Operator Notation

In class, we introduced the notation $D$, for derivative, and $I$ for the identity, that is

\[ D[f(x)] = f'(x) \quad \text{and} \quad I[f(x)] = f(x). \]

We also discussed that $D^2 = D \circ D$, that is $D^2[f(x)] = D[D[f(x)] = D[f'(x)] = f''(x)$. In the following questions, convert the given linear ODE to operator notation.

1. $y'' + 2y' + y = 0$.
2. $y''' + 5y'' + 2y' + y = 0$.
3. $y'' + 2xy' + x^2y = 0$.

Recall that, for constants $a$ and $b$,

\[ (D + a)(D + b)f(x) = (D + a)(f'(x) + bf(x)) \]
\[ = f''(x) + bf'(x) + af'(x) + abf(x). \]

In the following questions, calculate the given expression by taking the appropriate derivatives and adding them together as indicated.

4. $(D^2 + 2D + I)x^3$.
5. $(D + 2)(D + 3)\sin x$.
6. $(D - 3)(D + 2)(D - 1)e^{3x}$.

Solving Linear Homogeneous Constant Coefficient Second Order ODEs

Find the general solution to the following second order ODEs. When appropriate (i.e. when there are complex roots to the characteristic equation) also express the solution using trigonometric functions.

Solving Linear Constant Coefficient Second Order ODEs

Find the solutions for the following ODEs. Please split up your work into the following parts:

- Find the solution to the homogeneous equation.
- Find a particular solution of the nonhomogeneous equation.
- Combine and solve for the coefficients leading to the given initial condition.

7. $4y'' - 24y' + 45y = 0, \quad y(0) = 1, \quad y'(0) = 0$.
8. \( y'' - y' - 12y = 0, \quad y(1) = 1, \quad y(0) = 0. \)
   (Note: Read the initial conditions carefully here.)
9. \( y'' - 2y' + 5y = e^{5t}, \quad y(0) = 0, \quad y'(0) = \frac{1}{2}. \)
10. \( y'' - y' - 12y = t, \quad y(0) = 4, \quad y'(0) = 1. \)
11. \( y'' - 8y' + 16y = e^{4t}, \quad y(0) = 4, y'(0) = 0. \)
12. \( y'' + 9y = \sin(3t), \quad y(0) = 0, \quad y'(0) = 0. \)

Applications
13. A 12kg body attached to a spring stretches it 8cm. After it comes to rest, it is stretched an additional 5cm and released. Find the equation of motion of the body, the period, frequency, and amplitude of the motion. What is the position \( y(t) \) of the particle when it reaches its maximum velocity, and what is this velocity?
14. A spring is stretched 5cm by a 9kg object. The object is then removed, and a 6kg object is attached. After the system comes to rest, the spring is stretched an additional 10cm, and the object is then released by being thrown downward with a velocity of 10cm/sec.
   (a) Find the equation of motion.
   (b) Calculate its amplitude,
   (c) phase shift,
   (d) and natural frequency.
   (e) Sketch a rough graph of the behavior, and label the previous quantities on your graph.
15. A 8kg mass is attached to a spring-mass-dashpot system with a damping coefficient of \( r = 3 \), stretching the spring 5cm when it reaches equilibrium.
   (a) Is the system over-, under-, or critically damped?\(^1\)
   (b) Is the system stable? (i.e. does \( \lim_{t \to \infty} y(t) = 0 \)?)\(^2\)
   (c) Find the equation of motion if the mass is pulled down 2cm and released, and sketch a rough graph of the behavior.
   (d) Find the equation of motion if the mass is lifted up 2cm and released, and sketch a rough graph of the behavior.

\(^1\) You should not need to actually solve the system to determine this, although you can find out if you solve for the general solution.
\(^2\) You should be able to answer this from the general solution.