Laplace Transform

Here are the crucial facts to know about the Laplace Transform:

\[ L[af(t) + bg(t)](s) = aL[f(t)](s) + bL[g(t)](s) \]  
\( \text{(Linearity)} \)

\[ L[e^{at}f(t)](s) = L[f(t)](s - a) \]  
\( \text{(Exponential Shift)} \)

\[ L[f'(t)](s) = sL[f(t)] - f(0) \]  
\( \text{(Derivative Rule)} \)

\[ L[t^n](s) = \frac{n!}{s^{n+1}} \text{ for } \Re(s) > 0 \]  
\( \text{(Polynomial Evaluation)} \)

\[ L[u(t-a)f(t)](s) = e^{-as}L[f(t+a)](s) \]  
\( \text{(Heaviside Function)} \)

\[ L[\delta(t-a)](s) = e^{-as} \]  
\( \text{(Dirac Delta)} \)

Recall that the Derivative Rule required \( \lim_{t \to \infty} f(t)e^{-st} = 0 \).

Technically, by making use of the identities

\[ \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \]

this table is sufficient to answer any question you’ll be asked involving the Laplace Transform. That being said, it may be helpful to become familiar with the following shortcuts as well:

\[ L[\sin(at)](s) = \frac{a}{s^2 + a^2} \quad \text{if } a > 0 \]  
\( \text{(Sine Evaluation)} \)

\[ L[\cos(at)](s) = \frac{s}{s^2 + a^2} \quad \text{if } a > 0 \]  
\( \text{(Cosine Evaluation)} \)

\[ L[f(at)](s) = \frac{1}{a}L[f(t)]\left(\frac{s}{a}\right) \quad \text{if } a > 0 \]  
\( \text{(Scaling)} \)

Use these facts to calculate the Laplace Transform of the following expressions.

1. \( t^2 + t^3 \).
2. \( \cos 2t + 5t^8 \).
3. \( (1 + 2t + t^2)(e^{-(3+i)t} + e^{-(3-i)t}) \).
4. \( t^3 \sin(2t) \) (Hint: Use \( \sin t = \frac{e^{it} - e^{-it}}{2i} \)).
5. \( f(t) = \begin{cases} 
\sin 4t & t < \pi, \\
0 & \pi \leq t. 
\end{cases} \)

Inverse Laplace Transform

When solving ODEs, we have a need to go backwards, i.e. given a function \( F(s) \) we ask “is there a function \( f(t) \) such that \( L[f(t)](s) = F(s) \)?
The answer is known as the inverse Laplace transform, \( \mathcal{L}^{-1} \), and properties of this are given by reading the facts from above in reverse. The most useful results are the following:

\[
\mathcal{L}^{-1}[aF(s) + bG(s)](t) = a\mathcal{L}^{-1}[F(s)](t) + b\mathcal{L}^{-1}[G(s)](t) \quad \text{(Linearity)}
\]

\[
\mathcal{L}^{-1}[F(s)](t) = e^{at}\mathcal{L}^{-1}[F(s+a)](t) \quad \text{(Exponential Shift)}
\]

\[
\mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^n}{n!} \quad \text{(Polynomial Evaluation)}
\]

\[
\mathcal{L}^{-1}[e^{-as}F(s)](t) = h(t-a)\mathcal{L}^{-1}[F(s)](t-a) \quad \text{(Heaviside Function)}
\]

\[
\mathcal{L}^{-1}[1](t) = \delta(t) \quad \text{(Dirac Delta)}
\]

Again, technically this is all you need, but it is sometimes handy to have the following available as well:

\[
\mathcal{L}^{-1}\left[\frac{a}{s^2 + a^2}\right](s) = \sin(at) \quad \text{if } a > 0 \quad \text{(Sine)}
\]

\[
\mathcal{L}^{-1}\left[\frac{s}{s^2 + a^2}\right](s) = \cos(at) \quad \text{if } a > 0 \quad \text{(Cosine)}
\]

\[
\mathcal{L}^{-1}[F(as)](t) = \frac{1}{a}\mathcal{L}^{-1}[F(s)]\left(\frac{t}{a}\right) \quad \text{if } a > 0 \quad \text{(Scaling)}
\]

In the following questions find the inverse Laplace transform of the given expression. (Remember, you may need to use partial fractions first.)

6. \( \frac{5}{s} + \frac{2}{s^3} + \frac{12}{s^4} \).

7. \( \frac{2}{s^2 + 4} \).

8. \( \frac{3}{s-1} + \frac{9}{s^2-2s+1} \).

9. \( F(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2 + 1)} \).

**Initial Value Problems**

Find the particular solution corresponding to the given initial conditions by using the Laplace Transform.

10. \( y'' - 5y' - 6y = e^{3t}, \quad y(0) = 2, \quad y'(0) = 1 \).

11. \( y'' + y' + y = t^2 + t, \quad y(0) = y'(0) = 1 \).

12. \( y'' + 4y' + 4y = f(t), \quad y(0) = y'(0) = 0, \quad \text{where} \quad f(t) = \begin{cases} \sin 4t & \text{if } t < \pi, \\ 0 & \text{if } \pi \leq t. \end{cases} \)

13. \( y'' + 4y' + 5y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 3 \).