ (\alpha^* TM)]_1$.

On the other hand, an element $\phi$ of $\mathrm{Hom}(S, \Pi \otimes TM) = \mathrm{Hom}_{k-alg}[\mathcal{O}(\Pi \otimes TM),\ A] = \mathrm{Hom}_{k-alg}[\boldsymbol{s}(S^\bullet_{\mathcal{O}(M)}\Omega^1_M),\ A]$, consists of a map of algebras $\alpha : \mathcal{O}(M) \to A$ (the restriction of $\phi$ to $\mathcal{O}(M)$), and a map of $\mathcal{O}(M)$-modules $\beta : [\boldsymbol{s}(S^\bullet_{\mathcal{O}(M)}\Omega^1_M)]^1 \to A$, i.e., $\beta : \Pi \otimes \Omega^1_M \to A$. Now, a map of $\mathcal{O}(M)$-modules $\Omega^1_M \to \mathcal{O}(M)$ is the same as a section of $(TM)_0$ (an even vector field on $M$), a map of $\mathcal{O}(M)$-modules $\Omega^1_M \to \mathcal{O}(S)$ is a section of $(\alpha^* TM)_0$, and so $\beta : \Pi \otimes \Omega^1_M \to A$ is a section of $(\alpha^* TM)_1$.

*Remarks.* (1) The underlying topological space $|\Pi(TM)|$ is just $|M|$. If $M$ and $S$ were even then $\underline{\mathrm{Hom}}(\mathbb{A}^{0|1}, M)(S)$ are just the maps from $S$ to $M$.

(2) The statement (b) is just the odd version of the standard description of $TM$ as the moduli of all maps from a "double point" (or a "point with a tangent vector") to $M$, i.e., description of $TM$ as the space that represents the functor $\underline{\mathrm{Hom}}(\mathrm{Spec}(D), M)$ for the algebra of dual numbers $D = \Bbbk[\varepsilon]/\varepsilon^2$.

## 4. **Integration on super affine spaces**

Integrals of functions on super manifolds will be (ordinary) numbers. Integrals on affine spaces will be defined "by hand". In general the objects one can integrate are called *densities*, the correct replacements of top differential forms.

Our basic example of an integral is the fermionic Gaussian integral. We motivate Gaussian integrals as the simplest case of path integrals. Gaussian integral on a super point $\mathbb{R}^{0|q}$ turns out to be the Pffafian of the quadratic form on an odd vector space (if $q$ is even, otherwise it is zero).

4.0.2. *SUSY (supersymmetry).* Supersymmetry of a function $f$ on a super manifold is a vector field $\delta$ that kills it (i.e., $f$ is constant on the flow lines). The interesting case is when $\delta$ is odd, i.e., so it mixes even and odd stuff. If such $\delta$ can be interpreted as one of the coordinate vector fields then the integral of $f$ is zero. If this can be done generically – say everywhere except on some submanifolds $C_i$ of $|M|$ – then the integral will be given by contributions from submanifolds $C_i$.

*Example.* Our example will be the integral $\int_{\mathbb{R}} Dx\ P'(x)e^{-\frac{1}{2}P(x)^2}$ which has a super interpretation

$$\int_{\mathbb{R}^{1|2}} Dx\ d\psi^2\ d\psi^1\quad e^{-\frac{1}{2}P(x)^2+P'(x)\psi^1\psi^2}.$$

It has a super-symmetry $\delta$ which is a part of a coordinate system as long as one stays away from zeros of $P$. So the integral is a sum of contributions from zeros $b$ of $P(x)$. The contributions are $sign(P'(b))$, and then the integral is the degree of $P$ as a map from $S^1$ to itself. Actually this integral can be easily calculated by a substitution – what we got from the super picture is a localization of the integral around few critical points.

### 4.1. **Integration on affine spaces.**

4.1.1. *Integration on super points.* Integration of functions on a super point $\mathbb{A}^{0|m}$ is defined by using successively the formula

$$\int d\psi\ a + b\psi \quad \overset{\mathrm{def}}{=} \quad b.$$

So, in general the integral just takes the highest degree coefficient (with a correct sign)

$$\int d\psi^n \cdots d\psi^1 \sum_{I=\{i_1<\cdots<i_k\}} c_I \cdot \psi^I \stackrel{\text{def}}{=} c_{12\cdots m}.$$

We also denote it

$$\int d\psi^n \cdots d\psi^1\ f \stackrel{\text{def}}{=} [f : \psi^1 \cdots \psi^m].$$

4.1.2. *Covariance.* (1) Formula $\int d\psi\ a_0 + a_1\psi\ = a_1$ is somewhat reminiscent of integrals of holomorphic functions over a circle. There, $\int_{S^1} \sum_n\ b_n z^n\ dz =\ b_{-1}$ is the coefficient of $z^{-1}$ in the Laurent series expansion.

(2) However, only one of these formulas can work: a change $z = cu$ does not affect $\int_{S^1}\ z^{-1}\ dz$, but $\psi = c\phi$ seems to give nonsense: $\int\ d\psi\ \psi =\ c^2 \cdot \int\ d\phi\ \phi$. The reason is that $d\psi$ is really contravariant

$$d(c\psi)\ =\ c^{-1}\ d\psi.$$

We will deal with this in the next section when we tackle change of variable.[11]

4.1.3. *Integration on super affine spaces.* Integrals on $\mathbb{A}^{n|m}$ are evaluated so that one first integrates over the fermionic variables and then we are left with an ordinary integral. For example if $S[x, \psi^1, \psi^2] =\ U(x) + V(x)\psi^1\psi^2$ then

$$\int_{\mathbb{A}^{1|2}} dx\ d\psi^2\ d\psi^1\ e^{-S[x,\psi^1,\psi^2]} = \int_{\mathbb{A}^{1|2}} dx\ d\psi^2\ d\psi^1\ e^{-U(x)} \sum_k \frac{(-1)^k}{k!} V(x)^k (\psi^1\psi^2)^k$$

$$= -\int_{\mathbb{A}^{1|0}} dx\ e^{-U(x)} \int_{\mathbb{A}^{0|2}} d\psi^2\ d\psi^1\ V(x)\psi^1\psi^2 =\ -\int_{\mathbb{A}^{1|0}} dx\ V(x) e^{-U(x)}.$$

4.2. **Gaussian integrals.**

4.2.1. *Even Gaussian integrals.* On an ordinary real vector space $M$ we consider a quadratic form $S[x]$ and a choice of coordinates $x^i$.

We normalize the Lebesgue measure on $M$ with respect to coordinates $x^i$

$$Dx \stackrel{\text{def}}{=} \prod \frac{dx^i}{\sqrt{2\pi}},$$

and we consider the Gaussian integral

$$\int_M Dx\ e^{-\frac{1}{2}S(x)}.$$

[11] We will denote $d\psi =\ (\boldsymbol{d}\psi)^{-1}$, where quantity with $\boldsymbol{d}\psi$ is covariant.

*Lemma.* When we write $S$ in terms of the coordinates $S(x) = x^i A_{ij} x^j$, the integral is

$$\int_M Dx\ e^{-\frac{1}{2}(x^i A_{ij} x^j)} = (\det A)^{-\frac{1}{2}}.$$

4.2.2. *$\sigma$-model interpretation.* Consider a $\Sigma$-model, i.e., the moduli of maps from $\Sigma$ to $M$. In the simplest case when $\Sigma$ is just a point (hence a 0-dimensional manifold), the moduli of maps from $\Sigma$ to $M$ is just $M$. A positive definite quadratic form $S(x)$ on $M$ gives a path integral which is the above Gaussian integral.

4.2.3. *Fermionic Gaussian integrals give Pffafian.* Now let $M$ be a super point $(0, m)$. Quadratic functions on $M$ are function of the form $S[x] = \psi^i B_{ij} \psi^j$ for an anti-symmetric $B$ (so $\frac{1}{2} S[x] = \sum_{i<j} \psi^i B_{ij} \psi^j$). A fermionic Gaussian integral is

$$\int_M d\psi^m \cdots d\psi^1\ e^{\frac{1}{2}\ S[x]} = \int_M d\psi^m \cdots d\psi^1\ e^{\sum_{i<j}\ \psi^i B_{ij} \psi^j}.$$

4.2.4. *Odd Feynman amplitudes.* Let $\mathcal{P}(m)$ be the set of all *pairings* of the set $\{1, ..., m\}$, i.e., all partitions $\gamma$ of $A$ into 2-element subsets. To a pairing $\gamma$ one assigns the sign $\sigma_\gamma$ as the sign of any permutation $i_1, j_1, ..., i_q, j_q$ that one obtains by choosing an ordering $\{i_1 < j_1\}, ..., \{i_q < j_q\}$ on $\gamma$.

The $\gamma$-amplitude of a quadratic form $B(x) = \psi^i B_{ij} \psi^j$ is

$$F_\gamma(B) \stackrel{\text{def}}{=} \sigma_\gamma \cdot \prod_{\{i<j\} \in \gamma} B_{ij}.$$

Notice that the difference from the even case is that there is a sign $\sigma_\gamma$ attached to a Feynman graph $\gamma$.

4.2.5. *Fermionic Gaussian integrals give Pffafian.*

*Lemma.* The odd Gaussian integral on $\mathbb{R}^{0|m}$ is a sum over all pairings

$$\int_M d\psi^m \cdots d\psi^1\ e^{\frac{1}{2} \sum\ \psi^i B_{ij} \psi^j} = \sum_{\gamma \in \mathcal{P}(\{1,...,m\}} \sigma_\gamma \prod_{\{i<j\} \in \gamma} B_{ij} = \sum_{\gamma \in \mathcal{P}(\{m\}} F_\gamma(B).$$

*Proof.* The exponential power series is a finite sum

$$\int_M d\psi^m \cdots d\psi^1\ \sum_k \frac{1}{k!} (\sum_{i<j} B_{ij} \psi^i \psi^j)^k.$$

Since we get only the even degree terms, this is zero if $m$ is odd. If $m = 2q$ is even, this is

$$\int_M d\psi^m \cdots d\psi^1\ \frac{1}{q!}\ (\sum_{i<j} B_{ij} \psi^i \psi^j)^q,$$

and we get a contribution $\frac{1}{q!}\ \sigma_{i_1j_1\dots i_qj_q}B_{i_1j_1}\cdots B_{i_qj_q}$, whenever $i_1,j_1,...,i_q,j_q$ is a permutation of $1,...,m$, such that $i_k<j_k$. Therefore, the fermionic Gaussian integral is a sum over all such permutations

$$\int_M d\psi^m\cdots d\psi^1\ e^{-\frac{1}{2}\ \psi^i B_{ij}\psi^j} = \frac{1}{q!} \sum_{(i_1j_1\dots i_nj_n)\in\ S_m,\ i_k<j_k} \sigma_{i_1j_1\dots i_nj_n}\cdot B_{i_1j_1}\cdots B_{i_nj_n}.$$

However, $\sigma_{i_1j_1\dots i_nj_n}$ and $B_{i_1j_1}\cdots B_{i_nj_n}$ only depend on the associated pairing $\gamma=\{\{i_1<j_1\},...,\{i_q<j_q\}\}$. Moreover, all permutations $(i_1,j_1,...,i_n,j_n)$ over one pairing $\gamma$ form a $S_q$-torsor. so the RHS simplifies to the claim of the lemma.

*Remarks.* (0) The sum $\sum_{\gamma\in\mathcal{P}(\{1,...,m\}}\ \sigma_\gamma\ \prod_{\{i<j\}\in\gamma}\ B_{ij}$ is called the Pffafian $Pf(B)$ of the anti-symmetric matrix $B$.[(12)]

(1) The integral is a Feynman sum. So, Pffafian may be the first appearance of Feynman sums.[(13)]

(2) The Pffafian of an antisymmetric matrix of even size is a square root of its determinant:

$$Pf(B)^2 = \det(B).$$

This square root is normalized by $Pf=1$ on $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (and on block diagonal matrices with such blocks on the diagonal).

(3) Again we find that the odd part gives a contribution in the opposite direction since $(\det(A)^{-\frac{1}{2}}$ is replaced by $(\det(B)^{\frac{1}{2}}=Pf(B)$.

4.2.6. *Question.* Use integrals to prove (i) $Pf^2=\det$ and (ii) $Pf(ABA^{-1})=Pf(B)$ for orthogonal $A$.

A linear operator $A$ on an ordinary vector space $V$ gives a symmetric bilinear form $\mathcal{A}$ on the odd vector space $T^*(\Pi V)=\Pi[V\oplus V^*]$ by

$$\mathcal{A}(u\oplus\lambda, v\oplus\mu)\ \stackrel{\text{def}}{=}\ .$$

Notice that the space $T^*(\Pi V)=\Pi[V\oplus V^*]$ comes with a canonical volume element (5.1.1).

*Corollary.* The determinant of a linear operator $A$ on an even vector space $V$ can be calculated as the Gaussian integral for the form $\mathcal{A}$ (using the canonical volume element).

## 4.3. **Wick's theorem.**

[12]Here $Pf(B)$ is attached to a symmetric form $B$ on an odd vector space $M$ and a system of coordinates $\psi^1,...,\psi^m$ on $M$, i.e., precisely to the associated matrix $(B_{ij})$. However, one needs less – a volume form $dv=\ d\psi^1\cdots d\psi^m$ on $M$ rather then a system of coordinates.

[13]This raises a question of whether the even Gaussian $(\det(A)^{-\frac{1}{2}}$ is a Feynman sum in some way. This could be interesting for infinite dimensional spaces.

# 5. Integration on supermanifolds

## 5.1. Integration on superdomains.

5.1.1. *Volume elements $dv$ of a super vector space.* Recall that the Berezian $Ber(V)$ of a (finite dimensional) super vector space $V = V_0 \oplus V_1$ is the line[14]

$$Ber(V) \stackrel{\text{def}}{=} \Lambda^{\text{top}} V_0 \otimes S^{\text{top}}(V_1)^*.$$

By a *volume element* $dv$ of $V$, we mean any basis of $Ber(V)^*$.

To an oredered system of coordinates $x^i, \psi^j$ on $V$ one attaches volume elements on $V_0$

$$dv_0 = \ dx^1 \cdots dx^p \stackrel{\text{def}}{=} x^1 \wedge \cdots \wedge x^p \in \ \Lambda^{\text{top}} \ V_0^* = Ber(V_0^*)$$

and on $V_1$

$$dv_1 = \ d\psi^1 \cdots d\psi^q \stackrel{\text{def}}{=} (\psi^1)^* \cdots (\psi^q)^* \in \ S^{\text{top}} \ V_1 \ = Ber(V_1),$$

hence also a volume element $dv = dv_0 \otimes dv_1$ on $V$.

Notice that this notation is counterintuitive in the odd direction, so we also denote $\boldsymbol{d}\psi^j \stackrel{\text{def}}{=} \ \psi^j = \ (d\psi^j)^{-1}$ for odd coordinates $\psi$, and[15]

$$\boldsymbol{d}v_1 \stackrel{\text{def}}{=} \boldsymbol{d}\psi^1 \cdots \boldsymbol{d}\psi^q \stackrel{\text{def}}{=} \psi^1 \cdots \psi^q = \ (dv_1)^{-1},$$

so that $dv = dv_0 / \boldsymbol{d}v_1 = \frac{dx^1 \cdots dx^p}{\boldsymbol{d}\psi^1 \cdots \boldsymbol{d}\psi^q}$.

5.1.2. *Super domains.* To a super vector space $V$ one attaches a supermanifold $\mathbb{A} = \mathbb{A}_V$ with

$$\mathcal{O}(\mathbb{A}_V) \stackrel{\text{def}}{=} S^*(V^*) \cong S(V_0^*) \otimes \wedge^*(\mathcal{F}V_1^*).$$

A *super domain* in $\mathbb{A}_V$ is the restriction of the manifold structure to an open $U \subseteq V$, we denote it $U \times V_1 = \ \underline{U}$; so

$$\mathcal{O}(U \times V_1 \stackrel{\text{def}}{=} C^\infty(U) \otimes \wedge^*(\mathcal{F}V_1^*).$$

5.1.3. *Integrals of compactly supported functions on super domains.* One defines the integrals of a compactly supported function $f$ on a super domain $U \times V_1 \subseteq V$, and with respect to a volume element $dv$ as

$$\int_{U_{V_1}} f(v) \ dv \stackrel{\text{def}}{=} \int_U \langle f(v), {dv_1}^{-1} \rangle \ dv_0, \qquad f \in C_c^\infty(U_{V_1}) \stackrel{\text{def}}{=} C_c^\infty(U) \otimes S^*(V_1),$$

where

- $dv_i$'s are volume elements of $V_i$ such that $dv = dv_0 \cdot dv_1$, i.e., $dv = dv_0 / \boldsymbol{d}v_1$.

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[14] The order does not matter since the first factor is even.

[15] With this notation integrals over odd vector spaces are $\int \ d\psi(a + b\psi) = \ \int \ \frac{a+b\psi}{\boldsymbol{d}\psi} = \ b.$

- If $dv_1 = d\psi^m \cdots d\psi^1$ (i.e., $\boldsymbol{d}v_1 = \boldsymbol{d}\psi^m \cdots \boldsymbol{d}\psi^1 = \psi^m \cdots \psi^1$), then
$$\langle f(v), dv_1\rangle = [f(v) : \boldsymbol{d}v_1]$$
is the coefficient of $\boldsymbol{d}v_1 = \boldsymbol{d}\psi^m \cdots \boldsymbol{d}\psi^1 = \psi^m \cdots \psi^1$ in $f$.

5.1.4. *$dv$-unimodular coordinates systems on $V$.* A coordinate system consisting of $x^i \in V_0^*$ and $\psi_j \in V_1^*$, is said to be $dv$ unimodular if $dv_0 = dx^1 \cdots dx^p$ and $dv_1 = d\psi^1 \cdots d\psi_q$ satisfy $dv = dv_0 \cdot dv_1$. Then

$$\langle f(v), dv_1\rangle = [f(v) : \psi^1 \cdots \psi^q] \text{ is the coefficient of } \psi^1 \cdots \psi^q \text{ in } f(v), \text{ and}$$
$$\int_{U_{V_1}} f(v)\ dv = \int_U [f(v) : \psi^1 \cdots \psi^q]\ dx^1 \cdots dx^p.$$

5.2. **Change of variable formula on superdomains.**

5.2.1. *The determinant of the differential.* Let $F : M \to N$ be a map of super manifolds which is locally an isomorphism. Its differential is an isomorphism of locally free $\mathcal{O}_M$-modules $dF : \mathcal{T}_M \to F^*\mathcal{T}_N$, so it has a Beresian

$$Berd(dF) : Ber(\mathcal{T}_M) \xrightarrow{\cong} Ber(F^*\mathcal{T}_N) = F^*(Ber(\mathcal{T}_N)),$$

i.e.,

$$Berd(dF) \in \text{Hom}[Ber(\mathcal{T}_M), F^*(Ber(\mathcal{T}_N))] = \Gamma(M, \mathcal{H}\text{om}[Ber(\mathcal{T}_M), F^*(Ber(\mathcal{T}_N))]) = \Gamma[M,\ F^*(Ber(\mathcal{T}_N$$

If one has volume elements $dv_M$ and $dv_N$ then the Berezian $Berd(dF)$ trivializes to an invertible function $[Berd(dF) : F^*(dv_N) \otimes dv_M{}^{-1}]$ on $M$.

Now let $F : U' \times V_1' \to U'' \times V_1''$ is an isomorphism of superdomains in $\mathbb{A}_{V'}$ and $\mathbb{A}_{V''}$. A choice of coordinate systems $x_i, \xi_j$ and $y_p, \eta_q$ in $V'$ and $V''$, gives the Jacobian matrix[16] $\mathcal{J}(F)^{x_i,\xi_j}_{y_p,\eta_q}$ of $dF$ with respect to cooridnates, this is a block matrix of derivatives

$$dF_{00} = (\frac{\partial(y_p \circ F)}{\partial x_i}),\quad dF_{01} = (\frac{\partial(y_p \circ F)}{\partial \xi_j}),\quad dF_{10} = (\frac{\partial(\eta_q \circ F)}{\partial x_i}),\quad dF_{11} = (\frac{\partial(\eta_q \circ F)}{\partial \xi_j}).$$

If the coordinate systems are unimodular for volume elements $dv'$ and $dv''$ then the even function $Ber(dF) : F^*(dv'') \otimes dv'$ is denoted $Ber(dF)^{dv''}_{dv'}$.

5.2.2. *Theorem.* [Berezin] Given volume elements $dv', dv''$ on $V'$ and $V''$

$$\int_{U'' \times V''} g(v'')\ dv'' = \int_{U' \times V'} g(F(v')) \cdot |Ber(dF)^{dv''}_{dv'}|\ dv'.$$

Here, terms containing odd factors are neglected in the definition of the absolute value $| - |$.

[16]Sometimes called Berezian matrix.

## 5.3. **Integration of densities on super manifolds.**

5.3.1. *The sheaf $Den_X$ of densities.* Let $U$ be open in $\mathbb{R}^p$ so that $\mathcal{U} = U\times\mathbb{R}^{0|q}$ is a superdomain. First define compactly supported functions on $\mathcal{U}$

$$C_c^\infty(\mathcal{U}) \overset{\text{def}}{=} C^\infty(U)\otimes\mathcal{O}(\mathbb{R}^{0|q}.$$

Then define the space of densities $\mathcal{D}en(\mathcal{U})$ to consist of all linear functionals $\phi$ on $C_c^\infty(\mathcal{U})$ such that there exist functions $\phi_I \in C^\infty(U)$ with

$$\sum_I f_I\psi^I \overset{\phi}{\mapsto} \sum_I \int_U dt^1\cdots dt^p\ \phi_I\cdot f_I.$$

Now, compactly supported functions form a cosheaf $C^\infty_{\mathbb{R}^{p|q},c}$ on $\mathbb{R}^{p|q}$, and then densities form a sheaf $\mathcal{D}en_{\mathbb{R}^{p|q}}$ on $\mathbb{R}^{p|q}$.

Now one can define a sheaf of densities $\mathcal{D}en_M$ on any super manifold $M$.

5.3.2. *Lemma.* (a) $\mathcal{O}_M$-module $\mathcal{D}en_M$ is locally free of rank one and of parity the same as $q$.

(b) The above notion of the integral $\int_{\mathbb{R}^{p|q}}$ on a super affine space $\mathbb{R}^{p|q}$, gives an $\mathcal{O}_{\mathbb{R}^{p|q}}$-basis of the $\mathcal{O}_{\mathbb{R}^{p|q}}$-module of densities.

*Proof.* (a) follows from (b).

(1) A density on a manifold $M$ is given by
   (a) an atlas $M\supseteq M_i \underset{\cong}{\overset{\phi}{\to}} \underline{U_i}$ where $\underline{U_i}$ is a super domain in some $\mathbb{A}^{p|q} = \mathbb{R}^{p|q}$,
   (b) a systems $s = (s_i)_{i\in I}$ of functions $s_i \in C_M^\infty(M_i)$ on open pieces $M_i\subseteq M$, such that on intersections

$$s_j(m) = \ s_i(m)\cdot|Ber(d(\phi_i\circ\phi_j{}^{-1})\ (\phi_j(m))|, \quad m \in M_{ij}.$$

(2) Two such presentations $(M_i, U_i, \phi_i, s_i)_{i\in I}$ and $(\tilde{M}_j, \tilde{U}_j, \tilde{\phi}_j, \tilde{s}_j)_{j\in J}$ give the same density if ...

5.3.3. *Integration of compactly supported densities on manifolds.* For a compactly supported density $\omega$ on $M$, one picks a partition of unity $(M_i, f_i)i \in I$ and charts $\phi_i$ : $M_i \overset{\cong}{\to} \underline{U_i}\subseteq\mathbb{A}^{p|q}$ to calculate

$$\int_M \omega \overset{\text{def}}{=} \sum \int_{M_i} f_i\omega$$

with $\int_{M_i} f_i\omega$ calculated in $U_i$ via identification $\phi_i$.

5.4. **Summary.** Integration on a smooth manifold requires a top degree differntial form $\omega \in \ \Omega^{\text{top}}{}_c(M)$ and an orientation $or_M$.[17]

On a super manifold $M$ one needs an orientation $or_{|M|} = or_{M_{red}}$ of the unerlying smooth manifold, and a some $\omega$ which lies in the correct analogue of the differential forms of top degree. One again starts with the vector bundle (locally free sheaf) $\Omega^1_M$ of 1-forms on a super maniffold. The top exterior power of a vector bundle is however not the correct point of view (it does not even exist for super vector bundles), the correct construction is the Berzinian line bundle $Ber(\Omega^1_M)$. So one needs $\omega \in \underline{\Gamma}_c(M, Ber(\Omega^1_M)$ and $or_M \in \Gamma(|M|, Or_{|M|})$.

(1) *Differential forms on a supermanifold $M$* are defined by

$$\Omega^*_M \stackrel{\text{def}}{=} \wedge^*_{\mathcal{O}_M}(\Omega^1_M) \ \text{ for } \ \Omega^1_M \stackrel{\text{def}}{=} \mathcal{H}\text{om}_{\mathcal{O}_M}(\Omega^{-1}_M, \mathcal{O}_M).$$

This is
  (a) good for de Rham cohomology: $H^*(\Omega^*_M) = H^*(|M|, \mathbb{R})$,
  (b) but not for integration (for instance top form exists only for even $M$).
  (c) It is not super commutative. For example, $\mathbb{A}^{p|q} = \mathbb{A}^{p|0} \times |a^{0|q}$ gives a factorization $\Omega^*(\mathbb{A}^{p|q}) = \ \Omega^*(\mathbb{A}^{p|0}) \otimes \Omega^*(\mathbb{A}^{0|q})$. Here, $\Omega^*(\mathbb{A}^{p|0}$ is a super commutative algebra isomorphic to $\mathcal{O}_{\mathbb{A}^{p|p}}$. However, in $\Omega^*(\mathbb{A}^{0|q} \xrightarrow{\cong} \mathcal{O}(\mathbb{A}^{0|q}) \ \otimes \wedge^*(\oplus d\psi^j)$ the second factor is *not super commutative* though as an ordinary algebra $\mathcal{F}[\wedge^*(\oplus d\psi^j)] = \ S^*(\mathcal{F}[\oplus d\psi^j])$ is commutative.

(2) $dx^i$'s and $d\psi^j$'s **do not interact** since they live in separate factors of $Ber(V^* = \wedge^{\text{top}} \ V_0^* \otimes S^{\text{top}} \ V_1^*)$. So they commute with each other, but amongst themselves both groups *anticommute* in the sense that the algebras $\mathcal{F}(\wedge^{\text{top}} \ V_0^*)$ and $\mathcal{F}(S^{\text{top}} \ V_1^*)$ are both exterior powers of ordinary vector spaces.

(3) Let us put together the standard and the natural notation:[18]

$$\int f \ dx^1 \cdots dx^n \ d\psi^1 \cdots \psi^q \ = \ \int [f : d\psi^1 \cdots d\psi^q] \ dx^1 \cdots dx^p \ = \ \int f \ \frac{dx^1 \cdots dx^p}{\boldsymbol{d}\psi^1 \cdots \boldsymbol{d}\psi^q}.$$

(4) Remember that $d$ is odd:
  (a) $d(fg) = df \cdot g + (-1)^{\overline{f}} f \cdot dg$
  (b) $df \cdot g = (-1)^{(\overline{f}+1) \cdot \overline{g}} \ g \cdot df$.

The first line in (4) is a definition and the second follows, since one passes from $d(fg) = df \cdot g + (-1)^{\overline{f}} f \cdot dg$ to $d(gf) = dg \cdot f + (-1)^{\overline{g}} g \cdot df$ by multiplying with $(-1)^{\overline{f}\overline{g}}$.

[17] For instance when we integrate $\omega = dx$ on $[a, b] \subseteq \mathbb{R}$ to $b - a$ we are clearly using a preferred orientation of $\mathbb{R}$.

[18] It does not really matter whether we write $dx^i$'s and $d\psi^j$'s on the same line or on different lines since they do not interact.

## 6. **Super-symmetry of integrals**

**This material is provisional, key calculations are still incorrect.**

6.1. **Super-symmetry of the action.** An ordinary (i.e., continuous) symmetry of a function $S$ on $M$ is a flow $\Phi$ on $M$ (i.e., an action of $(\mathbb{R},+)$ on $M$), which preserves $S$: $S(\Phi_r(x)) = S(x)$. An infinitesimal symmetry is a vector field $\xi$ on $M$ which preserves $S$ in the sense that $\xi S = \langle dS, \xi\rangle$ vanishes. The same applies to super-spaces. In this setting the action will be an even function on a super manifold, and we say that a symmetry is bosonic or fermionic if the vector field is even or odd.

A super-symmetry (SUSY) is a symmetry (i.e., a vector field on a super manifold), which mixes even and odd, i.e., $x^i, \partial_{x^i}, \psi^i, \partial_{\psi^i}$. (However, this is likely to be an odd vector field!)

6.1.1. *Nondegenerate Fermionic symmetry kills integrals.* For instance, if $S(x, \psi^1, \psi^2)$ is independent of $\psi^2$ this is a fermionic symmetry – the vector field is $\frac{\partial}{\partial \psi^2}$ and the flow is the motion in the direction of $\psi^2$.

Observe that in that case the integral $\int_{\mathbb{A}^{1|2}} dx\ d\psi^2\ d\psi^1\ e^{-S[x,\psi^1,\psi^2]}$ vanishes. This is the source of simplicity in super-integrals:

*Lemma.* (a) If $f \in \mathcal{O}(M)$ has an odd symmetry $\partial$ which appears as a coordinate vector field then

$$\int_M d\mu\ f\ =\ 0.$$

(b) If $\partial$ is an odd vector field such that for some odd function $f$ one has $\partial(f) = 1$, then locally there is a coordinate system such that $f = \psi^1$ and $\partial = \partial_{\psi^1}$.

If $\partial = \sum f_i \partial_{x^i} + sum\ g_j \partial_{\psi^j}$, the condition is that the "odd-odd" part $\sum g_j \partial_{\psi^j}$ does not vanish.

*Proof.* (a) follows from the example above.

*Remarks.* Actually,

- The correct object to be integrated is not a function but a "measure" (more precisely a *density*), so the precise requirement is that $f{\cdot}\mu$ has $\partial$-symmetry. (For instance both $f$ and $\mu$ may be fixed by $\partial$.)
- As we see in (b) the condition that $\partial$ is a coordinate vector field, is locally a non-degeneracy condition – the part of $\partial$ that differentiates odd variables should not vanish.

6.2. **Example:** $\int_{\mathbb{R}} Dx\ P'(x)e^{-\frac{1}{2}P(x)^2}$ **via super-symmetry.** For an example, we will calculate the path integral for the action $S[x,\psi^1,\psi^2] = U(x)+V(x)\psi^1\psi^2$ when $U=\frac{1}{2}P^2$ and $V = -P'$ for a polynomial $P=P(x)$ on $\mathbb{A}^1$, and then we will see that the result can be explained using a SUSY of this action.

6.2.1. *Calculation by reduction to integration on* $\mathbb{A}^{1|0} = \mathbb{R}$. We will reduce the integral for $S[x,\psi^1,\psi^2] = \frac{1}{2}P(x)^2 - P'(x)\psi^1\psi^2$ to integration on the real line, hence to standard calculus – by 4.1.3

$$\int_{\mathbb{A}^{1|2}} Dx\ d\psi^2\ d\psi^1\ e^{-S} = \int_{\mathbb{R}} Dx\ P'(x)e^{-\frac{1}{2}P^2}.$$

Now substitution $u = P(x)$ gives a Gaussian integral

$$\int_{P(-\infty)}^{P(\infty)} Dx\ e^{-\frac{1}{2}u^2} = \deg(\tilde{P}).$$

Here $\tilde{P}$ is $P$ interpreted as a map from $S^1$ to itself – this can be done since $P(\pm\infty)$ is one of $\{\pm\infty\}$. The degree of $\tilde{P}$ is defined as the number of times $\tilde{P}$ winds up the circle onto itself in positive direction. Observe that if $\deg P$ is even then $\deg(\tilde{P})$ is 0 and this is also the value of the integral since $P(-\infty) = P(\infty)$. If $\deg P$ is odd then $\deg(\tilde{P})$ is the sign $\varepsilon=\pm 1$ of the highest coefficient and this is also the value of the integral since $P(\pm\infty) = \varepsilon\cdot\pm\infty$.

6.2.2. *SUSY.* The vector field

$$\delta = (\psi^1+\psi^2)\frac{\partial}{\partial x} + P(x)\frac{\partial}{\partial\psi^1} - P(x)\frac{\partial}{\partial\psi^2}$$

is an odd supersymmetry of the action $S$ and of the measure $dx\ d\psi^2\ d\psi^1$. So, it is a SUSY of the partition function.

*Proof.* (1) $\delta$ is clearly odd. We apply $\delta$ to the function $S$

$$\delta\, S = (\psi^1{+}\psi^2)(P(x)P'(x){-}P''(x)\psi^1\psi^2) - P(x)\ P'(x)\ \psi^2 - (-P(x))\ P'(x)\frac{\partial}{\partial\psi^2}\psi^1\psi^2 = 0.$$

We check the sign in the last term: $\frac{\partial}{\partial\psi^2}\psi^1\psi^2 = -\psi^1\frac{\partial}{\partial\psi^2}\psi^2 = -\psi^1$.

(2) Similarly, this is a SUSY of the measure $dx\ d\psi^2\ d\psi^1$. For this recall that a vector field acts on 1-forms by

$$\delta(df) \stackrel{\text{def}}{=} d(\delta f).$$

The action on $Ber(\Omega^1_M)$ is then clear, so

$$\delta\ (dx\ d\psi^2\ d\psi^1) = \delta(dx)\ d\psi^2\ d\psi^1 + dx\ \delta(d\psi^2)\ d\psi^1 + dx\ d\psi^2\ \delta(d\psi^1)$$
$$= d(\psi^1+\psi^2)\ d\psi^2\ d\psi^1 + dx\ dP(x)\ d\psi^1 + dx\ d\psi^2\ d(-P(x)) = 0$$

since in the last two terms one has $dx$ twice and in the first term one of $d\psi^i$'s.

6.2.3. *Local arguments in* $\mathbb{A}^{1|2}$. I only sketch the argument since I do not understand it completely. For simplicity assume that $P$ has only the simple zeros (since we are really doing the Morse theory on a circle $S^1$).

**(1)** *The vanishing of the integral in* $\mathbb{A}^{1|2}$ *but away from the zeros of* $P(x)$. Locally in $\mathbb{A}^{1|2}$, and as long as we stay away from the zeros of $P(x)$, we can get the the integral to vanish. This comes from supersymmetry.

Let $a \in A^1$, if $P(a) \neq 0$ then the "odd-odd" component of $\delta(a)$ is non-trivial. Then one can change the coordinates near $a$ to $\tilde{x}, \tilde{\psi}^1, \tilde{\psi}^2$ so that $\delta = \frac{\partial}{\partial\tilde{\psi}^2}$ and the measure does not change: $d\tilde{x}\ d\tilde{\psi}^1\ d\tilde{\psi}^2 = dx\ d\psi^1\ d\psi^2$ (i.e., the Jacobian is 1).

**(2)** *The special case* $P = \alpha{\cdot}x$. If $P = \alpha{\cdot}x$. the integral is the Gaussian integral

$$\int_{\mathbb{R}} Dx\ P'(x)e^{-\frac{1}{2}P^2} = \int_{\mathbb{R}} Dx\ \alpha e^{-\frac{1}{2}\alpha^2x^2} = \frac{\alpha}{\sqrt{\alpha^2}} = sign(\alpha) = sign(P'(0)).$$

**(3)** The first step reduces the calculation to neighborhoods of zeros $b$ of $P(x)$. There $P(x)$ is approximated by a line $\alpha{\cdot}(x-b)$. However, since $P = \alpha{\cdot}(x-b)$ has no other zeros, by the first step the integral near $b$ is the global integral and gives $sign(\alpha)$.

So the total integral is the sum of signs of $P'(x)$ at zeros of $P(x)$, i.e., we count $+1$ when $P$ crosses the $x$-axis upwards and $-1$ when $P$ crosses the $x$-axis downwards. The sum of these is the degree of the extension $\tilde{P}$ of $P$ to $S^1$.

6.2.4. *Remarks.* (1) One should still describe how the local contributions glue together. For instance, over an interval $\mathcal{I} \subseteq \mathbb{R}$, where $P$ does not vanish, integral is usually not zero in the inverse $\{x \in \mathcal{I}\}$ of $\mathcal{I}$ in $\mathbb{A}^{1|2}$, though it vanishes in the "slanted inverse" $\{\tilde{x} \in \mathcal{I}\}$.) So the integral can be large in a "small" difference of these spaces.

Possibly, we are not calculating the integral precisely, but only up to some precision – this would suffice if we would know that the integral is an integer. (The integral here is one of $-1, 0, 1$.)

(2) Physicists would describe a vector field $\delta$ by

$$\delta x = \varepsilon(\psi^1 + \psi^2), \quad \delta\psi^1 = \varepsilon P(x), \quad \delta\psi^2 = -\varepsilon P(x)$$

where $\varepsilon$ is an "infinitesimal real number", i.e., $\varepsilon^2 = 0$. They can also phrase it as an "infinitesimal change (transformation) of coordinates"

$$x \mapsto \widetilde{x} = x + \delta x = x + \varepsilon(\psi^1 + \psi^2), \quad \psi^1 \mapsto \widetilde{\delta}\psi^1 = \psi^1 + \varepsilon P(x), \quad \psi^2 \mapsto \widetilde{\delta}\psi^2 = \psi^2 - \varepsilon P(x).$$

## 7. **Supersymmetry in differential geometry [Verbitsky]**

7.0.5. *Question.* If you have additional structure, can one map some specific objects like we did with the super point?

7.1. **SUSY on manifolds.** The super-symmetry of a manifold $M$ is the action of the real super Lie algebra $\mathfrak{g}_?$ on the differential forms $\Omega^*_M$, i.e., on the functions on the super manifold $M^s \stackrel{\text{def}}{=} \underline{Map}(\mathbb{R}^{0|1}, M)$.

The Lie algebra depends on the structure of $M$:

- de Rham Lie algebra $\mathfrak{g}_{dR}$ is the Lie algebra of $\mathcal{S} = \mathrm{Aut}(\mathbb{A}^{0|1})$. It acts for any $M$ (actually so does $\mathcal{S}$).
- Kaehler Lie algebra $\mathfrak{g}_{ka}$ acts on form on Kaehler manifolds.
- Hyperkaehler Lie algebra $\mathfrak{g}_{hk}$ acts on form on hyperkaehler manifolds.

*Question.* In the first case

(1) $\mathcal{S}$ acts on the De Rham moduli $M_{dR} = Map(\mathbb{A}^{0|1}, M)$.
(2) $Rep(\mathcal{S}/\boldsymbol{s})$ is the category of complexes.

(1) What are the groups corresponding to other super Lie algebras?
(2) What are the categories of representations?
(3) (Complexes with some extra structure!)?
(4) How do these groups act on $M_{dR}$? (They act on $\mathcal{O}(M_{dR}$ hence also on $M_{dR}$.)

7.1.1. *de Rham super Lie algebra* $\mathfrak{g}_{dR}$. This is the semi-direct product $\mathfrak{g}_{dR} \stackrel{\text{def}}{=} \Bbbk{\cdot}d \ltimes \Bbbk{\cdot}h$, where $\Bbbk{\cdot}d$ is a one-dimensional odd abelian Lie algebra, and $so(1,\Bbbk) = \Bbbk{\cdot}h$ is the Lie algebra of the multiplicative group $G_m$ which acts on $\Bbbk{\cdot}d$ by weight one. (The representations of the corresponding algebraic super group are precisely the complexes of vector spaces.)

7.1.2. *Realization in differential operators and the canonical basis.* On $\Omega^*_M = \mathcal{O}_{M^s}$ one has two vector fields: the De Rham differential $d$ (odd) and the degree operator $h$ given by the grading on the differential forms. The resulting semi-direct product is the Lie algebra $\mathfrak{g}_{dR} = \Bbbk{\cdot}d \ltimes \Bbbk{\cdot}h$.

7.2. **de Rham complex of a manifold.** The de Rham complex of a manifold $M$ has to do with the action of the super group $\widetilde{G}_m = \Bbbk^{0|1} \ltimes G_m$ on the super manifold $M^s = \mathrm{Spec}(\Omega^*_M/M)$.

Observe that the closed forms form a subalgebra $\mathcal{Z}^*_M \subseteq \Omega^*_M = \mathcal{O}_{M^s}$ – these are the functions invariant under the vector field $d$, so they should correspond to a quotient of $M^s$ by the flow generated by $d$: $\mathcal{Z}^*_M = \mathcal{O}_{\overline{M^s}}$, i.e.,

$$\overline{M^s} \overset{\text{def}}{=} \operatorname{Spec}(\mathcal{Z}^*_M).$$

Next, the exact forms $\mathcal{B}^*_M$ form an ideal in closed forms: if $\beta$ is closed then $d\alpha\wedge\beta = d(\alpha\wedge\beta)$. So they define a subscheme $\overset{\bullet}{M}$ of $\overline{M^s}$ with

$$\overset{\bullet}{M} \overset{\text{def}}{=} \operatorname{Spec}(\mathcal{H}^*_M), \quad \textit{for} \quad \mathcal{H}^*_M \overset{\text{def}}{=} \mathcal{Z}^*_M/\mathcal{B}^*_M = \mathcal{O}_{\overset{\bullet}{M}}, \quad \textit{i.e.,} \quad \mathcal{I}_{\overset{\bullet}{M}} = \mathcal{B}^*_M \subseteq \mathcal{Z}^*_M = \mathcal{O}(\overline{M^s}).$$

7.2.1. *Sheaves $\mathcal{Z}^*, \mathcal{B}^*, \mathcal{H}^*$.* These are not $\mathcal{O}_X$-modules, however their geometric nature is clear in characteristic $p > 0$. In characteristic $p > 0$. these are coherent $\mathcal{O}_{X^{(1)}}$-modules.

Hodge theory seems to be precisely the description of the space $X/\mathcal{D}_X = X/FN_X(X^2)$. The standard approach is indirect and attempts to describe the sheaves on this space (Hodge sheaves).

7.2.2. *Question.* In characteristic $p > 0$, Cartier operator is a canonical isomorphism

$$C^{-1}: \quad \mathcal{H}^*[(Fr_{X/\Bbbk})_*\Omega^*_{X/\Bbbk}] \overset{\cong}{\to} \Omega^*_{X^{(1)}/\Bbbk}.$$

How can one explain this? $\underline{Map}(\Bbbk^{0|1}, X^{(1)}) = (X^{(1)})^s$ is described as a subquotient of $X^s = \underline{Map}(\Bbbk^{0|1}, X)$.

This seems to be a case of a general statement about the relation between forms on spaces $X$ and $Y$ related by a finite flat map, i.e., about the relation between spaces $X^s$ and $Y^s$.

Observe that in our case the map $(X^{(1)})^s \to X^s$ factors thru $X^{(1)} \to X$ since the differential of the Frobenius map- is zero.

7.3. **SUSY on Riemannian manifolds.** The super-symmetry of a Riemannian manifold $(M, g)$ is the action of the real super Lie algebra $\mathfrak{g}_R$ on the differential forms $\Omega^*_M$.

7.3.1. *Riemann-de Rham super Lie algebra $\mathfrak{g}_R$.* This is the semi-direct product $\mathfrak{g}_R \overset{\text{def}}{=} \mathfrak{h}_R \ltimes so(1, \mathbb{R})$ of the Heisenberg Lie algebra $\mathfrak{h}_R \overset{\text{def}}{=} \Pi(V) \oplus \mathbb{R}{\cdot}K$, where the $V = \mathbb{R}^2$ with the bilinear pairing $xy$, and $so(1, \mathbb{R}) = \mathbb{R}{\cdot}h$ is the special orthogonal group for this quadratic form.

7.3.2. *Realization in differential operators and the canonical basis.* De Rham differential $d$ has an adjoint $d^*$ on a Riemannian manifold $(M, g)$. Their anti-commutator $[d, d^*] = dd^* + d^*d$ is the Laplace operator $\Delta$. Moreover, the grading on the differential forms defines the degree operator $h$. (Now $V = \mathbb{R}^2 = \mathbb{R}d \oplus \mathbb{R}d^*$ and the central element $K$ acts on forms $\Omega^*(M)$ as the Laplace operator $\Delta$.)

Observe that $d, d^*$ and even $\Delta$ are vector fields on the associated super manifold $M^s$. So, a Riemannian structure $g$ on $M$ define a representation of the Riemann-de Rham super Lie algebra $\mathfrak{g}_R$ by vector fields on the super manifold $M^s$.

7.4. **SUSY on Kaehler manifolds.** Super-symmetry of a Kaehler manifold $M$ is the action of a complex super Lie algebra $\mathfrak{g}_{ka}$ on forms $\Omega^*_M$.

7.4.1. *Kaehler-de Rham super Lie algebra* $\mathfrak{g}_{ka}$. Here $\mathfrak{g}_{ka}$ is the semi-direct product $\mathfrak{h}_{ka} \ltimes sl_2$ of the Kaehler-Heisenberg Lie algebra $\mathfrak{h}_{ka}$ with $sl_2 = sl(V) = su(1,1;\mathbb{C})$. Here, $\mathfrak{h}_{ka} = \Pi(V\oplus V^*)\oplus\mathbb{C}\cdot K$ for the 2d representation $V = L(1)$ of $sl_2$.

So, $\dim_{\mathbb{C}}(\mathfrak{h}_{ka}) = 5$ and $\dim(\mathfrak{g}_{ka}) = (4,4)$.

7.4.2. Any representation $V$ of a Lie algebra $\mathfrak{g}$ defines $\mathfrak{h}_V = \Pi(V\oplus V^*)\oplus\mathbb{C}\cdot K$ and $\mathfrak{g}_V \stackrel{\text{def}}{=} \mathfrak{h}_V \ltimes \mathfrak{g}$. Moreover, a self-dual representation $W$ defines a smaller version $\mathfrak{h}(W) = \Pi(W)\oplus\mathbb{C}\cdot K$ and $\mathfrak{g}(W) \stackrel{\text{def}}{=} \mathfrak{h}(W) \ltimes \mathfrak{g}$.

7.4.3. *Canonical basis.*

7.4.4. *Realization in differential operators.* The central element $K$ acts on forms $\Omega^*(M)$ as the Laplace operator $\Delta$. $sl_2$ acts by the standard Lefschetz action on forms.

7.5. **SUSY on hyperkaehler manifolds.** Super-symmetry of a hyperkaehler manifold $M$ is the action of a real super Lie algebra $\mathfrak{g}_{hk}$ on forms $\Omega^*_M$.

7.5.1. *Hyperkaehler-de Rham super Lie algebra* $\mathfrak{g}_{hk}$. The hyperkaehler-de Rham super Lie algebra $\mathfrak{g}_{hk}$ is the semi-direct product $\mathfrak{h}_{hk} \ltimes so(1,4)$ of the hyperkaehler-Heisenberg Lie algebra $\mathfrak{h}_{hk}$ with $so(1,4) = su(1,1;\mathbb{H})$. Here,

- $\mathfrak{h}_{hk} = \Pi(W)\oplus\mathbb{C}\cdot K$ where
- $\mathbb{W} = \mathbb{H}\oplus\mathbb{H}$ has quaternionic Hermitian metric of signature $(1,1)$,
- and $so(1,4)$ appears as the unitary group $su(1,1;\mathbb{H})$ for this metric. for the 2d representation $V = L(1)$ of $sl_2$.

Here. $\dim\ so(1,4) = \dim\ so(5) = [5^2-5]/2 = 10$, while $\dim(\mathfrak{h}_{hk}) = (1,8)$, hence $\dim(\mathfrak{g}_{hk}) = (11,8)$.

## 8. **Homological algebra in terms of super geometry**

This section is an unfinished sketch of how the formalism of complexes arises naturally in super geometry.[19]

Complexes in homological algebra are representations of a certain super group $\mathcal{S}$ – the automorphism group of the super point. Its underlying super manifold is $S^{1|1}$.

More generally, homological algebra uses representations of semi-direct products $\mathfrak{s} \rtimes T$ where $T$ is a torus (even!) and $\mathfrak{s}$ is an odd representation of $T$.

### 8.1. **Search for Homological Algebra.**

The above explains the notion of complexes in terms of the simplest super point $\mathbb{A}^{0|1}$. However

8.1.1. *What is homological algebra?* The question is

*How does one make derived functors natural and clear?*

The first examples may be the Ext functors because (i) they have an existence without any homological algebra formalism, (ii) sheaf cohomology is an Ext functor $\mathrm{Ext}(\Bbbk_X, -)$. (So is group cohomology ...)

So one may want to fit them in the $\mathcal{S}$-framework and then construct RHom and then the derived category.

*Question.* What are the injectives/projectives in complexes?

8.1.2. *De Rham cohomology of flat connections.* One nice instance is the De Rham cohomology which is completely explained geometrically: manifold $M$ gives a moduli $\mathcal{M}(M) = Map(\mathbb{A}^{0|1}, M)$ and $\mathcal{S}$ acts on $\mathcal{M}(M)$ hence on the space of functions on $\mathcal{M}(M)$.

*Questions.*

(1) What exactly makes this work: say, there are no $C^0$ super manifolds, so maps into a topological space probably do not work. (What about schemes $M$?).
(2) What about the geometric construction of differential forms?
(3) Can one use this as a model for explaining other (co)homology theories?
(4) What is De Rham homology? Currents?
(5) Is singular (co)homology the simplicial version of the classifying space of a (non-commutative!) group $\mathcal{S}$?
(6) What is the algebraic structure of $BG$ if $G$ is $A \rtimes B$ for abelian $A, B$ ?
(7) What does one know about classifying spaces super groups?
(8) If one could answer this there would be more questions in waiting: explain K-theory and non-abelian cohomology.

[19] Moreover the same happens with the chiral De Rham complex, indicating that some parts of QFT may be relatives of homological algebra.

8.1.3. *Non-abelian cohomology.* One non-abelian generalization of $\mathcal{S}$ is the automorphism groups $\mathrm{Aut}(\mathbb{A}^{0|n})$ of larger supper points. Probably, $\mathrm{Aut}(\mathbb{A}^{0|n}) = (\mathbb{A}^{0|n}) \rtimes GL_n$.

*Question.* Could this be related to appearances of $GL_n$ in K-theory?

8.2. **The center of super linear algebra.**

8.2.1. *Group scheme $\mu_{2,\mathbb{Z}}$.* Let $\boldsymbol{s}$ be the group scheme $\boldsymbol{s} \stackrel{\text{def}}{=} \mu_{2,\mathbb{Z}}$.

*Lemma.* (a) The Hopf algebra of functions is $\mathcal{O}(\boldsymbol{s}) = \mathbb{Z}\oplus\mathbb{Z}x = \mathbb{Z}\oplus\mathbb{Z}y$ with $x^2 = 1$ and $\Delta(x) = x(u)tenx$, and in terms of $y = x-1$ one has $y(y+2) = 0$ and $\Delta(y) = y\otimes 1 + 1\otimes y + y\otimes y$.

(b) The group algebra (enveloping algebra) is the $\mathbb{Z}$-dual Hopf algebra

$$U(\boldsymbol{s}) \ = \ \mathbb{Z}\oplus\mathbb{Z}\zeta \quad \text{with} \quad \zeta^2 = \zeta \quad \text{and} \quad \Delta_\zeta = \zeta\otimes 1 + 1\otimes\zeta.$$

Here, $1,\zeta$ is a basis dual to $1,y$.

(c) $U(\boldsymbol{s})$ is the algebra $\mathbb{Z}p\oplus\mathbb{Z}q$ with orthogonal idempotents $q=\zeta$ and $p = 1-q$, and the Hopf structure $\Delta q = q\otimes 1 + 1\otimes q$ and $\Delta p = p\otimes 1 - 1\otimes q == p\otimes 1 + 1\otimes p - 1\otimes 1$.

*Proof.* $\mathcal{O}(\boldsymbol{s})$ is the quotient of $\mathcal{O}(G_{m,\mathbb{Z}}) = \mathbb{Z}[x,x^{-1}]$ by the relation $x^2 = 1$. This is the same as $\mathbb{Z}[x]/\langle x^2-1$. In terms of $y = x-1$ this is $0 = x^2-1 = y^2+2y = .$

The comultiplication in $G_m$ is $\Delta(x) = x(u)tenx$ (i.e., $\Delta(x)(u,v) = x(uv) = x(u){\cdot}x(v) = (x\otimes x)(u,v)$),and in terms of $y$ $\Delta(y) = \Delta(x0 - \Delta(1) = x\otimes x - 1\otimes 1 = y\otimes 1 + 1\otimes y + y\otimes y$.

The $\mathbb{Z}$-dual Hopf algebra $U(\boldsymbol{s})$ has a basis $1,\zeta$ dual to $1,y$. $\zeta^2=\zeta$ comes from $\langle\zeta^2,y\rangle = \langle\zeta\otimes\zeta,\Delta(y)\rangle = \langle\zeta\otimes\zeta, y\otimes 1 + 1\otimes y + y\otimes y\rangle = 1 = \langle\zeta,y\rangle$ and $\langle\zeta^2,1\rangle = 0$. Also, $\Delta_\zeta = \zeta\otimes 1 + 1\otimes\zeta$ since

$$\langle\Delta_\zeta, y^i\otimes y^j\rangle = \langle\zeta, y^{i+j}\rangle = \delta_{i+j,1}.$$

Since $q=\zeta$ is an idempotent, so is $p = 1-\zeta$. One has

$$\Delta(p) = \Delta(1) - \Delta(z) = 1\otimes 1 - (q\otimes 1 + 1\otimes q) = p\otimes 1 - 1\otimes q.$$

*Corollary.* Representations of $\boldsymbol{s}$ over a ring $\Bbbk$ form a tensor category of $\mathbb{Z}_2$-graded $\Bbbk$-modules.

The natural braided structure on $Rep_\Bbbk(\boldsymbol{s})$ is the super braiding.

8.2.2. *The center of the category of super $\Bbbk$-modules.*

8.2.3. *Lemma.* Let $\Bbbk$ be an even ring. The center of the category of super $\Bbbk$-modules is the enveloping algebra $Z(\Bbbk)(\boldsymbol{s})$ of the algebraic group $\boldsymbol{s} = \mu_{2,\mathbb{Z}}$ over the center $Z(\Bbbk)$ of $\Bbbk$.

Say, The center of the category of super abelian groups is the enveloping algebra of the (algebraic) group $\boldsymbol{s} \stackrel{\text{def}}{=} \mu_{2,\mathbb{Z}}$.

## 8.3. Super group $\mathcal{S} = \mathrm{Aut}(\mathbb{A}^{0|1})$ and complexes.

8.3.1. *Odd additive groups* $G_a^{0|q}$. For $q \geq 0$ the $(0,q)$-dimensional super-point $\mathbb{A}^{0|q}$ has a canonical structure of a supergroup which we denote $G = G_a^{0|q}$. Its algebra of functions is

$$\mathcal{O}(G) = S^*(\Psi^*)$$

for an odd q-dimensional vector space $\Psi^* = \oplus_1^q \Bbbk \cdot \psi^i$. So, if we forget parity

$$\mathcal{F}[\mathcal{O}(G)] = \overset{\bullet}{\wedge} \mathcal{F}(\Psi^*).$$

$G$ is a commutative group with

$$\Delta(\psi^i) = 1 \otimes \psi^i + \psi^i \otimes 1.$$

We can think of $G$ as the parity change of the vector space $(G_a)^q$.

The Lie algebra $\mathfrak{g} = \mathfrak{g}_a^{0|q}$ of $G$ is commutative, so it is the same as the underlying vector space $(\Psi^*)^* = \Psi$. It has a basis $\zeta_i$ dual to $\psi^i$'s. The enveloping algebra is therefore the symmetric algebra

$$U\mathfrak{g} = S^\bullet \Psi \quad \text{and} \quad \mathcal{F}(U\mathfrak{g}) = \overset{\bullet}{\wedge} \mathcal{F}(\Psi),$$

i.e., we get exterior algebra when we forget parity. One has duality of Hopf algebras $\mathcal{O}(G) = [U\mathfrak{g}]^*$.

8.3.2. *Group* $\mathcal{S}$. Let $\boldsymbol{g}$ be the super-point $G_a^{0|1} = \mathrm{Spec}(\Bbbk \oplus \Bbbk \psi)$ for $\psi$ odd and $\psi^2 = 0$. Let $\mathcal{S}$ be the semi-direct product $\boldsymbol{g} \ltimes G_m = G_a^{0|1} \ltimes G_m$, where $G_m$ acts on $\boldsymbol{g}$ by $s \bullet \psi \overset{\text{def}}{=} s^{-1} \cdot \psi$.

In terms of Lie algebras, $\mathfrak{g}^{1|0} \overset{\text{def}}{=} Lie(\boldsymbol{g}) = \Bbbk \cdot \zeta$ and $\mathfrak{g}_m = Lie(G_m) = \Bbbk \cdot i$, hence the lie algebra $\mathfrak{S} = Lie(\mathcal{S})$ is $\mathfrak{g}_a^{0|1} \ltimes \mathfrak{g}_m = \Bbbk \cdot \zeta \oplus \Bbbk \cdot i$. Here, $s \bullet \zeta = s \cdot \zeta$, $s \in G_m$, or equivalently, $[i, \zeta = \zeta]$.

8.3.3. *Lemma.* $\mathcal{S} = \underline{\mathrm{Aut}}(\mathbb{A}^{0|1})$. In particular, $\mathcal{S}$ is a *based group* in the sense that it comes with a canonical map $\mu_{2,\mathbb{Z}} \to \mathcal{S}$;

*Proof.* $G_a^{0|1}$ acts on $\mathbb{A}^{0|1}$ by translations and $G_m$ by homothety.

8.3.4. *Corollary.* (a) A complex of $\Bbbk$-modules is the same as an algebraic representation of the based super group $\mathcal{S}$ (i.e, a representation such that the action of $\mu_{2,\mathbb{Z}}$ given by $\mu_{P2,\mathbb{Z}} \to G_m$ is the standard action on $\mathbb{Z}_2$-graded $\Bbbk$-modules ($\overset{1}{=}$ representations of $\mu_{2,\mathbb{Z}}$).

(b) In particular, a dg-scheme is the same as a based action of of the super group $\mathcal{S}$ on a super scheme.

8.3.5. *Question.* What is $\underline{\mathrm{Aut}}(\mathcal{S})$ for the space $\mathcal{S}$ or a group $\mathcal{S}$? (Seemingly, $\underline{\mathrm{Aut}}_{groups}(\mathcal{S})$ is just $\underline{Inn}(\mathcal{S})$ which is $G_m$ acting on the odd part $(\boldsymbol{g}, +)$.)

8.3.6. *Notions of complexes corresponding to super groups* $\mathcal{T} \stackrel{def}{=} \mathfrak{s} \ltimes T$. More generally, one may be interested in representations of groups $\mathcal{T} \stackrel{\text{def}}{=} \mathfrak{s} \ltimes T$ where $T$ is a torus (even!) and $\mathfrak{s} \cong G_a^{0|1}$) is an "odd representation" of $T$.

For $\mathcal{T} = \mathcal{S}^2$ we get the notion of bicomplexes and the passage to the total complex is the restriction for $\Delta_{\mathcal{S}} \hookrightarrow \mathcal{S}^2$.

If $T = G_m$ acts on $\mathfrak{s} = \Bbbk\psi_+ \oplus \Bbbk\psi_-$ by $s \bullet \psi_\pm \stackrel{\text{def}}{=} s^{\pm 1} \cdot \psi_\pm$, we get the notion of mixed complexes (in the sense of Weibel).

For a subgroup $\mathcal{T} = \boldsymbol{g} \ltimes \pm 1 \subseteq \mathcal{S}$, (or just $\mathcal{T} = \boldsymbol{g} \subseteq \mathcal{S}$), we get the notion of a differential super module. (The notion of a differential module is not of a super nature!)

## 8.4. **Cohomology.**

8.4.1. *Subgroups of* $\mathcal{S}$. Some of the basic operations are now restrictions to a subgroup:

- graded vector spaces to $\mathbb{Z}_2$-graded ones: $G_m \supseteq \{\pm 1\}$.
- complex to a differential super module: $\mathcal{S} \supseteq \boldsymbol{g} = G_a^{0|1}$.
- complex to a graded vector space: $\mathcal{S} \supseteq G_m$.
- bicomplex gives the total complex: $\mathcal{S} \stackrel{\Delta}{\hookrightarrow} \mathcal{S}^2$.

8.4.2. *Cohomology.* One can pass to representations of the quotient $G_m$ of $\mathcal{S}$ by using invariants $C^* \mapsto Z^*(C^*)$ or coinvariants $C^* \mapsto C^*/B^*(C^*)$, with respect to the subgroup $\boldsymbol{g}$. Finally, one can combine the two to get

$$H^* \stackrel{\text{def}}{=} Im[Z^* \to id/B^*].$$

So one has the subcategory $Rep(G_m) \subseteq Rep(\mathcal{S})$ and two adjoints $Z^*$ and $B^*$ for the inclusion. Then $H^* \stackrel{\text{def}}{=} Im[Z^* \to id/B^*]$ is one modulo (the image of) the other. [Notice how this is different from the $!*$ construction which uses the image of $! \to *$.]

8.4.3. *Long Exact Sequences.* A "surprising fact": –

*A short exact sequence $0 \to A' \to A \to A'' \to 0$ in the abelian category $Rep(\mathcal{S}/\boldsymbol{s})$ (=complexes), gives a long exact sequence of cohomologies in the abelian category $Rep(G_m)$:*

$$\cdots \to H(A' \otimes \Pi^{-1}) \to H(A \otimes \Pi^{-1}) \to H(A'' \otimes \Pi^{-1}) H(A') \to H(A) \to H(A'') H(A' \otimes \Pi) \to H(A \otimes \Pi) \to H($$

*Remark.*

(1) Claim (1) should be a departing point for thinking of this setting. Here, $G_m$ is a quotient of $\mathcal{S}$ by $\boldsymbol{s}$ that one can think of as $\Pi$ (actually, $\Pi = Lie(\boldsymbol{s})$).

(2) (2) is really infinitely many copies of the standard long exact sequence of cohomologies.

8.5. **dg-schemes.** Recall that a dg-scheme is the same as a super scheme with a based action of of the super group $\mathcal{S}$.

We are interested in several forgetful functors from dg-schemes to schemes and super schemes and the corresponding induction functors

For even schemes the "odd tangent cone" $\underline{Map}(\mathbb{A}^{0|1}, N)$ is a right adjoint of the forgetful functor from dg-schemes to schemes $(|M|, \mathcal{O}_M^*, d) \mapsto (|M|, \mathcal{O}_M^0)$.

I have difficulties with what seems a simpler situation: the forgetful functor from dg-schemes to super schemes. (There are three more forgetful functors which may need adjoints $Z^*, id/B^*, H^*$ ?)

8.6. **De Rham moduli** $M_{dR} \stackrel{\mathbf{def}}{=} Map(\mathbb{A}^{0|1}, N)$**.** This is one standard construction of dg-schemes from schemes. Actually, this is induction $Ind_1^{\mathcal{S}}$ :

8.6.1. *Lemma.* The forgetful functor from dg-schemes to schemes $(|M|, \mathcal{O}_M^*, d) \mapsto (|M|, \mathcal{O}_M^0)$ has a right adjoint $N \mapsto \underline{Map}(\mathbb{A}^{0|1}, N)$.

*Proof.*

$$\mathrm{Hom}_{dg-schemes}[(|M|, \mathcal{O}_M^*, d),\ \underline{Map_{super\ schemes}}(\mathbb{A}^{0|1}, N)]$$

$$\cong \mathrm{Hom}_{dg-schemes}[(|M|, \mathcal{O}_M^*, d),\ \underline{Map_{dg-schemes}}(\mathbb{A}^{0|1}, N)] \cong \mathrm{Hom}_{dg-schemes}[(|M|, \mathcal{O}_M^*, d) \times \mathbb{A}^{0|1},\ N]$$

$$= \mathrm{Hom}_{super-schemes}[(|M|, \mathcal{O}_M^*, d) \times \mathbb{A}^{0|1},\ N]^{\mathcal{S}} = \mathrm{Hom}_{super-schemes}([(|M|, \mathcal{O}_M^*, d) \times \mathbb{A}^{0|1}]/\boldsymbol{g},\ N)^{G_m}$$

$$= \mathrm{Hom}_{super-schemes}[(|M|, \mathcal{O}_M^*),\ N]^{G_m} = \mathrm{Hom}_{schemes}[(|M|, \mathcal{O}_M^0),\ N].$$

8.6.2. *De Rham cohomology.* One can view it in two steps as

(1) $\Omega^\bullet(M) \mapsto \mathcal{Z}^\bullet(M) \mapsto H_{dR}^\bullet(M)$, or
(2) $\Omega^\bullet(M) \mapsto \Omega^\bullet(M)/\mathcal{B}^\bullet(M) \mapsto H_{dR}^\bullet(M)$.

Geometrically this means

(1) passing from the dg-scheme $M_{dR}$ to a quotient dg-scheme $\overline{M_{dR}}$) and then to its dg-subscheme $\mathcal{H}(M)$. Alternatively, one can
(2) first pass to a subscheme

$$\begin{array}{ccc} \Omega_M^\bullet & \xleftarrow{\subseteq} & \mathcal{Z}^\bullet(M) \\ \downarrow & & \downarrow \\ \Omega_M^\bullet/\mathcal{B}_M^\bullet & \xleftarrow{\subseteq} & H^\bullet(M) \end{array} \quad ,i.e., \quad \begin{array}{ccc} M_{dR} & \xrightarrow{\twoheadrightarrow} & \overline{M_{dR}} \\ \subseteq\uparrow & & \subseteq\uparrow \\ \dot{M}_{dR} & \xrightarrow{\twoheadleftarrow} & \mathcal{H}(M) \end{array} .$$

8.6.3. *Examples.* For $M=\mathbb{R}^n$, $M_{dR}\cong\mathbb{R}^{n|n}$.

(1) $\underline{M=\mathbb{R}}$. Here, $\mathcal{B}^*=\Omega^1[-1]$, hence $\Omega^*/\mathcal{B}^*=\Omega^0=C^\infty(\mathbb{R})$, hence $\overset{\bullet}{\mathbb{R}}=\mathbb{R}$ and then $\mathcal{H}(M)=pt$ is its quotient.

(2) $\underline{M=S^1}$. Here, $M_{dR}\cong S^{1|1}\overset{\text{def}}{=}S^1\times\mathbb{A}^{0|1}$. One has $\mathcal{B}^*=\{\omega\in\Omega^1(S^1),\ int\ \omega\ =\ 0\}$, hence $\Omega^1/\mathcal{B}^1\xrightarrow[\cong]{\int}\mathbb{R}$. So, $\Omega^*/\mathcal{B}^*=C^\infty(S^1)\oplus\mathbb{R}$, and $\overset{\bullet}{S}{}^1$ is a dg-scheme with a base $S^1$ and "very little in the odd direction". It is interesting to see how this "very little" is related to the 1-hole in the circle. Then, $\mathcal{H}(M)\cong\mathbb{R}^{1|1}$ as a super scheme,m is a quotient of $\overset{\bullet}{M}_{dR}$ in the "even direction"

8.7. **Chiral De Rham complex.** On the space of maps $\underline{Map}(\mathbb{A}^{0|1},N)$ we have an action of $\mathrm{Aut}(\mathbb{A}^{0|1})=\mathcal{S}$. On a larger space of maps $\underline{Map}(\mathcal{S},N)$ we have an action of $\mathrm{Aut}_{super\ schemes}(\mathcal{S})$. Observe that

$$\underline{Map}(\mathcal{S},N)=\ \underline{Map}(G_m^{1|1},N)=\ \mathrm{Spec}[\Omega^*_{\underline{Map}(G_m,N)}]$$

is the "chiral de Rham complex". Its symmetries come from symmetries of the space $G_m^{1|1}$. If we interpret it as the group $\mathcal{S}$ we get an action of $\mathcal{S}\times\mathcal{S}$ by left and right translations.

Observe that $\mathcal{S}$ is a very simple version of a super space in terms of physics, and the above is the usual super-space game – explaining the symmetries of a super space from its (non-commutative) group structure!

If we think that homological algebra deals with $\mathrm{Aut}(\mathbb{A}^{0|1})$, should we think of representations of $\mathrm{Aut}(S^{1|1})$ as a generalization of homological algebra?

8.8. **Homotopy.** Homotopies are morphisms of complexes which are produced from maps of the underlying (graded) objects by some kind of "integration over $\boldsymbol{g}$". The meaning of the integrations is just applying $d$ (since $d^2=0$ implies that $d(\phi)$ is $\boldsymbol{g}$-invariant).

8.8.1. *Homotopy category $Ho[Diff\mathfrak{m}(\Bbbk)]$ of differential $\Bbbk$-modules.* For two differential modules $M,N$, $\mathrm{Hom}_\Bbbk(M,N)$ is differential bimodule hence a differential module. Here $Z[\mathrm{Hom}_\Bbbk(M,N)]$ consists of maps of differential modules, and its subgroup $B[\mathrm{Hom}_\Bbbk(M,N)]\ =\ d\ \mathrm{Hom}_\Bbbk(M,N)$ consists of "special" homomorphisms called homotopies. So, $H[\mathrm{Hom}_\Bbbk(M,N)]$ is the space of morphisms in the homotopical category of differential modules $Ho[Diff\mathfrak{m}(\Bbbk)]$.

8.8.2. *Homotopy category $Ho[\mathcal{C}^*(\Bbbk)]$ of complexes of $\Bbbk$-modules.* For two complexes $M^*,N^*$ we get a bicomplex $\mathrm{Hom}(M^*,N^*)$ and $H^0[\mathrm{Hom}_\Bbbk(M,N)]$ is the space of morphisms in the homotopical category of complexes $Ho[\mathcal{C}^*(\Bbbk)]$.

8.9. **Spectral sequences.** A filtered complex $(C^*, F)$ is a $G_m$-equivariant vector bundle of complexes $\mathcal{C}^*$ over $\mathbb{A}^1$. Then $Gr_F(C^*)$ is the fiber at zero, $\mathcal{C}^*|_0$.

The spectral sequence gives a way of calculating the central fiber of the cohomology sheaf from the cohomology of the central fiber.

$H^*(\mathcal{C}^*)_0 = \; Gr_F H^*(\mathcal{C}^*)$ from $H^*(\mathcal{C}^*_0) = \; H^*(Gr_F \mathcal{C}^*)$.

Is there a SUSY in this spectral sequence?

8.10. **Cones.**

(1) The first surprising fact about the abelian category of complexes (representations of $\mathcal{S}$), is the map (isomorphism?)
$$\mathrm{Hom}(M, N) \to \mathrm{Ext}^1(M \otimes \Pi, N)$$
where $\Pi$ is the representation $Lie(\mathfrak{s})$ of $\mathcal{S}$.

(2) The second is that a short exact sequence in the abelian category $Rep(\mathcal{S})$ gives a long exact sequence in the abelian category $Rep(G_m)$.

8.11. **Derived functors.** ?

8.12. **Complexes and simplicial abelian groups.** The equality of the two may mean a simplicial set description of $BG$ for $G = \mathfrak{s}$? (or $\mathcal{S}$ ?).